2019.9.2

1. Measure theoritic foundation

   ·) construction of Measure

   ·) Random Variable, distribution, Integration

2. Weak Convergence

   · Characteristic functions

   · weak Convergence

3. Sums of Independent random variables

   注意区别 → a) weak Law of Large Numbers

   → b) strong law of Large numbers

   c) Central Limit Theorem

   d) Concentration inequalities

4. Dependent random variables

   a) Conditional Expectation

   b) Markov Chains

c) Stopping time

5. Martingale

→ a) Convergence theorem

b) Inequalities

6. Ergodic Theorem

-) sub-additive
ergodic therem

《Probability Theory》
COURANT

Preliminaries

outcomes : $\omega_1, \omega_2, \cdots, \omega_n$

$$p_i = P(\omega_i)$$

$$p_i = p(\omega_i)$$

$$p_i \geq 0$$

$$\sum_{i=1} p_i = 1$$

推到无限:

$$\{\omega_i\}_{i\geq 1}, \quad p_i = p(\omega_i), \quad p_i \geq 0$$

$$\sum_{i=1}^{\infty} p_i = 1$$

推广到 $\mathbb{R}$ Can we assign prob to $\mathbb{R}$?

$$\mathcal{E}_k = \{\omega : p(\omega) \geq \tfrac{1}{k}\}$$

How many can be $\mathcal{E}_k$? $|\mathcal{E}_k| \leq k$

$$\bigcup_{k=1}^{\infty} \mathcal{E}_k = \mathcal{E} = \{\omega : p(\omega) > 0\}$$

countable (countable union of countable set is countable)

Probability to groups of outcomes.

Prob{ out of experiment $\le \frac{1}{2}$ }

$\mathcal{E}$ → for which I would to assign probability

Ω → set of all outcomes at the experiment

$\mathcal{E}$ to be closed unions intersection
$A_1 \cup A_2$ $A_1 \cap A_2$

complement
A_1^c

A collection $\mathcal{E}$ that satisfies:

1. $\Phi \in \mathcal{E}$
2. $A \in \mathcal{E} \Rightarrow A^c \in \mathcal{E}$
3. $A_1, A_2 \in \mathcal{E} \Rightarrow A_1 \cup A_2 \in \mathcal{E}$

Its name is "algebra" or "field"

性质:
if $A_1 \in \mathcal{E}$, $A_2 \in \mathcal{E} \Rightarrow A_1 \cap A_2 \in \mathcal{E}$

$\because A_1 \cap A_2 = (A_1^c \cup A_2^c)^c$

A collection ξ that satisfies:

1. $\Phi \in \xi$
2. $A \in \xi \Rightarrow A^c \in \xi$
3. $\{A_i\}_{i \geq 1} \in \xi \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ — $\mathcal{F} \to \sigma$-algebra

countable union

Its name is σ-algebra.

Samples:

$\{\phi, \Omega\}$

$\{\phi, A, A^c, \Omega\}$

$(\Omega, \mathcal{F}) \to$ measure space.

Def measure $\mu : \mathcal{F} \to \mathbb{R}$ satisfies:

a) $\mu(\Phi) = 0$

b) $\mu(A) \geq 0$

c) if $\{A_i\}$ is p.w. disjoint (pair-wise: $A_i \cap A_j = \phi \ \forall i \neq j$)

then $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$

为什么 σ-algebra 不定义到 2^{Ω} 上？

∵ 是为了定义 "volume" of a set

deep theorem in math which says that you cannot simultaneously have all the following 4 properties:

1. Volume is preserved under rotation
2. if $A \cap B = \phi$, $\mu(A+B) = \mu(A) + \mu(B)$
3. All sets have a "volume"
4. We are in ZFC
   (Axiom of choice)

∴ 相较之下舍掉第 3 个更合理.

∴ 2^{Ω} 并不是所有的都有测度.

下面看为啥这四条不能同时成立

$[0,1]$

$x \sim y$ if $x - y \in Q$ (定义 equalevent)

$\{S_\alpha\}$

$R \to$ set of representatives (be Axiom of choice) of $\{S_\alpha\}$

$R_q = R + q$

$\mu(R_q) = \mu(R)$

R_q is a countable partition

which is pairwise

$$\bigcup_q R_q = [0,1]$$

$$\therefore \sum_q \mu(R_q) = 1$$

$$\mu(R_q) = C$$ contradiction!

第三条矛盾

$(\Omega, \mathcal{F}, P) \to$ probability space

<u>Exercise</u>

P.T. (prove that)

$A_i \in \mathcal{F}$

$$P\left(\bigcup_i A_i\right) \le \sum_i P(A_i)$$

先看 $P(A_1 \cup A_2) \le P(A_1) + P(A_2)$

if $B \subseteq A$, $A, B \in \mathcal{F}$

<u>proof</u> $A = B \cup (A \setminus B)$

显然 $A \setminus B \in \mathcal{F}$ $\because$ $\overset{\in \mathcal{F}}{A} \cap \overset{\in \mathcal{F}}{B^c} \in \mathcal{F}$

$$P(A) = P(B) + P(A \setminus B) \ge P(B)$$

同理：

$$B_i = \left(\bigcup_{j=1}^{i} A_j\right) \setminus \left(\bigcup_{j=1}^{i-1} A_j\right) \subseteq A_i$$

$$\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i \qquad P\left(\bigcup_i B_i\right) = \sum_i P(B_i) \leq \sum_i P(A_i)$$

$$\bigcup_i B_i = \bigcup_i A_i$$

2019. 9. 5

$(\Omega, F, P) \to$ probability space

Ω: set of all possible outcomes

F: collection of events $A \subseteq \Omega$, it's a σ-algebra

P: countably additive probability measure.

例:

$\Omega = \{0, 1\}$

$F = \{\phi, \{0\}, \{1\}, \{0, 1\}\}$

Borl σ-algebra.

$\Omega = \mathbb{R}$ want $\mu([a,b]) = b - a$ $\infty > b \geq a > -\infty$

Lemma: Given any collection of events, $\mathcal{A}$

there exists a minimal σ-algebra

containing it.

proof: 2^{Ω} countains it.

$\bigcap_{\alpha} F_{\alpha}$ is a σ-algebra

take all σ-algebra contains $\mathcal{A}$

then take intersection.

σ-algebra generated by the open sets

is called the Borel σ-algebra.

Caratheodory's Extension theorem.

Any countably additive probability measure on a field (algebra) $\mathcal{A}$ uniquely existance to a countably probability measure in $\sigma(\mathcal{A})$

proof

STEP 1: Construct an outmeasure

$$P^*(B) = \inf \sum_i P(A_i) \qquad \begin{array}{l} A_i \in \mathcal{A} \\ B \subset \bigcup_i A_i \end{array}$$

prop of P^*:

(i) if $B \subset \bigcup_i B_i$ then

$$P^*(B) \leq \sum_i P^*(B_i) \qquad \text{(sub-additivity)}$$

(ii) if $A \in \mathcal{A}$ then

$$P^*(A) \leq P(A)$$

(iii) if $A \in \mathcal{A}$ then

$$P^*(A) \geq P(A)$$

proof

(i) Given $\varepsilon > 0$, $\exists$ a cover $\{A_{ij}\} \in \mathcal{A}$

$\bigcup_j A_{ij} \supseteq B_i$ s.t. $P^*(B_i) \geq \sum_j P(A_{ij}) - \frac{\varepsilon}{2^i}$

$B \subseteq \bigcup_{ij} A_{ij}$

$$p^*(B) \le \sum_{i,j} P(A_{ij})$$

$$= \sum_i \left(\sum_j P(A_{ij}) \right)$$

$$\le \sum_{i=1}^{\infty} \left(p^*(B_i) + \frac{\varepsilon}{2^i} \right) .$$ 外测度定义

$$\le \sum_{i=1}^{\infty} p^*(B_i) + \varepsilon$$ inf 的定义.

← 级数 $\lim_{n\to\infty} \frac{1}{2} + \frac{1}{2^2} + \cdots \frac{1}{2^n}$

(ii) by definition

∴ A 本身是 A 自己.

(iii) Let $A_i \in \mathcal{A}$ be any cover of A.

$\exists\, B_i \subset \mathcal{A}$ s.t. $A \subseteq \bigcup_i B_i$, B_i's are p.w. disjoint $B_i \subseteq A_i$

$$B_i = A_i \setminus \bigcup_{i=1}^{i-1} A_i$$

$B_i \cap A = C_i$

$$P^*(A) \le \sum_i P(B_i)$$

$$P(A) = \sum_i P(B_i \cap A)$$ (∵ 因 countability additive prob measure)

$$\le \sum_i P(B_i)$$

STEP 2:

We say that a E is "measurable" if

$$p^*(B) \geq p^*(B \cap E) + p^*(B \cap E^c) \quad \forall B$$

$\mathcal{E}$: All measureable event, first we show it's a σ-algebra.

(i) $\phi \in \mathcal{E}$ ✓

(ii) if $E \in \mathcal{E}$ $E^c \in \mathcal{E}$ ✓

(iii) if $E_1, E_2 \in \mathcal{E}$, $E_1 \cup E_2 \in \mathcal{E}$

$$p^*(B) \geq p^*(B \cap E_1) + p^*(B \cap E_1^c)$$

$$\geq p^*(B \cap E_1) + p^*(B \cap E_1^c \cap E_2) + p^*(B \cap E_1^c \cap E_2^c)$$

$$\geq p^*(B \cap (E_1 \cup E_2)) + p^*(B \cap (E_1 \cup E_2)^c)$$

$$\varepsilon + p^*(B) \geq \sum_i p(A_i) = \sum_i p(A_i \cap A) + \sum_i p(A_i \cap A^c)$$

$$\geq p^*(B \cap A) + p^*(B \cap A^c)$$

$$(\because A_i \cap A \subset \mathcal{A},\ A_i \cap A^c \subset \mathcal{A})$$

以上证明了E是代数，下面证明是sigma 代数

Then $\bigcup_i E_i \in \mathcal{E}$ E_i are p.w. disjoint.

$\hat{E}_n = \bigcup_{i=1}^{n} E_i \in \mathcal{E}$ $\hat{E}_n^c \supseteq E^c$

$$P^*(B) \geq P^*(B \cap \hat{E}_n) + P^*(B \cap \hat{E}_n^c)$$

$$\geq P^*(B \cap \hat{E}_n \cap E_n) + P^*(B \cap \hat{E}_n \cap E_n^c) + P^*(B \cap \hat{E}_n^c)$$

$$= P^*(B \cap E_n) + P^*(B \cap \hat{E}_{n-1}) + P^*(B \cap \hat{E}_n^c)$$

$$\geq P^*(B \cap E_n) + P^*(B \cap \hat{E}_{n-1}) + P^*(B \cap E^c)$$

$$P^*(B) \geq P^*(B \cap E_n) + P^*(B \cap E_{n-1}) + P^*(B \cap E^c)$$

$$\geq \sum_{i=1}^{n} P^*(B \cap E_i) + P^*(B \cap E^c)$$

$$P^*(B) \geq \sum_{i=1}^{\infty} P^*(B \cap E_i) + P^*(B \cap E^c)$$

$$\geq P^*(B \cap E) + P^*(B \cap E^c)$$

Uniqueness

P_1, P_2 are countabily additive probability measure that agree on $\mathcal{A}$, the they must agree on $\sigma(\mathcal{A})$

$\mathcal{M} = \{B: P_1(B) = P_2(B)\}$

$\mathcal{M} \supseteq \mathcal{A}$

$\mathcal{M}$ is a monotone-class

monotone-class

$A_1 \subseteq A_2 \subseteq \cdots \subseteq \cdots$ $\quad$ $A_1 \supseteq A_2 \supseteq A_3 \cdots\cdots$

$\bigcup_i A_i$ $\quad$ $\bigcap_i A_i$

<u>proof</u> (Monoton class theorem)

if $\mathcal{A}$ is an algebra, then

$\mathcal{M}(\mathcal{A}) = \sigma(\mathcal{A})$

$\sigma(\mathcal{A}) \supseteq \mathcal{M}(\mathcal{A})$

$\mathcal{M}(\phi) \subseteq \sigma(\mathcal{A})$

$\phi \in \mathcal{M}(\mathcal{A})$

$B \in \mathcal{M}(\mathcal{A})$ then $B^c \in \mathcal{M}(\mathcal{A})$

$\mathcal{M}_0 = \{B \in \mathcal{M}(\mathcal{A}): B^c \in \mathcal{M}(\mathcal{A})\}$

$\mathcal{A} \in \mathcal{M}_0$

if $B_1, B_2 \in \mathcal{M}(\mathcal{A})$ then $B_1 \cup B_2 \in \mathcal{M}(\mathcal{A})$

$\mathcal{M}_A = \{B \in \mathcal{M}(\mathcal{A}): A \cap B \in \mathcal{M}(\mathcal{A})\}$ $\quad$ $A \in \mathcal{A}$

(1) $\mathcal{A} \subseteq \mathcal{M}_A$

(ii) $\mathcal{M}_A$ is monotone hence $\mathcal{M}_A = \mathcal{M}(\mathcal{A})$ $\forall A \in \mathcal{A}$

$\forall C \in \mathcal{M}(\mathcal{A})$

$$\mathcal{M}_C = \{ B \in \mathcal{M}(\mathcal{A}) : B \cap C \in \mathcal{M}(\mathcal{A}) \}$$

$$\mathcal{M}_C \supseteq \mathcal{A}$$

$$\mathcal{M}_C = \mathcal{M}(\mathcal{A})$$

Review: 2019.9.9.

(Ω, F, P)

probability measure satisfies:

(i) $P(\phi)=0$, $P(\Omega)=1$

(ii) $P(A)\geq 0$, $\forall A\in F$

(iii) $\{A_i\}$ are p.w. disjoint

then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

F: sigma-algebra

(i) $\phi \in F$

(ii) $A\in F \Rightarrow A^c \in F$

(iii) $\{A_i\}\in F \Rightarrow \bigcup_i A_i \in F$

countable union

如果是 finite union

则是 algebra.

Borel σ-algebra:

generated by open sets

Cantheodory's theorem

if there is a countable additive prob measure on $\mathcal{A}$, then, there exists a unique extension on a countablity additive probability measure on $\sigma(\mathcal{A})$

$$\mu^*(B) = \inf_{\substack{\{A_i\} \\ B\subset \bigcup_i A_i}} \sum_i P(A_i)$$

E is measureable if:

$\mu^*(B) \geq \mu^*(B \cap E) + \mu^*(B \cap E^c) \quad \forall B$

outer measure

$\mathcal{E} = \{E : E \text{ is measurable}\}$

(i) $A \subseteq \mathcal{E}$

(ii) $\mathcal{E}$ is a algebra, ...

(iii) Monotone class theorem to show uniquess

下面来看存在性.

Lebesgue

$\Omega = \mathbb{R}$

Algebra: finite disjoint unions of intervals of the type $(a, b]$ or $(-\infty, b]$ or (a, ∞)

proof $(-\infty, a_1] \cup (b_1, a_2] \cup (b_2, +\infty)$ in F

$\therefore A^c \subset F \quad \therefore$ is algebra.

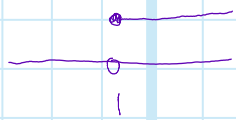
Consider functions, $F(x)$ that satisfies:

(i) if $\lim_{x \to -\infty} F(x) = 0$ if $\lim_{x \to \infty} F(x) = 1$

实际就是 CDF 函数.

(ii) $F(x)$ is non-decreasing

(iii) $F(x)$ is right continuous



Def: $P((-\infty, x]) = F(x)$

$P((a,b]) = F(b) - F(a)$

P is a countably additive prob measure

iff F is right-continuous

$\Rightarrow$:

Pf: $I_n = (x, x_n]$ $x_n \geq x$

$x_n \downarrow x$, $I_n \downarrow \phi$

if P is c.a. $P(I_n) \downarrow 0$

$\therefore \lim_{x_n \downarrow x} F(x_n) - F(x) = 0$

如果 countably additive 那么一定右连续.

$\therefore \lim_{x_n \downarrow x} F(x_n) = F(x)$

$\Leftarrow$:

Assume P is not countably additive

$\Leftrightarrow \exists A_n \downarrow \phi$ and $P(A_n) \geq \delta \ \forall n$ for some $\delta > 0$

$P(\bigcup_i \tilde{A}_i) = \sum_i P(\tilde{A}_i)$

$P(\bigcup_{i=1}^{\infty} \tilde{A}_i)$

$$B=\bigcup_i \tilde{A}_i$$

$$A_i=B\setminus\bigcup_{j=1}^{i}\tilde{A}_j$$

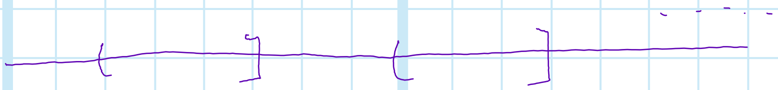
$\exists\, l$ s.t. $P([-l,l])\geq 1-\frac{\delta}{2}$

$$B_n=A_n\cap(-l,l]$$

$$B_n\downarrow\phi,\quad P(B_n)\geq\frac{\delta}{2}$$

$$\begin{aligned}\delta\leq P(A_n)=P(A_n)&=P(A_n\cap(-l,l])+P(A_n\cap(-l,l]^c)\\&=P(B_n)+P(A_n\cap(-l,l]^c)\\&\leq P(B_n)+P((-l,l]^c)\\&\leq P(B_n)+\frac{\delta}{2}\end{aligned}$$

B_n的形状：∵是A_n和( ]的交，A_n是一个algebra，形状是一堆区间



构造B_n是为了让其有界（紧）∴取一个$[-l,l]$让概率损失不太大。

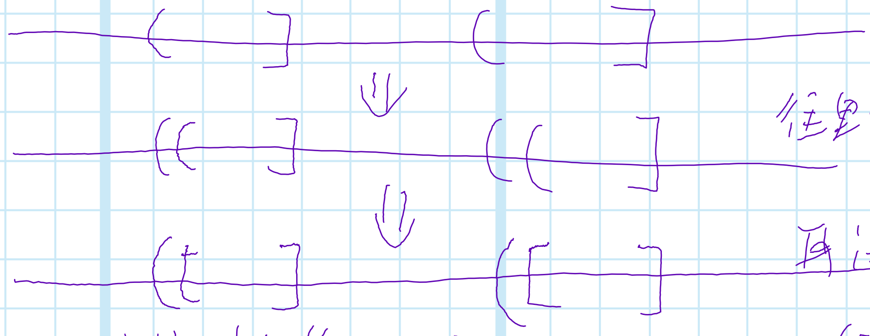
F_n is closed subsets of compact set

$$\bigcap_n F_n=\phi$$

Def compact set:

任一开覆盖都有有限子覆盖

B_n不是闭的，下面来根据B_n构造一个闭集



往里挪一点.

再闭上.

这样构造了 C_n

∵ 我们假设了 right countinous ∴ 可以这么操作.

$$P(B_n \setminus C_n) = \frac{\varepsilon}{3 \cdot 2^n}$$

D_n is the clousure of C_n.

(D_n is not algebra)

$$C_n \subset D_n \subset B_n \subset A_n$$

$$E_n = \bigcap_{j=1}^{n} C_j \qquad F_n = \bigcap_{j=1}^{n} D_j$$

$$E_n \subset F_n \subset D_n \subset B_n \subset A_n$$

$$P(E_n) \geqslant P(B_n \cap E_n)$$

$$P(B_n \cap E_n^c) + P(B \cap E_n) = P(B_n)$$

$$\Rightarrow P(B_n \cap E_n^c) + P(E_n) \geqslant P(B_n)$$

$$P\left(B_n \cap \left(\bigcup_{j=1}^{n} C_j^c\right)\right)$$

$$= P\left(\bigcup_{j=1}^{n} (B_n \cap C_j^c)\right) \leq P\left(\bigcup_{j=1}^{n} (B_j \cap C_j^c)\right) \quad (\because B_n \downarrow)$$

$$\leq \sum_{j=1}^{n} P(B_j \setminus C_j)$$

$$\le \sum_{i=1}^{n} \frac{\delta}{3n} \le \frac{\delta}{3}$$

$$\frac{\delta}{3} + P(E_n) \geq P(B_n)$$

$$P(E_n) \geq P(B_n) - \frac{\delta}{3} \geq \frac{\delta}{2} - \frac{\delta}{3} = \frac{\delta}{6}$$

Random Variable

(Ω, F) measurable space

$(\mathbb{R}, \mathbb{B}_R)$

$f: \Omega \to \mathbb{R}$, s.t. $\forall B \in \mathbb{B}_R$

$f^{-1}(B) = \{\omega : f(\omega) \in B\} \in F$

类似连续性的定义, inverse map of open set is also open set.

satisfies to verfy for any collection $\mathcal{A}$ that generates $\mathbb{B}_R$

$A_x = \{\omega : f(\omega) \le x\} \in \mathcal{F} \quad \forall x$

$(-\infty, q]\ q \in \mathbb{Q}$.

$f(\omega),\ X(\omega)$

$$1_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \in A^c \end{cases} \qquad A \in \mathcal{F}$$

$$1_A^{-1}(B) = \begin{cases} \emptyset & B \text{ does not contain } 0 \text{ or } 1 \\ A & B \text{ contain } 1 \text{ but not } 0 \\ A^c & B \text{ contains } 0 \text{ but not } 1 \\ \Omega & B \text{ contains both.} \end{cases}$$

$af_1 + bf_2$

$$\sum_{i=1}^{k} c_i 1_{A_i}(\omega) \qquad \text{(simple functions)}$$

Lemma, any bounded measurable function can be expressed as the uniform limit of simple functions,

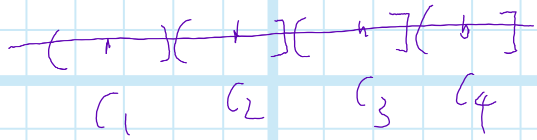
$f_n(x) \xrightarrow{\text{uniformly}} f(x)$ if $\forall \varepsilon > 0, \exists N_\varepsilon$
s.t. $\forall x$ $\quad \sup_x |f_n(x) - f(x)| \le \varepsilon \ \forall n \ge N_\varepsilon$

$-m < f(\omega) \le m$

Pf $\quad n \quad (-m, m]$

c_i^n : midpoint of the interval

$A_i^{(n)} = \{\omega : f(\omega) \in i\text{th interval}\}$



$$f^n = \sum_{i=1}^{n} c_i^n \mathbb{1}_{A_i^{(n)}} \qquad |f^n(\omega) - f(\omega)| \le \frac{m}{n}$$

2019.9.12

Review.

Random Variable

$(\Omega, \mathcal{F})$

Ω: Set of all outcomes

$\mathcal{F}$: Collection of subset of Ω

(σ-algebra)

$\mathbb{R}, \mathbb{B}$

σ-algebra generated by the open intervals

Random Variable is a mapping from

$f: \Omega \mapsto \mathbb{R}$, s.t.

$f^{-1}(B) \in \mathcal{F} \quad \forall B \in \mathbb{B}_R$

直接检验所有的B很复杂，可以等价只检验$(-\infty,x]$

Lemma suffices that $\{\omega: f(\omega) \le x\} \in \mathcal{F}, \forall x \in \mathbb{R}$

Pf. 构造个 Monotone to contain $(-\infty, x]$

$$\mathcal{M}_B = \{B \in \mathbb{B}_R : f^{-1}(B) \in \mathcal{F}\}$$

$$(-\infty, x] \in \mathcal{M}_B$$

$$f^{-1}(B^c) = (f^{-1}(B))^c$$

$$f^{-1}(\cup_i B_i) = \cup_i f^{-1}(B_i)$$

$$f^{-1}(\cup_i B_i) = \{\omega: f(\omega) \in \cup_i B_i\}$$

$\omega \in f^{-1}(\bigcup_i B_i) \Rightarrow \exists\, i_0(\omega)$ s.t.

$f(\omega) \in B_{i_0} \quad \omega \in f^{-1}(B_{i_0})$

$\omega \in \bigcup_i f^{-1}(B_0^i)$

$\Rightarrow f(\omega) \in \bigcup_i B_i$

Lemma: if f_1, f_2 are r.v., the $f_1 + f_2$ is a r.v.

<u>Pf</u>: 等价于证明:

$\{\omega : f_1(\omega) + f_2(\omega) < x\} \in \mathcal{F}$

$\{\omega : f_1(\omega) < q,\ f_2(x) < x - q\} \quad q \in \mathbb{Q}$

$A_q \in \mathcal{F} \quad \because \{\omega : f_1(\omega) < q\} \cap \{\omega : f_2(\omega) < x - q\}$

$$\bigcup_{q \in \mathbb{Q}} A_q = B$$

$\because \omega \in B \quad f_1(\omega) + f_2(\omega) = x - \delta$ for som $\delta > 0$

$$f_1(\omega) < q < f_1(\omega) + \frac{\delta}{3}$$

$$x - \delta + \frac{\delta}{3} - q > f_2(\omega) = x - \delta - f_1(\omega) > x - \delta - q$$

$$x - \frac{2\delta}{3} - q > f_2(\omega) > x - \delta - q$$

for $A \in \mathcal{F}$

$$1_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$$

for $A_i,\ 1 \le i \le n \in \mathcal{F}$

$\sum c_i \mathbb{1}_{A_i}(\omega)$ is a r.v. and is called a simple function

<u>Lemma</u> if f is a bounded r.v. Then
$\exists \{f_n\}$, $f_n \in S$ s.t. $f_n \to f$ (uniformly)

<u>Integration</u> (or Expectation) $\Omega, \mathcal{F}, P$

when $f = \mathbb{1}_A(\omega)$

$$\int \mathbb{1}_A(\omega)\, dp = P(A)$$

$$\int \sum_{i=1}^{n} c_i \mathbb{1}_{A_i}(\omega)\, dp = \sum_{i=1}^{n} c_i P(A_i)$$

if f is bounded, Let $f_n \in S \xrightarrow[\text{uniformly}]{} f$

$$\int f\, dP = \lim_{n\to\infty} \int f_n\, dP$$

有两个问题:

1、does the limit exist ?

2、does the sequace f_n matter ? (不同的序列是否收敛到不同极限)

先看第一个:

$\forall \varepsilon$, $\exists N_\varepsilon$, s.t. $\forall n \geq N_\varepsilon$

$$|f_n(\omega) - f(\omega)| < \varepsilon$$

$$|f_n(\omega)| < M + \varepsilon \quad \forall n > N_\varepsilon$$

$\forall\, n_1, n_2 > N_\varepsilon$

$$\left| \int f_{n_1} dP - \int f_{n_2} dP \right| \le \int |f_{n_1} - f_{n_2}| dP \le 2\varepsilon.$$

$\therefore$ is Cauchy sequence, & compact

$\therefore$ converge.

再看第二个:

$$f_n \quad g_n$$

$$|f_n - g_n| < \varepsilon.$$

$\therefore$ 这个是 well-defined.

For non-negative f

$$\int f\, dP = \sup\left[\int g\, dP;\ g \le f,\ g \text{ is bound} \right]$$

下面来看 任意函数 (有正有负)

We say that f is integrable if

$$\int |f|\, dP < \infty$$

$$f_+ = \max\{f, 0\}$$

$$f_- = \max\{-f, 0\}$$

$$f = f_+ - f_-$$

$$|f| = f_+ + f_-$$

$$\int f\,dP = \int f_+\,dP - \int f_-\,dP. \qquad \boxtimes$$

这个证明trick叫 standard machine

逐渐往上扩张,

We say that a sequence of $\{f_n\} \to f$

(i) pointwise if $f_n(\omega) \to f(\omega) \ \forall \omega$

(ii) almost-sure if $P(\{\omega : f_n(\omega) \to f(\omega)\}) = 1$

(iii) in measure if $\forall \varepsilon > 0$ 依测度收敛.

$$P(\{\omega : |f_n(\omega) - f(\omega)| > \varepsilon\}) \xrightarrow[n\to\infty]{} 0$$

举例说明强弱

$H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ $\quad H_n \to \infty$ as $n \to \infty$

$$r_n(x) = \begin{cases} 1, & \text{if } x \in [H_n, H_{n+1}] \\ 0, & \text{o.w.} \end{cases}$$

$\Omega = (0,1]$

$$c_k(\omega) = \sum_{n=1}^{\infty} r_k(\omega + n)$$

可见 $c_k(\omega) =$ 0 → no integer translate falls in the interval $[H_n, H_{n+1}]$

1 → o.w.

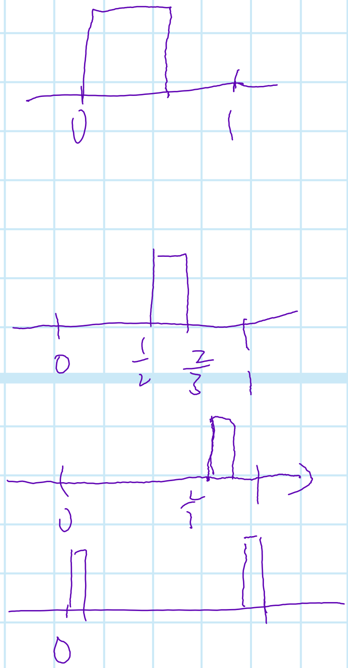
$E\left[l_k(\omega)\right]$

$\int l_k(\omega)\,dP = \frac{1}{k+1}$

$\limsup_{k} l_k(\omega) = 1$

$\liminf_{k} l_k(\omega) = 0$

$l_1(\omega)$, $l_2(\omega)$, $l_3(\omega)$, $l_4(\omega)$



$l_k \to 0$ is prob

but not almost sure.

<u>Lemma</u> if $f_n \to f$ a.s. (almost sure) then $f_n \to f$ in measure (in μ)

<u>Pf</u> $B = \{\omega : f_n(\omega) \to f(\omega)\}$, $P(B) = 1$

given $\varepsilon > 0$,

Let $A_n^\varepsilon = \{\omega : |f_n(\omega) - f(\omega)| > \varepsilon\}$

Define $B_n^\varepsilon = \bigcup_{m \geq n} A_m^\varepsilon = \{\omega : \sup_{m \geq n} |f_m(\omega) - f(\omega)| > \varepsilon\}$

$B_n^\varepsilon \downarrow B^\varepsilon = \{\omega : \limsup_{m} |f_m(\omega) - f(\omega)| > \varepsilon\}$

B_n 是 decreasing

$P(B^\varepsilon) = 0$

$P(B_m^\varepsilon) \downarrow P(B^\varepsilon) = 0$

$P(A_n^\varepsilon) \leq P(B_n^\varepsilon) \downarrow 0 \quad \therefore P(A_n^\varepsilon) \to 0$ $\blacksquare$

Lemma If $f_n \to f$ in μ, then $f_{n_i} \to f$ in a.s.

$\exists \{n_i\}$

$f_n \to f$ 不成立见上面反例，但是可以找一个子序列 $\{n_i\}$ f_{n_i} a.s.

Pf Given k,

We know $P\{\omega : |f_n(\omega) - f(\omega)| > \frac{1}{k}\} \xrightarrow[n\to\infty]{} 0$

$\therefore$ Can find n_k s.t.

$$P\{\omega : |f_{n_k}(\omega) - f(\omega)| > \frac{1}{k}\} \le \frac{1}{2^k}$$

$$A_k = \{\omega : |f_{n_k}(\omega) - f(\omega)| > \frac{1}{k}\}$$

$$B_k = \bigcup_{m \ge k} A_m \qquad B_k \downarrow B \qquad P(B) = 0.$$

$$P(B_k) \le \sum_{m \ge k} P(A_m) \le \frac{1}{2^{k-1}}$$

$$P(A_m) \le \frac{1}{2^m}$$

$$B \supseteq \{\omega : \limsup_k |f_{n_k}(\omega) - f(\omega)| > 0\}$$

Bounded Convergence Theorem

Let f be a bounded function and

f_n (is a uniformly bounded)

s.t. $f_n \to f$ in μ

then $\int f_n \, dP \to \int f \, dP$

Pf Let $A_n^{\varepsilon} = \{ \omega : |f_n(\omega) - f(\omega)| > \varepsilon \}$

$$|f_n - f| \leq \varepsilon \mathbb{1}_{(A_n^{\varepsilon c})} + 2M \mathbb{1}_{A_n^{\varepsilon}}$$

$$\left| \int f_n \, dP - \int f \, dP \right| = \left| \int (f_n - f) \, dP \right| \leq$$

$$\int |f_n - f| \, dP \leq \varepsilon P(A_n^{\varepsilon c}) + 2M P(A_n^{\varepsilon})$$

$$\therefore \limsup_n \left| \int f_n \, dP - \int f \, dP \right| = \varepsilon$$

Fatou's Lemma

Let $\{f_n\}$, f be non-negative measurable functions, $f_n \to f$ in measure. Then:

$$\int f \, dP \leq \liminf_n \int f_n \, dP$$

先來看一個弱的版本，如果 $f_n \to f$ pointwise

$$\int \liminf_n f_n \, dP \leq \liminf_n \int f_n \, dP$$

Pf. Let $0 \leq g \leq f$ ← bounded $\quad g_n = g \wedge f_n$ ← minimal

$f_n \to f$ in measure

$\Rightarrow g_n \to g$ in measure

$$\int g \, dP = \lim_{n\to\infty} \int g_n \, dP \quad \text{(BCT)}$$

$$\leq \liminf_n \int f_n \, dP$$

2019.9.16

Product space

$(\Omega_1, \mathcal{F}_1, P_1) \quad (\Omega_2, \mathcal{F}_2, P_2)$

$$\Omega_1 \times \Omega_2$$

$$\mathcal{F} = \sigma(\mathcal{F}_1 \times \mathcal{F}_2)$$

$A \times B \to$ measurable rectanble

$$A \in \mathcal{F}_1, \ B \in \mathcal{F}_2$$

Lemma. Finite disjoint union of measurable rectangle is an algebra. $\mathcal{A}$

$$(A_1 \times A_2)^c = (A_1^c \times A_2) \cup (A_1 \times A_2^c) \cup (A_1^c \times A_2^c)$$

$$\bigcup_i (A_{1i} \times A_{2i}) \bigcup \cup_j (B_{1j} \times B_{2j})$$

下面来证了这个属于 F
拆成一块一块的.

下面看如何在 $A \times B$ 上定义概率.

$A \times B \qquad P_1(A) P_2(B)$

For any $E \in \mathcal{A}$, $P(E) = \sum_i P_1(A_{1i}) P_2(A_{2i})$

$E = \bigcup_i A_{1i} \times A_{2i}$

P is a countability additive pseudo measure on $\mathcal{A}$

Assume o.w.

$\Rightarrow \exists E_n \downarrow \phi$

s.t. $P(E_n) \not\downarrow 0$

$$\bar{E}_{n,w_2} = \{ w_1 : (w_1, w_2) \in E_n \}$$

$E_{n,w_2} \downarrow \phi \qquad P_1(E_n, w_2) \downarrow 0$

$f_n(w_2) = P_1(E_n, w_2)$

$P(E) = \int f_n(w_2) \, dP_2$

<u>Lemma</u> BCI

if $\{A_i\}$ is any collection af events in F, s.t.

$$\sum_{i=1}^{\infty} P(A_i) < \infty$$

then, $P(A_n \text{ i.o.}) = 0$

A_n i.o. $= \{ w : w$ is present in infinite many A_is $\}$

i.o. = infinite often

Pf

$$B_n = \bigcup_{m \geq n} A_m$$

$$P(B_n) = \sum_{m=n}^{\infty} P(A_n) \xrightarrow[n \to \infty]{} 0$$

$B_n \downarrow \qquad B_n \supseteq B$

2014.4.19

Product Space & Fubini's Theorem

$(\Omega_1, \mathcal{F}_1, P_1)$ $(\Omega_2, \mathcal{F}_2, P_2)$

$\Omega = \Omega_1 \times \Omega_2$

$\mathcal{F} = \sigma(\mathcal{F}_1 \times \mathcal{F}_2)$

P : product measure

Lemma. Given any $A \in \mathcal{F}$, $A_{\omega_1} := \{\omega_2 : (\omega_1, \omega_2) \in A\}$

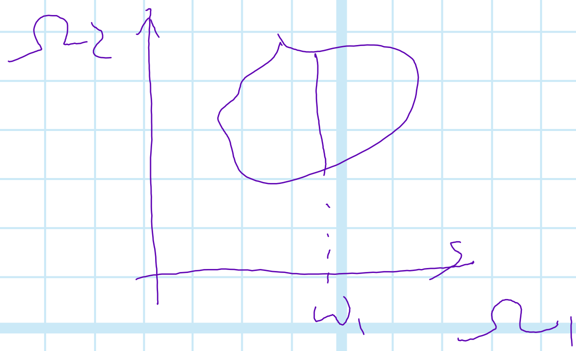
$B_{\omega_2} := \{\omega_1 : (\omega_1, \omega_2) \in A\}$

Then: (i) $A_{\omega_1} \in \mathcal{F}_2$, $\forall \omega_1$

and $B_{\omega_2} \in \mathcal{F}_1$, $\forall \omega_1$

(ii) $P(A) = \int P_2(A_{\omega_1})\, dP_1$

$= \int P_1(B_{\omega_2})\, dP_2$



Pf (i) $A = A_1 \times B_1$

$$A_{\omega_1} = \begin{cases} B_1, & \omega_1 \in A_1 \\ \Phi, & \omega_1 \notin A_1 \end{cases} \in F_2$$

$$\mathcal{M} = \{ A \in F, \text{ s.t. } A_{\omega_1} \in F_2 \ \forall \omega_1, \ B_{\omega_2} \in F_1 \ \forall \omega_2 \}$$

$$\mathcal{M} \supseteq A_1 \times B_1$$

下面逐一验证:

$\rightarrow$ $A \in \mathcal{M}$, then $A^c \in \mathcal{M}$

$$\because \bar{A}_{\omega_1} = \{\omega_2 : (\omega_1, \omega_2) \in A^c\}$$

$$= \{\omega_2 : (\omega_1, \omega_2) \in A\}^c = A_{\omega_1}^c \in F_2$$

$\rightarrow$ $A_i \in \mathcal{M}$ then $\bigcup_i A_i \in \mathcal{M}$

$$\left(\bigcup_i A_i\right)_{\omega_1} = \bigcup_i A_{i,\omega_1}$$

$$\{\omega_2 : (\omega_1, \omega_2) \in \cup_i A_i\}$$

$$= \bigcup_i \{\omega_2 : (\omega_1, \omega_2) \in A_i\}$$

下面证明 $P_2(A_{\omega_1})$ is measurble.

$$A_{\omega_1} = \begin{cases} \phi, & \omega_1 \notin A_1 \\ B_1, & \omega_1 \in A_1 \end{cases}$$

$$P_2(A_{\omega_1}) = \begin{cases} P_2(\emptyset), & \omega_1 \notin A_1 \\ P_2(B_1) & \omega_1 \in A_1 \end{cases}$$

2 1的证

$$\{\omega_1 : P_2(A_{\omega_1}) \le x\} \in \mathcal{F}_1$$

$$A_n^y \{\omega : f_n(\omega) \le y\} \qquad f_n \uparrow f$$

$$A^y \quad \{\omega : f(\omega) \le y\} \qquad A_n^y \downarrow A^y$$

$$\mathcal{M} = \{A : P_2(A_{\omega_1}) \ \& \ P_1(B_{\omega_2}) \text{ is monotone class } \forall \omega_1, \omega_2\}$$

Fubini's Theorem,

$f(\omega_1, \omega_2)$ is a measurable function.

$\forall \omega_1$: $g_{\omega_1}(\omega_2)$, $h_{\omega_2}(\omega_1)$ are measurable

and if f is integrable then $g_{\omega_1}(\omega_2)$ is a integrable for ω_1, $h_{\omega_2}(\omega_1)$ is integrable a.e.

$$G(\omega_1) = \int g_{\omega_1}(\omega_2)\, dP_2$$

$$H(\omega_2) = \int h_{\omega_2}(\omega_1)\, dP_1$$

$$\int f(\omega_1, \omega_2)\, dP = \int G(\omega_1)\, dP_1 = \int H(\omega_2)\, dP_2$$

<u>Pf</u> $A^x = \{(\omega_1, \omega_2) : f(\omega_1, \omega_2) \le x\}$

$$B_{\omega_1} = \{\omega_2 : g_{\omega_1}(\omega_2) \le x\}$$

$$= \{\omega_2 : f(\omega_1, \omega_2) \le x\}$$

$$= A^x_{\omega_1}$$

$$f = 1_A$$

$$G(\omega_1) = \int 1_{A_{\omega_1}}(\omega_2)\, dP_2$$

$$= P_2(A_{\omega_1})$$

$$H = P_1(B_{\omega_2})$$

例：$\Omega_1 \times \Omega_1 = \mathbb{N} \times \mathbb{N}$ 一个反例，说明可积重要性

$$f(i,j) = \begin{cases} 2^{i+j} & j = i+1 \\ -2^{i+j} & j = i-1,\ i \ge 2 \\ 0 & o.w. \end{cases}$$

$$p(i,j) = \frac{1}{2^{i+j}}, \quad i, j \geq 1 \qquad p_1(i) = \frac{1}{2^i} \qquad p_2(j) = \frac{1}{2^j}$$

$$G(i) = \sum_{j \geq 1} f(i,j) P_2(j)$$

$$= \sum_{j \geq 1} \frac{1}{2^j} f(i,j) = \frac{1}{2^{2\cdots}}$$

$$G(i) = \begin{cases} 0 & i \geq 2 \\ 2 & \end{cases}$$

$$H(j) = \begin{cases} 0 & j \geq 2 \\ -2 & j = 2 \end{cases}$$

$\therefore \int G(\omega_1) dP_1 = 1 \qquad \int H(\omega_2) dP_2 = -1$

$\therefore$ 如果 f 不可积 那么 Fubini 不成立

Characteristic functions

$f : \Omega \to \mathbb{R}$ 推广到复数

$f : \Omega \to \mathbb{C}$ complex number.

$$f(\omega) = f_R(\omega) + i f_I(\omega)$$

$$\int f(\omega)\,dP = \int f_R(\omega)\,dP + i\int f_I(\omega)\,dP$$

$$\left|\int f(\omega)\,dP\right| \le \int |f(\omega)|\,dP$$

$\Omega \to \mathbb{R}$

$X(\omega)$

$$\Phi(t) = \int e^{itX}\,dP$$

is measurable

$\because$ $\cos(tX(\omega))$ $\sin(tX(\omega))$

is measurable.

$|\Phi(t)| \le 1$

$\Phi(0) = 1$

Theorem $\Phi(t)$ is uniformly countinous function with $\Phi(0)=1$, $|\Phi(t)| \le 1$. which is positive definite.

$t_1 < t_2 < \cdots < t_n$

$$M_{ij} = \Phi(t_i - t_j) \succeq 0$$

$$x^* M x \ge 0$$

$$\phi(t_i - t_j) = \mathbb{E}\left[e^{i(t_i - t_j)X}\right]$$

$$x^* M x$$

$$= \sum_{k\,l} x_k^* E\left(e^{i(t_k - t_l)X}\right) x_l$$

$$= E\left[\sum_{k,l} x_k^* e^{i(t_k - t_l)X} x_l\right]$$

$$= E\left[\left|\sum_k x_k^* e^{it_k X}\right|^2\right]$$

2019.4.23

characteristic functions

1. properties & Definition
2. Fourier Inversion
3. Weak Convergence
4. Levy's Continuity theorem (one part)

$X(\omega)$: random variable.

$\Rightarrow e^{itX(\omega)}$: a complex valued random variable

$$|e^{itX(\omega)}| \leq 1$$

$\Phi(t) = E[e^{itX}]$ characteristic functions

性质:

1) $\Phi(0) = 1$

2) $|\phi(t)| \leq 1$

些 展式 sin, cos 表放.

3) $\phi(t)$ is a positive-definite function.

Given $\varepsilon > 0$,

$$|\phi(t) - \phi(s)| = |E[e^{itx}] - E[e^{isx}]|$$

$$= |E[e^{isx}(e^{i(t-s)x} - 1)]|$$

$$\leq E[|e^{isx}(e^{i(t-s)x} - 1)|]$$

$$= E[|e^{i(t-s)x} - 1|]$$

找一个δ使得当$t-s < \varepsilon$时$|\cdots| < \delta$

M_{ε_1}

$P(|x| > M) \leq \varepsilon_1$

$$|e^{i(t-s)x} - 1| \leq |t-s||x|$$

$$= \left|2\sin\left(\frac{(t-s)x}{2}\right)\right|$$

$$E[|e^{i(t-s)x} - 1|] \leq E[|t-s| M \mathbb{1}_{\{|x| \leq M\}} + 2\mathbb{1}_{\{|x| > M\}}]$$

$$\leq (t-s)M_{\varepsilon_1} + 2P(x > M) \qquad t-s = \frac{\varepsilon_1}{M_{\varepsilon_1}}$$

$$\leq (t-s)M_{\varepsilon_1} + 2\varepsilon_1 \leq 3\varepsilon_1$$

4) If X is integrable, then $\phi(t)$ is a countinously differentiable function of t

$$\frac{\phi(t+\delta)-\phi(t)}{\delta} = \frac{1}{\delta} E\left[e^{i(t+\delta)X} - e^{itX}\right]$$

$$= E\left[e^{itX}\left[\frac{e^{i\delta X}-1}{\delta}\right]\right]$$

$$= E\left[e^{itX}\left[\frac{e^{itX}-1}{i\delta X}\right]iX\right]$$

$$\lim_{\delta\to 0}\left[\frac{\phi(t+\delta)-\phi(t)}{\delta}\right] = E\left[e^{itX}iX\right]$$

$$\left\{\begin{array}{l} Y_{\delta_n} = e^{itX}\left[\frac{e^{i\delta_n X}-1}{i\delta_n X}\right]iX \\ Y_{\delta_n} \to e^{itX}iX = Y \\ \delta_n \to 0 \\ \lim_{n\to\infty} E[Y_{\delta_n}] = E[Y] \\ \cancel{x} \quad |Y_{\delta_n}| \le |X| \end{array}\right.$$

Lemma $\left| e^{ix} - \sum_{m=0}^{n} \frac{(ix)^m}{m!} \right| \le \min\left\{ \frac{|x|^{n+1}}{(n+1)!}, \frac{2(x)^n}{n!} \right\}$

Fourier Inversion formula

Let $a<b$ be continuity points of the distribution function.

$$F(b) - F(a) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \phi(t) \left[\frac{e^{-itb} - e^{-ita}}{-it} \right] dt$$

$$F(x) = P(X(w) \le x)$$

continuity points are countable.

Let $A_n = \{ x : F(x) - F_-(x) \ge \frac{1}{n} \}$

$|A_n| \le n$

Pf

$$\frac{1}{2\pi} \int_{-T}^{T} E[e^{itx}] \left[\frac{e^{-itb} - e^{-ita}}{-it} \right] dt$$

$$= \frac{1}{2\pi} \int_{-T}^{T} E\left[\frac{e^{it(x-b)} - e^{it[x-a]}}{-it} \right] dt$$

$$= \frac{1}{2\pi} \int_{-T}^{T} \left(\int \frac{e^{it(x-b)} - e^{it(x-a)}}{-it} \, dp \right) dt$$

$$\left[\begin{array}{l} |e^{ix}-e^{iy}| \le x-y \\ \therefore \text{中间那个积分是有界的.} \end{array}\right.$$

$$= \frac{1}{2\pi}\int\left(\int_{-T}^{T}\left(\frac{e^{it(x-b)}-e^{it(x-a)}}{-it}\right)dt\right)dP$$

利用 Fubini

$$g(\Omega, \mathbb{R}) = \frac{g(\omega)\mathbb{1}_{[-T,T]}}{2T}$$

Let $U(T,x) = \frac{1}{2\pi}\int_{-T}^{T}\frac{e^{itx}}{it}\,dt$

原式 $= \int\left[U(T,x-a) - U(T,x-b)\right]dP$

$U(T,x)$ is bounded for every x

$$|U(T,x)| \le 2$$

Dirichlet Integral

$$\lim_{T\to\infty} U(T,x) = \begin{cases} \frac{1}{2} & x>0 \\ -\frac{1}{2} & x<0 \\ 0 & x=0 \end{cases} = R(x)$$

$$= \int\left(R(x-a) - R(x-b)\right)dP$$

$$= \int\left[\frac{1}{2}\left(\mathbb{1}_{x>a} - \mathbb{1}_{x<a}\right) - \frac{1}{2}\left(\mathbb{1}_{x>b} - \mathbb{1}_{x<b}\right)\right]dP$$

$$= \frac{1}{2}\left(P(X>a) - P(X<a) - P(X>b) + P(X<b) \right)$$

$$= \frac{1}{2} P(X \in [a,b)) + \frac{1}{2} P(X \in (a,b])$$

$$= \frac{1}{2}\left[P(X=a) - P(X=b) + F(b) - F(a) + F(b) - F(a) \right]$$

几个特例

a) $X(\omega) = 0$ $\qquad E[e^{itX}] = 1$

b) $X \sim N(\mu, \sigma^2)$ $\qquad E[e^{itX}] = e^{-\frac{t^2\sigma^2}{2} + i\mu t}$

Weak Convergence

$X_1, \cdots, X_n \cdots$ are random variables

We say $X_n \xrightarrow{\text{weak}} X_*$ $\qquad P_n \overset{w}{\Longrightarrow} P$ weak * topology

Let $F_n(x)$ be the distribution function of X_n

$F(x)$ be the distribution function of X_*

if $F_n(x) \to F(x)$ p.w. $\forall x \in$ continuity points of $F(x)$

例: $X_n(\omega) = 1 + \frac{1}{n}$

$F_n(1) = 0 \ \forall n \qquad F(1) = 1 \qquad F_n(1) \not\to F(1)$

Levy's continuity theorem

The following are equivalent

(i) $f_n \overset{w}{\Rightarrow} f$ or $P_n \overset{w}{\Rightarrow} P$

(ii) $\int f_b \, dP_n \to \int f_b \, dP$ $\forall f_n$ if bounded and countions

(iii) $\Phi_n(t) \to \Phi(t)$ pointwise $\forall t$.

证明方式: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i)

(ii) $\Rightarrow$ (iii)

Just let $f_b = e^{itX}$

(i) $\Rightarrow$ (ii)　　　利用 compact 性质.

Pick $\varepsilon > 0$

Can find continuity points a and b s.t.

$F(a) < \varepsilon$, $F(b) > 1 - \varepsilon$

pick $\delta > 0$ and divide $[a, b]$ into disjoint parts
$b = a_{N_\delta}$

$[a, a_1]$, $(a_1, a_2]$, ..., $(a_{N_\delta - 1}, a_{N_\delta}]$

in each interval, the function fluctuates by at most δ w.l.o.g a_i is countinity points of f

为什么可行？∵紧集的开覆盖存在有限个覆盖

$$\hat{f} = \sum_{i=1}^{N_\delta} a_i \mathbb{1}_{I_i}$$

$$\int f \, dP = \lim_{n\to\infty} \int f_b \, dP_n =$$

传递顺序：

$$\boxed{\int f_b \, dP_n} \quad \int \hat{f} \, dP_n \quad \int \hat{f} \, dP \quad \int f_b \, dP$$

$$\limsup_{n\to\infty} \left| \int f_b \, dP_n - \int \hat{f} P_n \right| = \left| \int (f_b - \hat{f}) \, dP_n \right|$$

$$\leq \int_\Omega |f_b - \hat{f}| \, dP_n$$

$$\leq M P_n([a,b)') + \delta \left[F(a_{N_\delta}) - F_b(a) \right]$$

$$\leq 2N_\delta + \delta$$

下面从 $\int \hat{f} \, dP_n$ 到 $\int \hat{f} \, dP$

$$\int \hat{f} \, dP_n = \sum_{i}^{N_\delta} f_b(a_i) \left[f_1(a_i) - f_1(a_i) \right]$$

$$\int \hat{f} \, dP = \sum_{i}^{N_\delta} f_b(a_i) \left[f(a_i) - f_1(a_i) \right]$$

下面从 $\int \hat{f} \, dP$ 到 $\int f_b \, dP$

$$\limsup_{n\to\infty} \left| \int f_b \, dP_n - \int f_b \, dP \right| \leq 2N_\delta + 2\delta$$

2014.9.26.

Levy's continuity Theorem

The following are equivelent

(i) $P_n \overset{w}{\Longrightarrow} P$ and/or $F_n \overset{w}{\Longrightarrow} F$

(ii) $\forall$ bounded contioas functions f_n

$$\int f_b \, dP_n \to \int f_b \, dP$$

(iii) $\phi_n(t) \to \phi(t)$

when $\phi_i(t)$ and $\phi(t)$ are the ch. fns of P_i & P

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$$

Prokhorov's theorem

If $x_1, x_2, \cdots$ is any sequence of r.v.

then $\exists$ a subsequence n_k s.t.

$$P_{n_k} \overset{w}{\Longrightarrow} Q$$

where Q is non-decreasing

$$0 \le \phi(x) \le 1$$

Q is right countinous

这里Q和CDF稍微有点区别

只有 $Q(-\infty)\geq 0$, $\phi(\infty)\leq 1$ 的约束.

而CDF是等式约束.

Lemma

Let $\phi_n(t)$ be the characteristic function of $P_n(X_n/F_n)$ if $\phi_n(t)\to\psi(t)$ and $\psi(0)=1$ & $\psi(t)$ is continous at $t=0$, then $\psi(t)$ is the characteristic function of some distribution G and $P_n\overset{w}{\Rightarrow}G$

这个Lemma比较强.

$q\in Q$ Let $q_1, q_2, \cdots$ be an enumeration of Q

$F_1(q_1)$ $F_2(q_1)$ $F_3(q_1)$ $\cdots$

$\exists$ $n_k^{(1)}$ s.t. $F_{n_k^{(1)}}(q_1)$ converges. 有界数列存在收敛子序列.

$F_{n_1^{(1)}}(q_1)$, $F_{n_2^{(1)}}(q_1)$, $F_{n_3^{(1)}}(q_1)$ $\cdots$ 收敛.

新序列. $F_{n_1^{(1)}}(q_2)$, $F_{n_2^{(1)}}(q_2)$, $F_{n_3^{(1)}}(q_2)$ 存在收敛子序列:

$F_{n_1^{(2)}}(q_2)$, $F_{n_2^{(2)}}(q_2)$, $F_{n_3^{(2)}}(q_2)$

同样能得到：

$F_{n_i^{(k)}}(q_j)$ converges $1 \le j \le k$

∵是不停取子序列，∴最后的每个序列

对所有的 $q_1 \cdots q_j$ 都收敛。

$G_k(q) := F_{n_k^{(k)}}(q)$

Observation

$G_k(q)$ converges $\forall q \in Q$ (diagnolization)

$\to G(q)$

例如：

G_l, $G_l(q_l) := F_{n_l^{(l)}}(q_l)$

$G_{l+1}(q_l) = F_{n_{l+1}^{(l+1)}}(q_l)$

$G_{l+2}(q_l) = F_{n_{l+2}^{(l+2)}}(q_l)$

∵ $F_{n_{l+k}^{(l+k)}}(q_l)$ 都是 $F_{n_l^{(l)}}(q_l)$

中抽出的子序列

收敛。

$\exists$ a subsequence

$F_{n_k^{(k)}}(q) := F_{m_k}(q) \to G(q) \quad \forall q \in Q$

$\forall x \in \mathbb{R} \setminus Q$

$G(x) := \inf_{\substack{q > x \\ q \in Q}} G(q)$ 从 Q 扩展到 $\mathbb{R}$.

∵ G 是 non-decreasing

∴ 用 inf 来表出来.

<u>Lemma</u> $G(x)$ is right-contious.

pf let $x_n \downarrow x$, $G(x_n) \geq G(x)$

$\Rightarrow \quad \lim G(x_n) \geq G(x)$

suppose $q > x$, then $\exists N$, s.t. $x \leq x_n \leq q$ $\forall n \geq N$ ∴ $G(q) \geq G(x_n)$ $\forall n \geq N$

$G(q) \geq \lim_n G(x) \geq G(x)$

$G(x) = \inf_{q > x} G(q) \geq \lim_n G(x) \geq G(x)$

<u>Lemma</u> $G_k(x) \to G(x)$ at all contiously point at G.

<u>Proof</u> Let $x_k \to x$.

$r > x \quad r \in Q$

$$G_k(r) \geq G_k(x)$$

$$\downarrow$$

$$G(r)$$

$$G(r) \geq \limsup_k G_k(x)$$

$$G(x) = \inf_{r>x} G(r) \geq \limsup_k G_k(x) \qquad ①$$

$y < x$

then take r

$$y < r < x$$

$$G_k(x) \geq G_k(r) \geq G_k(y)$$

$$\liminf_k G_k(x) \geq G(r) \geq \liminf_k G_k(y)$$

$$G(r) \geq G(y)$$

$$\liminf_k G_k(x) \geq G(y) \quad \forall\, y < x \qquad ②$$

由①②

∴ if x is contiuous point of G,

as $y \to x$, $G(y) \to G(x)$.

$$\phi_{n_k}(t) = \int e^{itx}\, dF_n$$

$$\frac{1}{2T}\int_{-T}^{T} \phi_{n_k}(t)\, dt = \frac{1}{2T}\int_{-T}^{T}\left[\int e^{itx}\, dF_{n_k}\right]$$

$$\left(\int e^{itX_{n_k}} dP_{n_k} = \int e^{itx} dF_{n_k}\right)$$

change of variable

$$(\Omega_1, F_1, P) \xrightarrow{X} (\Omega_2, F_2, Q_2) \xrightarrow{Q}$$

$$\int Q(X^{-1})\, dQ = \int T\, dP$$

$$= \int\left(\frac{1}{2T}\int_{-T}^{T} e^{itx}\, dt\right) dF_{n_k} \quad (\text{Fubini})$$

$$= \int_{\mathbb{R}} \frac{1}{2T}\left(\frac{e^{ixT} - e^{-ixT}}{ix}\right) dF_{n_k}$$

$$= \int_{\mathbb{R}} \frac{\sin(xT)}{xT}\, dF_{n_k}$$

$$\left|\int_{\mathbb{R}} \frac{\sin(xT)}{xT}\, dF_{n_k}\right|$$

$$\leq \int \left| \frac{\sin(xT)}{xT} \right| dF_{n_k}$$

$$\leq \int_{(-l,l)} \left| \frac{\sin xT}{xT} \right| dF_{n_k} + \int_{[-l,l]^c} \left| \frac{\sin(xT)}{xT} \right| dF_{n_k}$$

$$\leq \int_{(-l,l)} 1 \, dF_{n_k} + \frac{1}{Tl} \int_{(-l,l)^c} dF_{n_k}.$$

$$= F_{n_k}(l) - F_{n_k}(-l) + \frac{1}{Tl}\left[1 - F_{n_k}(l) + F_{n_k}(-l)\right]$$

w.l.o.G. l is a contious point.

$$\lim_{k\to\infty} \frac{1}{2T} \int_{-T}^{T} \phi_k(t)\, dt = \frac{1}{2T} \int_{-T}^{T} \phi(t)\, dt$$

$$\frac{1}{2T} \int_{-T}^{T} \phi(t)\, dt = G(l) - G(-l) + \frac{1}{Tl}\left[1 - G(l) + G(-l)\right]$$

$$\Rightarrow F_{n_k}(l) \to G(l) \qquad F_{n_k}(-l) \to G(-l)$$

$T = \frac{1}{\sqrt{l}}$ 代入

$$\frac{\sqrt{l}}{2} \int_{-\frac{1}{\sqrt{l}}}^{\frac{1}{\sqrt{l}}} \phi(t)\, dt = G(l) - G(-l) + \frac{1}{\sqrt{l}}\left[1 - G(l) + G(-l)\right]$$

当 $l \to \infty$ 时 该积分 $\to 1$.

$\because$ $\phi(t)$ is contions at $t=0$ & $\phi(0)=1$

$\forall \varepsilon>0, \exists \delta$ if $|t|<\delta$ then

$$|\phi(t)-1|<\varepsilon$$

$\forall l: \frac{1}{\sqrt{l}} < \delta$

$$\frac{\sqrt{l}}{2}\int_{-\frac{1}{\sqrt{l}}}^{\frac{1}{\sqrt{l}}}\phi(t)\,dt = \underbrace{\frac{\sqrt{l}}{2}\int_{-\frac{1}{\sqrt{l}}}^{\frac{1}{\sqrt{l}}} 1\,dt}_{=1} + \underbrace{\frac{\sqrt{l}}{2}\int(\phi(t)-1)\,dt}_{\le\varepsilon}$$

$$G(\infty)-G(-\infty)=1$$

$\because G(\infty)\le 1$, $G(-\infty)=0$

$\therefore$ 只能 $G(\infty)=1$, $G(-\infty)=0$

$\therefore G$ 是分布函数.

下面回到 Prohorov's Threom 来说明 G 唯一

$F_{n_k} \xrightarrow{w} G$ 若 $F_{m_k} \not\Rightarrow G$. 假设存在子序列不收敛到 G.

$\hookrightarrow$ sub-sub $\hat{F}_k \Rightarrow G'$

Suppose $F_n \not\Rightarrow G(x)$ $\forall$ contious points of G

then $\exists x_0 \to$ a contiuous point subsequence

$$\liminf |F_{n_k}(x_0)-G(x_0)| > \varepsilon$$

就是说 如果找到一个不收敛到 G 的序列,

我们可以找一个子序列收敛到G，这矛盾.

2019. 9. 30

→ Independent of random variable

→ Characterization

→ weak law of large number

→ Lemma's about consistency

We define event A and B to be "independent" if $P(A \cap B) = P(A)P(B)$

We define random variables X_1, X_2 to be independent if $\forall A, B \in \mathbb{B}_{\mathbb{R}}$

$$P(\{w : X_1(w) \in A, X_2(w) \in B\}) = P(\{w : X_1(w) \in A\}) \cdot P(\{w : X_2(w) \in B\})$$

<u>Lemma</u>: X_1 & X_2 are independent iff the distribution induced by the mapping $(\Omega \times \Omega) \xrightarrow[(X_1, X_2)]{} \mathbb{R}^2$ $X_1(\omega_1), X_2(\omega_2)$ is the product measure.

$(\omega_1, \omega_2) \longrightarrow X_1(\omega_1)\ X_2(\omega_2)$

$A \in \mathcal{B}_{\mathbb{R}^2} \quad P_1(\{(\omega_1, \omega_2) : X_1(\omega_1), X_2(\omega_2)\} \in A\}$

$A = A_1 \times B_1$

$P(\{(\omega_1, \omega_2) \in A_1 \times B_1\})$

$= P_1(A_1) P_2(B_1)$

$= P(\{\omega_1 : X_1(\omega_1) \in A_1\}) P(\{\omega_2 : X_2(\omega_2) \in B_1\})$

$X_1, \dots X_n$ are mutually independent if

$\forall A_1, \dots A_n$

$P(\{\omega : X_1(\omega) \in A_1, X_2(\omega) \in A_2, \dots, X_n(\omega) \in A_n\})$

$= \prod_{i=1}^{n} P(\{\omega : X_i(\omega) \in A_i\})$

Exercise :

if $X_1\ X_2$ are independent and integrable.

then $E(X_1 X_2) = E[X_1]\,E[X_2]$

要证 $\int X_1(\omega) X_2(\omega)\, dP(\omega) = \left(\int X_1(\omega) dP\right)\left(\int X_2(\omega) dP\right)$

用 standard machine

$X_1 = \sum_i c_i \mathbb{1}_{A_i}$

$X_2 = \sum_j d_j \mathbb{1}_{B_j}$

$X_1 X_2 = \sum_{i,j} c_i d_j \mathbb{1}_{A_i \cap B_j}$.

$$\sum c_i d_j P(A_i \cap B_j)$$
$$= \sum c_i d_j P(A_i) P(B_j)$$
$$= \left(\sum_i c_i P(A_i)\right)\left(\sum_j d_j P(B_j)\right)$$

standard machine 就是从 simple function 开始逐步往高级的推广

Lemma X_1, X_2 are independent, the $f(X_1)$ $f(X_2)$ are independent.

Corollary $E\left(e^{it(X_1+X_2)}\right] = E[e^{itX_1}]E[e^{itX_2}]$

X_1, X_2 are independent.

$$E[e^{itX_1} e^{itX_2}] = E(e^{itX_1}) E[e^{itX_2}]$$

Remark Converse is not true

Corollary X_1 & X_2 are independent iff

$$\phi(t_1, t_2) = E\left[e^{(t_1 X_1 + t_2 X_2)}\right]$$
$$= E[e^{it_1 X}] E[e^{it_2 X_2}] \quad \forall t_1, t_2$$

Example: $f(x_1, x_2) = \begin{cases} \frac{1}{4}(1 + x_1 x_2 (x_1^2 - x_2^2)), & |x_1| \le 1, \ |x_2| \le 1 \\ 0, & \text{o.w.} \end{cases}$

举例说明上面那个 converse is not true

边缘分布: $f_{X_1}(x_1) = \int_{\mathbb{R}} f(x_1, x_2)\, dx_2$

$$f_{X_1}(x_1) = \begin{cases} \frac{1}{2}, & |x_1| \le 1 \\ 0, & \text{o.w.} \end{cases}$$

$$f_{X_2}(x_2) = \begin{cases} \frac{1}{2}, & |x_2| \le 1 \\ 0, & \text{o.w.} \end{cases}$$

$$f_{X_1+X_2}(y) = \begin{cases} \frac{1}{4}(2+y) & -2 \le y \le 0 \\ \frac{1}{4}(2-y) & 0 \le y \le 2 \end{cases}$$

$$\phi_{X_1+X_2}(t) = \frac{\sin^2(t)}{t^2}$$

下面来证弱大数定律.

Weak Law of large number.

Theorem VI.

Let $X_1, X_2, X_3, \cdots$ be mutually independent random variable. (zero-mean) with

$$\sup_i E[X_i^2] \le C < \infty$$

Then $\frac{X_1 + \cdots + X_n}{n} \to 0$ in measure.

$\forall \varepsilon > 0 \quad P_r\left\{ \omega : \left| \frac{X_1 + \cdots X_n}{n} \right| > \varepsilon \right\} \to 0 \quad \text{as } n \to \infty.$

Pf: let $S_n = X_1 + \cdots X_n$

$$P_r\left\{ \left| \frac{S_n}{n} \right| > \varepsilon \right\} \le \frac{E\left[\left(\frac{S_n}{n}\right)^2\right]}{\varepsilon^2}$$

$$\varepsilon^2 \mathbb{1}_{\left|\frac{S_n}{n}\right| > \varepsilon} \le \left(\frac{S_n}{n}\right)^2 \mathbb{1}_{\left|\frac{S_n}{n}\right| > \varepsilon} \le \left(\frac{S_n}{n}\right)^2$$

$$\frac{E[S_n^2]}{n^2 \varepsilon^2} = \frac{E\left[\left(\sum_{i=1}^n X_i\right)^2\right]}{n^2 \varepsilon^2}$$

$$= \frac{E\left[\sum_{i=1}^n X_i^2 + 2 \sum_{i<j} X_i X_j\right]}{n^2 \varepsilon^2}$$

$$= \frac{\sum_{i=1}^n E[X_i^2]}{n^2 \varepsilon^2} + \frac{2 \sum_{i<j} E[X_i X_j]}{n^2 \varepsilon^2}$$

$$\le \frac{nC}{n^2 \varepsilon^2} = \frac{C}{n \varepsilon^2}$$

下面来把 $\sup_i E[X_i^2] \le C$ 放松到一阶.

Suppose X is integrable, 即
$E[|X|] < \infty$

$$Y_M = |X|\,\mathbb{1}_{|X|>M} \qquad X_M = |X|\,\mathbb{1}_{|X|\le M}$$

$$|X| = X_M + Y_M$$

$$E[|X|] = E[X_M] + E[Y_M]$$

$$X_M \uparrow |X| \qquad E[X_M] \uparrow E[|X|]$$

Let $\lim\limits_{M\to\infty} E[Y_M] = 0$.

We call a collection of random variable to be uniformly integrable if

$$\lim_{M\to\infty} \sup_{\alpha} E[|X_\alpha| \cdot \mathbb{1}_{|X_\alpha|>M}] = 0.$$

Theorem V2

Let $X_1, X_2, \dots X_n$ be zero-mean, mutually independent uniformly integrable (U-I) collection of random variable. Then,

$$\frac{X_1 + \cdots + X_n}{n} \to 0 \quad \text{in measure.}$$

相比于 V_1，这个条件更宽松。

$\because \sup\limits_i E[X_i^2] \le C < \infty \Rightarrow$ U-I.

$\because E[|X|^\alpha]^{\frac{1}{\alpha}}$ is increasing in α

先证这个 T

$\alpha > \beta$

$E[|X|^\alpha]^{\frac{1}{\alpha}} \geq E[|X|^\beta]^{\frac{1}{\beta}}$

Let $|X|^\beta = Y$

$E[|Y|^{\frac{\alpha}{\beta}}]^{\frac{\beta}{\alpha}} \geq E[|Y|]$ $\quad \frac{\alpha}{\beta} > 1$

$f(x) = x^{\frac{\alpha}{\beta}}$ is convex by Jessen-inequal

$E[f(|Y|)] \geq f(E[|Y|])$

$E[|Y|^{\frac{\alpha}{\beta}}] \geq E[|Y|]^{\frac{\alpha}{\beta}}$

下面来证如下引理.

Lemma $\sup_i E[|X_i|^{1+\varepsilon}] \leq C$ then U-I

Pf

$$E\left[|X_i| \mathbb{1}_{|X_i| > M}\right]$$

$$E[M^\varepsilon |X_i| \mathbb{1}_{|X_i|>M}] \leq E\left[|X_i|^{1+\varepsilon} \mathbb{1}_{|X_i|>M}\right]$$

$$\leq E\left[|X_i|^{1+\varepsilon}\right] \leq C \qquad \boxtimes$$

Exercise

$\exists \{X_i\}$ $\sup_i E[|X_i|] \leq C < \infty$ but not U-I

$|X_i|^{1+\varepsilon}$ $\varepsilon > 0$ 的时候都是 U-I 但当 $\varepsilon = 0$ 的时候就不是 U-I]

(3) 증) Theorem V2.

Pf Let $\hat{X}_i^M = X_i \mathbb{1}_{|X_i| \le M}$

$\hat{Y}_i^M = X_i \mathbb{1}_{|X_i| > M}$

Given $\delta > 0$ $\exists M$, s.t. U.I. X_i

$$\sup_i E[|X_i| \mathbb{1}_{|X_i| > M}] < \delta$$

$$a_i = E[\hat{X}_i^M] = -E[\hat{Y}_i^M]$$

$$E[\hat{X}_i^M + \hat{Y}_i^M] = E[X_i] = 0$$

$$\left|E[\hat{Y}_i^M]\right| \le E[|\hat{Y}_i^M|]$$

$$= E[|X_i| \mathbb{1}_{|X_i| > M}] \le \delta \quad \forall i$$

$$S_n = \sum_{i=1}^n X_i = \sum_{i=1}^n (\hat{X}_i^M - a_i) + \sum_{i=1}^n (\hat{Y}_i^M + a_i)$$

$$E\left[\left|\frac{S_n}{n}\right|\right] \le E\left[\frac{\left|\sum_{i=1}^n (\hat{X}_i^M - a_i)\right|}{n}\right] + \sum_{i=1}^n \frac{1}{n} \underbrace{E\left[|\hat{Y}_i^M| + |a_i|\right]}_{2\delta}$$

$$E\left[\frac{\left|\sum_{i=1}^n (\hat{X}_i^M - a_i)\right|}{n}\right] \le \sqrt{E\left(\frac{\sum_i \hat{X}_i^M - a_i}{n^2}\right)^2} = \sqrt{\frac{\sum_i E((\hat{X}_i^M - a_i)^2)}{n^2}}$$

$$\le \frac{M+\delta}{\sqrt{n}}$$

$$E\left((Y_1+Y_2+\cdots+Y_n)^2\right)=\sum_i E(Y_i^2)$$ 最后一步

$$E\left[(\hat{X}_i^m-a_i)^2\right]\le(m+\delta)^2$$

$$\limsup_{n\to\infty} E\left[\left|\frac{S_n}{n}\right|\right]\le 2\delta$$

$$\limsup_{n\to\infty} E\left[\left|\frac{S_n}{n}\right|\right]=0.$$

下面看如何推到 ∞

$X_1,\cdots,X_n \quad \Omega_1\times\cdots\times\Omega_n,$

cylinder $\times\mathbb{R}\times\mathbb{R}\cdots$

$(\mathbb{R},\mathbb{B}_{\mathbb{R}},P_1),\ (\mathbb{R}^2,\mathbb{B}_{\mathbb{R}^2},P_2),\cdots(\mathbb{R}^m,\mathbb{B}_{\mathbb{R}^m},P_m)$

they are consistent if $P_m(\underset{\mathbb{B}_{\mathbb{R}^k}}{\underset{\cap}{A}}\otimes\mathbb{R}^{m-k})=P_k(A)$

于是往低维上投影后的概率不变.

Lemma Any set $A\in\mathbb{B}_{\mathbb{R}^n}$ $\exists\ K\subseteq A$ which is close and bounded. $\varepsilon>0$ $P(A\setminus K)\le\varepsilon$

$n=1$ $(a,b]\xleftarrow[n\to\infty]{}[a+\frac{1}{n},b]$

$$P([a+\tfrac{1}{n}, b]) = F(b) - F(a+\tfrac{1}{n})$$

$$F(a+\tfrac{1}{n}) \to F(a) \quad n \to \infty$$

right countiounty

下面证 $\mathcal{A}$ 是 monotone.

$$\bigcup_{i=1}^{\infty} (a_i, b_i]$$

$$\mathcal{A} = \{A : A \subseteq \mathbb{B}^{R}, \exists K \subseteq A, K \text{ "cpt"}, P(A \setminus K) \le \varepsilon \ \forall \varepsilon > 0\}$$

$A_i \uparrow A_\infty \quad A$

$A_i \in \mathcal{A} = | \ A_\infty \in \mathcal{A}$

$P(A_i) \uparrow P(A) \quad P(A \setminus A_n) \le \frac{\varepsilon}{2}$

2019.10.3

1. Kolmogorov's consistency theorem

2. Strong law of large number

Sequence n

$(\mathbb{R}^n, \mathbb{B}_{\mathbb{R}^n}, P_n)$: finite dimensional consistant distributions.

Then $\exists$ P on $(\mathbb{R}^\infty, \mathbb{B}_{\mathbb{R}^\infty})$ s.t.

$$P(\pi_n^{-1}(A)) = P_n(A) \quad \forall A \in \mathbb{R}_{\mathbb{R}^n}$$

$\uparrow$ sylindar

consistant: 就是在维度不断增加的过程中，投影概率相同.

<u>Proof</u>: collection of finite-dimensional cylinders is a field $\pi_n^{-1}(A)$ for some n, $A \in \mathbb{B}_{\mathbb{R}^n}$

先证 $\tilde{A} \cup \tilde{B}$ 是 finite measure.

$\exists n_1, A_1 \in \mathbb{B}_{\mathbb{R}^{n_1}}$ s.t. $\tilde{A} = \pi_{n_1}^{-1}(A_1)$

$\exists n_2, B_1 \in \mathbb{B}_{\mathbb{R}^{n_2}}$ s.t. $\tilde{B} = \pi_{n_2}^{-1}(B_1)$

$\pi_{n_2}^{-1}(A_1 \times \mathbb{R}^{n_2 - n_1}) = \tilde{A}$

例: $\tilde{A} = (x_1, x_2, \cdots,)$

where $x_1 \in [-1, 1]$

$\tilde{B}_2 = (x_1, \cdots)$

$x_1 \in [0,2]$ & $x_2 \in [1,3]$

下面定义: countably additive prob on the field

$$P(\tilde{A}) = P_n(A)$$

<u>Lemma</u> P is countably additive on the field.

<u>Pf</u> 反证, Assume not.

$A_n \downarrow \phi$, $P(A_n) \geq \delta \ \forall n$

$\uparrow$

$\cancel{\phi}$

Assume $k(n)$ is increase.

$$B_{k(n)} \in \mathcal{B}_{\mathbb{R}^{k(n)}} \quad A_n = \pi^{-1}_{(n)}(B_{k(n)}) \ \forall n$$

$\exists K_{k(n)} \subseteq B_{k(n)}$

s.t. $P(B_{k(n)} \setminus K_{k(n)}) \leq \frac{\delta}{2^{n+2}}$

$$C_n = \pi^{-1}_{k(n)}(K_{k(n)}) \subset A_n$$

$$D_n = \bigcap_{m=1}^{n} C_m \subset A_n$$

$$P(A_n \setminus D_n) = P\left(A_n \cap \left(\bigcap_{m=1}^{n} C_m\right)^c\right)$$

$$= P\left(A_n \cap \left(\bigcup_{m=1}^{n} C_m^c\right)\right)$$

$$= P\left(\bigcup_{m=1}^{n} A_n \cap C_m^c\right)$$

$$\le P\left(\bigcup_{m=1}^{n} (A_m \cap C_m^c)\right)$$

$$\le \sum_{m=1}^{n} P(A_m \cap C_m^c)$$

$$= \sum_{m=1}^{n} P_{K(m)}\left(B_{K(m)} \setminus K_{K(m)}\right)$$

$$\le \sum_{m=1}^{n} \frac{\delta}{2^{K(m)+2}}$$

$$\le \frac{\delta}{4} \sum_{m=1}^{\infty} \frac{1}{2^m}$$

$$= \frac{\delta}{4}$$

$P(D_n) > \frac{3\delta}{4}$ not empty

和 $D_n \downarrow \emptyset$ 矛盾

下面看强大数定律.

V1. Let $X_1, X_2, \cdots, X_n, \cdots$ be zero mean independent random variables s.t.

$$E[X_i^4] \le C \quad \forall i.$$

Then $\dfrac{X_1 + \cdots + X_n}{n} \to 0$ a.s.

$S_n = X_1 + \cdots + X_n$

<u>Pf</u>

$$P_r\left(\left|\frac{S_n}{n}\right| \geq \varepsilon\right) \le \frac{E[S_n^4]}{\varepsilon^4 n^4}$$

Markov inequality

$$E[S_n^4] = E\left[\left(\sum_{i=1}^{n} X_i\right)^4\right] = \sum_{i=1}^{n} E[X_i^4] + 6\sum_{i<j} E[X_i^2 X_j^2]$$

$$\left(\because E[X_i^2 X_j^2] \le \sqrt{E(X_i^4)E(X_j^4)} \le C\right)$$

$$\le nC + 6\binom{n}{2}C$$

$$= n + 3(n(n-1))$$

$$\le 6n^2 C$$

$$P\left(\left|\frac{S_n}{n}\right| \geq \varepsilon\right) \le \frac{6C}{\varepsilon^4 n^2}$$

$$A_n = \left\{\omega : \left|\frac{S_n(\omega)}{n}\right| > \varepsilon\right\}$$

$$P(A_n) \le \frac{6C}{\varepsilon^4 n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} P(A_n) < \infty$$ i.o. : infinite often

$$\Rightarrow P(A_n \text{ i.o.}) = 0.$$

$$P\left(\left\{ \omega : \left|\frac{S_n(\omega)}{n}\right| > \varepsilon \text{ i.o.}\right\}\right) = 0$$

$$\Rightarrow P\left(\left\{ \omega : \limsup_n \left|\frac{S_n(\omega)}{n}\right| \le \varepsilon\right\}\right) = 1.$$

$$A_\varepsilon = \left\{ \omega : \limsup_n \left|\frac{S_n(\omega)}{n}\right| \le \varepsilon \right\}$$

$$P(A_\varepsilon) = 1 \quad \forall \varepsilon > 0$$

$$\varepsilon = \frac{1}{m}$$

$$A_{\frac{1}{m}} \downarrow A_0 \quad P(A_{\frac{1}{m}}) = 1 \ \forall m \quad P(A_0) = 1$$

$$\therefore P\left(\left\{ \omega : \limsup_n \left|\frac{S_n(\omega)}{n}\right|\right\}\right) = 0$$

V2,

Let $X_1, X_2, \cdots, X_n, \cdots$ be independent identically distribution random variable non-negative.

证 Step 1. $Y_n = X_n 1_{X_n \le n}$

$$Z_n = X_n - Y_n = X_n 1_{X_n > n}$$

$$P(Z_n > 0) = P_r(X_n > n)$$

$$\sum_{n=1}^{\infty} P(Z_n > 0) = \sum_{n=1}^{\infty} P(X_n > n)$$

$$= \sum_{n=1}^{\infty} P(X > n) < \infty$$

$$\sum_{i=1}^{\infty} \int 1_{\{X > n\}} \, dP$$

$$\leq \int \left(\sum_{n=1}^{\lfloor X \rfloor} 1 \right) dP$$

$$= E(\lfloor X \rfloor) \leq E[X] < \infty$$

$$P_r(Z_n > 0 \;\; i.o.) = 0.$$

$$T_n = Y_1 + \cdots + Y_n$$

$$\hat{T}_n = Z_1 + \cdots + Z_n$$

$$A_n = \{ w : \quad Z_n = 0 \;\; \forall n \geq m \}$$

$$A_m \uparrow A_\infty \qquad P(A_\infty) = 1$$

$$\frac{\hat{T}_n(w)}{n} = \frac{Z_1(w) + \cdots + Z_n(w)}{n} = \sum_{k=1}^{\min(n,m)} \frac{Z_k(w)}{n}$$

$$\limsup_{n} \frac{\hat{T}_n(\omega)}{n} \le \frac{\sum_{k=1}^{m} Z_k(\omega)}{n} = 0$$

$$\therefore \frac{\hat{T}_n(\omega)}{n} \to 0 \quad a.s.$$

$$E[Z_n] = E[X 1_{X>n}]$$

$\because$ X is integrable $\therefore$ $1_{X>n} \xrightarrow[n\to\infty]{} 0$

$$E[Z_n] \to 0$$

$$E\left[\frac{\hat{T}_n}{n}\right] \to 0$$

$a_n \geq 0 \quad a_n \to 0$ then $A_n = \frac{\sum_{i=1}^{n} i}{n} \to 0$

$\forall \varepsilon > 0 \quad \exists N_\varepsilon \quad a_n < \varepsilon \quad \forall n \geq N_\varepsilon$

$$\limsup A_n \le \limsup \frac{(a_1 + \cdots a_{N_\varepsilon}) + (n - N(\varepsilon))\varepsilon}{n}$$

$$= \limsup_{n\to\infty} \frac{a_1 + \cdots a_{N(\varepsilon)}}{n} + \varepsilon = \varepsilon$$

(Cesàro-Sum lemma)

$$\frac{T_n - E[T_n]}{n} \to 0 \quad a.s.$$

$$S_n = T_n - E[T_n] \to 0 \quad a.s.$$

$$\frac{T_n - E[T_n]}{n} \to 0 \quad a.s.$$

step 2. Take $\alpha > 1$, $k(n) = \lfloor \alpha^n \rfloor$

待证: $$\frac{T_{k(n)}^{(w)} - E[T_{k(n)}]}{k(n)} \to 0 \text{ a.s.}$$

step 3. $k(n_0) \le m < k(n_0+1)$

$$\frac{T_{k(n_0)}^{(w)} - E[T_{k(n_0)}] - \left(E(T_{k(n_0+1)} - T_{k(n_0)})\right)}{m} \le$$

$$T_{k(n_0)}^{(w)} - E[T_{k(n_0+1)}] \Big] \le T_m(w) - E[T_m] \le \frac{T_{k(n_0+1)}^{(w)} - E[T_{k(n_0)}]}{m}$$

$$\frac{\alpha^{n_0+1} - \alpha^{n_0}}{\alpha^{n_0}} = \alpha - 1$$

2019.10.14

1. strong law of large number

2. Kolmogorov's 0-1 law

3. Toward CLT

<u>Theorem</u> (Etemadi) Let $X_1, X_2, \cdots$ be a sequence of non-negative identically distributed and pairwise-independent r.v. s.t. $E[X] < \infty$

Then $\dfrac{X_1 + X_2 + \cdots X_n}{n} \longrightarrow E[X]$ a.s.

<u>prof</u>

Step 1: Trunction 切掉尾巴

Step 2: Sub-sequence

Step 3: Saidwich.

Step 1: Def $Y_n = X_n \mathbb{1}\{X_n \le n\}$

$$Z_n = X_n - Y_n$$

$$T_n = (Y_1 + \cdots + Y_n)$$

$$S_n = (X_1 + \cdots X_n)$$

$$\frac{X_1 + \cdots + X_n}{n} = \frac{Y_1 + \cdots Y_n}{n} + \frac{Z_1 + \cdots + Z_n}{n}$$

$$A_n := \{\omega : Z_n(\omega) > 0\}$$

$$\sum_n P(A_n) = \sum_n P(Z_n > 0) = \sum_n P(X > n) < \infty$$

$\sum_n P(X > 0)$ is finite

$$\sum_{n=1}^{\infty} P(X > n)$$

$$= \sum_{n=1}^{\infty} \left(\int 1_{\{X > n\}} \, dP \right)$$

$$= \int \left(\sum_{n=1}^{\lfloor X \rfloor} 1 \right) dP$$

$$\leq E(\lfloor X \rfloor) \leq E[X] < \infty$$

$$\therefore P(A_n \text{ i.o.}) = 0$$

i.o. : $\bigcap_{m=1}^{\infty} \left(\bigcup_{n=m}^{\infty} A_n \right)$

A : A_n i.o. $\quad P(A^c) = 1$

$$B_m = \{\omega : Z_n = 0,\ \forall n \geq m\}$$

$$B_n \uparrow A^c$$

$$P(B_m) \uparrow 1$$

for $\omega \in B_m$, $\dfrac{Z_1(\omega) + \cdots + Z_n(\omega)}{n} \leq \dfrac{\sum_{k=1}^{m-1} Z_k(\omega)}{n}$

$$\therefore \limsup_{n\to\infty} \frac{Z_1(\omega) + \cdots + Z_n(\omega)}{n} = 0$$

$$\therefore \forall \omega \in A^c \quad \limsup_{n\to\infty} \frac{Z_1(\omega) + \cdots + Z_n(\omega)}{n} = 0$$

$$\limsup_{n\to\infty} E\left[\frac{Z_1 + \cdots + Z_n}{n}\right] \to 0$$

$$\frac{1}{n} \sum_{i=1}^{n} Z_i$$

$$E[Z_i] = a_i$$

$$a_i \geq 0$$

$$\lim_{i\to\infty} a_i = 0 \qquad \frac{\sum_{i=1}^{n} a_i}{n} \xrightarrow{\text{cesaro-sum lemma.}} 0$$

$$a_i = E[Z_i] = E[X_i 1_{X > i}]$$

以上完成了第一步

Step 2.

以下証明

$$\frac{\bar{T}(n)}{n} = \frac{Y_1 + \cdots + Y_n - E[Y_1] - E[Y_n]}{n} \to 0 \quad a.s.$$

Take $\alpha > 1$,

$$k(n) = \lfloor \alpha^n \rfloor \qquad \bar{Y}_i = Y_i - E[Y_i]$$

$$P\left(\left| \frac{\bar{T}(k(n))}{k(n)} \right| > \varepsilon \right) \leq \frac{E[\bar{T}(k(n))^2]}{\varepsilon^2 k^2(n)} = \frac{\sum_{i=1}^{k(n)} E[\bar{Y}_i]}{\varepsilon^2 k^2(n)}$$

Goal:

$$\sum_{n=1}^{\infty} P\left(\left| \frac{\bar{T}(k(n))}{k(n)} \right| > \varepsilon \right) < \infty$$

want:

$$\sum_{n=1}^{\infty} \sum_{i=1}^{k(n)} \frac{E[\bar{Y}_i]^2}{\varepsilon^2 k^2(n)} < \infty$$

$$= \sum_{i=1}^{\infty} \frac{E[\bar{Y}_i^2]}{\varepsilon^2} \sum_{n=i}^{\infty} \frac{1}{n^2}$$

$$\propto \sum_{i=1}^{\infty} \frac{1}{i} \frac{E[\bar{Y}_i^2]}{\varepsilon^2}$$

want

$$= \sum_{i=1}^{\infty} \frac{E[\bar{Y}_i^2]}{\varepsilon^2} \left(\sum_{n: k(n) \geq i} \frac{1}{k(n)} \right) \qquad \text{(Fubini)}$$

$$n : \alpha^n \geq i$$

$$n \geq \left\lceil \frac{\log i}{\log \alpha} \right\rceil = i_0$$

$$= \sum_{i=1}^{\infty} \frac{E[\bar{Y}_i^2]}{\varepsilon^2} \left(\sum_{n \geq i_0} \frac{1}{\lfloor \alpha^n \rfloor^2} \right)$$

$$\leq \sum_{i=1}^{\infty} \frac{E[\bar{Y}_i^2]}{\varepsilon^2} \left(\sum_{n \geq \lceil i \rceil} \frac{2}{\alpha^{2n}} \right)$$

$$\alpha^{2n} \leq 2 \lfloor \alpha^n \rfloor^2$$

$$\alpha^{2n} \leq 2(\alpha^2 - 1)^2$$

$$\alpha^{2n} \leq 2\alpha^{2n} - 4\alpha^n + 2$$

$$4\alpha^n \leq \alpha^{2n} + 2$$

$$0 \leq (\alpha^2 - 1)^2 + 1$$

$$\frac{E[\bar{Y}_i^2]}{\alpha^{2 i_0}} \leq \sum_{i=1}^{\infty} \frac{\alpha^2}{(\alpha^2 - 1)\varepsilon^2} \frac{E[\bar{Y}_i^2]}{i^2}$$

Lemma. $X \geq 0$, if $E[X] < \infty$,

$$Y_i = X \mathbb{1}_{\{X \leq i\}}$$

$$\sum_{i=1}^{\infty} \frac{Var(Y_i)}{i^2} \leq 4E[X]$$

Step 3.

$$\frac{Y_1 + \cdots + Y_{k(n_0)}}{k(n_0)} \leq \frac{Y_1 + \cdots + Y_n}{n} \leq \frac{Y_1 + \cdots + Y_{k(n_0+1)}}{k(n_0+1)}$$

$$\frac{k(n_0)}{n} \qquad\qquad \frac{k(n_0+1)}{n}$$

$$\frac{k(n_0)}{n} \geq \frac{1}{\alpha}$$

$$\frac{k(n_0+1)}{n} \leq \alpha$$

$$\frac{1}{\alpha} E[X] \leq \liminf_{n\to\infty} \frac{Y_1 + \cdots + Y_n}{n}$$

$$\leq \limsup_{n\to\infty} \frac{Y_1 + \cdots Y_n}{n} \leq E[X]\alpha \qquad \forall\, \omega \in B_\alpha$$

$$\sum_i \frac{Var(Y_i)}{i^2} \le \sum_i \frac{E[Y_i^2]}{i^2}$$

$$\le \sum_i \frac{1}{i^2} \int Y_i^2 \, dP$$

$$= \sum_i \frac{1}{i^2} \int \left[\int_0^{Y_i} 2t \, dt \right] dP$$

Fubini

$$= \sum_i \frac{1}{i^2} \int_0^i 2t \, P(X > t) \, dt$$

$$= \int_0^\infty 2t \, P(X>t) \left(\sum_{i \ge \max\{t,1\}} \frac{1}{i^2} \right) dt$$

Claim: $\forall t$

$$2t \cdot \sum_{i \ge \max(t,1)} \frac{1}{i^2} \le 4$$

$t \le 1$ $\quad \frac{2t\pi^2}{6}$

$1 < t \le 2$ $\quad 2t\left(\frac{\pi^2}{6} - 1\right) \le 4\left(\frac{\pi^2}{6} - 1\right)$

$t > 2$ $\quad 2t \sum_{i \ge \lceil t \rceil} \frac{1}{i^2} \le \frac{2t}{\lceil t \rceil} \le 2$

$$\le \int_0 4 \, P(X > t) \, dt = 4 E[X]$$

Kolmogorov's 0-1 Law.

Suppose $X_1, X_2, \cdots$ are independent r.v.

$F^n = \sigma(X_n, X_{n+1}, \cdots)$

$F^n \downarrow F^\infty$ called tail σ-algebra.

$A \in F^\infty \Rightarrow P(A)$ is 0 or 1

2019.10.17

1. Kolmogorov's 0-1 law

2. Central Limit Theorem

0-1 law.

(Ω, F, P)

Let $X_1, X_2, \cdots X_n, \cdots$ be a sequence of independent r.v.

$F \supseteq F^i := \sigma(X_i, X_{i+1}, \cdots)$

$F^i \downarrow F^\infty \subseteq F$

Then $\forall A \in F^\infty$, $P(A) = 0$ or 1

Pf Let $G_i = \sigma(X_1, \cdots, X_i)$

We know that $G_i \perp F^{i+1}$

$G = \bigcup_i G_i$ is a field

$A \in F^\alpha \Rightarrow A \perp B$ for any $B \in G \Rightarrow$

$\exists\ i_0$ s.t. $B \in G_{i_0}$ $G_{i_0} \perp F^{i_0+1} \supseteq F^\infty$

$F^\infty \subseteq \sigma(G)$

$\forall A \in F^\alpha$ define two countably additive measure on $\sigma(G)$ according to

$B \in \sigma(G)$

$$P_1^A(B) = \frac{P(A\cap B)}{P(A)}, \qquad P_2^A(B) = P(B)$$

On G, $P_1(B) = P_2(B)$

Lemma about complex numbers

Lemma 1.

$$|e^{iz} - 1 - z| \le |z|^2 \quad \text{when } |z| \le 1$$

Pf
$$e^{iz} - 1 - z = \sum_{n=2}^{\infty} \frac{z^n}{n!}$$

$$\Rightarrow |e^{iz} - 1 - z| \le \sum_{n=2}^{\infty} \frac{|z|^n}{n!} \le \sum_{n=2}^{\infty} \frac{|z|^n}{2^{n-1}} \le |z|^2 \sum_{n=2}^{\infty} \frac{1}{2^{n-1}}$$
$$= |z|^2$$

$$2^{n-1} \le n! \text{ for } n \ge 2$$

Lemma 2.

$$\left| e^{ix} - \sum_{k=0}^{n} \frac{(ix)^k}{k!} \right| \le \min\left\{ \frac{(x)^{n+1}}{(n+1)!}, \ \frac{2(x)^n}{n!} \right\}$$

Pf claim

$$e^{ix} - \sum_{k=0}^{n} \frac{(ix)^k}{k!} = \frac{i^{n+1}}{n!} \int_0^z (x-s)^n e^{is} ds$$

$$\frac{i^{n+1}}{n!} \int_0^x (z-s)^n e^{is} ds = \frac{i^{n+1}}{n!} \left. \frac{(x-s)^n x^s}{i} \right|_0^z + \frac{i^{n+1}}{n!} \int_0^x n(x-s)^{n-1} \frac{e^{is}}{i} ds$$

$$= \frac{-i^n z^n}{n!} + \frac{i^n}{(n-1)!}\int_0^x (x-s)^{n-1} e^{is}\, ds$$

$$\Rightarrow \left| e^{ix} - \sum_{k=0}^{n} \frac{(ix)^k}{k!} \right| \leq \frac{1}{n!}\int_0^x |x-s|^n ds$$

$$= \frac{1}{n!}\frac{|x|^{n+1}}{n+1} = \frac{|x|^{n+1}}{(n+1)!}$$

$$\because \left| e^{ix} - \sum_{k=0}^{n-1} \frac{(ix)^k}{k!} \right| \leq \frac{2|x|^n}{n!}$$

Lemma 3.

if $|z_i|, |w_i| \leq \theta \qquad 1 \leq i \leq n$

then, $\left| \prod_{i=1}^{n} z_i - \prod_{i=1}^{n} w_i \right| \leq \theta^{n-1} \sum_{i=1}^{n} |z_i - w_i|$

pf $\quad n=4$ 时:

$$|z_1 z_2 z_3 z_4 - z_1 z_2 z_3 w_4| + |z_1 z_2 z_3 w_4 - z_1 z_2 w_3 w_4|$$
$$+ |z_1 z_2 w_3 w_4 - z_1 w_2 w_3 w_4|$$
$$+ |z_1 w_2 w_3 w_4 - w_1 w_2 w_3 w_4|$$

$$\leq \theta^3 |z_4 - w_4| + \theta^3 |z_3 - w_3| + \theta^3 |z_2 - w_2| + \theta^3 |z_1 - w_1|$$

Central Limit Theorem (v1)

If $X_1, X_2, \cdots, X_n, \cdots$ are iid with

$$E[X]=0, \quad E[X^2]=\sigma^2$$

then $\frac{X_1+X_2+\cdots X_n}{\sqrt{n}} \overset{w}{\Longrightarrow} N(0,\sigma^2)$

Proof 利用特征函数. $\Phi_{\frac{X_1+\cdots X_n}{\sqrt{n}}}(t) \xrightarrow[n\to\infty]{} e^{-\frac{t^2\sigma^2}{2}}$

$$\Phi_{\frac{X_1+\cdots X_n}{\sqrt{n}}}(t) = \Phi_X\left(\frac{t}{\sqrt{n}}\right)^n$$

$$E\left[e^{\frac{itX}{\sqrt{n}}}\right]^n = E\left[1-\frac{t^2X^2}{2n} + e^{\frac{itX}{\sqrt{n}}} - 1 + \frac{t^2X^2}{2n}\right]^n$$

If I show that

$$E\left[e^{\frac{itX}{\sqrt{n}}} - 1 + \frac{t^2X^2}{2n}\right] = o\left(\frac{1}{n}\right)$$

即:

$$\lim_{n\to\infty} nE\left[e^{\frac{itX}{\sqrt{n}}} - 1 + \frac{t^2X^2}{2n}\right] = 0$$

observe that

$$nE\left[e^{\frac{itX}{\sqrt{n}}} - 1 + \frac{t^2X^2}{2n}\right]$$

$$= nE\left[e^{\frac{itx}{\sqrt{n}}} - 1 - \frac{itx}{\sqrt{n}} - \frac{\left(\frac{itx}{\sqrt{n}}\right)^2}{2!} \right]$$

$$\leq E\Big[\underbrace{\left| n\left(e^{\frac{itx}{\sqrt{n}}} - 1 - \frac{itx}{\sqrt{n}} - \left(\frac{tx}{\sqrt{n}}\right)^2 \frac{1}{2!} \right) \right|}_{u} \Big]$$

$$U = U 1_{\left|\frac{tx}{\sqrt{n}}\right| \leq \varepsilon} + U 1_{\left|\frac{tx}{\sqrt{n}}\right| > \varepsilon}$$

$$\leq n \frac{\left|\frac{tx}{\sqrt{n}}\right|^3}{6} 1_{\left|\frac{tx}{\sqrt{n}}\right| \leq \varepsilon} + n\left(\frac{tx}{\sqrt{n}}\right)^2 1_{\left|\frac{tx}{\sqrt{n}}\right| > \varepsilon}$$

$$u \leq \frac{|tx|^3}{6\sqrt{n}} 1_{\left|\frac{tx}{\sqrt{n}}\right| \leq \varepsilon} + t^2 x^2 1_{\left|\frac{tx}{\sqrt{n}}\right| > \varepsilon}$$

$$\therefore E[u] \leq E\Big[\underbrace{\frac{|tx|^3}{6\sqrt{n}} 1_{\left|\frac{tx}{\sqrt{n}}\right| \leq \varepsilon}}_{G_n} \Big] + t^2 E\Big[\underbrace{x^2 1_{\left|\frac{tx}{\sqrt{n}}\right| > \varepsilon}}_{H_n} \Big]$$

当$\varepsilon < 1$时：

$$E\left[x^2 1_{\left|\frac{tx}{\sqrt{n}}\right| > \varepsilon} \right]$$

$$= E\left[x^2 1_{|x| > \frac{\varepsilon\sqrt{n}}{|t|}} \right] \longrightarrow 0 \quad \text{as } n \to \infty$$

$$H_n \to 0 \quad \text{as} \quad n \to \infty$$

$$H_n \leq x^2 \qquad \lim_{n\to\infty} E[H_n] \to E\left[\lim_{n\to\infty} H_n\right] = 0$$

$$\frac{|tx|^3}{6\sqrt{n}} 1_{\left|\frac{tx}{\sqrt{n}}\right| \le \varepsilon} \le \frac{|tx|}{\sqrt{n}} \frac{t^2x^2}{6} 1_{\left|\frac{tx}{\sqrt{n}}\right| \le \varepsilon} \le \frac{\varepsilon t^2 x^2}{6}$$

Central Limit Theorem (Lindeberg-Feller)

for every n, $X_{n,m}$ $1 \le m \le n$ is a sequence of independent r.v. satisfies $E[X_{n,m}] = 0, \forall n, m$

(1) $$\sum_{m=1}^{n} E(X_{n,m}^2) \to \sigma^2 \text{ as } n \to \infty$$

(2) $$\forall \varepsilon > 0, \sum_{m=1}^{n} E(X_{n,m}^2 1_{|X_{n,m}| > \varepsilon}] \to 0 \text{ as } n \to \infty$$

Then
$$\sum_{m=1}^{n} X_{n,m} \xrightarrow{\omega} N(0, \sigma^2)$$

Pf 即证:
$$E\left[\prod_{m=1}^{n} e^{itX_{n,m}}\right] \to e^{-\frac{t^2\sigma^2}{2}}$$

$$= \prod_{m=1}^{n} E\left[e^{itX_{n,m}}\right] \to e^{-\frac{t^2\sigma^2}{2}}$$

估計:

$$\prod_{m=1}^{n} E\left[e^{itX_{n,m}}\right] - \prod_{m=1}^{n} E\left[1 - \frac{t^2 X_{n,m}^2}{2}\right]$$

$$a_n = \sup_{1\le m\le n} E[X_{n,m}^2]$$

$$a_n = \sup_{1\le m\le n} E[X_{n,m}^2]$$

$$\le \sup_{1\le m\le n}\left\{ E[\varepsilon^2 1_{|X_{n,m}|\le\varepsilon}] + \sum_{m=1}^{n} E[X_{n,m}^2 1_{|X_{n,m}|\ge\varepsilon}]\right\}$$

$$\therefore \left|\prod_{m=1}^{n} E[e^{itX_{n,m}}] - \prod_{m=1}^{n} E\left[1-\frac{t^2X_{n,m}^2}{2}\right]\right| \le$$

$$\sum_{m=1}^{n}\left| E\left[e^{itX_{n,m}} - 1 + \frac{t^2X_{n,m}^2}{2}\right]\right|$$

$$= \sum_{m=1}^{n}\left| E\left(e^{itX_{n,m}} - 1 + itX_{n,m} + \frac{t^2X_{n,m}^2}{2}\right]\right|$$

$$\le \sum_{m=1}^{n} E\left(\frac{t^3|X_{n,m}|^3}{3!} 1_{|tX_{n,m}|<\varepsilon} + t^2X_{n,m}^2 1_{|X_{n,m}|>\frac{\varepsilon}{|t|}}\right]$$

$$\le \sum_{m=1}^{n} \varepsilon E\left(\frac{|t|^2|X_{n,m}|^2}{3!}\right] + t^2E\left[X_{n,m}^2 1_{X_{n,m}>\varepsilon/|t|}\right]$$

$$\le \frac{\varepsilon t^2\sigma^2}{6}$$

hence $\left| \qquad \right| \xrightarrow[n\to\infty]{} 0$

$$\left|\prod_{m=1}^{n} E\left[1-\frac{t^2X_{n,m}^2}{2}\right] - \prod_{m=1}^{n} e^{-\frac{t^2\sigma_n^2}{2}}\right|$$

$$E[X_{n,m}^2] := \sigma_{n,m}^2$$

$$\leq \sum_{m=1}^{n} \left| 1 - \frac{t^2\sigma_{n,m}^2}{2} - e^{-\frac{t^2\sigma_{n,m}^2}{2}} \right|$$

$$\leq \sum_{m=1}^{n} \frac{|t^4\sigma_{n,m}^4|}{4}$$

$$\leq \left[\max_{1\leq m\leq n} (\sigma_{n,m}^2)\right] \frac{t^4}{4} \left(\sum_{m=1}^{n} \sigma_{n,m}^2\right)$$

$$= \frac{a_n^2 t^4}{4} \left(\sum_{m=1}^{n} \sigma_{n,m}^2\right) \to 0.$$

$\square$

Borel - Cantelli 2

BC1 if $\sum_i P(A_i) < \infty$

then $P(A_n \text{ i.o.}) = 0$

BC2 if events $A_1, \cdots, A_n, \cdots$ are mutually independent. and $\sum_i P(A_i) = \infty$

then $P(A_n \text{ i.o.}) = 1$

pf.

$$A_n \text{ i.o.} = \bigcap_m \bigcup_{i=m} A_k$$

let $B_m = \bigcup_{k=m}^{\infty} A_k$

$$P(B_m^c) = \prod_{k=m}^{\infty} P(A_k^c) = \prod_{k=1}^{\infty} (1 - P(A_k))$$

$P(B_m^c) = 0 \quad \forall m.$

$1 - x \le e^{-x}$

$$\le \prod_{k=m}^{\infty} e^{-P(A_k)} = e^{-\sum_{n=m}^{\infty} P(A_k)} = 0$$

2019.10.21

1. Signed Measure

2. Random-Nikodym Theorem

3. Conditional Expectation

Signed Measure

$(\Omega, \mathcal{F})$

A finite signed measure $\lambda : \mathcal{F} \to \mathbb{R}$ s.t.

$\sum_i \lambda(A_i) = \lambda(\bigcup_i A_i)$ whenever A_i is pairwise disjoint, and $\lambda(\phi) = 0$

Lemma 1. If λ is a finite signed measure and $A_n \uparrow A$, then $\lambda(A_n) \to \lambda(A)$

proof $B_n = A_n \setminus A_{n-1}$

$\because$ B_n's are pairwise disjoint

$$\lambda(A) = \sum_{i=1}^{\infty} \lambda(B_i) = \lim_{n\to\infty} \sum_{i=1}^{\infty} \lambda(B_i)$$

$$= \lim_{n\to\infty} \lambda\left(\bigcup_{i=1}^{n} B_i\right) = \lim_{n\to\infty} \lambda(A_n)$$

<u>Lemma 2</u> If λ is a finite signed measure

then $\sup\limits_{\substack{A\in\Omega \\ A\in F}} |\lambda(A)| < \infty$

<u>proof</u>. Define $\lambda_+(A) = \sup\limits_{B\subset A} |\lambda(B)|$

If $\exists$ A s.t. $\lambda_+(A)$ and $\lambda_+(A^c)$ are finite.

then $\lambda_+(\Omega) < \infty$

$\because \forall B \subseteq \Omega$ $\lambda(B) = \lambda(A\cap B) + \lambda(A^c \cap B)$

$$\therefore |\lambda(B)| \le |\lambda(A\cap B)| + |\lambda(A^c\cap B)|$$

$$\le \lambda_+(A) + \lambda_+(A^c)$$

$$\Rightarrow \sup_{B\subseteq\Omega} |\lambda(B)| \le \lambda_+(A) + \lambda_+(A^c)$$

$$= \lambda_+(\Omega)$$

Let $\lambda_+(A) = \infty$ $\Rightarrow \forall m\ \exists n_0$ s.t. $a_{n_0} > M$

claim:

$\exists A_1 \subseteq A$ s.t. $|\lambda(A_1)| \ge 2(1 + |\lambda(A)|)$

and $\lambda_+(A_1) = \infty$

以下证明:

$\because \lambda_t(A) = \infty$

$\exists B \subseteq A$, s.t. $|\lambda(B)| \geq k(1 + |\lambda(A)|)$

$$\lambda(A) = \lambda(B) + \lambda(A \setminus B)$$

$$|\lambda(A \setminus B)| \geq |\lambda(B)| - |\lambda(A)|$$

$$= k + (k-1)|\lambda(A)| \geq (k-1)(1 + |\lambda(A)|)$$

$\therefore \exists A_2 \subseteq A_1$, s.t. $|\lambda(A_2)| \geq 2(1 + |\lambda(A_1)|)$

$$\geq 4(1 + |\lambda(A)|)$$

$\because \lambda(A_n) \downarrow \lambda(A_*)$

$|\lambda(A_n)| \uparrow \infty$ $\qquad \therefore$ contradiction.

A set A is called "totally positive" if

$\forall B \subseteq A$, $\lambda(B) \geq 0$.

- - - - "totally negative" - - - -

"totally zero": both totally positive and totally negative.

<u>Lemma 3.</u>

If $\lambda(A) = l \geq 0$, then $\exists$ a totally positive

set $A_+ \subseteq A$ s.t. $\lambda(A_+) \geq l$

Pf let $m = \inf_{B \subseteq A} \lambda(B)$ if $m = 0$. we are done

if $m < 0$,

$\exists$ B_1 s.t. $\lambda(B_1) < \frac{m}{2}$

define $A_1 = A \setminus B_1$

$\lambda(A_1) = \lambda(A) - \lambda(B_1) \geq l - \frac{m}{2} \geq l$

$m_1 := \inf_{B \subseteq A_1} \lambda(B)$

claim $m_1 \geq \frac{m}{2}$

$\because$ $\lambda(\tilde{B}) < \frac{m}{2}$

$\lambda(\tilde{B} \cup B_1) = \lambda(\tilde{B}) + \lambda(B_1) < m$

$\exists B_2$ s.t. $\lambda(B_2) < \frac{m_1}{2} < \frac{m}{4}$

继续 $A_2 = A_1 \setminus A_2$ ...... $\lambda(A_n) \geq l \ \forall n$

$A_n \downarrow$

$m_n = \inf_{B \subseteq A_n} \lambda(B)$

$m_n \leq \frac{m}{2^n}$

Lemma 3.1, countably unions of totally positive sets are totally positive.

Pf. $m = \sup_{B' \subseteq \Omega} \lambda(B')$

$\exists B_1$ s.t. $\lambda(B_1) \geq \frac{m}{2}$

$\Rightarrow A_1 \subseteq B_1$ s.t. A_1 is t.p. and $\lambda(A_1) \geq \frac{m}{2}$

Given any set A, $A \cap \Omega_+ = A_+$ $A \cap \Omega_- = A_-$

Lemma 4. Ω can be partition into Ω_+ Ω_- where Ω_+ is totally possitive and Ω_- is totally negative.

$(\Omega, \mathcal{F}, \mu)$ be a probability space and

f be an integrable function

Then $\lambda(A) = \int f \mathbb{1}_A \, d\mu = \int_A f \, d\mu$

is a finite signed measure.

pf. $f_n = f \mathbb{1}_{\bigcup_{i=1}^{n} A_i}$

$f_n \to f \mathbb{1}_{\bigcup_i A_i}$

$|f_n| \le |f|$

$\Rightarrow \int f_n \, d\mu \to \int f \mathbb{1}_A \, d\mu$ dominate t.

$= \lambda(A)$.

If $\mu(A) = 0$, then $\lambda(A) = 0$.

$g_n = |f| \wedge n$.

$\int_A n \, d\mu \ge \int_A (|f| \wedge n) \, d\mu \ge 0$.

$\Rightarrow \int_A (|f| \wedge n) \, d\mu = 0 \ \forall n$

by dominated v.c. (

$$\int_A |f| \, d\mu = 0 \geq \left| \int_A f \, d\mu \right| = 0$$

Definition: λ is said to be "absolutely contious" w.r.t μ ($\lambda \ll \mu$) if whenever $\mu(A)=0 \Rightarrow \lambda(A)=0$

<u>Theorem</u> (Radon-Nikodym) λ is a finite signed measure that is abs. contious w.r.t μ (non-negative prob measure)

Then $\exists$ an integrable function f s.t.

$\lambda(A) = \int_A f \, d\mu \quad \forall A \in \mathcal{F}$. and f is called Radon-Nikodym derivative of λ w.r.t. μ.

<u>Pf</u> for $q \in \mathbb{Q}$, $\lambda_q(A) = \lambda(A) - q\mu(A)$

定义一个 signed-measure 记为.

$\lambda_q(A)$ 时将 Ω 分为 Ω^q_+, Ω^q_-

I can assume that Ω^q_+ is decreasing in q

∵ q 是 countable ∴ 可以拆掉一些 让其单调

$$f(\omega) = \sup \{ q : \omega \in \Omega_+^q \}$$

$$\{\omega : f(\omega) > x\} = \bigcup_{q > x} \Omega_+^q \in \mathcal{F}.$$

$\Rightarrow f$ is "measurable"

finite almost surely $\quad f(\omega) = \infty$

$$\omega \in \Omega_+^q \quad \forall q \quad A = \bigcap_q \Omega_+^q$$

$$A \subseteq \Omega_+^q \Rightarrow \lambda_q(A) \geq 0 . \quad \forall q$$

$$\Rightarrow \lambda(A) - q\mu(A) \geq 0 \quad \forall q$$

$$\Rightarrow \lambda(A) \geq q\mu(A) \quad \forall q$$

$\Rightarrow \mu(A) = 0 .$ ∵ $\mu(A)$ 乘任一 q 可以是 ∞

∴ $\mu(A)$ 只能是 0.

if $f(\omega) = -\infty$ then

Let $B = \{\omega : f(\omega) = -\infty\}$ $\quad B \in \Omega_-^q \quad \forall q .$

$$\Rightarrow \lambda(B) - q\mu(B) \leq 0 \quad \forall q .$$

$$\lambda(B) \leq q\mu(B)$$

$$\omega \to -\infty \Rightarrow \mu(B) = 0$$

Given $h > 0$,

$$I_n = \{\omega : f(\omega) \in [nh, (n+1)h)\}$$

for $\omega \in A \cap I_n$

$$\lambda(A \cap I_n) - (n+1)h\mu(A \cap I_n) \le 0$$

and $\lambda(A \cap I_n) - nh\mu(A \cap I_n) \ge 0$

$$\lambda(A \cap I_n) - h\mu(A \cap I_n) \le nh\mu(A \cap I_n) \le$$

$$\int_{A \cap I_n} f \, d\mu \le (n+1)h\mu(A \cap I_n) \le \lambda(A \cap I_n) + h\mu(A \cap I_n)$$

$$A_+ = \{\omega : f(\omega) \ge 0\}$$

$$A = A_+ \sqcup A_-$$

$$\lambda(A_+ \cap I_n) - h\mu(A_+ \cap I_n) \le nh\mu(A_+ \cap I_n) \le$$

$$\int_{A_+ \cap I_n} f \, d\mu \le (n+1)h\mu(A_+ \cap I_n) \le$$

$$\lambda(A_+ \cap I_n) + h\mu(A_+ \cap I_n)$$

$$\lambda(A_+) - h\mu(A_+) \le \int_{A_+} f \, d\mu$$

$$\lambda(A_-) - h\mu(A_-) \le \int_{A_-} f \, d\mu \le \lambda(A_-) + h\mu(A_-)$$

$$= \int_{\Omega} f \, d\mu$$

$\Rightarrow f$ is integrable.

$$\lambda(A) - h\mu(A) \le \int_A f\,d\mu$$
$$\le \lambda(A) + h\mu(A)$$

Conditional Expection

Ω, F, measurable mappings random variable

如果只有 $G \subseteq F$.

Let f be a integrable function measurable w.r.t. F and $G \subseteq F$.
then G is called $E[f|G]$ if g is integrable and G-measurable and satisfies:

$$\int_A g\,d\mu = \int_A f\,d\mu \quad \forall A \in G.$$

$g : \Omega \to \mathbb{R}$.

$$\{w : g(w) \le x\} \in G.$$

define a finite signed measure on G

$$\int_A g\,d\mu = \lambda(A) = \int_A f\,d\mu.$$

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Conditional Expection

properties

Conditional Probabilities

(Ω, F, μ) f a measurable r.v. measurable w.r.t. F $G \subseteq F$.

$g := E[f|G]$ is a G-measurable integrable random variable that satisfies:

$$\int_A g \, d\mu = \int_A f \, d\mu \quad \forall A \in G.$$

$\forall A \in G$, $\lambda(A) = \int_A f \, d\mu$. $\lambda << \mu$

$\exists$ an integrable g G-measurable s.t.

$$\int_A g \, d\mu = \lambda(A) = \int_A f \, d\mu$$

if g_1 and g_2 are both conditional expections

then $g_1 = g_2$ a.s.

g_1, g_2

$g_1 - g_2 \qquad A_\varepsilon = \{\omega : g_1 - g_2 \geq \varepsilon\} \in G$

$$0 = \int_{A_\varepsilon} (g_1 - g_2)\, d\mu \geq \varepsilon \mu(A_\varepsilon)$$

$$\Rightarrow \mu(A_\varepsilon) = 0 \quad \forall \varepsilon > 0$$

$$\mu(A_{-\varepsilon}) = 0 \quad \forall \varepsilon > 0.$$

Properties of conditional expectation

1) $E[f] = E[E[f \mid G]]$

$$g = E[f \mid G]$$

$$\int_\Omega f\, d\mu = \int_\Omega g\, d\mu$$

2) $f \geq 0 \Rightarrow g \geq 0$. a.s.

$$A_\varepsilon = \{\omega : g(\omega) \leq -\varepsilon\} \in G$$

$$\int_{A_\varepsilon} g\, d\mu \leq -\varepsilon \mu(A_\varepsilon)$$

$$\| $$

$$\int_{A_\varepsilon} f\, d\mu \geq 0$$

$$\Rightarrow \mu(A_\varepsilon) = 0.$$

3) $E[af_1 + bf_2 \mid G] = aE[f_1 \mid G] + bE[f_2 \mid G]$ a.s.

$$\int_A \hat{g}\, d\mu = \int_A (af_1 + bf_2)\, d\mu = a\int_A f_1\, d\mu + b\int_A f_2\, d\mu$$

$\forall A \in G$

$$= a\int_A g_1\, d\mu + b\int_A g_2\, d\mu$$

$$= \int_A (ag_1 + bg_2)\, d\mu$$

5) If h is bounded and G-measurable, then

$E[hf \mid G] = hE[f \mid G]$ a.s.

pf

$$\int_A hf\, d\mu = \int_A hg\, d\mu \qquad \forall A \in G.$$

Standard machine

$h = 1_B \quad B \in G$

$$\int_A 1_B f\, d\mu = \int_A 1_B g\, d\mu$$

$$\Leftrightarrow \int_{A\cap B} f\, d\mu = \int_{A\cap B} g\, d\mu$$

$$\int_A s_n f\, d\mu = \int_A s_n g\, d\mu \to \int_A hf\, d\mu = \int_A hg\, d\mu$$

If $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \mathcal{F}$,

$$\underbrace{E[f|\mathcal{G}_1]}_{\hat{g}_1} = E[\underbrace{\underbrace{E[f|\mathcal{G}_2]}_{g_2}|\mathcal{G}_1]}_{\hat{g}_1} \quad a.s.$$

(Tower property)

$$\int_A \hat{g}_1 \, d\mu = \int_A f \, d\mu \quad \forall A \in \mathcal{G}_1$$

$$\int_A \hat{g}_2 \, d\mu = \int_A g_2 \, d\mu \, \forall A \in \mathcal{G}_1 = \int_A f \, d\mu.$$

$$\int_B g_2 \, d\mu = \int_B f \, d\mu$$

$\forall B \in \mathcal{G}_2$

4) Jensen's Inequality

Let Φ be a convex function

$$E[\Phi(f)|\mathcal{G}] \geq \Phi(E[f|\mathcal{G}]) \quad a.s.$$

Pf $\Phi(f) \geq a_\alpha f + b_\alpha, \quad \alpha \in A.$

$$E[\phi(f)|\mathcal{G}] \geq a_\alpha E[f|\mathcal{G}] + b_\alpha.$$

hence $\Phi(f) \geq a_\alpha f + b_\alpha$

$$E[\phi(f)|G] \geq \sup_\alpha [a_\alpha E[f|G] + b_\alpha] = \Phi(E[f|G])$$

$$E[|f||g] \geq |E[f|g]| \text{ a.s.}$$

(Ω, F, μ) $G \in F$.

和期望定义类似，$E[A]$ 用 $E[1_A]$ 来定义.

$E[f|G]$ 用 $E[1_A|G]$ 来定义.

A mapping $\mu : \Omega \times F \to [0,1]$ is called a "regular conditional properties" if it satisfy

(a) For every $A \in F$, $\mu(\cdot w, A) = E[1_A|G]$ a.s.

(b). For almost every w: $\mu(w, A)$ to be a probability measure on (Ω, F).

$A \subset B$. $A, B \in F$.

$$\begin{cases} \mu(w,A) = E[1_A | G] \\ \mu(w,B) = E[1_B | G] \end{cases}$$

$C = \{\omega : \mu(\omega, A) > \mu(\omega, B)\}$

$P(C) = 0.$

Pf $\Omega = [0,1]$, $F = \mathcal{B}_R$

$$A_q = (-\infty, q) \qquad q \in Q$$

$$\mu(\omega, A_q) = E[1_{A_q} \mid G]$$

There is a set C s.t. $P(C) = 1$

where $\mu(\omega, A_{q_1}) \le \mu(\omega, A_{q_2}) \quad \forall q_1 < q_2$

whenever $\omega \in C$

$$\mu(\omega, A_y) = \inf_{q > y} \mu(\omega, A_q)$$

$q < 0$: $\mu(\omega, A_q) = E[A_{(-\infty, q]} \mid G] = 0.$

$q \ge 1$: $\mu(\omega, A_q) = 1.$

$$\mu(\omega, A_y) = \inf_{q > y} \mu(\omega, A_q)$$

define a pseudo distribution on Ω.

$$\int 1_A \, dP = \int \mu(\omega, A) \, dP \qquad \forall A \in G$$

Martingale

A sequence : $\cdots, X_{-1}, X_0, X_1, X_2 \cdots$

$F_n = \sigma(X_n, X_{n-1}, X_{n-2}, \cdots)$

$F_n \uparrow$: called filtration

F_n, X_n F_n = a filtration

X_n is adapted to the filtration

X_n is F_n measurable.

$E[X_n \mid F_{n-1}] = X_{n-1}$ a.s.

construct martingale to prove things.

David Williams

"Probability with martingale"

2019.10.28

Martingale

1. Definition, properties
2. Inequalities
3. Optimal stopping time theorem.

Martingales.

(Ω, F)

Defn A sequence of σ-algebras $\{F_i\}_{i \in \mathbb{N}}$ is said to be a filtration if $F_i \subseteq F_j$, where $i < j$.

Defn A sequence of r.v. $\{X_i\}_{i \in \mathbb{N}}$ is said to be 'adapted' to the filtration $\{F_i\}$ if X_i is F_i-measurable.

Defn A sequence of r.v. $\{X_i\}_{i \in \mathbb{N}}$ is said to be "predictable" w.r.t $\{F_i\}$ if X_{i+1} is F_i-measurable

Defn: A sequence of r.v. $\{X_i\}$, adapted to filtration $\{F_i\}$ is said to be a "martingale" if

$$E[X_i \mid F_{X_{i-1}}] = X_{i-1} \quad a.s. \quad \forall i$$

sub martingale $E[X_i \mid F_{i-1}] \geq X_{i-1}$ a.s.

sup martingale $E[X_i \mid F_{i-1}] \leq X_i$ a.s.

Example of martingales

(a) Let X be a integrable r.v.

Let $Y_1, Y_2, \cdots Y_n$ be a sequence of r.v.

$F_i := \sigma(Y_1, \cdots Y_n)$

$\hat{X}_i = E[X \mid F_i)$ is a martingale-sequence.

Pf $E[\hat{X}_i \mid F_{i-1}) = E[E[X \mid F_i] \mid F_{i-1})$

$= E[X \mid F_{i-1}) = \hat{X}_{i-1}$

(b) Let $X_1, X_2, \cdots, X_n$ be a sequence of zero-mean independent r.v. let $S_n = \sum_{i=1}^{n} X_i$

Pf

$S_n = S_{n-1} + X_n$

$$= E[S_n \mid F_{n-1}] = E[S_{n-1} + X_n \mid F_{n-1}]$$

$$= S_{n-1} + E[X_n \mid F_{n-1}] = E[X_n] \quad \text{a.s.}$$

Martingale Transformations.

(i) let (X_n, F_n) be a martingale-sequence. and that ϕ be a convex function s.t. $\phi(X_n)$ is integrable $\forall n$.

then $\{\varphi(X_n), F_{(n)}\}$ is a . . . submartingale

by Jesson:

$$E[\varphi(X) \mid F_{n-1}] \geq \varphi(E(X_n \mid F_{n-1})) = \phi(X_{n-1})$$

(iii) let $\{X_n, F_n\}$ be a martingale-sequence, be a predictable martingale

$$((C \circ X)_n = \{ \sum_{m \in n}^{n} C_m (X_m - X_{m-1})$$

If C_n is uniformly bounded their $((C \circ X))_n$ is a martingale.

$(C \circ X)_n = (C \circ X)_{n-1} + C_n (X_n - X_{n-1})$

$E[(C \circ X)_n \mid F_{n-1}] = (C \circ X)_{n-1} + C_n E[X_n - X_{n-1} \mid F_{n-1}]$

Martingale Inequalities.

Let $\{X_n, F_n\}$ be a sub-martingale.

Let $A = \{w : \sup_{1 \le i \le m} X_i(w) \ge l\}$

Then $P(A) = \frac{1}{l} \int_A X_n \, dp \le \frac{1}{l} E(X_n)$

Pf $A = \overset{A_1}{\{w : X_1(w) \ge l\}} \cup \overset{A_2}{\{w : X_1(w) < l, X_2(w) \ge l\}}$

$\cup \cdots \cup \{X : X_1(w) < l, \cdots, X_{n-1}(w) < l, X_n(w) \ge l\}$

$A_i \in F_i$

$E[X_n \mid F_{n-1}] \ge X_{n-1}$

$E[X_n \mid F_j] \ge X_j \quad j \le n$

$\Rightarrow \int_{A_j} X_j \, dp \le \int_{A_j} X_n \, dp$

$\int_{A_j} X_n \, dp = \int_{A_j} E[X_n \mid F_j] \, dp \ge \int_{A_j} X_j \, dp$

$$E(X_n \mid F_{j-1}) \to E[X_n \mid F_j] \geq X_j$$

$$= E[E[X_n / F_j] \mid F_{j-1}] \geq E[X_j \mid F_{j-1}] \geq \cancel{X_j}_{j-1}$$

$$\int_{A_j} dP \leq \frac{1}{l} \int_{A_j} X_j \, dP \leq \frac{1}{l} \int_{A_j} X_n \, dP$$

$$\sum_{j=1}^{n} \int_{A_j} 1 \, dP = P(A) \leq \frac{1}{l} \int_A X_n \, dP$$

Let X, Y be non-negative r. v.

s.t. $P(Y \geq \lambda) \leq \frac{1}{l} \int_{Y \geq l} X \, dP, \ \forall l \geq 0$

Then, $\forall p > 1$, $E[Y^p] \leq \left(\frac{p}{p-1}\right)^p E[X^p]$

$F_Y(y)$ be the distribution function and

$T_Y(y)$ be $1 - F_Y(y)$

$\underline{Pf}$ $\int Y^p \, dP = \int_0^\infty y^p \, dF = -\int_0^\infty y^p \, dT$

$= -y^p T \Big|_0^\infty + \int_0^\infty p y^{p-1} T \, dy$

$\int y^p dT = y^p T - \int p \, y^{p-1} \, T$

$= -y^p T \Big|_0^\infty + \int_0^\infty p \, y^{p-1} T \, dy$

$$= \int_0^\infty p\, y^{p-1} P(Y > y)\, dy$$

$$\le \int_0^\infty p\, y^{p-1} \left(\frac{1}{y} \int_{Y > y} X \, dP \right) dy$$

$$= p \int X \left(\int_0^Y y^{p-2} \, dy \right) dP$$

$$= \frac{p}{p-1} \int X \, Y^{p-1} \, dP$$

$$\le \frac{p}{p-1} \left(\int X^p dP \right)^{\frac{1}{p}} \left(\int Y^p dP \right)^{\frac{p-1}{p}}$$

(Corollary) If $\{X_n, F_n\}$ be a sub-martingale then let $S_n = \sup_{1 \le i \le n} |X_i|$

$$\|S_n\|_p \le \frac{p}{p-1} \|X_n\|_p$$

<u>Definition</u>

Given a filtration F_n, a random variable $T \in \mathbb{N}$ is called a stopping time if $\{ w : T(w) \le n \}$ is F_n-measurable

<u>Theorem</u> Let $\{X_n, F_n\}$ be a super martingal and T be a stopping time. Then, $T \wedge n$ is a stopings time and

$$E[X_{T \wedge n}] \le E[X_1] \quad n \ge 1.$$

$T = T_1 \wedge T_2$.

$\{ w : T(w) > n \} = \{ w : T_1(w) > n, T_2(w) > n \}$

Pf. $1 \le k \le n-1$ $\quad A_k = \{ w : T(w) = k \}$

$A_n = \{ w : T(w) \geq n \}$

$E[X_{T\wedge n}] = \int_{A_1} dp + \int_{A_2} X_2 \, dp + \dots + \int_{A_n} X_n \, dp$

$$\boxed{\int_{A_n} X_n \, dP} \le \int_{A_n} X_{n-1} \, dP$$

$\le \dots + \int_{A_{n-1} \cup A_n} X_{n-1} \, dP = E[X_{T\wedge(n-1)}]$

Theorem. (Doob's optimal stopping time theorem)

" Let X_n be a super-martingale and T be a stopping time if any of the following conditions:

(a) T is bounded

(b) X_n is uniformly bounded and T is finite a.s.

(c). $E[T] < \infty$ and $|X_n - X_{n-1}|$ is uniformly bounded

then $E[X_T] \leq E[X_0]$

pf

(a) $T \leq N$

$E[X_{T\wedge N}] \leq E[X_0]$

$T\wedge N = T$.

(b) $Y_n = Y_{T\wedge n}$.

$Y_n \xrightarrow[n\to\infty]{} Y_N$ a.s.

$E[Y_n] \leq E[X_0]$

$E[\lim_n Y_n] \leq E(X_0)$

$E[\lim_{n} Y_n) \leq E[X_0)$

bounded convergence theory

$\lim_n E[Y_n] = E[\lim_n Y_n)$

c) $X_{T\wedge n} - X_0 = X_{T\wedge n} - X_{T\wedge(n-1)} + X_{T\wedge(n-1)} - X_{T\wedge(n-2)})$

$\therefore |X_{T\wedge n} - X_0| \leq |X_{T\wedge n} - X_{T\wedge n-1}| + \cdots + |X_{T\wedge 1} - X_0|$

2019.10.31

Optimal stopping time theorem

Thm 1 Let $\{X_n\}$ be a super-martingale and T be a stopping time.
Then $X_n^T(\omega) := X_{T(\omega)\wedge n}(\omega)$ is a super-martingale and consequently

$$E[X_{T\wedge n}] \leq E[X_0]$$

Thm 2 (i) Let $\{X_n\}$ be a super-martingale and T be a stopping time if any of the conditions hold:

(a) T is bounded

(b) X_n is bounded and T is finite a.s.

(c) $|X_n - X_{n-1}|$ is bounded and $E[T] < \infty$

then $E[X_T] \leq E[X_0]$.

(ii) If $\{X_n\}$ is a martingale and T is a stopping time If any of the previous conditions holds:

then $E[X_T] = E[X_0]$

(iii) Let $\{C_n\}$ be a bounded predictable sequence and X_n is a martingale s.t. $X_n - X_{n-1}$ is uniformly bounded then $E[(C\circ X)_T] = 0$ & $E[T] < \infty$

(iv) If $\{X_n\}$ is a non-negative super martingale and T is finite a.s then $E[X_T] \leq E[X_0]$

Proof

Thm 1

$$E[X_n^T \mid F_{n-1}] \leq X_{n-1}^T$$

$$E[X_{T\wedge n} \mid F_{n-1}] \leq X_{T\wedge n-1}$$

$$X_{T\wedge n} = X_{T\wedge n} 1_{T\geq n} + X_{T\wedge n} 1_{T<n}$$

$$= X_n 1_{T\geq n} + X_{T\wedge(n-1)} 1_{T<n}$$

$$E[X_{T\wedge n} | F_{n-1}] = E[X_n 1_{T\geq n} | F_{n-1}] + E[X_{T\wedge(n-1)} 1_{T<n} | F_{n-1}]$$

$$= 1_{T\geq n} E[X_n | F_{n-1}] + 1_{T<n} X_{T\wedge(n-1)} \quad a.s$$

$$\leq 1_{T\geq n} X_{n-1} + 1_{T<n} X_{T\wedge n-1}$$

$$= 1_{T\geq n} X_{T\wedge n-1} + 1_{T<n} X_{T\wedge n-1}$$

$X_{T\wedge n}$ is measurable w.r.t. F_n.

$$\{\omega : X_{T\wedge n} \leq x\} \in F_n.$$

$$\{\omega : X_{T\wedge n} \leq x\} = \{\omega : X_n \leq x, T(\omega) \geq n\}$$

$$\coprod_{m=1}^{n-1} \{\omega : \underbrace{X_m \leq x,}_{\in F_m \in F_n} T(\omega) = m\}$$

proof (Thm 2)

(a) $T \leq N$ $\qquad T\wedge N = \min(T, N)$

Then $X_{T\wedge N} = X_T$

$\therefore$ From Thm 1, $E[X_{T\wedge N}] \leq E[X_0]$

(b) $X_{T\wedge n} \xrightarrow{n\to\infty} X_T$ a.s.

$$E[X_{T\wedge n}] \le E[X_0]$$

$$E[\lim_n X_{T\wedge n}] \overset{BCT}{=} \lim_n E[X_{T\wedge n}] \le E[X_0]$$

$$= E[X_T]$$

(c) $$X_{T\wedge n} - X_0 = \sum_{m=1}^{n} (X_{T\wedge m} - X_{T\wedge m-1})$$

$$= \sum_{m=1}^{n\wedge T} (X_{T\wedge m} - X_{T\wedge m-1})$$

$$\therefore |X_{T\wedge n} - X_0| \le \sum_{m=1}^{n\wedge T} |X_{T\wedge m} - X_{T\wedge m-1}|$$

$$\le k(T\wedge n) \le kT$$

$$E[|X_{T\wedge n} - X_0|] \le kE[T] < \infty$$

$$\Rightarrow \lim_{n\to\infty} E[X_{T\wedge n} - X_0] \overset{dce}{=} E[\lim_{n\to\infty}(X_{T\wedge n} - X_0)]$$

$$= E[X_T - X_0]$$

(ii) $Y_n = -X_n$

$$E[X_T] \le E[X_0]$$

(iii) $$Y_n = \sum_{m=1}^{n} C_m (X_m - X_{m-1})$$

$$|Y_n - Y_{n-1}| = |C_n(X_n - X_{n-1})| \le K \cdot M$$

$E[T] < \infty$

$$E[Y_T] = E[Y_1] = E[C_1(X_1 - X_0)]$$

$$E[C_1(X_1 - X_0)]$$

$$= E[E[C_1(X_1 - X_0) | F_0]]$$

$$= E[C_1 E[X_1 - X_0 | F_0]]$$

$$= E[C_1[E[X_1 | F_0] - X_0]]$$

$$= E[C_1(X_0 - X_0)]$$

(iv) $X_{T \wedge n} \to X_T$

by Fatou:

$$E[\liminf_n X_{T\wedge n}] \le \liminf_n E[X_{T\wedge n}] \le \liminf_n E[X_0]$$

$$= E(X_0)$$

$$\|$$

$$E[X_T]$$

Example.

$X_0 = 0$

$X_1, X_2 \ldots$ are iid $\mathrm{Ber}(\frac{1}{2})$ r.v.'s

$S_n = X_1 + \ldots X_n$ $S_0 = 0$.

$$E[S_n \mid F_{n-1}] = E[S_{n-1} + X_n \mid F_{n-1}]$$
$$= S_{n-1} + \underbrace{E[X_n \mid F_{n-1}]}_{=0} \quad \because E[X_n] \text{ a.s. } X_n \perp F_{n-1}$$
$$= S_{n-1}$$

$\therefore S_n$ is a martingale.

$$F_n = \sigma(X_1, \cdots, X_n)$$

$$E[X \mid G] = E[X] \quad \text{a.s.}$$

If X is s.t. $E[X^2] < \infty$

$$E[X \mid G] = g$$

$$E[(X-g)^2] \le E[(X-Y)^2] \quad \text{for all } Y \text{ that } G\text{-measurable}$$

$$E[Y^2] < \infty$$

Pf $X - Y = (X-g) - (Y-g)$

$$E[(X-Y)^2] = E[(X-g)^2 - 2(X-g)(Y-g) + (Y-g)^2]$$

For $E[(X-g)(Y-g)]$

$$= E[E[(X-g)(Y-g) \mid G]]$$

$E[fg \mid G] = g E[f \mid G]$ a.s. if fg, g, f are integrable.

$$= E[(Y-g) E[X-g \mid G]]$$

$T = \inf\{n \geq 1 : S_n = 0\}$

$\{w : T(w) = 2k\}$

$= \{w : X_1 + \cdots X_{2k} = 0,$

$X_1 + \cdots X_i \neq 0, 1 \leq i \leq 2k-1\}$

what is $E[T]$?

$P(T = 2k) = \frac{\binom{2k-2}{k-1}}{k} \frac{1}{2^{2k}}$ Catalan number

$\sim \mathcal{O}\left(\frac{1}{k^{3/2}}\right)$

$S_n^2 - n$ is a martingale.

$E[S_T^2 - T] = 0$

$E[S_T^2] = E[T]$ $\quad E[T] = 0.$

<u>Thm</u> Let T be a stopping time, s.t.

$\exists N, \varepsilon > 0$ $P(T \leq n+N \mid F_n) \geq \varepsilon$ $\forall n,$

then $E[T] < \infty$

证. $Pr(T > kN) = E[1_{T > kN}]$

$= E[1_{T > kN} \; 1_{T > (k-1)N}]$

$= E[1_{T > (k-1)N} E[1_{T > kN} | F_{(k-1)N}]]$

$\leq E[(1-\varepsilon) 1_{T > (k-1)N}]$

$\leq (1-\varepsilon) P(T > (k-1)N)$

$\leq (1-\varepsilon)^k$

$E(T] = \sum_{k=1}^{\infty} P(T \geq k) < \infty$

Monkey typing Problem.

$X_1, X_2 \cdots$ is Monkey typing

$T = \inf\{n \geq 1 : (X_n, X_{n-1}, X_{n-2}, X_{n-3}, X_{n-4}, X_{n-5}, X_{n-6})$

$= (B, A, B, A, A, B, A)$

$E[T] < \infty$

首先构造鞅:

$X_1, X_2, \cdots X_{n-1}, X_n, \cdots$

$$Y_{n,1} = \begin{cases} 26 & \text{if } X_n = \text{'B'} \\ 0 & \text{o.w.} \end{cases}$$

$\vdots$

$\vdots$

$$Y_{n,2} = \begin{cases} 26^2 & \text{if } X_{n-1}, X_n = BA \\ 0 & \text{o.w.} \end{cases}$$

$$Y_{n,3} = \begin{cases} 26^3 & \text{if } BAB \\ 0 & \text{o.w.} \end{cases}$$

$\vdots$

$$Y_{n,7} = \begin{cases} 26^7 & \text{if } BABABA \\ 0 & \end{cases}$$

$Z_n = Y_{n,1} + Y_{n,2} + \cdots Y_{n,7} + Y_{n,8} + \cdots Y_{n,n} + n$ is martingale.

$$E[Z_n | F_{n-1}] = E[Y_{n,1} + \cdots + Y_{n,n} | F_{n-1}]$$

$$= 1 + Y_{n-1,1} + \cdots + Y_{n-1,6} + Y_{n-1,7} + \cdots Y_{n-1,n-1}$$

$E[Y_{n,1} | F_{n-1}] = 26 \times \frac{1}{26} = 1$

$E[Y_{n,2} | F_{n-1}]$

$Y_{n,2} = Y_{n-1,1}\, 26 \cdot 1_{X_n = 1}$

$Y_{n,7} = Y_{n-1,6}\, 26 \cdot 1_{X_n = 1}$

$Y_{n,8} = Y_{n-1,7}$

$(Z_n - Z_{n-1}) \leq 7 \times 26^7$

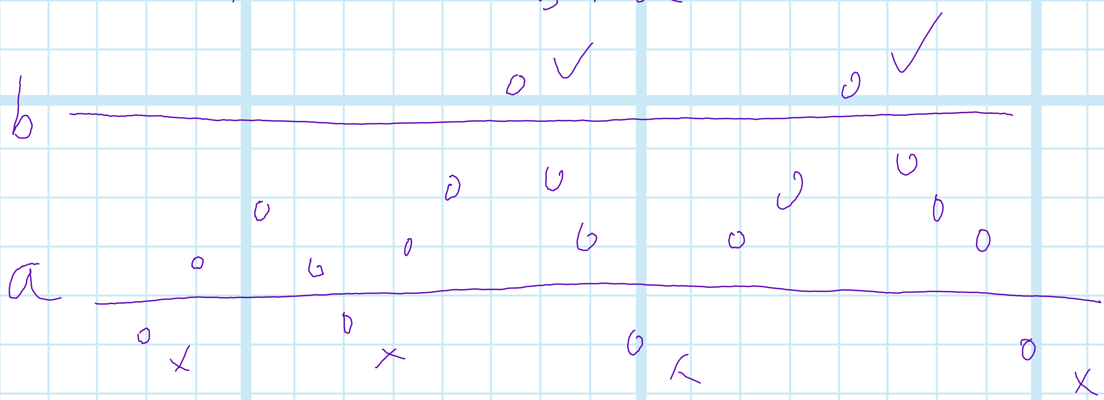
2019.11.9

Doob's upcrossing Inequality

Martingale convergence Threom

Doob's upcrossing Inequality

$X_0, X_1, \cdots$ $\qquad b > a$



$U_n(a,b)$: number of upcrossings of the process until time n.

$$U_n(a,b) = \max_k \{ \exists\ 0 \le s_1 < t_1 < s_2 \cdots < s_k < t_k \le n,\ X_{s_i} < a,\ X_{t_i} > b \}$$

$s_1 = 1 \quad t_1 = 7 \quad s_2 = 9 \quad t_2 = 12$

$U_n(a,b)$ is F_n-measurable.

Theorem 1) Let X_n be a super martingale

Then $(b-a) E[U_n(a,b)] \le E((X_n - a)_-)$

$\le E[|X_n|] + |a|$

2) Let $\{X_n\}$ be a submartingale, Then:

$$(b-a)E[U_n(a,b)] \le E[(X_0-a)_+] - E[(X_n-a)_-] \le E[|X_n|] + |a|$$

$z_- := \max\{0, -z\}$ 一般有 $z = z_+ - z_-$

Pf Define a predictable process C_n.

$C_0 = 0$

$n \geq 1$

$$C_n = 1_{C_{n-1}=0} 1_{X_{n-1} \le a} + 1_{C_{n-1}=1} 1_{X_{n-1} \le b}$$

$$Y_n = (C \circ X)_n$$

$$\sum_{m=1}^{n} C_m (X_m - X_{m-1})$$

$$Y_n \geq (b-a) U_n(a,b) - (X_n - a)_-$$

$$E[Y_n] \le 0$$

2) $Z_n = (X_n - a)_+ + a$

$\phi(x) = (x-a)_+ + a$

Z_n is a sub-martingale.

$$Y_n = (C \circ Z)_n$$

$$Y_n \geq (b-a) U_n(a,b)$$

$$\tilde{Y}_n = ((1-C) \circ Z)_n$$

$$Y_n + \tilde{Y}_n = Z_n - Z_0 = (X_n - a)_+ - (X_0 - a)_+$$

$$E[\tilde{Y}_n] \geq 0$$

$$E[Y_n] \leq E[(X_n - a)_+] - E[(X_0 - a)_+]$$

$$\vee\!/$$

$$(b-a) E[U_n(a,b)]$$

Doob's martingale convergence theorem.

If $\{X_n\}$ is a super/ sub/ martingale with $\sup_n E[|X_n|] < \infty$ then $\lim_n X_n$ exists a.s. and is finite.

Pf $\forall\ a<b$,

$$P(\{\omega : U_\infty(a,b) = \infty\}) = 0$$

$$P\Big(\bigcup_{\substack{a<b \\ a,b\in\mathbb{Q}}} \{\omega : U_\infty(a,b) = \infty\}\Big) = 0$$

If $\lim_n X_n(w)$ doesn't exist

$\exists\ a, b \in \mathbb{Q}$

$$\liminf_n X_n(w) < a < b < \limsup_n X_n(w)$$

Corollary If $\{X_n\}$ is non-negative supermartingale then $X_n \to$ a.s. a finite Limit.

Thereom. If X_n is martingale and

$|X_n - X_{n-1}| < K$ (uniform over n)

$C = \{w : \lim_n X_n$ exists and is finite$\}$

$D = \{w : \liminf_n X_n = -\infty, \limsup_n X = +\infty\}$

then $P(C \cup D) = 1$

Pf. Let $T_M = \inf\{n : X_n < -M\}$ is a stopping time

$Y_n = X_{n \wedge T_M} + M + K$ is a non-negative super martingale.

Hence

on $\{\omega : \liminf_n X_n > -M\}$ $X_n \to$ to a finite lim

$B_m = \{\omega : \liminf_n X_n > -M$ and $\lim_n X_n$ doesn't exist$\}$

$P(\bigcup_M B_m) = 0$

$\{\omega : \liminf_n X_n > -\infty\}$

$\{\omega : \liminf_n X_n > -\infty$ or $\limsup_n X_n < \infty$
and lim does't exist$\} = 0$

<u>Theorem</u> $X_1, \cdots, X_n \cdots$ is a martingale

If $\sup_n E[X_n^2] < \infty$ then $X_n \to X_\infty$ a.s.

and $E[(X_n - X_m)^2] \to 0$ Further $E[X_\infty^2] < \infty$

and $X_n = E[X_\infty | F_n]$ a.s.

$X_0 = 0$

w.l.o.g assume zero mean

$X_n \to X_\infty$ a.s. (Doob)

$Y_n = X_n - X_{n-1}$

$X_n = \sum_{m=1}^{n} Y_m$ $\quad E[Y_i Y_k]$ $i \neq k$ $\quad i > k$

$$E\left[(X_i - X_{i-1})(X_k - X_{k-1})\right]$$

$$E\left[X_i X_k \mid F_k\right] = X_k E\left[X_i \mid F_n\right] = X_n .$$

$$E\left[X_n^2\right] = \sum_{m=1}^{n} E\left[Y_m^2\right]$$

$$E\left[(X_\infty - X_n)^2\right] \geq E\left[(X_\infty \mid F_n)\right] - X_n$$

$$E\left[(X_\infty - X_m)^2\right] \underset{\text{Jenson}}{\geq} E\left[E\left[(X_\infty - X_m) \mid F_n\right]^2\right] \geq$$

$$E\left[\left[E\left[X_\infty \mid F_n\right] - X_n\right]^2\right]$$

2019.11.7

Lemma If $\{X_n\}$ is U.I and X is intergable & further $X_n \to X$ in measure then $E[|X_n - X|] \to 0$.

proof

define $X_n^k = X_n \mathbb{1}_{\{|X_n| < k\}} + k\, sgn(X_n) \mathbb{1}_{\{|X_n| \geq k\}}$

$$X^k = X \mathbb{1}_{\{|X| < k\}} + k\, sgn(X) \mathbb{1}_{\{|X| \geq k\}}$$

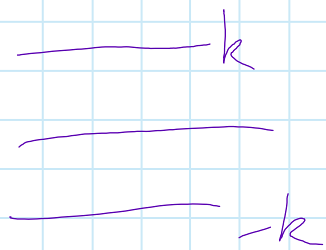
$$X_n - X = X_n - X_n^k + X_n^k - X^k + X^k - X.$$

$$|X_n - X| \leq |X_n - X_n^k| + |X_n^k - X^k| + |X^k - X|$$

If $X_n \to X$ in measure

then $X_n^k \to X^k$ in measure

$\because |X_n^k - X^k| \leq |X_n - X|$



$$\limsup E\left[|X_n - X|\right] \le \limsup_n E\left[|X_n - X_n^k|\right] +$$

$$\underbrace{\limsup_n E\left[|X_n^k - X^k|\right]} + E\left(|X - X^k|\right)$$

$$|X_n^k - X^k| \le \varepsilon + 2k \mathbb{1}_{|X_n^k - X^k| > \varepsilon}$$

$$E\left[|X_n^k - X^k|\right] \le \varepsilon + 2k\, P\left(|X_n^k - X^k| > \varepsilon\right)$$

$$\limsup_n E\left[|X_n - X|\right] \le \limsup_n E\left[|X_n| \mathbb{1}_{|X_n| \ge k}\right] + E\left[|X| \mathbb{1}_{|X| > k}\right]$$

$$\le \lim_{k \to \infty} \sup_n E\left[|X_n| \mathbb{1}_{|X_n| \ge k}\right] + \lim_{k \to \infty} E\left[|X| \mathbb{1}_{|X| > k}\right]$$

$$0$$

Theorem If $\{X_n\}$ is a martingale s.t. $\sup_n E[X_n^2] < \infty$ then $X_n \to X_\infty$ a.s. and $E[X_\infty^2] < \infty$, $X_n = E[X_\infty | F_n]$ a.s.

Theorem If $\{X_n\}$ is a martingale, and $\{X_n\}$ is U.I. then $X_n \to X_\infty$ a.s.

$E[|X_\infty|] < \infty$, $E[|X_n - X_\infty|] \to 0$, $X_n = E[X_\infty | F_n]$ a.s.

Fatou. Lemma.

Proof

$u.I. \Rightarrow \sup_n E[|X_n|] < \infty$

$m \geq n$

$$E\left[\left|E[X_m - X_\infty | F_n]\right|\right] = E\left[\left|X_n - E[X_\infty | F_n]\right|\right]$$

$$\leq E\left[E[|X_m - X_\infty| \,|\, F_n]\right] = E[|X_m - X_\infty|] \to 0$$

$\because E[|Z|] = 0 \Rightarrow Z = 0$ a.s.

Theorem If $\{X_n\}$ is a martingale, and $\sup_n \|X_n\|_p < \infty$ for $p > 1$ then, $X_n \to X_\infty$ a.s., $\|X_\infty\|_p < \infty$, $\|X_n - X_\infty\|_p \to 0$, $X_n = E[X_\infty | F_n]$ a.s.

$$\|X_n\|_p \triangleq E[|X_n|^p]^{\frac{1}{p}}$$

$$\|X_n - X_\infty\|_p \leq \underbrace{\|X_n - X_n^k\|_p}_{\to 0} + \underbrace{\|X_n^k - X^k\|_p}_{\to 0} + \underbrace{\|X_\infty - X_\infty^k\|_p}_{\to 0}$$

Doob's Decomposition

Let $\{F_n\}$ be a filtration and $\{X_n\}_{n\geq 0}$ be adopted to the filtration and X_n is integrable Then $X_n = X_0 + M_n + A_n$ where M_n is a martingale, A_n is predictable.

and is unique a.s.

Pf If such a decomposition exists, then:

$$E[(X_n - X_{n-1}) | F_{n-1}] = A_n - A_{n-1}$$

$$A_n = \sum_{m=1}^{n} E[X_m - X_{m-1} | F_{m-1}]$$ 这就是 A_n 的定义.

$$X_n = X_0 + M_n + A_n$$

$$E[X_n - X_0 - A_n | F_{n-1}]$$

$$= E[X_n | F_{n-1}] - X_0 - A_n$$

$$= E[X_n | F_{n-1}] - X_0 - A_{n-1} - E[X_n - X_{n-1} | F_{n-1}]$$

$= X_{n-1} - X_0 - A_{n-1}$

$\langle X_n \rangle$

X_n is a square-integrable martingale $X_0 = 0$

$E[X_n^2] < \infty$

then $X_n^2 = M_n + A_n$ ← sub-martingale

$A_n \hookrightarrow \langle X_n \rangle$

Theorem $\{X_n\}$ be square integrable martingale ($X_0 = 0$) Let $A_n = \langle X_n \rangle$, Then.

(i) If $A_\infty(\omega) < \infty$, then $\lim_n X_n$ exists a.s.

(ii) If X_n has uniformly bounded increment

if $\lim_n X_n$ exists then $A_\infty(\omega) < \infty$ a.s.

pf

$S_k(\omega) = \inf\{n : A_{n+1} > k\}$ stopping time

$X_n^2 = M_n + A_n$

$X_{n \wedge S_k}^2 = M_{n \wedge S_k}^2 + A_{n \wedge S(k)}$

$E[X_{n \wedge S(k)}^2] = E[A_{n \wedge S(k)}] < k$

$$Y_n = X_{n \wedge S(k)}$$

$$Y_n \to Y_\infty \quad a.s.$$

$$E[Y_n^2] \leq k.$$

$$X_{n \wedge S(k)} \to Y_\infty$$

$\lim_n X_n$ exists on $\omega : S_k(\omega) = \infty$

$$T_k(\omega) = \inf\{n : |X_n| > k\}$$

$$X^2_{n \wedge T(k)} = M_{n \wedge T(k)} + A_{n \wedge T(k)}$$

$$E[A_{n \wedge T(k)}] = E[X^2_{n \wedge T(k)}]$$

$$\leq (M + k)^2$$

Mid-Term.

Theorem

Let $\{X_n\}$ iid r.v. distrubuted as $N(0,1)$

$S_n = \sum_{m=1}^{n} X_m$, Then $\limsup_n \frac{S_n}{\sqrt{2n \log\log n}} = 1$ a.s

proof:

Let $h(n) = \sqrt{2n \log \log n}$, $n \geq 3$.

$$E[e^{\theta S_n}] = e^{\frac{1}{2}\theta^2 n} \quad \text{(Gaussian)}$$

$$e^{\theta S_n} = e^{\theta S_{n-1}} + e^{\theta X_n}$$

$$E[e^{\theta S_n} | F_{n-1}] = e^{\theta S_{n-1}} E[e^{\theta X_n} | F_{n-1}]$$

$$\geq e^{\theta S_{n-1}} e^{\theta E[X_n | F_{n-1}]} = e^{\theta S_{n-1}} \text{ a.s.}$$

$e^{\theta S_n}$ is a sub-martingal

$$\therefore \quad P\left(\sup_{1 \leq k \leq n} S_k \leq l\right) \leq e^{\frac{1}{2}\theta^2 n} e^{-\theta l} = e^{\frac{1}{2}(\theta^2 n - \theta l)}$$

$$= e^{-\frac{1}{2}\frac{l^2}{n}}$$

(Doob's inequality)

$$P\left(\sup_{1 \leq k \leq n} S_k \leq l\right) \leq e^{-\frac{1}{2}\frac{l^2}{n}}$$

$k > 1$

$$l_n = k h(k^{n-1})$$

$$P\left(\sup_{1 \leq m \leq k^n} S_m \geq l_n\right) \leq e^{-\frac{1}{2}\frac{k^2 h^2(k^{n-1})}{k^n}}$$

$$= e^{-\frac{1}{2}\frac{k^2 \, 2k^{n-1} (\log \log k^{n-1})}{k^n}}$$

$$= e^{-k\log[\log k^{n-1}]}$$

$$= \frac{1}{[(n-1)\log k]^k}$$

eventually (a.s.) $\quad \sup_{1\le m\le k^n} S_m \le k\, h(k^{n-1})$

$$\frac{\sup_{1\le m\le k^n} S_m}{h(k^n)} \le \frac{k\, h(k^{n-1})}{h(k^n)} \quad \text{eventually}$$

$$= \frac{k\sqrt{2k^{n-1}\log\log k^{n-1}}}{\sqrt{2k^{n}\log\log k^{n}}}$$

$$\le k$$

$$\frac{S_m}{\sqrt{2n\log\log n}} \le \frac{\max_{1\le m\le h(k^n)} S_m}{h(k^{n-1})} = \frac{\max_{1\le m\le h(k^n)} S_m}{h(k^n)} \cdot \frac{h(k^n)}{h(k^{n-1})}$$

$$h(k^{n-1}) < m < h(k^n)$$

$$\le k\,\frac{h(k^{n-1})}{h(k^n)} \cdot \frac{h(k^n)}{h(k^{n-1})}$$

$$= k$$

$N > 1$

$$\frac{S(N^{n+1}) - S(N^n)}{\sqrt{N^{n+1} - N^n}} \text{ is Gaussian,}$$

$$P\left(\frac{S(N^{n+1})-S(N^n)}{\sqrt{N^{n+1}-N^n}} \geq (1-\varepsilon)\frac{h(N^{n+1}-N^n)}{\sqrt{N^{n+1}-N^n}}\right)$$

$$\geq c(n\log N)^{-(1-\varepsilon)^2}$$

$$\boxed{P(X > x) \geq \left(x + \frac{1}{x}\right)\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}$$

infinity often, $\dfrac{S(N^{n+1}) - S(N^n)}{\sqrt{N^{n+1} - N^n}} \geq$

$$(1-\varepsilon)\frac{h(N^{n+1}-N^n)}{\sqrt{N^{n+1}-N^n}}$$

$$S(N^{n+1}) \geq (1-\varepsilon)h(N^{n+1}-N^n) - Kh(N^n)$$