北京大学 PEKING UNIVERSITY

傅里叶变换与球谐函数
Fourier Transform & Spherical Harmonics

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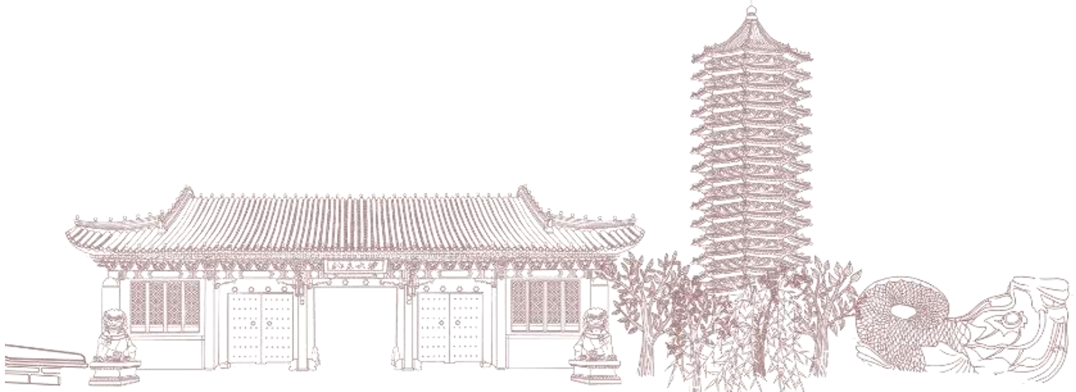
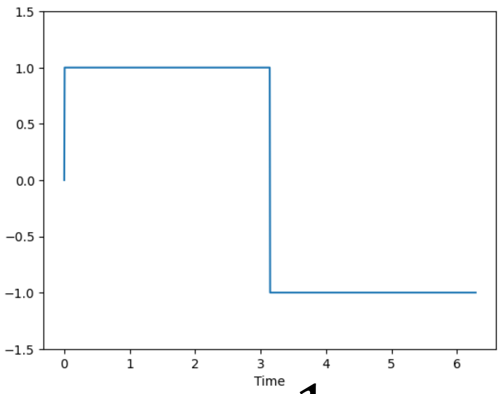
2024.4.22



傅里叶变换
Fourier Transform

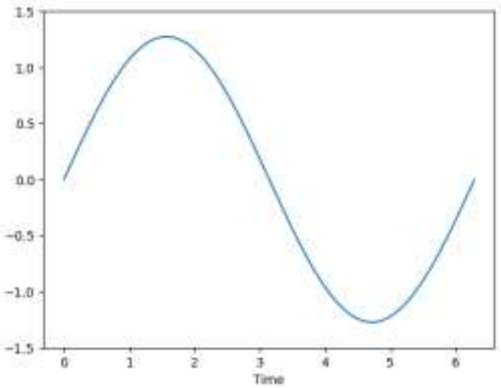
傅里叶展开 (Fourier Expansion)

频率增高，振幅变小



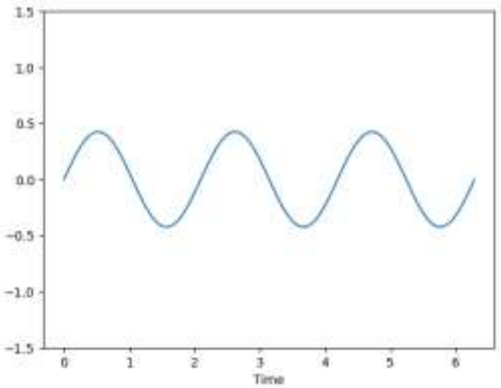
$$f = \frac{1}{T}$$

=



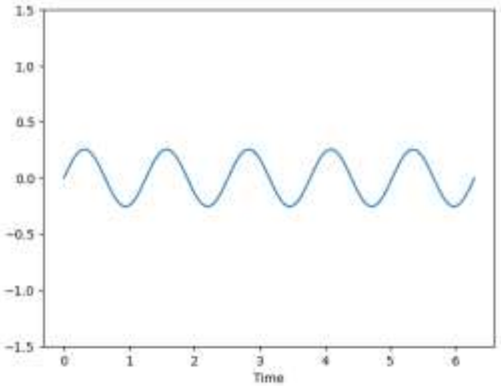
$$f = \frac{1}{T}$$

+



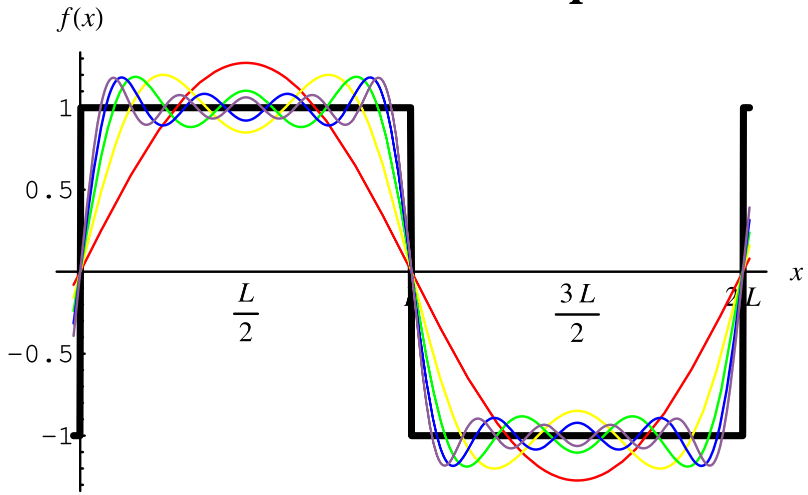
$$f = \frac{3}{T}$$

+

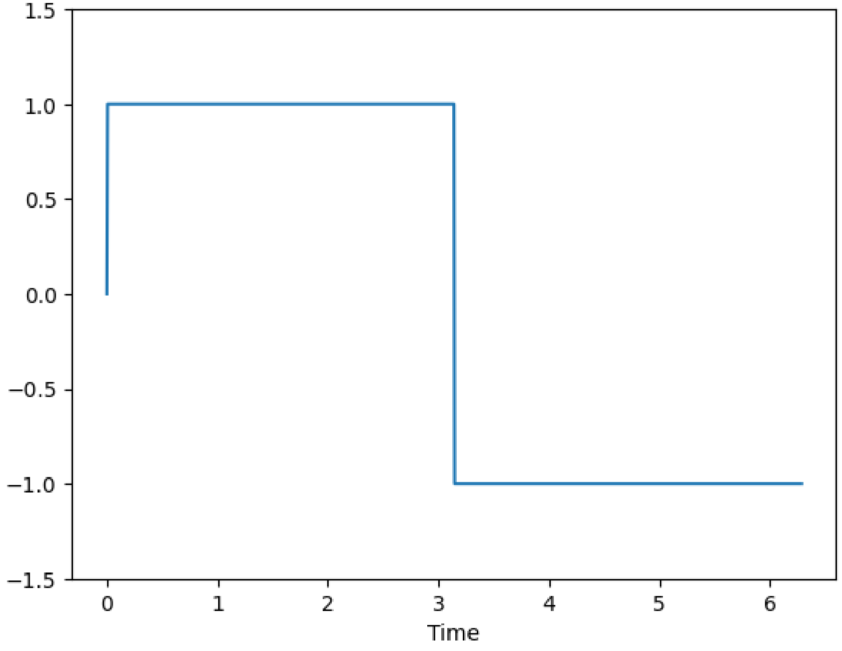


$$f = \frac{5}{T}$$

$+\cdots$

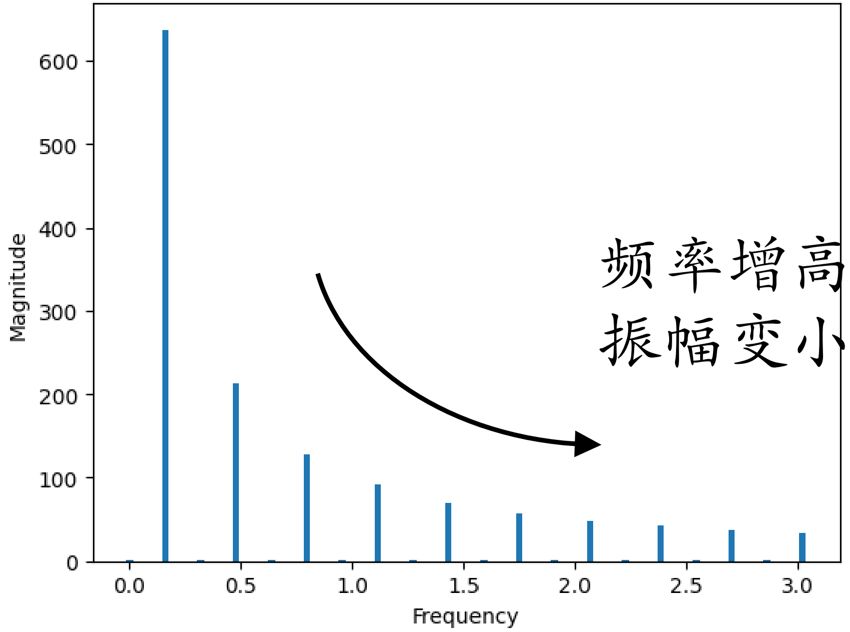


傅里叶展开 (Fourier Expansion)



原函数

傅里叶展开



频谱：振幅作为频率的函数

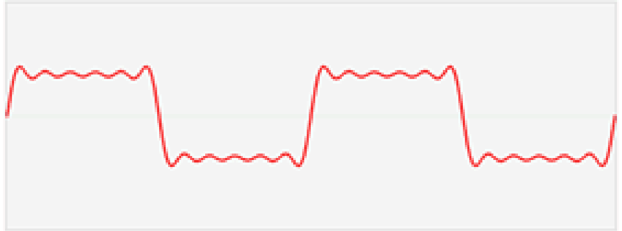
frequency spectrum

傅里叶展开 (Fourier Expansion)

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + b_n \sin(n\omega t)$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)\,dt\,, \qquad b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)\,dt$$

周期为T的函数$f(t)$，可以展开为$\cos\omega t\,,\sin\omega t\,,\cos 2\omega t\,,\sin 2\omega t\,...$这样一系列三角函数的线性组合，其中$\omega = \frac{2\pi}{T}$



傅里叶展开的指数形式

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + b_n \sin(n\omega t)$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)\,dt\,, \qquad b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)\,dt$$

$$e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow$$

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}\,, \qquad c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-in\omega t}\,dt}$$

复数，实部对应余弦，虚部对应正弦

非周期函数

$$f(t)=\sum_{n=-\infty}^{\infty}c_n e^{in\omega t},c_n=\frac{1}{T}\int_{-T/2}^{T/2}f(t)e^{-in\omega t}dt$$

- 一般的函数$f(t)$可以视为周期为∞的周期函数
- 此时$\omega=\frac{2\pi}{T}$为无穷小，求和可以化为积分

$$\lim_{\Delta x\to 0}\sum_n f(n\Delta x)=\int f(x)dx$$

傅里叶变换 (Fourier Transform)

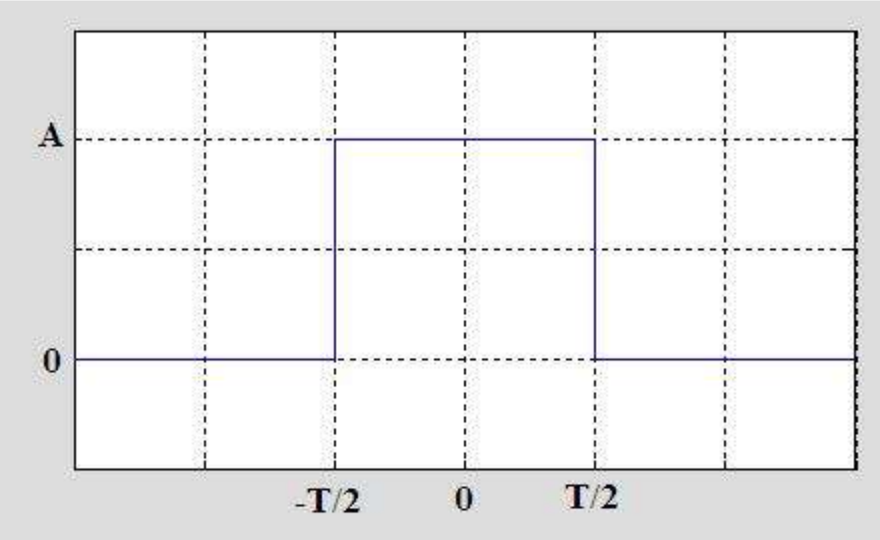
$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}, \qquad c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-in\omega t} dt$$

$$\Downarrow$$

$$\boxed{f(t) = \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega, \qquad F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt}$$

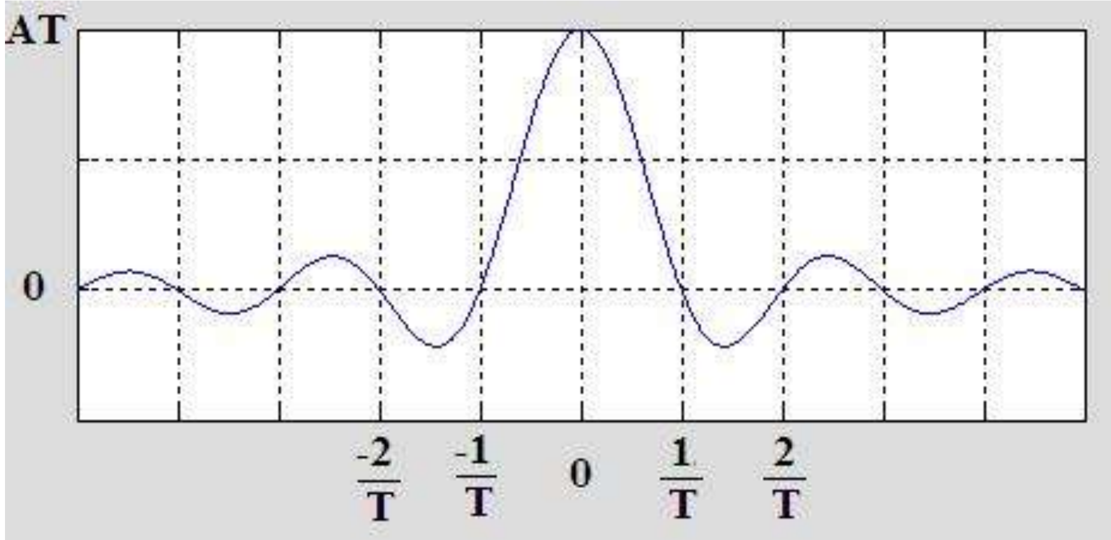
傅里叶变换 (Fourier Transform)

$$f(t) = \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega, \qquad F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$



$$\mathcal{F}(f) \rightarrow$$

$$\leftarrow \mathcal{F}^{-1}(F)$$



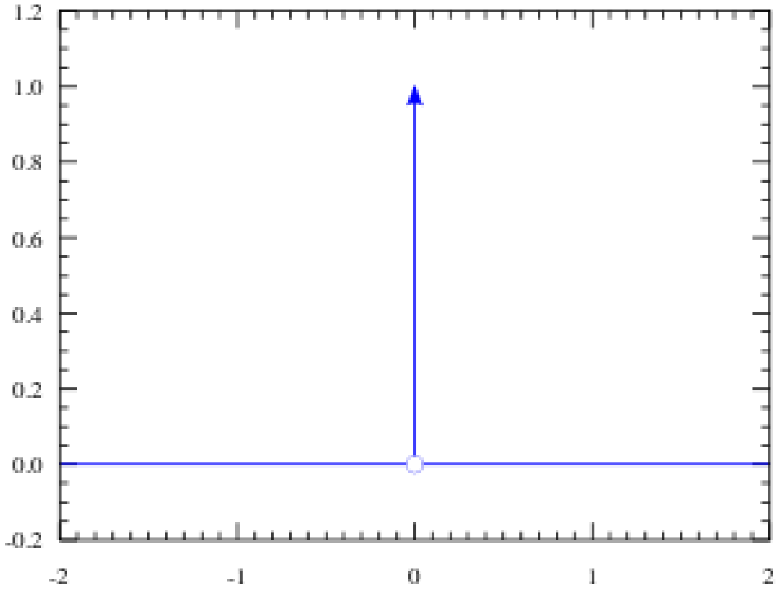
$f(t)$ $F(\omega)$

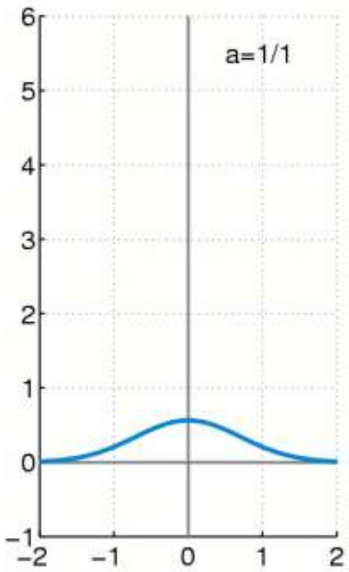
理解傅里叶变换

$$f(t) = \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega \qquad \boldsymbol{x} = \sum_i a_i\, \boldsymbol{e}_i$$

- 傅里叶变换为我们提供了另外一个视角理解函数（信号），称为频域 (Frequency Domain)
- 用线性代数来理解，我们其实是使用了另外一组基$\{\boldsymbol{e}_i\}$来表达向量$\boldsymbol{x}$，只不过现在“向量”都是函数
- $\{e^{i\omega t} | \omega \in \mathbb{R}\}$构成了一组基（有无穷多个向量），$F(\omega)$就是对应每个基的组合系数
- $f(t)$原来的基是什么？

Delta函数





$$\delta(t) = \begin{cases} \infty, t = 0 \\ 0, t \neq 0 \end{cases}, \qquad \int_{-\infty}^{\infty} \delta(t) dt = 1$$

$\delta(t-a)$是在a处的一个“尖”，满足积分结果为1

Delta函数积分恒等式

$$\int_{-\infty}^{\infty} f(a)\delta(t-a)da$$

$a \neq t$时$f(a)\delta(t-a)=0$，因此只需要考虑t的邻域$(t-\varepsilon, t+\varepsilon)$内的积分，此时

$$\int_{-\infty}^{\infty} f(a)\delta(t-a)da = \lim_{\varepsilon\to 0}\int_{t-\varepsilon}^{t+\varepsilon} f(a)\delta(t-a)da$$

$$= f(t)\lim_{\varepsilon\to 0}\int_{t-\varepsilon}^{t+\varepsilon} \delta(t-a)da = f(t)$$

$$\boxed{\int_{-\infty}^{\infty} f(a)\delta(t-a)da = f(t)}$$

Delta函数作为基函数的线性展开

$$f(t) = \int_{-\infty}^{\infty} f(a)\delta(t-a)da \qquad \boldsymbol{x} = \sum_i a_i\, \boldsymbol{e}_i$$

- 在线性代数的视角下，$\{\delta(t-a) | a \in \mathbb{R}\}$是一组基，每一个基向量$\delta(t-a)$是一个作为$t$的函数，$f(a)$是对应基向量$\delta(t-a)$的系数
- 这里注意的是，系数$f(a)$其实就是函数$f(t)$本身
- 类比于三维空间，一个向量$\boldsymbol{v}$可以表示为三维坐标(v_x, v_y, v_z)，默认情况下我们可以直接用(v_x, v_y, v_z)指代向量$\boldsymbol{v}$本身，而没有强调这是在三维笛卡尔坐标系下的分解
- 对应到函数的线性空间，当我们把函数写为$f(t)$的时候，其实也是默认了在以$\{\delta(t-a) | a \in \mathbb{R}\}$为基底进行分量展开，直接给出展开的系数

时域与频域的关系

时域

频域

$$f(t) = \int_{-\infty}^{\infty} f(a)\delta(t-a)da$$

$$f(t) = \int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$$

Delta函数：$\delta(t-a)$

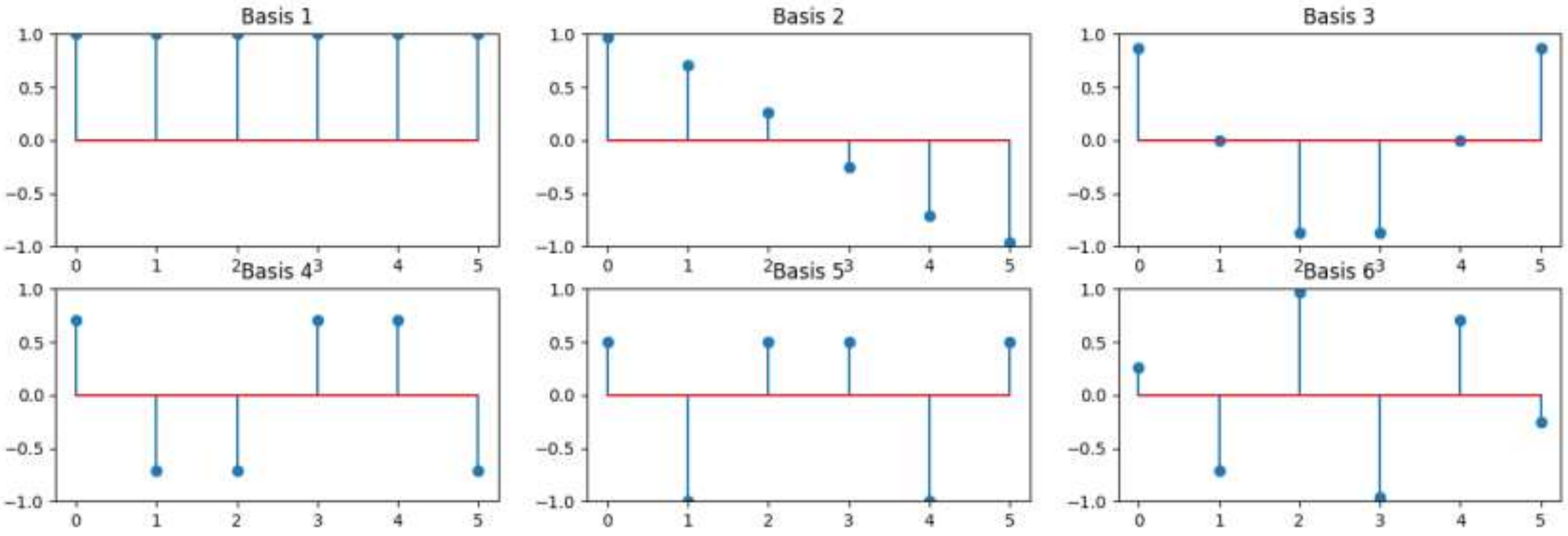
三角函数：$e^{i\omega t}$

傅里叶变换

离散傅里叶变换

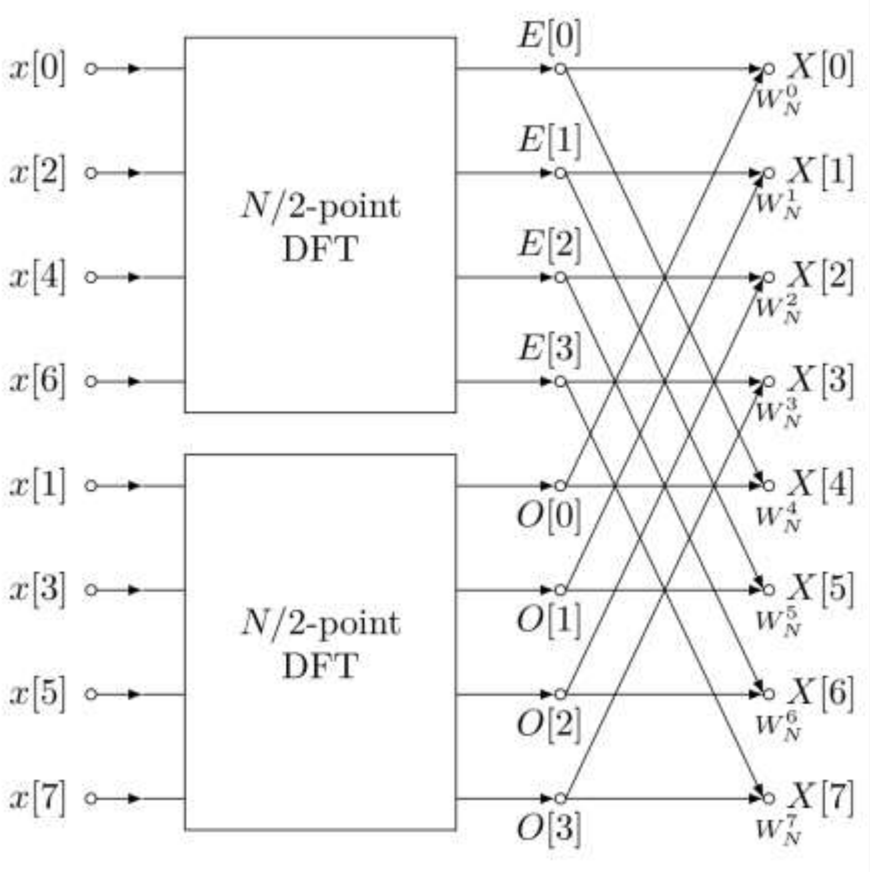
$$f(t) = \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega, \qquad F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$\boxed{x[p] = \frac{1}{N} \sum_k X[k] e^{i\frac{2\pi}{N}kp}, \qquad X[k] = \sum_p x[p] e^{-i\frac{2\pi}{N}kp}}$$

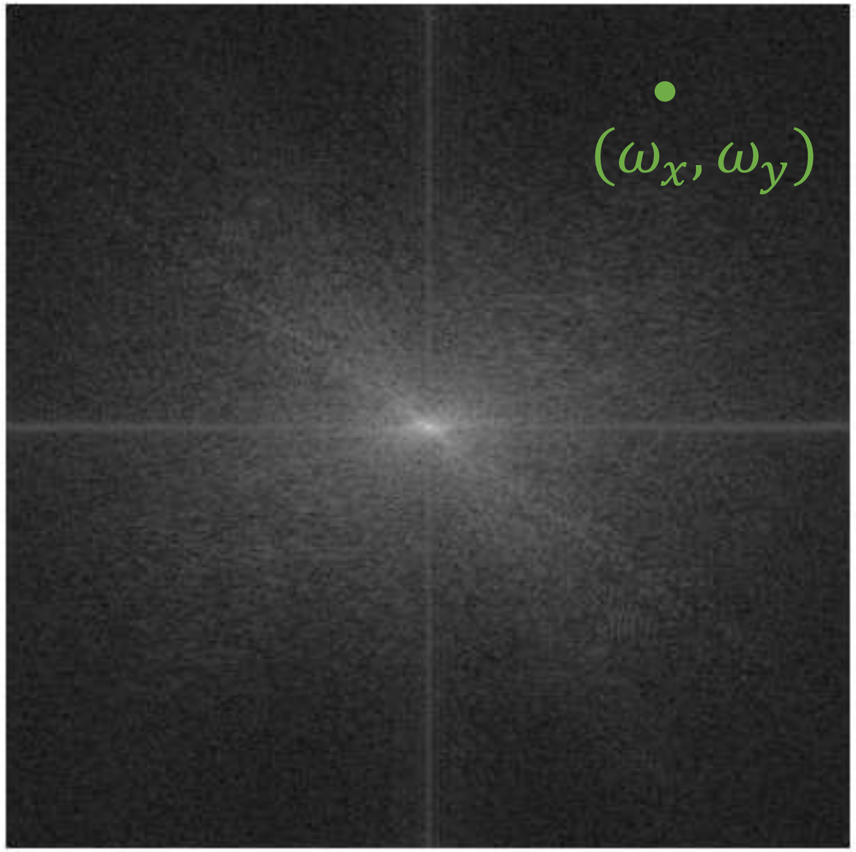


Fast Fourier Transform (FFT)

利用分治实现离散傅里叶变换的快速算法

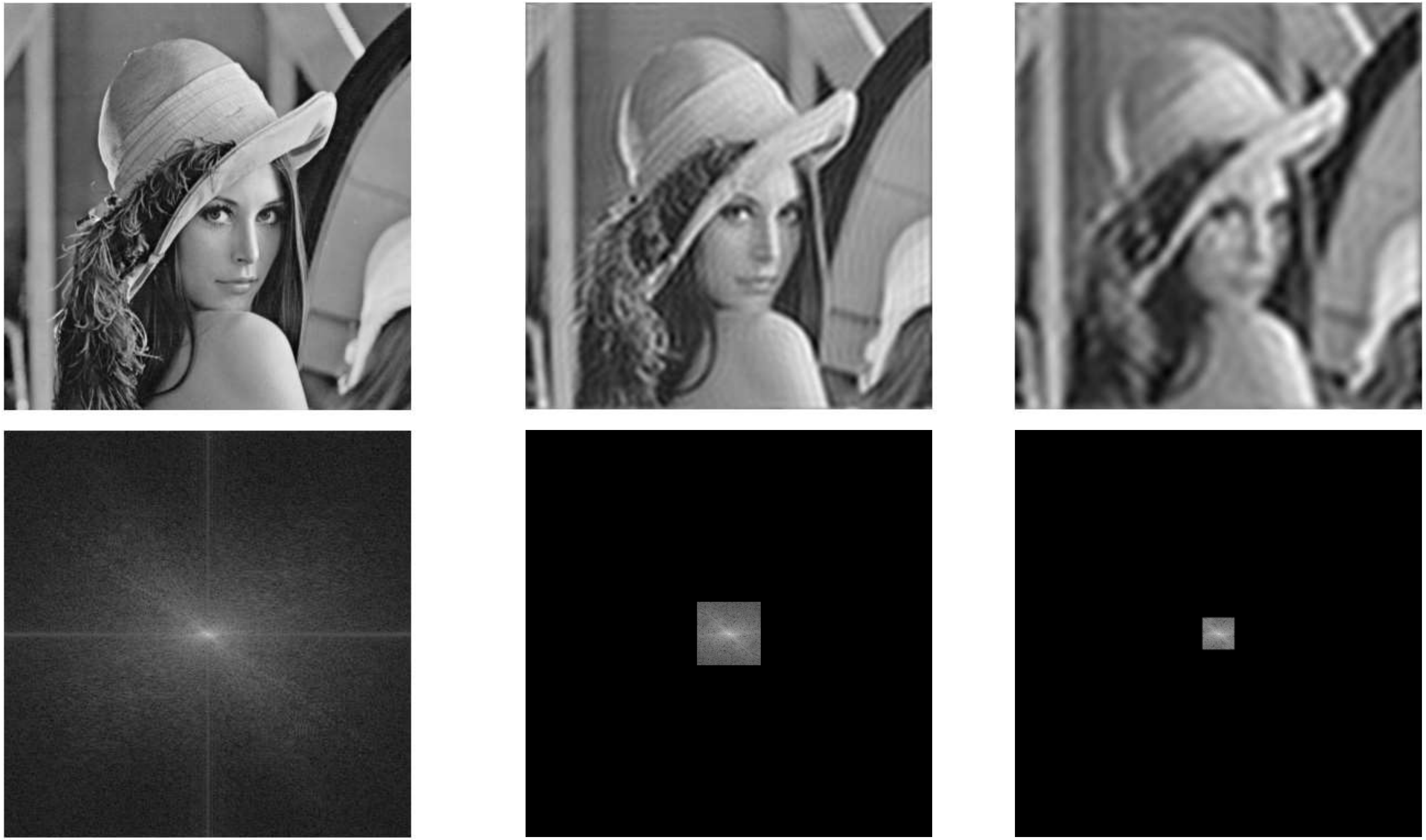


应用：图形去噪/压缩

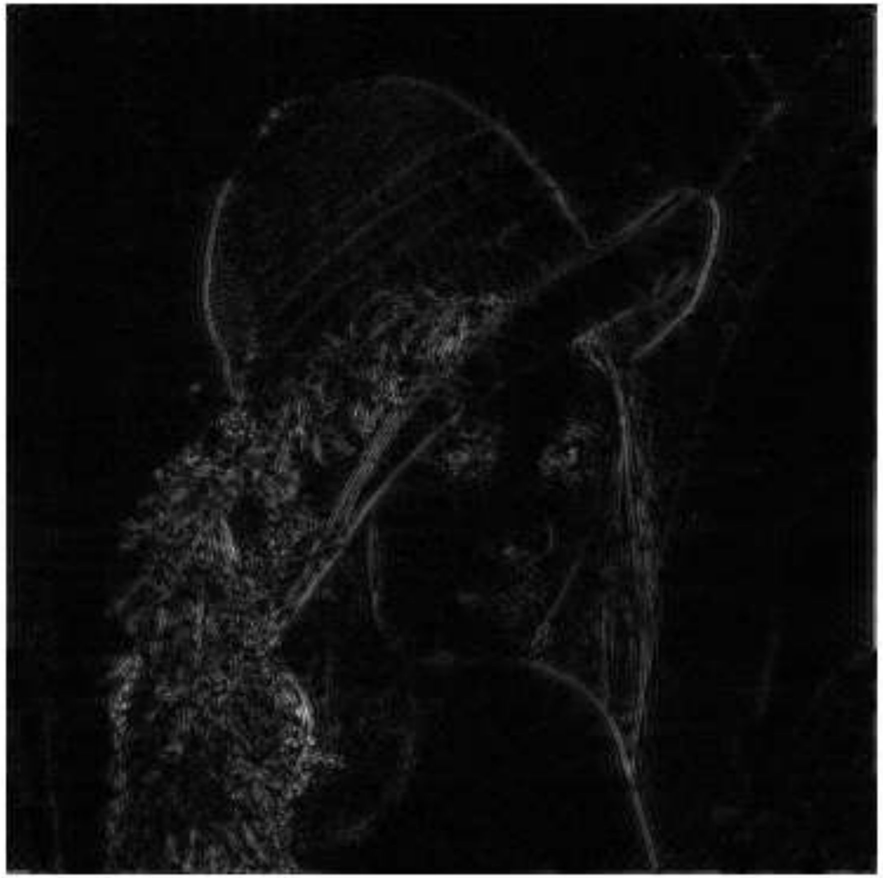


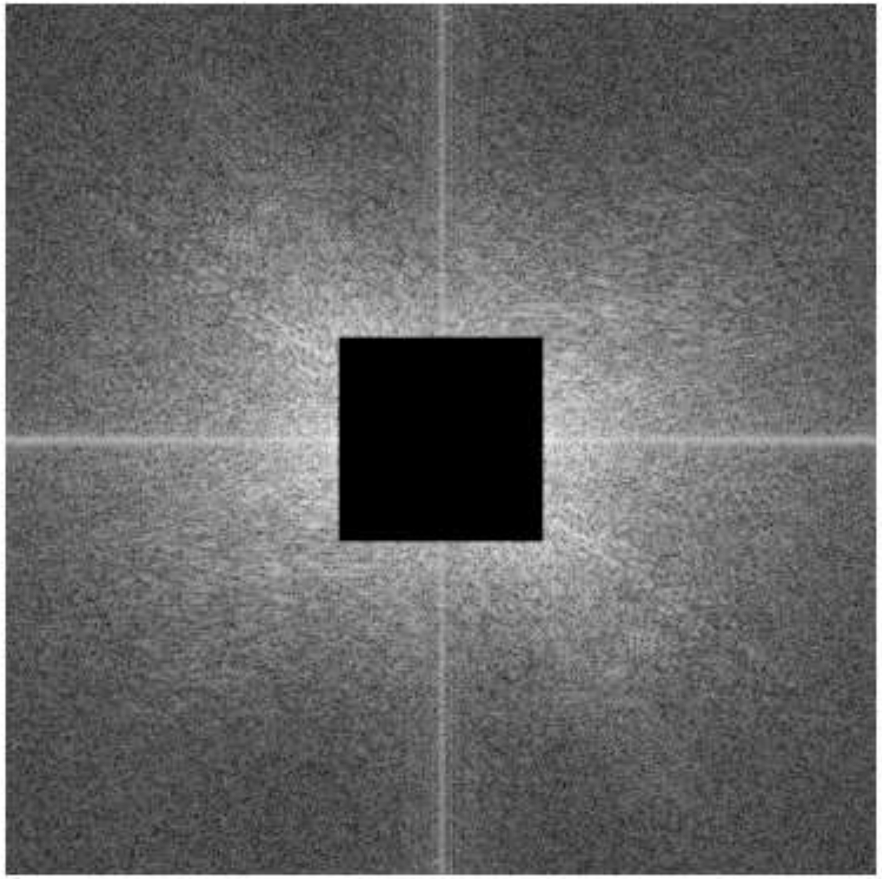
在x,y两个方向进行傅里叶变换

低通滤波



边缘提取

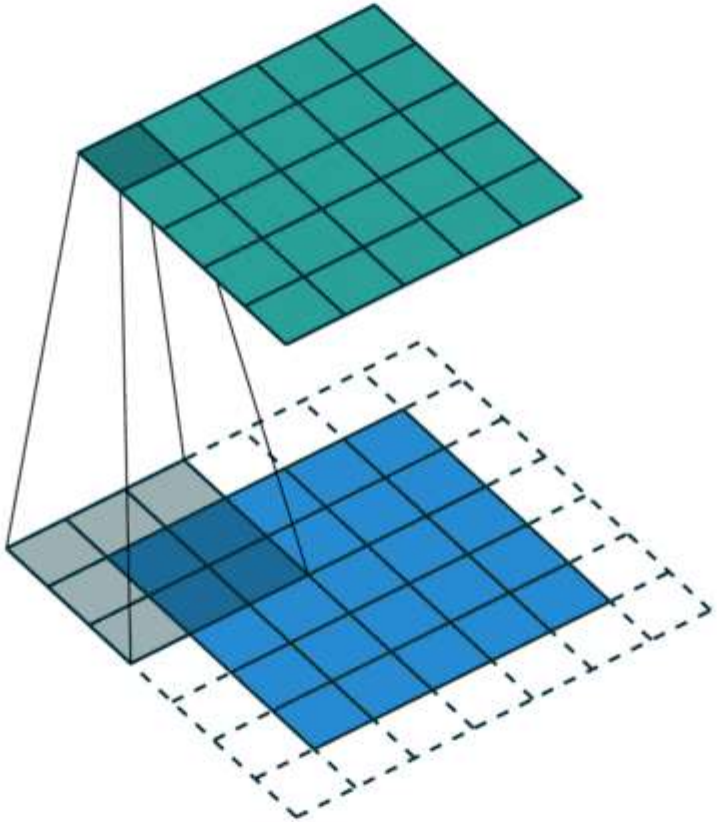




卷积
Convolution

卷积 (Convolution)

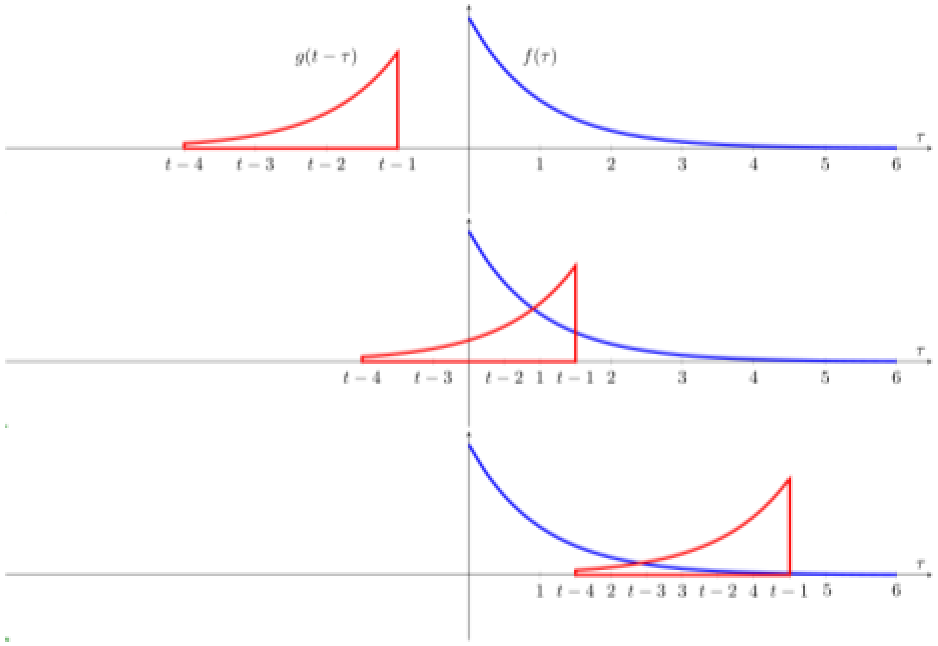
$$* \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} =$$

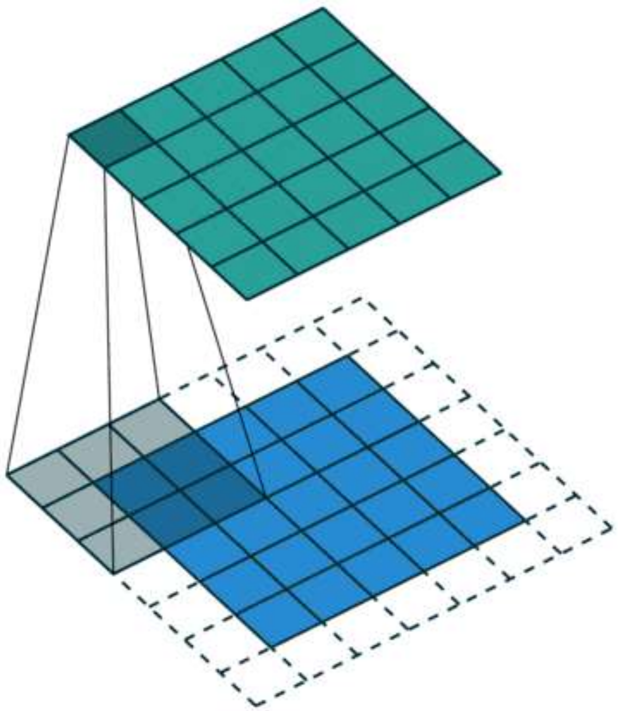


From GAMES101

卷积的定义

$$(f * g)(t) = \int f(\tau)g(t-\tau)d\tau$$





$g(t-\tau)$作为τ的函数是$g(\tau)$以$t/2$为中点的镜像

卷积与傅里叶变换

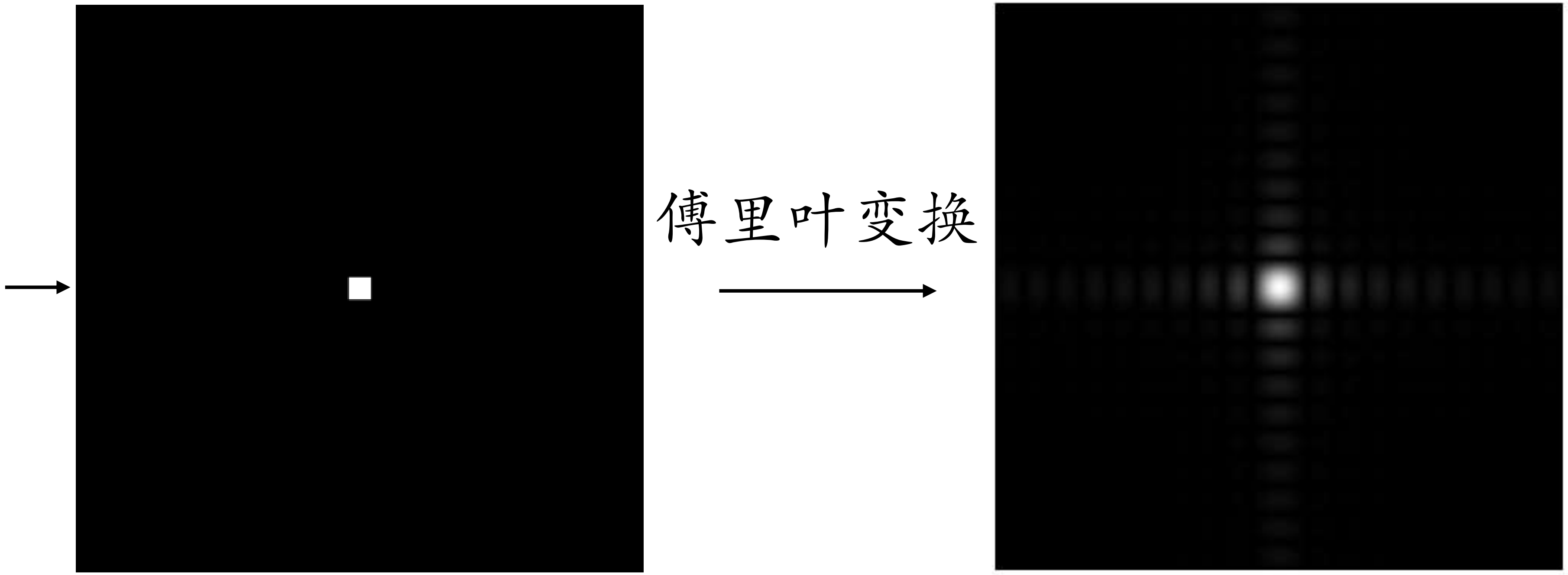
卷积定理:

函数卷积的傅里叶变换是函数傅里叶变换的乘积。即一个域中的卷积对应于另一个域中的乘积，例如时域中的卷积对应于频域中的乘积。

$$\mathcal{F}\{f * g\} = \mathcal{F}(f) \cdot \mathcal{F}(g)$$
$$\mathcal{F}\{f \cdot g\} = \mathcal{F}(f) * \mathcal{F}(g)$$

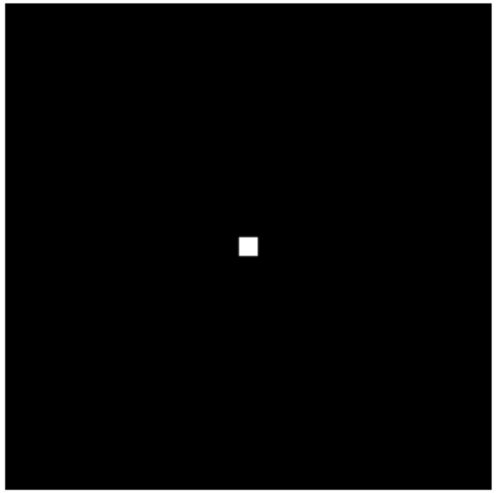
卷积核的傅里叶变换

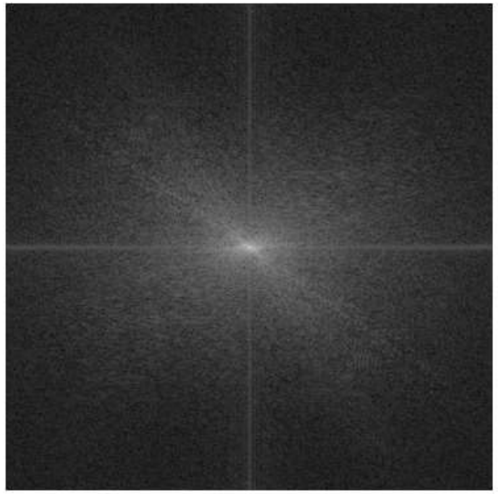
$$\frac{1}{9}\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$$



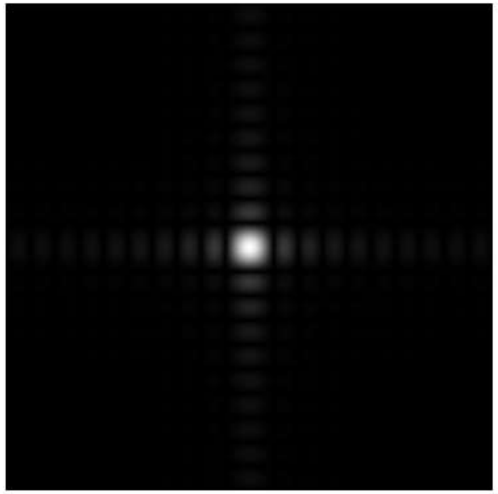
傅里叶变换视角下的卷积

卷积





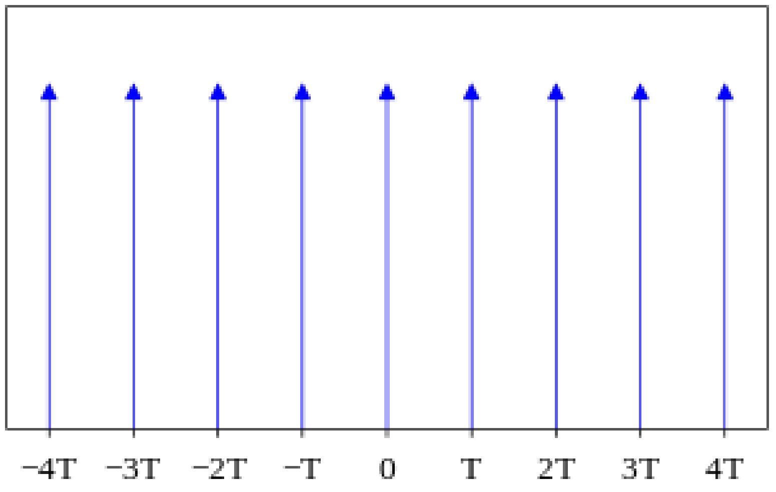
相乘

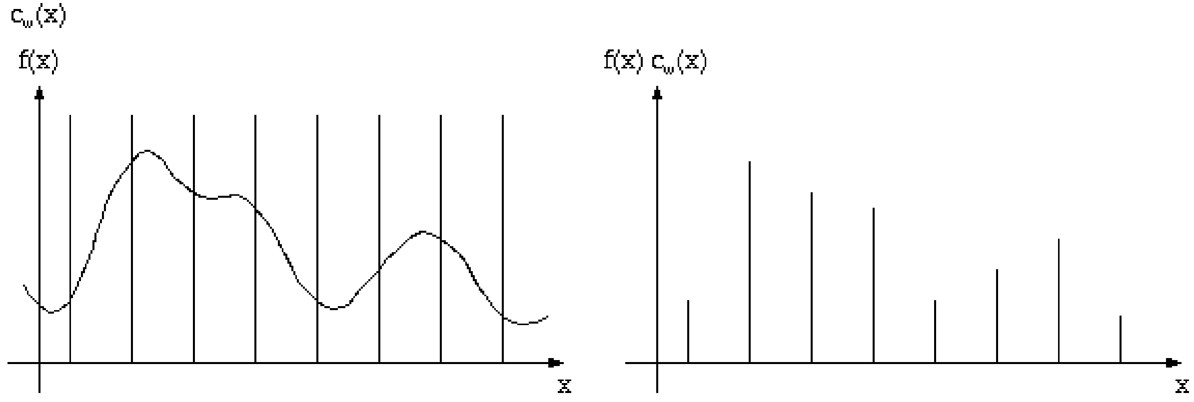


采样 (Sampling)

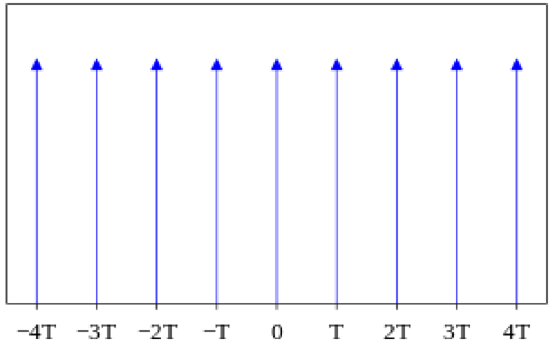
- 采样可以看成一个乘法过程：$f(t)C_T(t)$
- 梳状函数 (Dirac Comb)：

$$C_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$





梳状函数的傅里叶变换



- 梳状函数是周期函数，可以求傅里叶**级数**

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} \mathrm{C}_T(t)e^{-i\frac{2n\pi}{T}t}dt = \frac{1}{T}$$

此处利用性质$\mathrm{C}(t)$在$[-\frac{T}{2},\frac{T}{2})$内只在零点处有值，于是有

$$C_T(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T}e^{i\frac{2n\pi}{T}t}$$

也就是说$C_T(t)$只有离散的频率分量，都是$2\pi/T$的整数倍

梳状函数的傅里叶变换

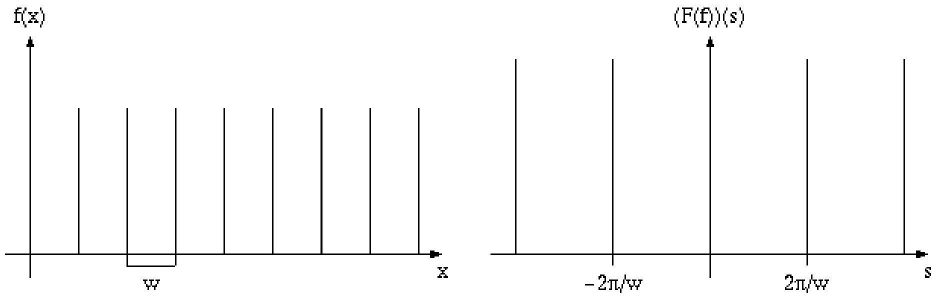
对于周期性的求和，我们可以用梳状函数改写

$$\sum_{n=-\infty}^{\infty} \frac{1}{T} e^{i\frac{2\pi}{T}nt} = \frac{1}{T}\int_{-\infty}^{\infty} C_{\frac{2\pi}{T}}(\omega)\, e^{i\omega t} d\omega$$

注意这里导出的梳状函数周期是$\frac{2\pi}{T}$

对比傅里叶变换的形式，我们就得到了周期为T的梳状函数的傅里叶变换是周期为$2\pi/T$的梳状函数

$$\mathcal{F}\big(C_T(t)\big) = \frac{1}{T} C_{\frac{2\pi}{T}}(\omega)$$



傅里叶变换下的采样

- 时域的采样乘法对应频域的卷积
- $F(\omega)$卷积上一个Delta函数

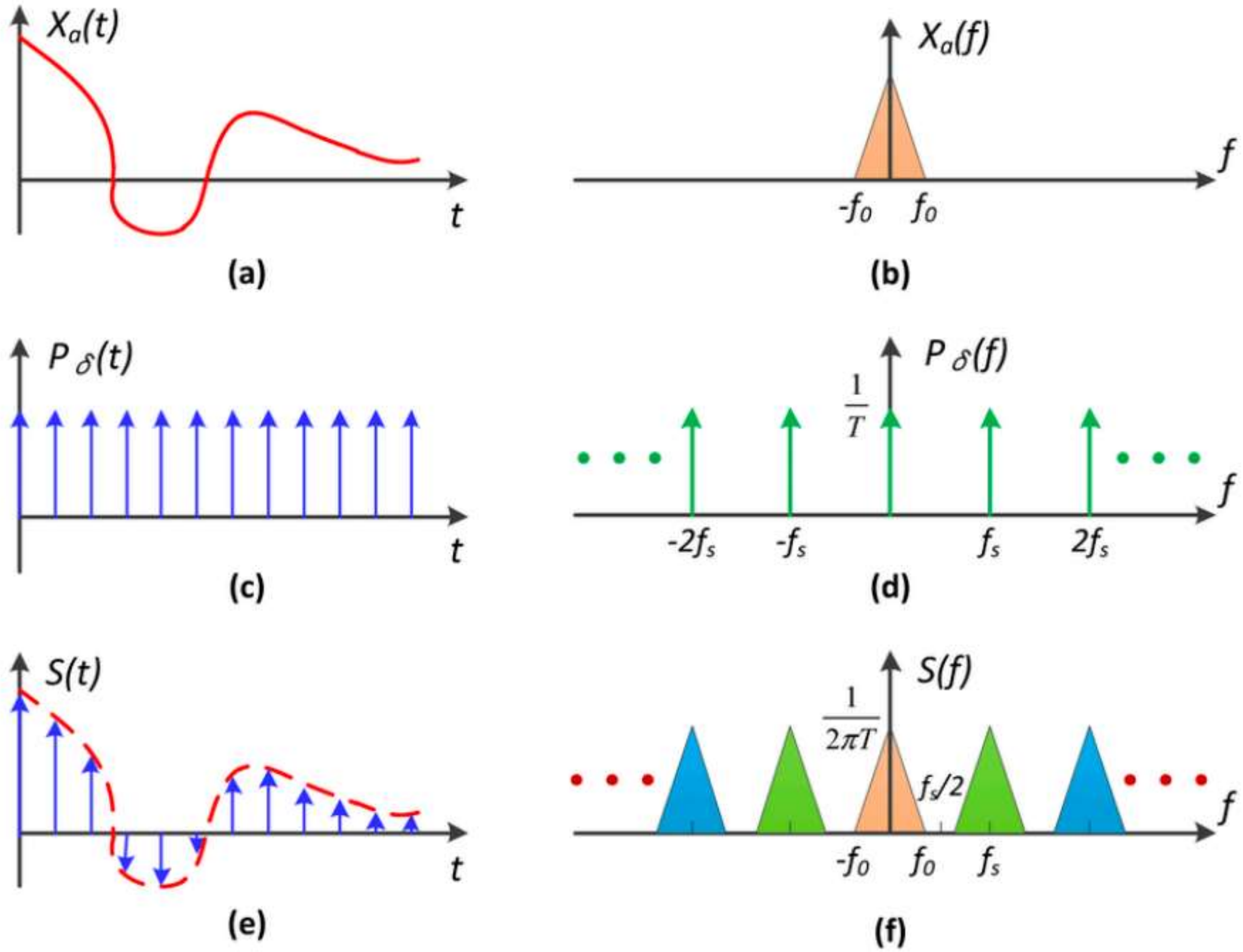
$$F(\omega) * \delta(\omega - \omega') = \int F(a)\delta(\omega - \omega' - a)da = F(\omega - \omega')$$

得到的$F(\omega - \omega')$是将$F(\omega)$平移到ω'的结果

- $F(\omega)$卷积梳状函数$C_{\frac{2\pi}{T}}(\omega)$：将$F(\omega)$每隔$\frac{2\pi}{T}$平移复制一份再叠加

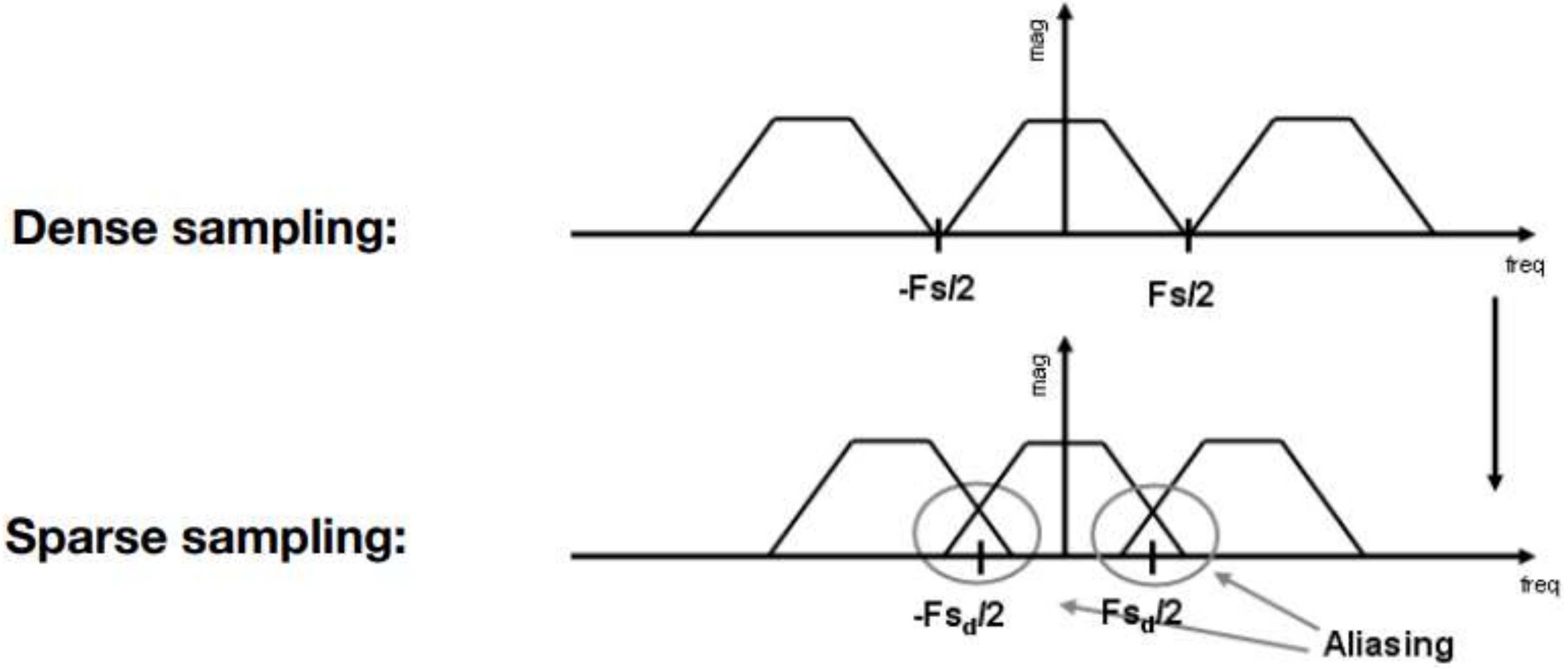
采样频率

傅里叶变换下的采样



https://www.researchgate.net/figure/The-evolution-of-sampling-theorem-a-The-time-domain-of-the-band-limited-signal-and-b_fig5_301556095

走样 (Aliasing)

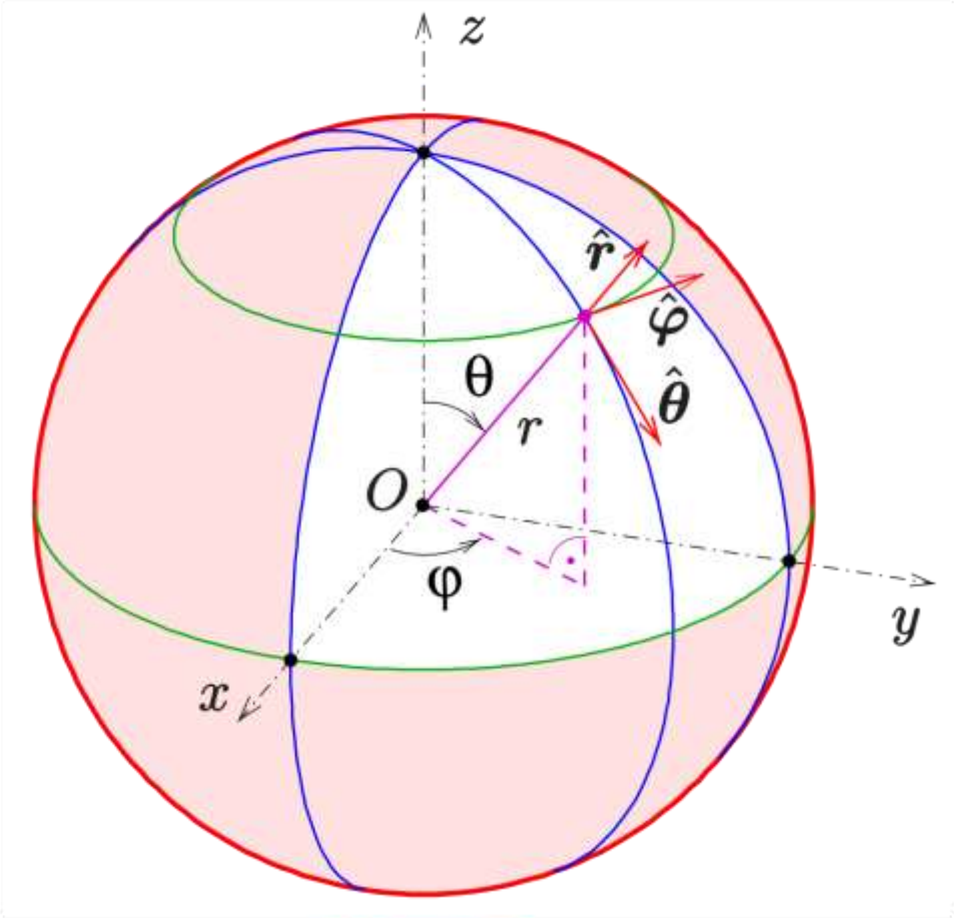


Dense sampling:
mag
freq
-Fs/2
Fs/2
Sparse sampling:
mag
freq
-Fs_d/2
Fs_d/2
Aliasing

球谐函数
Spherical Harmonics

球坐标 (Spherical Coordinate)

$$\varphi \in [0, 2\pi), \theta \in [0, \pi]$$

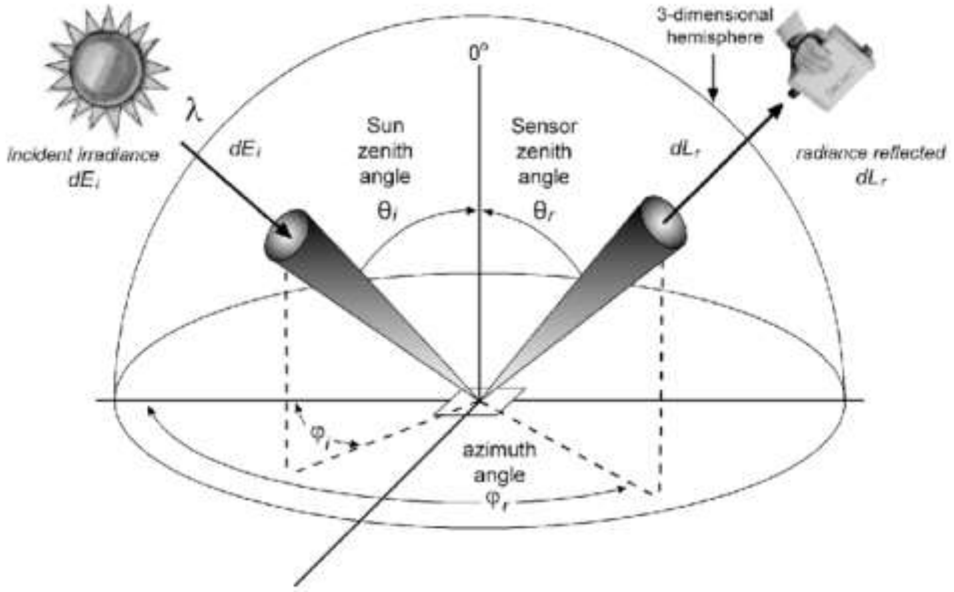


在渲染中

$$f(\theta, \varphi)$$

环境球

Environment Map



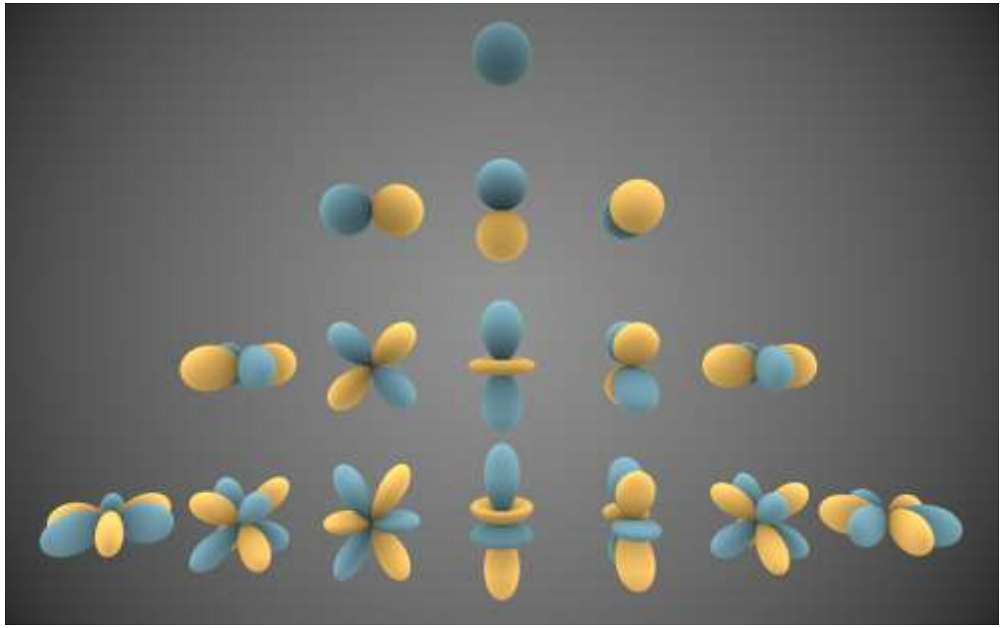
反射分布函数

BRDF

球谐函数 (Spherical Harmonics)

$$Y_l^m(\theta,\varphi) = \underbrace{\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}}_{\text{常数}}\underbrace{P_l^m(\cos\theta)}_{\cos\theta\text{的伴随勒让德多项式}}\underbrace{e^{im\varphi}}_{\varphi\text{的三角函数}}$$

l增大



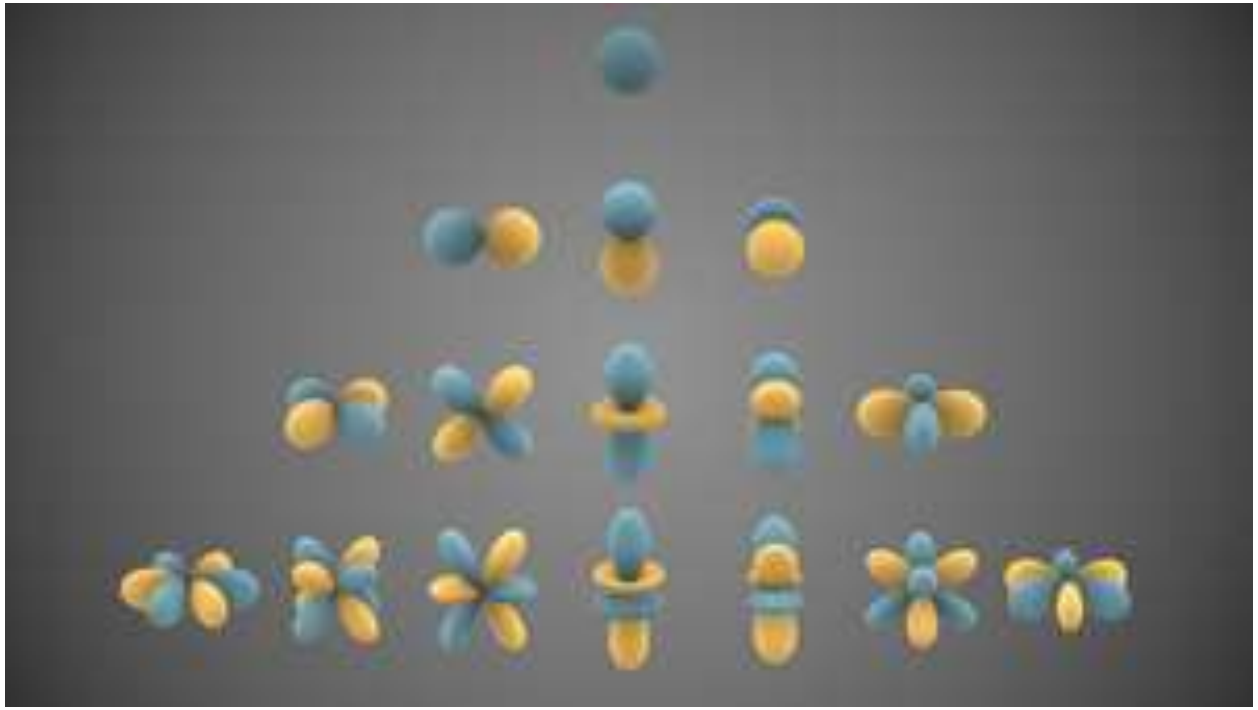
m增大

令(θ,φ)方向射出的长度为$Y_l^m(\theta,\varphi)$所作的可视化

球谐函数 (Spherical Harmonics)

球谐函数是一组球面上的函数$Y(\theta,\varphi)$，可以看成是在球面上做傅里叶展开的基底，也就是三角函数$e^{in\omega t}$在球面上的对应

“频率”增大



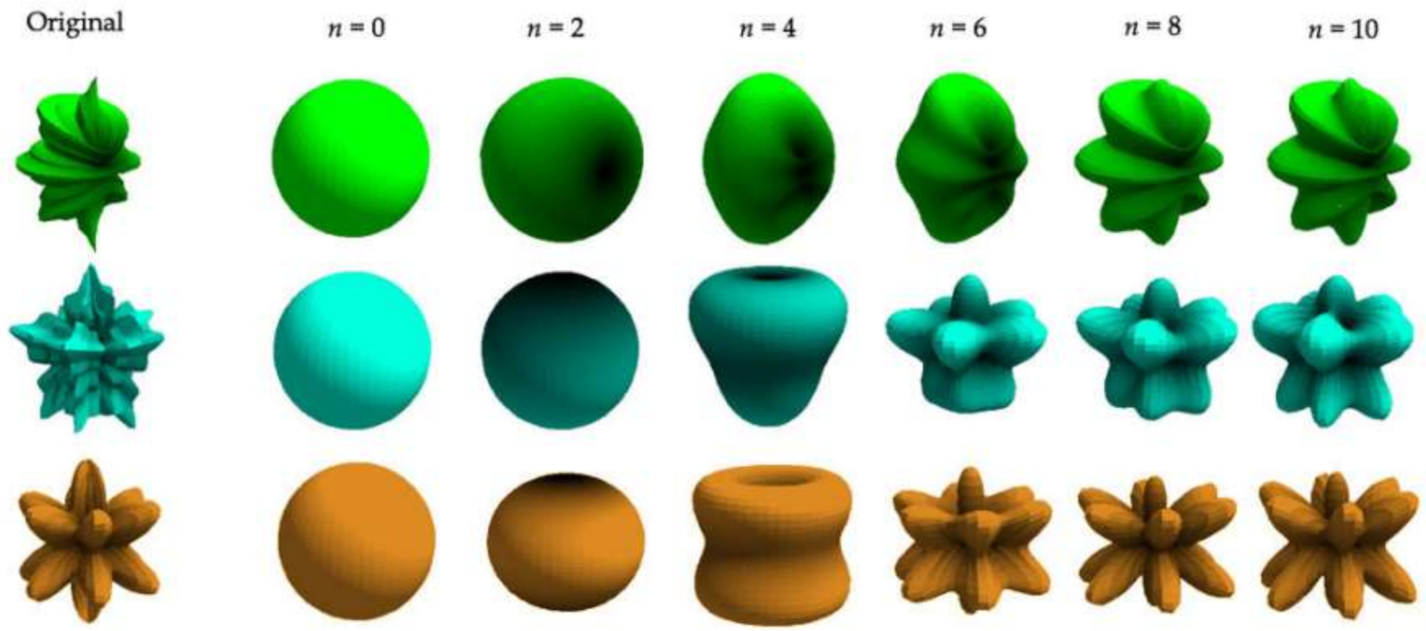
不同的“相位”，正弦，余弦···

球谐函数展开

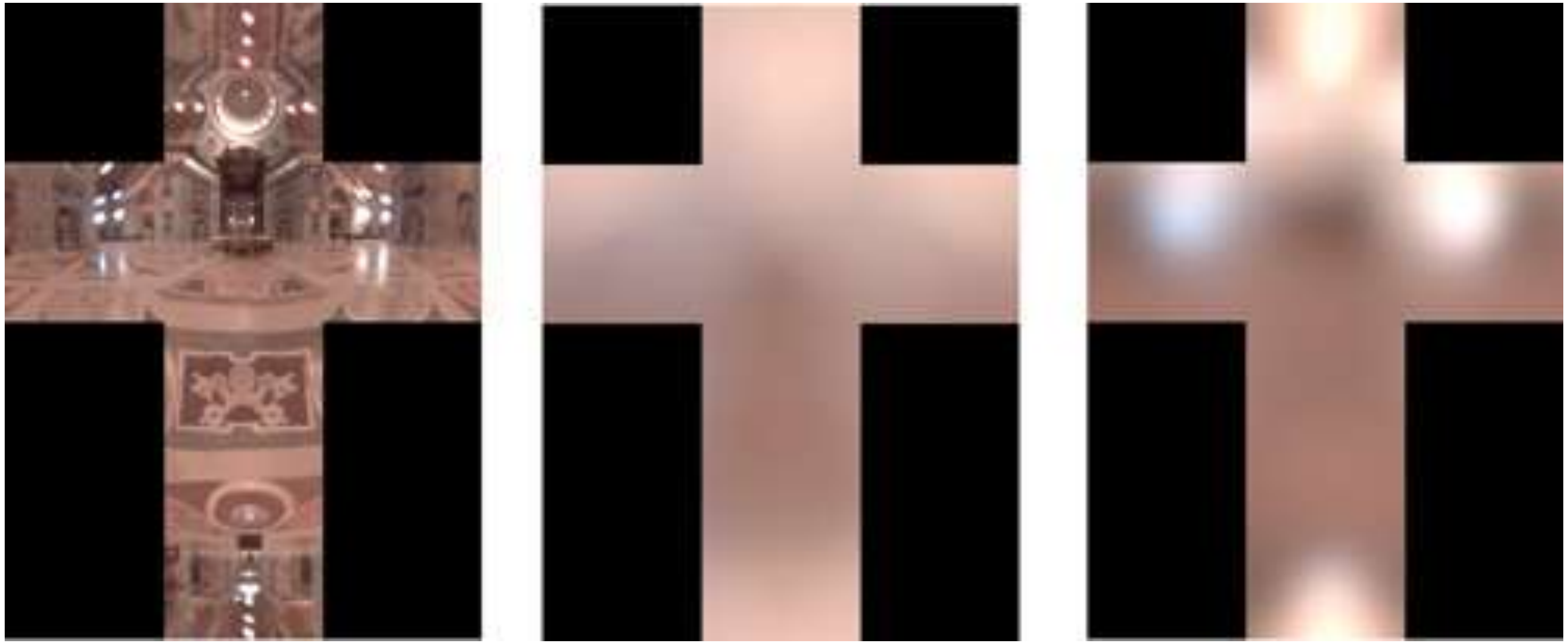
$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}, \qquad c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-in\omega t} dt$$

$$f(\theta,\varphi) = \sum_{l=0}^{n} \sum_{m=-l}^{l} C_l^m Y_l^m(\theta,\varphi)$$

$$C_l^m = \int_0^{2\pi}\int_0^{\pi} f(\theta,\varphi) Y_l^m(\theta,\varphi) \sin\theta \, d\theta d\varphi$$



环境球压缩



只保留低阶球谐展开中的权重C_l^m

与傅里叶展开的联系

- 拉普拉斯算子

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

- 一维傅里叶展开的基函数 $e^{i\omega x}$ 是一维拉普拉斯算子的特征向量

$$\frac{d^2}{dx^2}\left(e^{i\omega x}\right) = -\omega^2 e^{i\omega x}$$

- 球谐函数是球坐标下的拉普拉斯算子的特征向量

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}$$

总结

- 傅里叶变换是从时域基底换到频域基底的线性变换
- 时域上的相乘等价于频域上的卷积，反之频域上的相乘等价于时域上的卷积
- 时域上的采样等价于频域上的周期重复
- 球谐函数是傅里叶展开在球坐标下的基底

谢谢