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Méthodes et modèles probabilistes en physique statistique

Polymères dirigés en milieu aléatoire

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Avant-propos

Le but de ce mini-cours est d'introduire et d'illustrer des méthodes pour l'étude des systèmes désordonnés en mécanique statistique, sur l'exemple de modèles de polymères dirigés en interaction avec un environnement aléatoire.

Le polymère est décrit comme une marche aléatoire simple sur le réseau, collectant des récompenses aléatoires disposées sur son chemin (marche aléatoire dans un potentiel dépendant du temps et de l'espace). Le modèle est donné par une mesure de Gibbs, qui interpole entre la loi uniforme sur l'ensemble des trajectoires possibles, et la géodésique du problème de percolation orientée de dernier passage. Si les récompenses sont suffisamment fortes, le polymère adopte un comportement totalement différent de celui de la marche aléatoire sous-jacente, afin de profiter des environnement favorables. Cela donne lieu à une transition de phase, entre phases localisée et délocalisée.

Si le modèle est compliqué sur le réseau $\mathbb{Z}^d$, il est explicitement résoluble sur l'arbre.

Une application combinatoire sera donnée dans le cadre de la ρ-percolation: en chaque site du réseau $\mathbb{N}^{d+1}$ est plaçé un 0 ou un 1 aléatoirement, et l'on s'intéresse au nombre $N(n)$ de chemins orientés (arètes nord ou est) de longueur n pour lesquels la densité de 1 est au moins ρ ($1 > \rho > p$ avec p la probabilité de tirer un 1). Nous déduirons des estimations de N_n.

Quelques mots clés:

1- Mesures de Gibbs, energie (libre), entropie, diagramme de phases, transition de phases, percolation, polymères.

2- Méthode du second moment, martingales, inégalités de concentration, séries génératrices, inégalités de corrélation FKG-Harris.

Chapter 1

Directed polymers in random environment

1.1 Introduction

1.1.1 A polymer model

We start with an informal description of the situation we will discuss in these notes. Consider a hydrophilic polymer chain wafting in water. Due to the thermal fluctuation, the shape of the polymer should be understood as a random object. We now suppose that the water contains randomly placed hydrophobic molecules as impurities, which repel the hydrophilic monomers which the polymer consists of. The question we address here is:

How does the impurities affect the global shape of the polymer chain?

We try to answer this question in a mathematical framework. However, as is everywhere else in mathematical physics, it is very difficult to do so without compromising with a rather simplified picture of the initial problem. Here, our simplification goes as follows. We first suppress entanglement, self-intersections and U-turns of the polymer; we shall represent the polymer chain as a graph $\{(j, \omega_j)\}_{j=1}^n$ in $\mathbb{N} \times \mathbb{Z}^d$, so that the polymer is supposed to live in $(1+d)$-dimensional discrete lattice and to stretch in the direction of the first coordinate. Such a model is called *directed*. Each point $(j, \omega_j) \in \mathbb{N} \times \mathbb{Z}^d$ on the graph stands for the position of j-th monomer in this picture. Secondly, we assume that, the transversal motion $\{\omega_j\}_{j=1}^n$ performs a simple random walk in $\mathbb{Z}^d$, if the impurities are absent. We then define the energy of the path $\{(j, \omega_j)\}_{j=1}^n$ by the formula (1.1.3) at some (inverse) temperature $\beta > 0$, where $\{\eta(n, x) : n \geq 1,\ x \in \mathbb{Z}^d\}$ is a field of real i.i.d. random variables, with $\eta(n, x)$ describing the presence (or strength) of an impurity at site (n, x). The typical shape $\{(j, \omega_j)\}_{j=1}^n$ of the polymer is then given by the one that minimizes the energy (1.1.3). Let us suppose for example that $\eta(n, x)$ takes two different values $+1$ ("presence of a water molecule at (n, x)") and -1 ("presence of the hydrophobic impurity at (n, x)"). Then, the energy of the polymer is decreased by β each time a monomer is in contact with the impurity ($\eta(j, \omega_j) = -1$). Therefore, the typical shape of the polymer for each given configuration of $\{\eta(j,\ x)\}$ is given by the one which tries to avoid the impurities as much as possible.

The purpose of these lecture notes is to introduce rigorous results which answer the above question roughly as follows: (i) If the space dimension is large and the temperature is high, the impurities do not affect the global shape of the polymer; (ii) If the dimension is small or the environment is strong, then the impurities change the global shape of the polymer drastically.

The informal description given below has been put into a mathematical framework by the formalism of a Gibbs measure, in the framework of statistical mechanics.

1.1.2 Simple random walk model for directed polymers

The model we consider here is defined as a random walk in a random potential. We first fix the notations for the random walk and the random environment, and then introduce the polymer measure.

• *The random walk:* $(\{S_n\}_{n\geq 0}, P^x)$ is a simple random walk on the d-dimensional integer lattice $\mathbb{Z}^d$ starting from $x \in \mathbb{Z}^d$. More precisely, we let Ω_ω be the path space $\Omega_\omega = \{\omega = (\omega_n)_{n\geq 0}; \omega_n \in \mathbb{Z}^d, n \geq 0\}$, $\mathcal{F}$ be the cylindrical σ-field on Ω, and, for all $n \geq 0$, $S_n : \omega \mapsto \omega_n$ be the projection map. We consider the unique probability measure P^x on $(\Omega_\omega, \mathcal{F})$ such that $S_1 - S_0, \dots, S_n - S_{n-1}$ are independent and

$$P^x\{S_x = 0\} = 1, \quad P^x\{S_n - S_{n-1} = \pm e_j\} = (2d)^{-1}, \quad j = 1, 2, \dots, d,$$

where $e_j = (\delta_{kj})_{k=1}^d$ is the j-th vector of the canonical basis of $\mathbb{Z}^d$. In the sequel, $P^x[X]$ denotes the P^x-expectation of a r.v.(random variable) X on $(\Omega_\omega, \mathcal{F}, P^x)$,and P^0 will be simply written by P.

• *The random environment:* $\eta = \{\eta(n,x) : n \in \mathbb{N},\ x \in \mathbb{Z}^d\}$ is a sequence of r.v.'s which are real valued, non-constant, and i.i.d.(independent identically distributed) r.v.'s defined on a probability space $(\Omega_\eta, \mathcal{G}, Q)$ such that

$$Q[\exp(\beta\eta(n,x))] < \infty \quad \text{for all } \beta \in \mathbb{R}. \tag{1.1.1}$$

Here, and in the sequel, $Q[Y]$ denotes the Q-expectation of a r.v. Y on $(\Omega_\eta, \mathcal{G}, Q)$. We will take $\Omega_\eta = \mathbb{R}^{\mathbb{N}^* \times \mathbb{Z}^d}$ the canonical space for definiteness.

• *The polymer measure:* For any $n > 0$, define the probability measure $\mu_{n,\beta}^\eta$ on the path space $(\Omega_\omega, \mathcal{F})$ by

$$\mu_{n,\beta}^\eta(d\omega) = \frac{1}{Z_{n,\beta}^\eta} \exp\{\beta H_n(S)\}\ P(d\omega), \tag{1.1.2}$$

where $\beta > 0$ is a parameter (the inverse temperature), where

$$H_n(\omega) = H_n^\eta(\omega) = \sum_{1\leq j\leq n} \eta(j, \omega_j) \tag{1.1.3}$$

and

$$Z_n = Z_{n,\beta}^\eta = P\left[\exp\left(\beta \sum_{1\leq j\leq n} \eta(j, S_j)\right)\right] \tag{1.1.4}$$

is the normalizing constant, so-called partition function. Of course, in the present context, the above expectation is simply a finite sum,

$$Z_{n,\beta}^\eta = \sum_\omega (2d)^{-n} \exp\left(\beta H_n(\omega)\right)$$

where ω ranges over the $(2d)^n$ possible paths of length n for the simple random walk.

The polymer measure μ_n can be thought of as a Gibbs measure on the path space $(\Omega_\omega, \mathcal{F})$ with the Hamiltonian H_n. We stress that the random environment η is contained in both $Z_{n,\beta}^\eta$ and μ_n without being integrated out, so that they are r.v.'s on the probability space $(\Omega_\eta, \mathcal{G}, Q)$. The polymer is attracted to sites where the random environment is positive, and repelled by sites where the environment is negative.

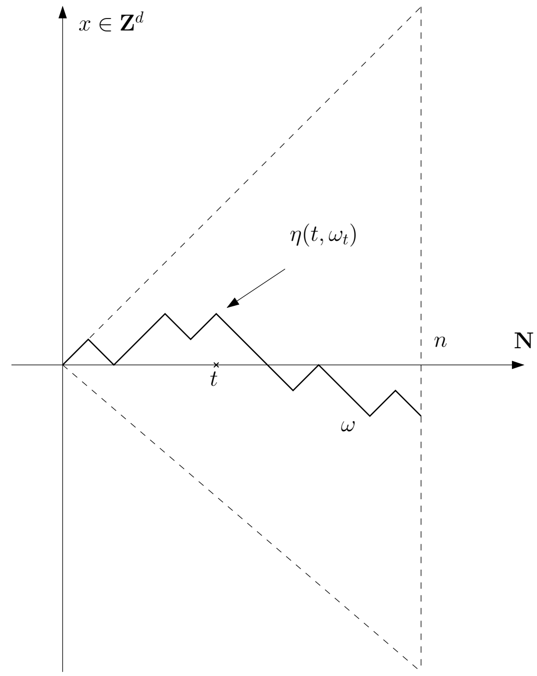

Figure 1.1: The walk picks up all the environment variables it meets up to time n

In view of our assumptions, an important quantity for this model is the logarithmic moment generating function λ of $\eta(n, x)$,

$$\lambda(\beta) = \ln Q[\exp(\beta\eta(n, x))], \quad \beta \in \mathbb{R}. \tag{1.1.5}$$

The function $\lambda(\beta)$ can be explicitly computed for some typical choice of the distribution of $\eta(n, x)$. For example, $\lambda(\beta) = \ln(pe^{-\beta} + (1-p)e^{\beta})$ for the Bernoulli environment and $\lambda(\beta) = \frac{1}{2}\beta^2$ for the Gaussian environment.

History, litterature

This model was originally introduced in physics literature [34] to mimic the phase boundary of Ising model subject to random impurities. Later on, the model reached the mathematics community [35, 7], where it was reformulated as above.

• A physical realization: torn paper sheets [39] A rectangular sheet of paper is tightened by a machine. The strain is applied on two opposite sides, and it is slowly increased. A small notch is made on a third side to initiate the tear. The fracture is governed by the random geometry of the fiber network. The fracture line is highly correlated the weakest bonds in the sheet, as can be checked by microdensiometry. Directed polymers with $d = 1$ and its zero temperature counterpart, oriented first passage percolation, are natural model for the fracture line.

• Particle in a random potential: In the case $\eta(t, x) \leq 0$, we can interpret the model in terms of a walker moving among deadly obstacles. A walker starts at the origin at time 0, it jumps at integer times and immediately after, it dies or survives. It moves according to a simple random walk when alive, but it has a probability $\exp \beta\eta(t, x) \in (0, 1]$ to die at time t if it is still alive at time $t-1$ and has jumped at location x at time t, and probability $1 - \exp \beta\eta(t, x)$ to survive the obstacle. Then, the polymer measure $\mu^{\eta}_{n,\beta}$ is equal to the law of the path conditionned to be alive at time n.

• Corresponding to $\beta = \infty$, the ground states, i.e. the paths ω maximizing H_n, are the geodesics of last passage percolation. Relations to percolation, last passage percolation, and all the related models (tandem queueing systems, totally asymmetric exclusion process,...)

• Model for normal growth with deposition, see Kardar-Parisi-Zhang equation.

1.2 Thermodynamics

We start with some preliminaries. On the space Ω_η of environments, define for $i \geq 1, x \in \mathbb{Z}^d$, the **shift operator** $\theta_{i,x} : \Omega_\eta = \mathbb{R}^{\mathbb{N}^* \times \mathbb{Z}^d} \to \Omega_\eta$ given by $\eta \mapsto \theta_{i,x}\eta$,

$$(\theta_{i,x}\eta)_{t,y} = \eta(i+t, x+y) \ . \tag{1.2.6}$$

Observe that $(\theta_{i,x}\eta)$ is simply the field of environment variables which is seen from the "point" (i,x). Its law is the same as the law of η itself, i.e., the product law Q.

Markov property and the partition function. For $n, m \geq 1$, $x \in \mathbb{Z}^d$, the random variable

$$Z_{m,\beta}^{\theta_{n,x}\eta} = P^x \left[\exp \left(\sum_{1 \leq t \leq m} (\beta\eta(t+n, S_t)) \right) \right], \tag{1.2.7}$$

is the partition function of the polymer of length m starting at x at time n. Since η and its shift $\theta_{n,x}\eta$ have the same law, $Z_{m,\beta}^{\theta_{n,x}\eta}$ has the same law as $Z_{m,\beta}^{\eta}$. By the Markov property, we can also write it in the form of a conditional expectation given $\mathcal{F}_n = \sigma\{S_t, t \leq n\}$,

$$Z_{m,\beta}^{\theta_{n,x}\eta} = P\left[\exp \beta\{H_{n+m}(S) - H_n(S)\} | \mathcal{F}_n\right] \quad \text{on the event } \{S_n = x\}.$$

For $n, m \geq 1$, we can express the partition function of the polymer of length $n+m$ by conditioning:

$$\begin{aligned}
Z_{n+m,\beta}^{\eta} &= P\left[e^{\beta H_{n+m}(S)}\right] \\
&= \sum_{x \in \mathbb{Z}^d} P\left[e^{\beta H_{n+m}(S)}; S_n = x\right] \\
&= \sum_{x \in \mathbb{Z}^d} P\left[e^{\beta H_n(S)} P\left[e^{\beta\{H_{n+m}(S) - H_n(S)\}} | \mathcal{F}_n\right]; S_n = x\right] \\
&= \sum_{x \in \mathbb{Z}^d} P\left[e^{\beta H_n(S)} P\left[e^{\beta\{H_{n+m}(S) - H_n(S)\}} | \mathcal{F}_n\right]; S_n = x\right] \\
&= \sum_{x \in \mathbb{Z}^d} P\left[e^{\beta H_n(S)}; S_n = x\right] \times Z_{m,\beta}^{\theta_{n,x}\eta},
\end{aligned} \tag{1.2.8}$$

using the previous observation. This important property is usually called Markov property. It can be reformulated as

$$\begin{aligned}
Z_{n+m,\beta}^{\eta} &= Z_{n,\beta}^{\eta} \sum_x \mu_{n,\beta}^{\eta}\{S_n = x\} Z_{m,\beta}^{\theta_{n,x}\eta} \\
&= Z_{n,\beta}^{\eta} \times \mu_{n,\beta}^{\eta}\left[Z_{m,\beta}^{\theta_{n,S_n}\eta}\right] \ .
\end{aligned} \tag{1.2.9}$$

1.2.1 Free energy

As it is well known in statistical mechanics, one can understand the Gibbs polymer measure by studying the (specific) free energy[1]

$$p_n = p^\eta_{n,\beta} = \frac{1}{n} \ln Z^\eta_{n,\beta} \ ,$$

more precisely to know if it converges as $n \to \infty$, and, if this is the case, how the limit depends on the environment η.

Theorem 1.2.1 *As* $n \to \infty$,

$$p^\eta_{n,\beta} \longrightarrow p(\beta) = \sup_n Q p^\eta_{n,\beta}$$

Q-a.s. and in L^p-norm, for all $p \in [1, \infty)$.

The theorem states that the sequence $p^\eta_{n,\beta}$ converges a.s. to a limit, the limit is deterministic and given by as a supremum over the polymer length.

□ The proof splits in two steps, with the first one showing that mean value converge, and the second one showing that random fluctuations are neglegible.
• *Step 1:* We first consider expected values and show that:

$$\lim_{n\to\infty} Q p^\eta_{n,\beta} = \sup_{n\in\mathbb{N}} Q p^\eta_{n,\beta} \in \mathbb{R}$$

For $m, n \geq 1$, recall the identity (1.2.9), and also that $Z^\eta_{m,\beta}$ and $Z^{\theta_{n,x}\eta}_{m,\beta}$ have the same law. By Jensen's inequality, we obtain

$$\ln Z^\eta_{n+m,\beta} \geq \ln Z^\eta_{n,\beta} + \sum_x \mu^\eta_{n,\beta}\{\omega_n = x\} \ln Z^{\theta_{n,x}\eta}_{m,\beta} \ .$$

Let $\mathcal{G}_n$ be the σ-field generated by $\eta(t, \cdot), t \leq n$. Taking expectation and using independence of the $\eta(i, y)$'s, we obtain

$$\begin{aligned}
Q[\ln Z^\eta_{n+m,\beta}] &\geq Q[\ln Z^\eta_{n,\beta}] + \sum_x Q[\mu^\eta_{n,\beta}\{\omega_n = x\}] Q[\ln Z^{\theta_{n,x}\eta}_{m,\beta}] \\
&= Q[\ln Z^\eta_{n,\beta}] + Q[\ln Z^\eta_{m,\beta}] \sum_x Q[\mu^\eta_{n,\beta}\{\omega_n = x\}] \qquad (Z^{\theta_{n,x}\eta}_{m,\beta} \overset{\text{law}}{=} Z^\eta_{m,\beta}) \\
&= Q[\ln Z^\eta_{n,\beta}] + Q[\ln Z^\eta_{m,\beta}]
\end{aligned}$$

i.e., $Q[\ln Z^\eta_{n,\beta}]$ is super-additive. From the super-additive Lemma (see Appendix, section 3.2), we see that

$$\lim_{n\nearrow\infty} \frac{1}{n} Q[\ln Z^\eta_{n,\beta}] = \sup_n \frac{1}{n} Q[\ln Z^\eta_{n,\beta}].$$

Now, the finiteness of p follows from the annealed bound (1.2.12) below.

• *Step 2:* We will apply to $X = \ln Z^\eta_{n,\beta}$ a concentration inequality, given in Lemma 3.3.4 in the Appendix. We prove below that $X = \ln Z^\eta_{n,\beta}$ satisfies the estimate (3.3.8) with some $B \in (0, \infty)$ and some $\delta > 0$: by Borel-Cantelli lemma, this implies that $\limsup_n \big|p_n - Q[p_n]\big| \leq \epsilon$ Q-a.s., for all $\epsilon \leq B\delta$. Hence,

$$\limsup_{n\to\infty} \big|p_n - Q[p_n]\big| = 0 \qquad Q-\text{a.s.},$$

[1]In the physics litterature, p_n is rather called the pressure; the specific free energy is defined as $-\beta^{-1}p_n$, it has the same unit as the energy $-H_n$. The name specific means that it has been normalized by the number n of monomers.

which, together with Step 1 above, completes the proof of almost sure convergence in Theorem 1.2.1. To get L^p convergence, one checks from the concentration inequality that the sequence $(p^\eta_{n,\beta})_n$ is uniformly integrable.

All what is left is to prove (3.3.8) for $X = \ln Z^\eta_{n,\beta}$. For that, we introduce a probability measure $\mu_{n,j}$ on $(\Omega_\omega, \mathcal{F})$ by

$$\mu_{n,j}(d\omega) = \frac{1}{Z_{n,j}} \exp(\beta H_{n,j}) P(d\omega), \quad j = 1, ..., n,$$

where $H_{n,j} = \sum_{1\le k\le n, k\ne j} \eta(k, \omega_k)$. We also introduce the σ-field $\mathcal{G}_j$ generated by $\eta(t,x)$ for $t \le j$ and $x \in \mathbb{Z}^d$. We fix arbitrary n and check (3.3.7) for

$$X = \ln Z^\eta_{n,\beta}, \quad X_j = \ln Z_{n,j}, \quad j = 1, ..., n.$$

Since X_j does not depend on $\eta(\cdot, j)$, we have $Q^{\mathcal{G}_{j-1}}[X_j] = Q^{\mathcal{G}_j}[X_j]$. Note on the other hand that the function $u \mapsto u^\delta$ is convex for $\delta \in \mathbb{R}\backslash(0,1)$, and then

$$\begin{aligned} \exp(\delta(X - X_j)) = \left(\frac{Z^\eta_{n,\beta}}{Z_{n,j}}\right)^\delta &= \mu_{n,j}[\exp(\beta\eta(\omega_j, j))]^\delta \\ &\le \mu_{n,j}[\exp(\beta\delta\eta(\omega_j, j))]. \end{aligned}$$

Now, the random measure $\mu_{n,j}$ is measurable with respect to $\mathcal{G}'_j \overset{\text{def.}}{=} \sigma[\eta(\cdot, k)\ ;\ k \ne j]$. Therefore,

$$\begin{aligned} Q^{\mathcal{G}'_j}[\exp(\delta(X - X_j))] &\le Q^{\mathcal{G}'_j}\left[\mu_{n,j}[\exp(\beta\delta\eta(\omega_j, j))]\right] \\ &= \mu_{n,j}\left[Q^{\mathcal{G}'_j}[\exp(\beta\delta\eta(\omega_j, j))]\right] \\ &= e^{\lambda(\delta\beta)}. \end{aligned}$$

This, together with $\mathcal{G}_{j-1} \subset \mathcal{G}'_j$, implies

$$Q^{\mathcal{G}_{j-1}}[\exp(|X - X_j|)] \le A := e^{\lambda(\beta)} + e^{\lambda(-\beta)}.$$

■

We have shown that a concentration (self-averaging) property holds in all generality:

Corollary 1.2.2 *With $B = 2\sqrt{6}\big(e^{\lambda(\beta)} + e^{\lambda(-\beta)}\big)^2$ we have for all $\varepsilon \le B$ and all $n \ge 1$,*

$$Q\left[|p^\eta_{n,\beta} - Qp^\eta_{n,\beta}| \ge \varepsilon\right] \le 2\exp\{-n\varepsilon^2/4B\}. \tag{1.2.10}$$

1.2.2 Upper bounds on the free energy

By Jensen inequality,

$$Q[p^\eta_{n,\beta}] = \frac{1}{n} Q \ln Z^\eta_{n,\beta} \le \frac{1}{n} \ln Q Z^\eta_{n,\beta} = \lambda(\beta)\ , \tag{1.2.11}$$

hence

$$p(\beta) \le \lambda(\beta) \tag{1.2.12}$$

This bound is rather universal in the realm of random medium, it is known as the **annealed bound**.

For other upper bounds, we can use standard monotonicity properties from thermodynamics:

Proposition 1.2.3 .
(i) $p^\eta_{n,\beta}$ *is a smooth convex function,* $p^\eta_{n,0}=0$*;* p *is convex with* $p(0)=0$*.*
(ii) $\beta\mapsto\beta^{-1}p^\eta_{n,\beta}$ *is increasing.*
(iii) $\beta\mapsto\beta^{-1}[p^\eta_{n,\beta}+\ln(2d)]$ *is decreasing.*

□ By differentiation, one gets

$$\frac{d}{d\beta}np^\eta_{n,\beta}=\mu^\eta_{n,\beta}[H_n]\ ,\quad \frac{d^2}{d\beta^2}np^\eta_{n,\beta}=\mathrm{Var}_{\mu^\eta_{n,\beta}}[H_n]>0\ ,$$

and all properties in (i) follow easily. By convexity, $\beta^{-1}p^\eta_{n,\beta}=\beta^{-1}[p^\eta_{n,\beta}-p^\eta_{n,0}]$ is non-decreasing. Turning to (iii), we have the identity

$$\frac{d}{d\beta}\frac{1}{\beta}\,[p^\eta_{n,\beta}+\ln(2d)]=-\frac{1}{\beta^2}\,[p^\eta_{n,\beta}+\ln(2d)]+\frac{1}{n\beta}\,\mu^\eta_{n,\beta}[H_n]=\frac{1}{n\beta^2}\,h(\mu^\eta_{n,\beta})\ ,$$

where $h(\mu_n)$ is the Boltzmann entropy of a probability measure on the n steps path space,

$$h(\nu):=\sum_\omega\nu(\omega)\ln\nu(\omega)\ . \tag{1.2.13}$$

Clearly, $h(\nu)\le 0$ for all ν, which ends the proof. ■

Remark 1.2.4 *In fact, the Boltzmann entropy is related to the relative entropy as defined in the Appendix, Section 3.1:* $h(\nu)$ *is equal to the entropy* $H(\nu|\mu)$ *of the probability* ν *with respect to the uniform probability measure* μ *on the* n *steps path space, i.e.,* μ *is the law of* $(\omega_i)_{i\le n}$ *under* P*. Since* $h(\nu)\le\ln\left(\sum_\omega\nu(\omega)^2\right)$*,* $h(\nu)$ *is non-positive, and negative if* ν *is not a Dirac mass.*

Remark 1.2.5 *At this point the reader wonders whether the strict inequality may hold in the annealed bound (1.2.12) or not. In this remark, we answer with the positive, in the particular case of the standard gaussian distribution* $\mathcal{N}(0,1)$ *for* η*. In this case,* $\lambda(\beta)=\beta^2/2$*, and for all* ω*,* $H_n(\omega)\sim\mathcal{N}(0,n)$*. Recall the standard Gaussian tail estimate: if* X *has density* $g(x)=(2\pi)^{-1/2}\exp -x^2/2$*,*

$$\frac{x}{1+x^2}g(x)\le\mathbf{P}(X>x)\le\frac{1}{x}g(x)\ ,\quad x>0\ ,$$

*– see (**??**)–. Then, we have for all* $a>\sqrt{2\ln(2d)}$*,*

$$\begin{aligned}\sum_{n\ge1}Q\left(\max_\omega H_n(\omega)>na\right)&\le\sum_{n\ge1}\sum_{\omega:\text{length }n}Q\left(H_n(\omega)>na\right)\\&\le\sum_{n\ge1}(2d)^n\frac{1}{a\sqrt{n}}\exp\{-na^2/2\}\\&<\infty\ .\end{aligned}$$

By Borel-Cantelli's lemma, we see that a.s.,

$$n^{-1}\ln Z^\eta_{n,\beta}\le n^{-1}\beta\max_\omega\{H_n(\omega)\}\le\beta a$$

for n *large enough. Hence,*

$$p\le\beta\sqrt{2\ln(2d)}\ . \tag{1.2.14}$$

Since this bound is of smaller order than $\lambda(\beta)$ *as* $\beta\to\infty$*, we conclude that* $p(\beta)<\lambda(\beta)$ *for* β *large enough.*

We derive an upper bound, which is sharper than the annealed bound (1.2.12). Its relevance is that it yields a sufficient condition for the strict inequality to hold in the annealed bound. It improves on the argument in the Remark 1.2.5, being however less transparent than the above argument. It is essentially of the same nature, in the sense that it does not take into account the correlation structure of the random vector $(H_n(\omega))_\omega$.

To simplify the notation, we introduce

$$\gamma(\beta) = \beta\lambda'(\beta) - \lambda(\beta), \quad \varphi(\beta) = \frac{\lambda(\beta) + \ln(2d)}{\beta}. \tag{1.2.15}$$

Note that $\varphi'(\beta) = (\gamma(\beta) - \ln(2d))/\beta^2$ and that $\gamma(\beta)$ is increasing in $\beta \geq 0$. Moreover,

$$\begin{aligned} Q[p^\eta_{n,\beta} + \ln(2d)] &\overset{\text{prop. 1.2.3 (iii)}}{=} \inf_{m\in]0,1]} \frac{1}{m} Q[p^\eta_{n,m\beta} + \ln(2d)] \\ &\overset{(1.2.11)}{\leq} \beta \inf_{m\in]0,1]} \varphi(m\beta) \\ &= \beta \inf_{\beta'\in]0,\beta]} \varphi(\beta') , \end{aligned} \tag{1.2.16}$$

which is depicted in figure 1.2. This gives the bound (1.2.17) below. Now, the optimal m is

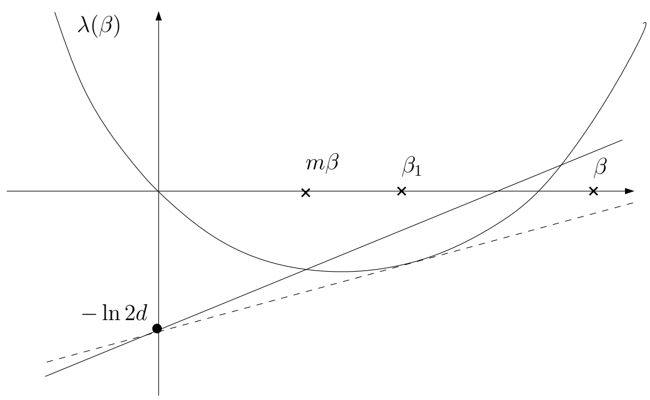

Figure 1.2: An upper bound.

equal to 1 if $\gamma(\beta) \leq \ln(2d)$; In this case we recover the annealed bound. Assume there exists $\beta_1 \in (0, \infty)$ such that $\gamma(\beta_1) = \ln(2d)$. When $\beta > \beta_1$, the optimal m is such that $m\beta = \beta_1$ and we find the bound

$$Qp^\eta_{n,\beta} \leq \beta\varphi(\beta_1) - \ln(2d) < \lambda(\beta)$$

where the last inequality is strict by strict convexity of λ. We summarize the above discussion in the next proposition.

Proposition 1.2.6 *We have*

$$p(\beta) \leq \inf_{m\in]0,1]} \frac{\lambda(m\beta) + \ln(2d)}{m} - \ln(2d) . \tag{1.2.17}$$

Hence, under Condition (T),

$$(\mathbf{T}) : \qquad \beta\lambda'(\beta) - \lambda(\beta) > \ln(2d) , \tag{1.2.18}$$

we have

$$p(\beta) < \lambda(\beta).$$

More precisely, if there exists a positive root β_1 to the equation $\beta\lambda'(\beta) = \ln(2d) + \lambda(\beta)$, then for all $\beta > \beta_1$ it holds

$$p(\beta) \le \frac{\beta}{\beta_1}\big[\lambda(\beta_1) + \ln(2d)\big] - \ln(2d) < \lambda(\beta)\ . \tag{1.2.19}$$

Example 1.2.7 *We consider again the Gaussian case, $\eta(i,x) \sim \mathcal{N}(0,1)$. We easily compute $\beta_1 = \sqrt{2\ln(2d)}$, we check that the bound in (1.2.19) is equal to $\beta\sqrt{2\ln(2d)} - \ln(2d)$, and therefore is strictly smaller than the one from (1.2.14).*

We now look for conditions ensuring $p(\beta) < \lambda(\beta)$ for large β, in terms of the marginal distribution of q of η.

Corollary 1.2.8 *Set $q(dh) = Q(\eta(x,n) \in dh)$ and $s = \sup\operatorname{supp}[q]$. If $q(\{s\}) < \frac{1}{2d}$, then, there exists $\beta_1 \in (0,\infty)$ such that $p(\beta) < \lambda(\beta)$ for $\beta > \beta_1$.*

□ exercise ■

Remark 1.2.9 *Lower bounds are less useful. We can use the formula as a supremum from theorem 1.2.1, and the simplest application leads to*

$$p(\beta) \ge Qp_1(\beta) = Q\ln P[\exp\{\beta\eta(1,\omega_1)\}]$$

which is already better than using Jensen inequality

$$p(\beta) \ge \ln P[\exp\{\beta Q\eta(1,\omega_1)\}] = \beta Q\eta(1,\omega_1).$$

But all these bounds correspond to local optimization in comparison with the polymer measure which is in fact highly non local. ■

1.2.3 Monotonicity and phase diagram

The function p being convex, its left and right derivatives $p'_g(\beta), p'_d(\beta)$, are non-decreasing. The function λ is increasing. Their difference has a nice monotonicity property.

Proposition 1.2.10 *The functions $\beta \mapsto \lambda(\beta) - Qp^\eta_{n,\beta}$ and $\beta \mapsto \lambda(\beta) - p(\beta)$ are non-decreasing on $\mathbb{R}^+$.*

□ With $\zeta_n = \exp\beta H_n$, it is straightforward to check

$$\begin{aligned}
\frac{\partial}{\partial\beta} Q\ln Z^\eta_{n,\beta} &= Q\frac{\partial}{\partial\beta}\ln Z^\eta_{n,\beta} \\
&= Q[(Z^\eta_{n,\beta})^{-1}\frac{\partial}{\partial\beta}Z^\eta_{n,\beta}] \\
&= P\left[Q[(Z^\eta_{n,\beta})^{-1}H_n\zeta_n]\right]
\end{aligned}$$

At this point, we will use the fact that independent variables are positively associated, and satisfy the Harris-FKG inequality given in the appendix.

For any fixed path ω, the probability measure $\zeta_n e^{-n\lambda(\beta)}dQ$ is product, and therefore the family η satisfies the FKG inequality. Note that the function H_n is increasing in η, while $(Z^\eta_{n,\beta})^{-1}$ is a decreasing for $\beta \geq 0$. We apply Proposition 3.4.2 for fixed ω, and we find

$$\begin{aligned} Q[(Z^\eta_{n,\beta})^{-1}H_n\zeta_n] &\leq e^{-n\lambda(\beta)}Q[(Z^\eta_{n,\beta})^{-1}\zeta_n] \times Q[H_n\zeta_n] \\ &= Q[(Z^\eta_{n,\beta})^{-1}\zeta_n] \times n\lambda'(\beta) \end{aligned}$$

using that

$$Q[\eta(t,x)e^{\beta\eta(t,x)}] = \lambda'(\beta)e^{\lambda(\beta} \ . \tag{1.2.20}$$

Integrating with respect to P, we get

$$\begin{aligned} \frac{\partial}{\partial\beta}Q\ln Z^\eta_{n,\beta} &\leq n\lambda'(\beta)P\left[Q[(Z^\eta_{n,\beta})^{-1}\zeta_n]\right] \\ &= n\lambda'(\beta)Q\left[(Z^\eta_{n,\beta})^{-1}P[\zeta_n]\right] \\ &= n\lambda'(\beta) \end{aligned}$$

which yields the desired result, since $p^\eta_{n,\beta}$ and λ are both equal to zero when $\beta = 0$. ■

Theorem 1.2.11 (Critical temperature) *There exists* $\beta_c = \beta_c(Q,d) \in [0,\infty]$ *such that*

$$\begin{cases} p(\beta) = \lambda(\beta) & \text{if } \beta \leq \beta_c, \\ p(\beta) < \lambda(\beta) & \text{if } \beta > \beta_c \end{cases} \tag{1.2.21}$$

□ This is a direct consequence of Proposition 1.2.10 ■

We call *high temperature region* (or low β region) the set of β's such that $p = \lambda$, and the *low temperature region* (or large β region) the set of β's such that $p < \lambda$. One expects that the polymer measure has completely different behavior in these two regions. If both have a nonempty interior, the function p is nonanalytic at $\beta = \beta_c$. (Indeed, it is given by $p(\beta) = \lambda(\beta)$ for $\beta \in [0, \beta_c]$, which analytic continuation on $\mathbb{R}$ is $\lambda(\beta)$, under the assumption (1.1.1).) The value β_c is called **critical**.

Remark 1.2.12 (i) We have trivially $p(0) = 0 = \lambda(0)$, showing that $\beta = 0$ is in the low temperature region.
(ii) We have seen sufficient conditions for $\beta_c < \infty$, e.g., condition (T) in (1.2.18).
(iii) Theorem 1.2.11 implies the absence of reentrant phase transition in the phase diagram of the model. Of course, in complete generality, we may have $\beta_c = 0$ or ∞, i.e., absence of one of the two regimes in the interval $(0, \infty)$.
(iii) For more information on FKG inequality, see [48, p.77–83]. ■

1.3 The martingale approach

Martingale theory is a powerful tool to study random sequences. In this section, we start to use in our context. Firs of all, it is efficient for proving that equality can hold in (1.2.12).

1.3.1 Weak disorder versus strong disorder

Classical considerations from thermodynamics and common sense made us consider $\ln Z_{n,\beta}^{\eta}$, a rather difficult quantity to study directly since we cannot even compute its expectation! It is far easier to consider the partition function $Z_{n,\beta}^{\eta}$ itself, for which we easily see that $QZ_{n,\beta}^{\eta} = \exp(n\lambda)$. All through, we will consider the *normalized partition function*, defined by

$$W_n = Z_{n,\beta}^{\eta} \exp(n\lambda(\beta)) \ , \quad n \geq 1. \tag{1.3.22}$$

This variable has expectation 1, which explains why we call it "normalized". To keep notations simple, we drop the subscripts n, β, and the superscript η.

Fix a path ω of the simple random walk. Then, the sequence $(H_k(\omega))_k$ is the sum of independent identically distributed real random variables on $(\Omega_\eta, \mathcal{G}, Q)$ – the randomness coming from the environment –, it is itself a random walk. The corresponding exponential martingale is

$$\bar{\zeta}_n = \bar{\zeta}_n(\omega) = \exp\Big(\beta H_n(\omega) - n\lambda(\beta)\Big), \tag{1.3.23}$$

for $\beta \in \mathbb{R}$: this is a positive, mean 1 martingale on $(G, \mathcal{G}, Q)$, with respect to the filtration $(\mathcal{G}_n)_n$, where

$$\mathcal{G}_n = \sigma\{\eta(j,x) \ ; \ j \leq n, \ x \in \mathbb{Z}^d\} \ ,$$

This holds for all path ω. By making a linear combination of those martingales indexed by ω, we will get another martingale. In particular,

$$W_n = P(\bar{\zeta}_n) \quad \text{is a positive martingale.}$$

This is much stronger a property than $Q[W_n] = 1$, and it will make the sequence $Z_{n,\beta}^{\eta}$ much easier to study than $\ln Z_{n,\beta}^{\eta}$ itself, a fact which was used first by Bolthausen in [7].

By Doob's martingale convergence theorem [70](corollary 11.7), the limit W_∞ exists Q-a.s., and is non-negative.

It is easy to see that the event $\{W_\infty = 0\}$ is measurable with respect to the tail σ-field

$$\mathcal{T} = \bigcap_{n\geq 1} \mathcal{T}_n \ , \quad \mathcal{T}_n = \sigma\{\eta(j,x) \ ; \ j \geq n, \ x \in \mathbb{Z}^d\} \ .$$

Indeed, we can write

$$\begin{aligned} W_{n+m} &= P[\bar{\zeta}_n Z_{m,\beta}^{\theta_{n,S_n}\eta} e^{-m\lambda(\beta)}] \ , \\ &= P\big[\bar{\zeta}_n \times \lim_{m\to\infty} (Z_{m,\beta}^{\theta_{n,S_n}\eta} e^{-m\lambda(\beta)})\big] \qquad \text{(finite sum)} \\ &= Z_{n,\beta}^{\eta} \times \sum_{x\in\mathbb{Z}^d} \mu_{n,\beta}^{\eta}(S_n = x) \lim_{m\to\infty} (Z_{m,\beta}^{\theta_{n,x}\eta} e^{-m\lambda(\beta)}) \ . \end{aligned}$$

For all n, by strict positivity of $\bar{\zeta}_n$, the event under consideration is equal to

$$\{W_\infty = 0\} = \Big\{ \lim_{m\to\infty} Z_{m,\beta}^{\theta_{n,x}\eta} e^{-m\lambda(\beta)} = 0 \ ; \quad x \in \mathbb{Z}^d, P(S_n = x) > 0 \Big\} \ .$$

We conclude that $\{W_\infty = 0\} \in \mathcal{T}_n$, and then $\{W_\infty = 0\} \in \mathcal{T}$. By Kolmogorov's zero-one law [26], every event in the tail σ-field $\mathcal{T}$ has probability 0 or 1. Summarizing all this, we can state the following.

Theorem 1.3.1 *The limit*

$$W_\infty = \lim_{n \nearrow \infty} W_n \tag{1.3.24}$$

exists Q-a.s. Moreover, there are only two possibilities for the positivity of the limit;

$$Q\{W_\infty > 0\} = 1 \ , \tag{1.3.25}$$

or

$$Q\{W_\infty = 0\} = 1 \ . \tag{1.3.26}$$

The above contrasting situations (1.3.25) and (1.3.26) will be called the **weak disorder** phase and the **strong disorder** phase, respectively. Again, observe that, for $\beta = 0$, $W_n = 1$ for all n, so that weak disorder takes place.

We will see later that the polymer is diffusive in the regime (1.3.25), as well as other consequences.

Remark 1.3.2 (i) It is an interesting question to find a characterization of (1.3.25) (or (1.3.26)) in terms of the distribution of $\eta(n, x)$. It will be addressed in section 1.4.1 below.

(ii)The question of whether the positive martingale W_n vanishes or not as $n \to \infty$, has somewhat similar flavor to some other topics in the probability theory such as Kakutani's dichotomy for infinite product measure (e.g.,[26, page 244]), nontriviality of the limit of the normalized Galton-Watson process [3] and of multiplicative chaos [37].

(iii) It is not difficult to see that

$$W_\infty > 0 \implies p(\beta) = \lambda(\beta) \ . \tag{1.3.27}$$

Indeed, we have a.s.

$$p(\beta) = \lim_{n\to\infty} n^{-1} \ln Z^\eta_{n,\beta} = \lambda(\beta) + \lim_{n\to\infty} n^{-1} \ln W_n \ ,$$

where $\lim_{n\to\infty} \ln W_n = \ln W_\infty$ is finite if $W_\infty > 0$.

(iv) Except for $\beta = 0$, the random variable W_∞ is not $\mathcal{T}$-measurable in the weak disorder phase. To see this, observe that it has to depend in a significant manner of $\eta(1, x)$ with x nearest neighbor of the origin.

Proposition 1.3.3 *There exists $\bar\beta_c = \bar\beta_c(Q, d) \in [0, \infty]$ such that*

$$\begin{cases} W_\infty > 0 & a.s. \quad \text{if } \beta \in \{0\} \cup (0, \bar\beta_c), \\ W_\infty = 0 & a.s. \quad \text{if } \beta > \bar\beta_c \end{cases} \tag{1.3.28}$$

Remarks analogous to 1.2.12 can be formulated here.

$\square$ Let $\delta \in (0, 1)$ arbitrary. Since $QW_n = 1$, the sequence $(W_n^\delta)_n$ is uniformly integrable. In addition to a.s. convergence, this imply that

$$\lim_{n\to\infty} QW_n^\delta = QW_\infty^\delta \ ,$$

which is either 0 in the strong disorder case, or strictly positive in the weak disorder case. We claim that

$$\beta \mapsto QW_n^\delta \quad \text{is non} - \text{increasing on } \mathbb{R}_+ \ . \tag{1.3.29}$$

This will imply that QW_∞^δ also is non-increasing on the positive half-line. Then, we obtain the proposition by putting $\bar{\beta}_c = \inf\{\beta \ge 0 : QW_\infty^\delta = 0\}$ (with the convention $\inf \emptyset = +\infty$). It only remains to prove (1.3.29). We have

$$\begin{aligned}
\frac{d}{d\beta} Q[W_n^\delta] &= Q\left[\frac{d}{d\beta} W_n^\delta\right] \\
&= \delta Q\left[W_n^{\delta-1} P\{(H_n - n\lambda')\bar{\zeta}_n\}\right] \\
&= \delta P\left[Q\left\{\bar{\zeta}_n W_n^{\delta-1}(H_n - n\lambda')\right\}\right] \\
&\le \delta P\left[Q\left\{\bar{\zeta}_n W_n^{\delta-1}\right\} Q\left\{\bar{\zeta}_n(H_n - n\lambda')\right\}\right] && \text{(by FKG)} \\
&= 0 && \text{(by 1.2.20)}
\end{aligned}$$

The FKG inequality from Proposition 3.4.2 was applied above to the product measure $\bar{\zeta}_n dQ$ for fixed ω, and to the decreasing function $W_n^{\delta-1}$ of η and the non-decreasing one $H_n - n\lambda'$. ■

Open problem: Is $\beta_c = \bar{\beta}_c$? The inequality $\beta_c \ge \bar{\beta}_c$ holds trivially. Does (1.3.26) in a neighborhood of β_0 implies $p(\beta) < \lambda(\beta)$ in a neighborhood of β_0 ? In other words, do we have a reciproque to Remark 1.3.2, (iii) ? We will see in Theorem 1.5.6 that the answer is yes when $d = 1$, and when $d = 2$ in [Lacoin]. But this will come out by finding explicitely the set of β's such that $W_\infty = 0$, and the set of β's such that $p(\beta) < \lambda(\beta)$. The question is open in dimension $d \ge 3$.

1.3.2 The second moment method and the L^2 region

In this section, we show how to prove that weak disorder holds for some values of the parameters d and β. The proof will be based on a second moment computation. The second moment of the normalized partition function can be written explicitly in terms of the expectation of a function of two independent copies of the random walk, the function being the exponent of the number of intersections between the walks.

We first recall the following fact about the return probability π_d for the simple random walk,

$$\pi_d \stackrel{\text{def.}}{=} P\{S_n = 0 \text{ for some } n \ge 1\} \quad \text{is} \quad \begin{cases} = 1 & \text{if } d \le 2, \\ < 1 & \text{if } d \ge 3. \end{cases} \tag{1.3.30}$$

More precisely, it is known that $\pi_{d+1} < \pi_d$ for all $d \ge 3$ (e.g., [59, Lemma 1]) and that $\pi_3 = 0.3405...$ [66, page 103]. In particular, $\pi_d \le 0.3405...$ for all $d \ge 3$.

Theorem 1.3.4 *[7] Suppose that $d \ge 3$ (hence $\pi_d < 1$) and that condition (L2) holds:*

$$(\mathbf{L2}) \qquad \gamma_1(\beta) \stackrel{\text{def.}}{=} \lambda(2\beta) - 2\lambda(\beta) < \ln(1/\pi_d). \tag{1.3.31}$$

Then, $W_\infty > 0$ a.s.

Note first that $\gamma_1(\beta)$ is continuous with $\gamma_1(0) = 0$, so that, for $d \ge 3$, the condition (L2) does hold if β is small, whatever the distribution of the environment is. More precisely, since λ is increasing, $\gamma_1'(\beta) = 2[\lambda'(2\beta) - \lambda'(\beta)]$ is positive for $\beta \ge 0$, and so $\gamma_1(\beta)$ is increasing on $\mathbb{R}^+$; similarly, it is decreasing on $\mathbb{R}^-$. Hence, for $d \ge 3$, the condition (L2) is equivalent to β in some open interval around 0. In particular, $p = \lambda$ holds for β in this interval.

□ Proof of Theorem 1.3.4. We compute the L^2-norm of the martingale W_n. To do so, we consider on the product space $(\Omega^2, \mathcal{F}^{\otimes 2})$, the probability measure $P^{\otimes 2} = P^{\otimes 2}(d\omega, d\widetilde{\omega})$, that we will view as the distribution of the couple $(S,\widetilde{S})$ with $\widetilde{S}=(\widetilde{S}_k)_{k\geq 0}$ an independent copy of $S=(S_k)_{k\geq 0}$.

$$\begin{aligned} Q[W_n^2] &= Q\left[P^{\otimes 2}\prod_{t=1}^n e^{\beta[\eta(t,S_t)+\eta(t,\tilde{S}_t)]-2\lambda(\beta)}\right] \\ &= P^{\otimes 2}\left[\prod_{t=1}^n \left(e^{\lambda(2\beta)-2\lambda(\beta)}\mathbf{1}_{S_t=\tilde{S}_t}+\mathbf{1}_{S_t\neq\tilde{S}_t}\right)\right] \\ &= P^{\otimes 2}\left[e^{\gamma_1(\beta)N_n}\right] , \end{aligned}$$

with N_n the number of intersections of the paths $S,\tilde{S}$ up to time n,

$$N_n=N_n(S,\tilde{S})=\sum_{t=1}^n \mathbf{1}_{S_t=\tilde{S}_t} \tag{1.3.32}$$

As $n\to\infty$, $N_n\nearrow N_\infty$, and by monotone convergence $Q[W_n^2]\nearrow P^{\otimes 2}\left[e^{\gamma_1(\beta)N_\infty}\right]$. It is easy to see that N_∞ is the number of visit to 0 of the simple random walk starting from 0. Hence, N_∞ is geometrically distributed with success probability π_d, and

$$\sup_n Q[W_n^2]<\infty \iff \gamma_1+\ln\pi_d<0 ,$$

i.e., iff (1.3.31) is fulfiled. Then, the martingale W_n is bounded in L^2, and by a classical convergence result [70], it converges in L^2 to a limit, which is necessarily equal to W_∞. So $QW_\infty=\lim_n QW_n=1$, which excludes the possibility that the limit vanishes in theorem 1.3.1.
■

Remark 1.3.5 Finer sufficient conditions for weak disorder, improving on (1.3.31), were obtained: [4] making use of size-biasing; [9] by a comparison of environment entropy and lattice entropy, following the approach of [28]. ■

Corollary 1.3.6 *Assume the η is bounded, and let $s=\sup\mathrm{supp}[q]<+\infty$, with $q(dh)=Q(\eta(x,n)\in dh)$ the law of η. If $q((-\infty,s))<1-\frac{1}{\pi_d}$, then, the condition (L2) (1.3.31) holds for all $\beta\geq 0$.*

□ In view of Theorem 1.3.4, it is enough to show that

$$\gamma_1(\beta)\overset{\beta\nearrow\infty}{\longrightarrow}\begin{cases}\infty, & \text{if } s=\infty\\ -\ln q(\{s\}) & \text{if } s<\infty.\end{cases} \tag{1.3.33}$$

Interpolating $\lambda(2\beta)$ and $2\lambda(\beta)$ by $f(\theta)=2^{1-\theta}\lambda(2^\theta\beta)$, $\theta\in[0,1]$, we have $f'(\theta)=(2^{1-\theta}\ln 2)\gamma(2^\theta\beta)$ with $\gamma(\beta)=\beta\lambda'(\beta)-\lambda(\beta)$ as in the proof of Corollary 1.2.8, and then

$$\gamma_1(\beta)=\ln 2\int_0^1 2^{1-\theta}\gamma(2^\theta\beta)d\theta . \tag{1.3.34}$$

We know from the proof of Corollary 1.2.8 that

$$\gamma(\beta)\overset{\beta\nearrow\infty}{\longrightarrow}\begin{cases}\infty, & \text{if } s=\infty,\\ -\ln q(\{s\}) & \text{if } s<\infty.\end{cases}$$

This, together with (1.3.34), implies (1.3.33). ■

Example 1.3.7 *Gaussian environment. If η is standard gaussian $\mathcal{N}(0,1)$, then $\gamma_1(\beta) = \beta^2$ and hence (1.3.31) holds if $\beta < \sqrt{\ln(1/\pi_d)}$.*

Example 1.3.8 *Absence of strong disorder regime. Consider the case of Bernoulli environment, where $\eta(t,x) = 1$ or 0 with probability p and $1-p$ respectively. By Corollary 1.3.6, (1.3.31) holds for all $\beta \geq 0$ if $p > \pi_d$. Theorem 1.3.4 shows that, in this case, weak disorder holds for all $\beta \geq 0$.*

We call L^2 **region**, the set of parameters β such that (1.3.31) holds. In this region, the natural martingale is bounded in L^2, and it allows second moment computations. Therefore, a number of results are known, we will see some in the next sections. As observed below Theorem Theorem 1.3.4, the intersection of the L^2 region with $(0,+\infty)$ is an interval $(0,\beta_1)$ with some $\beta_1 \in [0,\infty]$.

1.3.3 Diffusive behavior in L^2 region

All through this section we assume that $d \geq 3$, and that β belongs to the L^2 region.

The next theorem states that, in this region, the random environment do not change the transversal fluctuations of the polymer for large d and small enough β.

Theorem 1.3.9 *[35, 7, 64]) Under the assumptions of Theorem 1.3.4, we have*

$$\lim_{n\nearrow\infty} \mu^{\eta}_{n,\beta}[|S_n|^2]/n = 1 \quad Q\text{-a.s.}\ , \tag{1.3.35}$$

and for all $f \in C(\mathbb{R}^d)$ with at most polynomial growth at infinity

$$\lim_{n\nearrow\infty} \mu^{\eta}_{n,\beta}\left[f\left(S_n/\sqrt{n}\right)\right] = (2\pi)^{-d/2}\int_{\mathbb{R}^d} f\left(x/\sqrt{d}\right)\exp(-|x|^2/2)dx, \quad Q\text{-a.s.} \tag{1.3.36}$$

In particular, with Z a d-dimensional gaussian vector $Z \sim \mathcal{N}_d(0, d^{-1}I_d)$, we have

$$\mu^{\eta}_{n,\beta}\left(\frac{S_n}{\sqrt{n}} \in \cdot\right) \longrightarrow \mathbf{P}(Z \in \cdot\,) \quad Q - a.s.$$

Remark 1.3.10 *The first rigorous proof of (1.3.35) was obtained by Imbrie and Spencer [35] in the case of Bernoulli environment. The fact that the polymer is diffusive in some regime was much of a surprise. Soon afterwards, a more transparent proof based on the martingale analysis was given by Bolthausen [7]. The martingale proof was then extended to general environment under condition (1.3.31) by Song and Zhou [64]. The diffusive behavior (1.3.35) follows from (1.3.36) by choosing $f(x) = |x|^2$. In [7], (1.3.36) is obtained for the Bernoulli environment only. However, with the help of the observation made in [64], it is not difficult to extend the central limit theorem to general environment under the assumption in Theorem 1.3.9. In [2] Albeverio and Zhou proved, under the assumptions of Theorem 1.3.4, that under the polymer measure μ_n, the path S satisfies the invariance principle for almost every realization of the environment.* ■

We will not prove theorem 1.3.9, but we rather give a elementary proof of (1.3.36), in a weaker version with convergence in probability instead of almost sure. (The reader interested in the full proof of Theorem 1.3.9 is refered to Bolthausen [7] and later taken up by Song and Zhou [64]; the arguments are based on the L^2 analysis of certain martingales on $(\Omega_\eta, \mathcal{G}, Q)$.)

□ Writing the gaussian law $\nu = \mathcal{N}_d(0, d^{-1}I_d)$, we will prove that for all bounded continuous function $g : \mathbb{R} \to \mathbb{R}$,

$$\mu_n \left[g\left(n^{-1/2} S_n\right)\right] \to \nu(g) \tag{1.3.37}$$

in Q-probability as $n \to \infty$. We let $\nu_n(\cdot) = P[n^{-1/2}S_n \in \cdot]$. By the central limit theorem, $\nu_n \to \nu$ weakly as $n \to \infty$.

$$\begin{aligned}
& Q\left(\left| \mu_n \left[g\left(n^{-1/2} S_n\right)\right] - \nu_n(g) \right|^2 W_n^2 \right) \\
= \; & P^2 Q\left(e^{\beta H_n(S) + \beta H_n(\tilde S) - 2n\lambda} \left[g\left(n^{-1/2} S_n\right) - \nu_n(g)\right] \left[g\left(n^{-1/2} \tilde S_n\right) - \nu_n(g)\right]\right) \\
= \; & P^2 \left(e^{\gamma_1 N_n} \left[g\left(n^{-1/2} S_n\right) - \nu_n(g)\right] \left[g\left(n^{-1/2} \tilde S_n\right) - \nu_n(g)\right]\right) \qquad (1.3.38)
\end{aligned}$$

We know that, under P^2, the r.v. N_n converges to N_∞ a.s., and that $n^{-1/2}S_n$ – and similarly $n^{-1/2}\tilde S_n$ – converges to ν in law. Now, we claim that, under P^2, the triple

$$(N_n, n^{-1/2}S_n, n^{-1/2}\tilde S_n) \xrightarrow{\text{law}} (N, Z, \tilde Z) \tag{1.3.39}$$

with $(N, Z, \tilde Z)$ an independent triple where N has the same law as N_∞, Z and $\tilde Z$ have the law ν. The proof of this fact makes use of the observation that

$$\sup_{n \geq m} P^2(N_n \neq N_m) \to 0 \ \text{ as } m \to \infty$$

since $N_n \nearrow N_\infty < \infty$ a.s. Fix $m \geq 1$ and $f, g, \tilde g$ continuous and bounded. For all $n \geq m$, we write

$$\begin{aligned}
& P^2 \left[f(N_n) g\big(n^{-1/2} S_n\big) \tilde g\big(n^{-1/2} S_n\big) \right] \\
= \; & P^2 \left[f(N_n) g\big(n^{-1/2} S_n\big) \tilde g\big(n^{-1/2} S_n\big) \mathbf{1}_{N_n = N_m} \right] + \varepsilon(n, m) \\
= \; & P^2 \left[f(N_m) g\big(n^{-1/2} S_n\big) \tilde g\big(n^{-1/2} S_n\big) \mathbf{1}_{N_n = N_m} \right] + \varepsilon(n, m) \\
= \; & P^2 \left[f(N_m) g\big(n^{-1/2} (S_n - S_m)\big) \tilde g\big(n^{-1/2} (\tilde S_n - \tilde S_m)\big) \mathbf{1}_{N_n = N_m} \right] + \varepsilon'(n, m) \\
= \; & P^2 \left[f(N_m) g\big(n^{-1/2} (S_n - S_m)\big) \tilde g\big(n^{-1/2} (S_n - S_m)\big) \right] + \varepsilon''(n, m) \\
= \; & P^2[f(N_m)] \times P \left[g\big(n^{-1/2} (S_n - S_m)\big) \right] \times P \left[\tilde g\big(n^{-1/2} (S_n - S_m)\big) \right] + \varepsilon''(n, m) \; ,
\end{aligned}$$

which equalities define the terms $\varepsilon(n,m), \varepsilon'(n,m), \varepsilon''(n,m)$ on their first occurence. Here,

$$|\varepsilon(n,m)| \leq \|f\|_\infty \|g\|_\infty \|\tilde g\|_\infty P(N_n \neq N_m)$$

tends to 0 as $m \to \infty$ uniformly in $n \geq m$, $\varepsilon'(n,m) - \varepsilon(n,m) \to 0$ as $n \to \infty$ for all fixed m, and $\sup_{n \geq m} \varepsilon''(n,m) \to 0$ as $m \to \infty$. The last equality comes from independence in the increments of the random walks, and of the two random walks S and $\tilde S$. Hence, letting $n \to \infty$ and then $m \to \infty$, we get

$$P^2 \left[f(N_n) g\big(n^{-1/2} S_n\big) \tilde g\big(n^{-1/2} S_n\big) \right] \to P^2[f(N_\infty)] \times \nu[g] \times \nu[\tilde g]$$

which proves (1.3.39). Coming back to (1.3.38), and since $P^2(e^{\gamma N_n}) < \infty$ for some small enough $\gamma > \gamma_1$, (1.3.39) implies that

$$Q\left(\left| \mu_n \left[g\left(n^{-1/2} S_n\right)\right] - \nu_n(g) \right|^2 W_n^2 \right) \to P^2(e^{\gamma_1 N_\infty}) \left[\nu(g) - \nu(g)\right]^2 = 0$$

Since W_n^{-2} converges to a finite limit, it is bounded in probability, so the previous limit yields (1.3.37). ■

1.4 The semimartingale approach

The next step in our martingale analysis is to consider $\ln W_n$ as a semimartingale and to write its Doob's decomposition. Although this is quite natural, it was realized only recently. Viewed as a "conditional second moment" method, this step is most a natural continuation of chapter 1.3.

1.4.1 Semimartingale decomposition and overlap

It is convenient to introduce some more notation. For a sequence $(a_n)_{n\geq 0}$ (random or non-random), we set $\Delta a_n = a_n - a_{n-1}$ for $n \geq 1$. Let us now recall Doob's decomposition in this context [70]: any $(\mathcal{G}_n)$-adapted process $X = \{X_n\}_{n\geq 0} \subset L^1(Q)$ can be decomposed in a unique way as

$$X_n = M_n(X) + A_n(X), \quad n \geq 1,$$

where $M(X)$ is an $(\mathcal{G}_n)$-martingale and

$$A_0 = 0, \;\; \Delta A_n = Q[\Delta X_n | \mathcal{G}_{n-1}], \quad n \geq 1.$$

$M_n(X)$ and $A_n(X)$ are called respectively, the martingale part and compensator of the process X. If X is a square integrable martingale, then the compensator $A_n(X^2)$ of the process $X^2 = \{(X_n)^2\}_{n\geq 0} \subset L^1(Q)$ is denoted by $\langle\, X \,\rangle_n$ and is given by the following formula:

$$\Delta\langle\, X \,\rangle_n = Q[(\Delta X_n)^2 | \mathcal{G}_{n-1}]$$

Here, we are interested in the Doob's decomposition of $X_n = -\ln W_n$, whose martingale part and the compensator will be denoted M_n and A_n respectively

$$-\ln W_n = M_n + A_n. \tag{1.4.40}$$

To compute M_n and A_n, we introduce

$$U_n = \mu^{\eta}_{n-1,\beta}[e^{\beta\eta(n,\omega_n)-\lambda(\beta)}] - 1 \, .$$

It is then clear that

$$W_n/W_{n-1} = 1 + U_n \tag{1.4.41}$$

and hence that

$$\Delta A_n = -Q\big[\ln(1+U_n) | \mathcal{G}_{n-1}\big] \, , \tag{1.4.42}$$

$$\Delta M_n = -\ln(1+U_n) + Q\big[\ln(1+U_n) | \mathcal{G}_{n-1}\big] \, . \tag{1.4.43}$$

A key role in the asymptotics of the model is played by the following random variables on $(\Omega_\eta, \mathcal{G}, Q)$,

$$I_n = \sum_{x\in\mathbb{Z}^d} \mu^{\eta}_{n-1,\beta}\{\omega_n = x\}^2 \, . \tag{1.4.44}$$

We now mention to an interpretation of I_n. On the product space $(\Omega_\omega^2, \mathcal{F}^{\otimes 2})$, we consider the probability measure

$$\mu^{\eta\otimes 2}_{n,\beta} = \mu^{\eta}_{n,\beta} \otimes \mu^{\eta}_{n,\beta}(d\omega, d\widetilde{\omega}) \, ,$$

that we will view as the distribution of the couple $(\omega, \widetilde{\omega})$ with $\widetilde{\omega} = (\widetilde{\omega}_k)_{k\geq 0}$ an independent copy of $\omega = (\omega_k)_{k\geq 0}$ with law μ_n. We then have that

$$I_n = \mu^{\eta\otimes 2}_{n-1,\beta}(\omega_n = \widetilde{\omega}_n) \, . \tag{1.4.45}$$

Hence, the summation

$$\sum_{1 \le k \le n} I_k \tag{1.4.46}$$

is the expected amount of the overlap up to time n of two independent polymers in the same (fixed) environment. This can be viewed as an analogue to the so-called *replica overlap* often discussed in the context of disordered systems, e.g. mean field spin glass, and also of directed polymers on trees [21].

The large time behavior of (1.4.46) and the normalized partition function W_n are related as follows.

Theorem 1.4.1 *Let $\beta \neq 0$. Then,*

$$\{W_\infty = 0\} = \left\{ \sum_{n \ge 1} I_n = \infty \right\}, \quad Q\text{-a.s.} \tag{1.4.47}$$

Moreover, if $Q\{W_\infty = 0\} = 1$, there exist $c_1, c_2 \in (0, \infty)$ such that Q-a.s.,

$$c_1 \sum_{1 \le k \le n} I_k \le -\ln W_n \le c_2 \sum_{1 \le k \le n} I_k \qquad \text{for large enough } n\text{'s}. \tag{1.4.48}$$

□ Proof of Theorem 1.4.1: To conclude (1.4.47) and (1.4.48), it is enough to show the following (1.4.49) and (1.4.50):

$$\{W_\infty = 0\} \subset \left\{ \sum_{n \ge 1} I_n = \infty \right\}, \quad Q\text{-a.s.} \tag{1.4.49}$$

There are $c_1, c_2 \in (0, \infty)$ such that

$$\left\{ \sum_{n \ge 1} I_n = \infty \right\} \subset \{(1.4.48) \text{ holds}\}, \quad Q\text{-a.s.} \tag{1.4.50}$$

In view of the second line in (1.4.42), and since the variance is bounded by the second moment (conditionally on $\mathcal{G}_{n-1}$),

$$\Delta\langle\, M \,\rangle_n \le Q[\ln^2(1 + U_n)|\mathcal{G}_{n-1}]. \tag{1.4.51}$$

We now claim that there is a constant $c \in (0, \infty)$ such that

$$\frac{1}{c} I_n \le \Delta A_n \le c I_n, \quad \Delta\langle\, M \,\rangle_n \le c I_n. \tag{1.4.52}$$

Indeed, both follow from (1.4.42), (1.4.51) and Lemma 1.4.2 below; $\{e_i\}$, $\{\alpha_i\}$ and Q in the lemma play the roles of $\{e^{\beta\eta(n,z) - \lambda(\beta)}\}_{|z|_1 \le n}$, $\{\mu^\eta_{n-1,\beta}(\omega_n = z)\}_{|z|_1 \le n}$ and $Q[\,\cdot\,|\mathcal{G}_{n-1}]$.

We now conclude (1.4.49) from (1.4.52) as follows (the equalities and the inclusions here being understood as Q-a.s.):

$$\begin{aligned}
\left\{ \sum_{n \ge 1} I_n < \infty \right\} \quad &\subset \quad \{A_\infty < \infty,\ \langle\, M \,\rangle_\infty < \infty\} \\
&\subset \quad \{A_\infty < \infty,\ \lim_{n \nearrow \infty} M_n \text{ exists and is finite}\} \\
&\subset \quad \{W_\infty > 0\}\,.
\end{aligned}$$

Here, on the second line, we have used a well-known property for martingales, e.g. [26, page 255, (4.9)]: a square integrable martingale converges a.s. on the event $\{\langle M \rangle_\infty < \infty\}$.

Finally we prove (1.4.50). By (1.4.52), it is enough to show that

$$\{A_\infty = \infty\} \subset \left\{ \lim_{n\nearrow\infty} -\frac{\ln W_n}{A_n} = 1 \right\}, \quad Q\text{-a.s.} \tag{1.4.53}$$

Thus, let us suppose that $A_\infty = \infty$, and consider two cases. If $\langle M \rangle_\infty < \infty$, then again by [26, page 255, (4.9)], $\lim\limits_{n\nearrow\infty} M_n$ exists and is finite and therefore (1.4.53) holds. If, on the contrary, $\langle M \rangle_\infty = \infty$, then we will use the law of large numbers for martingales, see [26, page 255, (4.10)]: $M_n/\langle M \rangle_n \to 0$ a.s. on the event $\{\langle M \rangle_n = \infty\}$. In this case we see that

$$-\frac{\ln W_n}{A_n} = \frac{M_n}{\langle M \rangle_n}\frac{\langle M \rangle_n}{A_n} + 1 \longrightarrow 1 \quad Q\text{-a.s.}$$

by (1.4.52). This completes the proof of Theorem 1.4.1. $\square$

Lemma 1.4.2 *Let e_i, $1 \le i \le m$ be positive, non-constant i.i.d. random variables on a probability space $(H, \mathcal{G}, Q)$ such that*

$$Q[e_1] = 1, \quad Q[e_1^3 + \ln^2 e_1] < \infty.$$

For $\{\alpha_i\}_{1\le i\le m} \subset [0,\infty)$ such that $\sum_{1\le i\le m} \alpha_i = 1$, define a centered random variable $U > -1$ by $U = \sum_{1\le i\le m} \alpha_i e_i - 1$. Then, there exists a constant $c \in (0,\infty)$, independent of m and of $\{\alpha_i\}_{1\le i\le m}$, such that

$$\frac{1}{c}\sum_{1\le i\le m} \alpha_i^2 \;\le\; Q\left[\frac{U^2}{2+U}\right], \tag{1.4.54}$$

$$\frac{1}{c}\sum_{1\le i\le m} \alpha_i^2 \;\le\; -Q\left[\ln(1+U)\right] \le c\sum_{1\le i\le m} \alpha_i^2, \tag{1.4.55}$$

$$Q\left[\ln^2(1+U)\right] \le c\sum_{1\le i\le m} \alpha_i^2. \tag{1.4.56}$$

The readers are invited to try the proof of this lemma as an interesting exercise. A solution can be found in [16]. Here, we prove it under the more restrictive assumption of bounded η's.

$\square$ Proof of lemma 1.4.2, when $|\eta(t,x)| \le K$ a.s. Then, for fixed β, U_n stays in a fixed interval $\mathcal{I}$ which is bounded away from -1 and $+\infty$, and there exist constants $C_\pm \in (0,\infty)$ such that

$$u - C_- u^2 \le \ln(1+u) \le u - C_+ u^2\,, \quad u \in \mathcal{I}\,.$$

Recalling the first line of (1.4.42), we have by in one direction:

$$\begin{aligned}
\Delta A_n &= -Q^{\mathcal{G}_{n-1}}[\ln(1+U_n)]\,,\\
&\le -Q^{\mathcal{G}_{n-1}}[U_n|] + C_- Q^{\mathcal{G}_{n-1}}[U_n^2]\,,\\
&= C_- \mu_{n-1,\beta}^{\eta\otimes 2} Q^{\mathcal{G}_{n-1}}\left[(e^{\beta\eta(n,\omega_n)-\lambda(\beta)}-1)(e^{\beta\eta(n,\tilde\omega_n)-\lambda(\beta)}-1)\right]\\
&= C_- \sum_x \mu_{n-1,\beta}^{\eta\otimes 2}(\omega_n = \tilde\omega_n = x) Q[(e^{\beta\eta(n,x)-\lambda(\beta)}-1)^2]\\
&= C_-(e^{\gamma_1(\beta)}-1)\sum_x \mu_{n-1,\beta}^{\eta\otimes 2}(\omega_n = \tilde\omega_n)\\
&= \text{Cst } I_n\,.
\end{aligned}$$

Similarly, one gets the other direction $\Delta A_n \ge \text{Cst } I_n$, which proves (1.4.55). It is clear that $\ln^2(1+u) \le Cu^2$ for $u \in \mathcal{I}$ with some constant finite C, which is enough to get (1.4.56). $\blacksquare$

Corollary 1.4.3 *Q-a.s.,*

$$p(\beta) = \lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} Q^{\mathcal{G}_{t-1}} \ln \mu^{\eta}_{t-1,\beta}[e^{\beta\eta(t,\omega_t)}]$$

Observe that $p^{\eta}_{n,\beta} = (1/n)\sum_{t=1}^{n} \ln(Z_t/Z_{t-1})$, and recall that $p^{\eta}_{n,\beta}$ converge in L^1 to a deterministic limit. We have

$$p(\beta) = \lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} Q\left[\ln \mu^{\eta}_{t-1,\beta}[e^{\beta\eta(t,\omega_t)}]\right] = \text{a.s.}- \lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \ln \mu^{\eta}_{t-1,\beta}[e^{\beta\eta(t,\omega_t)}]\,.$$

1.4.2 Weak disorder and diffusive regime

Remark: (L^2 region) It is quite instructive to make a heuristic computation in the L^2 region:

$$\begin{aligned} I_n &= \sum_x \mu^{\eta}_{n,\beta}(\omega_n = x)^2 \\ &\simeq \sum_x [\, W_\infty \circ \theta^{\leftarrow}_{n,x}]^2 P[\omega_n = x]^2 \\ &\simeq Q[W_n^2] \times \sum_x P[\omega_n = x]^2 \\ &= \mathcal{O}(n^{-d/2}) \end{aligned} \tag{1.4.57}$$

by arguing successively the local limit theorem, and the ergodic theorem. At a rigorous level, only a slower polynomial decay has been so far achieved for I_n ((1.17) in [16]). In view of this heuristic computation, a natural question is whether $\lim_n QW_n^2 = \infty$ implies that $\sum I_n = \infty$ (and therefore $W_\infty = 0$) ? The answer is no, as we have seen in Remark 1.3.5. ■

Observation: (weak disorder region). In the weak disorder region, it is not difficult to see that the polymer measure is very similar to the simple random walk. Indeed, when $W_\infty > 0$, for any $A_n \in \mathcal{F}_n$ such that $P(A_n) \to 1$ as $n \to \infty$, we have

$$\mu^{\eta}_{n,\beta}(A_n) \longrightarrow 1 \quad \text{in } Q - \text{probability.}$$

This follows from

$$\mu^{\eta}_{n,\beta}(A_n^c) = W_n^{-1} P(e^{\beta H_n - n\lambda(\beta)}; A_n^c) \longrightarrow W_\infty^{-1} \times 0 = 0$$

in Q-probability, since $P(e^{\beta H_n - n\lambda(\beta)}; A_n^c) \to 0$ in L^1-norm.

This applies for instance to the set $A_n = \{|\omega_n| \in [a_n, b_n]\}$ with any positive sequences a_n, b_n such that $a_n = o(n^{1/2}), n^{1/2} = o(b_n)$. This shows that the polymer does not spread out much more than the simple random walk.

It is natural to expect that diffusive behavior takes place in the whole weak disorder region, not only under the stronger assumption (1.3.31).

Theorem 1.4.4 *[19] Assume $d \geq 3$ and weak disorder (1.3.25). Then, for all bounded continuous function F on the path space,*

$$\lim_n \mu^{\eta}_{n,\beta}[F(\omega^{(n)})] = \mathbf{E}F(B)$$

in probability, where $\omega^{(n)}$ is the rescaled path defined by $\omega^{(n)} = (\omega_{nt}/\sqrt{n})_{t\geq 0}$ and B is the Brownian motion with diffusion matrix $d^{-1}I_d$. In particular, this holds for all $\beta \in [0, \beta_c)$.

Exponents: Incidently, we see that that the scaling relation between exponents does hold in the full weak disorder region, with $\xi = 1/2$ and $\chi = 0$.

In the proof of theorem 1.4.4 convergence of the series $\sum I_n$ is used as a main technical quantitative ingredient.

1.4.3 Localization and delocalization

We want to characterize the following phenomenom which can be observed experimentally or numerically: For large β the polymer concentrates around the n-geodesics, i.e. the maximizers of H_n. For instance we could try to study $\mu^\eta_{n,\beta}(\omega \in G_n)$ for G_n a neighborhood of the set of the n-geodesics. A difficulty is that little is known on the geodesics. For the case $d = 1$ the reader can refer to Part ?? of [56]. We will also reduce our ambition in considering only the ending point of the path. A simpler quantity is the random variable J_n, which is the probability of the favourite site for the polymer at time n,

$$J_n = \max_{x \in \mathbb{Z}^d} \mu^\eta_{n-1,\beta}\{\omega_n = x\} \ . \tag{1.4.58}$$

Indeed, J_n is small when the measure is spread out – for instance if $\beta = 0$, $J_n = \mathcal{O}(n^{-d/2})$ –, but J_n should be much larger when $\mu^\eta_{n,\beta}$ concentrates on a small number of paths ($J_n \leq 1$). The advantage is that we don't need to know where is (are) located the favourite point(s) ! The shift in the time index is harmless up to a constant factor, we could have taken in the definition on I_n the maximum of $\mu^\eta_{n-1,\beta}\{\omega_{n-1} = x\}$ without changing its essence, but the present one is more natural.

In fact, J_n can be compared to $I_n = \sum_x \mu^\eta_{n-1,\beta}(\omega_n = x)^2$,

$$J_n^2 \leq I_n \leq J_n \ , \tag{1.4.59}$$

as can be seen by keeping only the biggest term in the sum for the lower bound, and using that $\sum_x \mu^\eta_{n-1,\beta}(\omega_n = x) = 1$ for the upper bound. It follows that, I_n vanishes if and only if J_n does.

In view of the above discuccion, the following definition from [12], [16], is most natural:

Definition 1.4.5 *We say that the polymer is* **localized** *if*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} J_t > 0, \quad Q\text{-}a.s. \tag{1.4.60}$$

and that the polymer is **delocalized** *if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} J_t = 0 \quad Q\text{-}a.s. \tag{1.4.61}$$

Roughly, delocalization and localization correspond to J_n vanishing or not as $n \to \infty$. The following result shows that necessarily one of the two cases happens. But it is all the more a *criterion* for localization and delocalization.

Theorem 1.4.6 (localization transition) *Let $\beta \neq 0$. The polymer is*

- *localized if and only if $p < \lambda$,*
- *delocalized if and only if $p = \lambda$.*

□ In view of (1.4.59), Theorem 1.4.6 directly follows from Theorem 1.4.1 and (1.4.47). ■

Remark: From this and from (1.4.59), we observe that I_n and J_n have Cesaro limit of the same nature. Either $n^{-1}\sum_{t=1}^{n} J_t$ and $n^{-1}\sum_{t=1}^{n} J_t$ have both a.s. positive limits (superior and inferior), or they vanishes a.s. as $n \to \infty$.

At this point we recall well known facts [66] for the simple random walk, i.e. the behavior of I_n and J_n in the case $\beta = 0$:

$$\max_{x \in \mathbb{Z}^d} P\{\omega_n = x\} = \mathcal{O}(n^{-d/2}) , \tag{1.4.62}$$

$$P^{\otimes 2}\{\omega_n = \tilde{\omega}_n\} = \mathcal{O}(n^{-d/2}) , \tag{1.4.63}$$

as $n \nearrow \infty$. The decay rate $n^{-d/2}$ in (1.4.62) can be understood as the position of ω_n being roughly uniformly distributed over the euclidean ball in $\mathbb{Z}^d$ with radius const. $\times \sqrt{n}$.

For $\beta \neq 0$ but in the weak disorder region, we first note from the convergence of $\sum I_n$ that $J_n \to 0$. In this region, can still prove (1.4.62) in some specific models – see e.g. see e.g.(1.4.57) –, but, in general, only in a weaker form with a smaller exponent – see e.g. (1.17) in [16] –. Anyway the picture remains similar, with the position ω_n of the polymer being widely spread out, or "delocalized".

We summarize this in a table (α is some constant in $(0, d/2]$).

order of magnitude	$\beta = 0$	*weak disorder* L^2		*strong disorder*	$p < \lambda$
I_n	$n^{-d/2}$	$\leq n^{-\alpha}$	$\to 0$	$\sum I_n = \infty$	1
J_n	$n^{-d/2}$	$\leq n^{-\alpha/2}$	$\to 0$	$\sum J_n = \infty$	1

1.5 Low dimensions

Dimensions $d = 1$ and 2 are special, due to recurrence of the simple random walk – more precisely, due to recurrence of the difference $\omega_n - \tilde{\omega}_n$ under the the product measure $P^{\otimes 2}$, which enforce the interactions between the polymer and its environment. One of our main results here is that strong disorder holds for all non-zero β in dimension 1 and 2.

Theorem 1.5.1 *Assume $d = 1$ or $d = 2$. For all $\beta \neq 0$, $W_\infty = 0$.*

We first develop an approach based on overlap estimates. It will allow us for an elementary proof of the theorem. In the second section, we develop a more sophisticated method to estimate the fractional moments of the partition function. It will yield a more complicated proof of the theorem, but it will give some quantitative estimates that will be crucial in the next section, which shows that localization holds in dimension 1 for all non-zero β. (Note: The theorem 1.5.1 is due to Carmona and Hu [12] for gaussian environment, and to [16] in the general case.)

1.5.1 Overlap estimates

Overlap estimate are based on the following elementary observation. For all $z \in \mathbb{Z}^d$,

$$\begin{aligned}
\mu_{t-1,\beta}^{\eta\otimes 2}(\omega_t = \widetilde{\omega}_t + z) &= \sum_x \mu_{t-1,\beta}^{\eta}(\omega_t = x)\mu_{t-1,\beta}^{\eta}(\omega_t = x+z) \\
&\leq \left(\sum_x \mu_{t-1,\beta}^{\eta}(\omega_t = x)^2 \times \sum_x \mu_{t-1,\beta}^{\eta}(\omega_t = x+z)^2\right)^{1/2} \\
&= \mu_{t-1,\beta}^{\eta\otimes 2}(\omega_t = \widetilde{\omega}_t) \\
&= I_t
\end{aligned}$$

where the inequality is from Cauchy-Schwarz.

□ First proof of theorem 1.5.1.

Dimension 1: We start with a simple computation which shows that, in dimension $d = 1$, the series $\sum I_n$ diverge. Indeed,

$$1 = \sum_{z: z=0[\mathrm{mod}2], |z|\leq 2t} \mu_{t-1,\beta}^{\eta\otimes 2}(\omega_t = \widetilde{\omega}_t + z) \leq (2t+1) I_t \ .$$

Hence, $I_t \geq 1/(2t+1)$ and $\sum_t I_t = \infty$, which shows that $W_\infty = 0$ when $d = 1$. This is an easy proof of the theorem when $d = 1$.

Dimension 2: We prove the theorem by contradiction. Assume that $W_\infty > 0$ almost surely. Consider on the path space the event

$$A_n = \{|\omega_n^{(1)}| \leq K\sqrt{n \ln n}, |\omega_n^{(2)}| \leq K\sqrt{n\ln n}\}$$

where the two coordinates of ω_n are smaller in absolute value than $K\sqrt{n\ln n}$. Let

$$X_n = P(e^{\beta H_{n-1} - (n-1)\lambda(\beta)}; A_n^c) \ .$$

By Markov inequality, for large n,

$$\begin{aligned}
Q\left(X_n \geq e^{-\frac{K^2}{4}\ln n}\right) &\leq e^{\frac{K^2}{4}\ln n} Q(X_n) \\
&= e^{\frac{K^2}{4}\ln n} P(A_n^c) \\
&\leq 4e^{-\frac{K^2}{4}\ln n} \ .
\end{aligned}$$

In the last line we have used Chernov's bound (**??**) for the random walk in the following way:

$$P(\pm\omega_n^{(1)} > K\sqrt{n\ln n}) \leq \exp\{-n\gamma^*(K\sqrt{n\ln n})\} \ ,$$

with γ^* the convex dual of γ,

$$\gamma(u) := \ln P(e^{u\omega_n^{(1)}}) = \ln \frac{1+\cosh u}{2} \leq \ln \frac{1+e^{u^2/2}}{2} \leq u^2/2 \ ,$$

implying that $\gamma^*(v) = \sup_u(uv - \gamma(u)) \geq v^2/2$. Taking $K > 2$, we get $X_n \to 0$ Q-almost surely by Borel Cantelli lemma. Then,

$$Y_n := \mu_{n-1,\beta}^{\eta}(A_n^c) \longrightarrow \frac{0}{W_\infty} = 0 \quad Q - a.s..$$

Hence, denoting by $\mathcal{C}(n,K)$ the cube $\mathcal{C}(n,K)=[-K\sqrt{n\ln n},K\sqrt{n\ln n}]^2$,

$$\begin{aligned}(1-Y_n)^2 &= \sum_{x,y\in\mathcal{C}(n,K)} \mu_{n-1,\beta}^{\eta\otimes 2}(\omega_n=x,\tilde\omega_n=y)\\ &\le \sum_{z\in\mathcal{C}(n,2K)} \mu_{n-1,\beta}^{\eta\otimes 2}(\omega_n=\tilde\omega_n+z)\\ &\le (4K\sqrt{n\ln n})^2 I_n\end{aligned}$$

Therefore, Q-a.s., we have $I_n \ge 1/(17K^2 n\ln n)$ ultimately, so $\sum_n I_n=\infty$, contradicting $W_\infty>0$. This ends the proof. ∎

1.5.2 Fractional moments estimates

We give another proof of theorem 1.5.1. We follow the strategy of proof of [16], which yields additional information on the decay of W_n. This will be crucial in the sequel.

The proof is carried out by estimating fractional moment, and by using the following

Lemma 1.5.2 *Suppose that there exist constants $c\in(0,\infty)$, $\theta\in(0,1)$ and a sequence $a_n\nearrow\infty$ such that*

$$Q[W_n^\theta]\le c\exp(-a_n),\quad n\ge 1. \tag{1.5.64}$$

Then $Q\{W_\infty=0\}=1$. If moreover

$$\sum_{n\ge 1}\exp(-\delta a_n)<\infty \quad \textit{for some } \delta\in(0,1),$$

then there exists $c>0$ such that

$$\varlimsup_{n\to\infty}\frac{1}{a_n}\ln W_n\le -c \qquad Q-a.s. \tag{1.5.65}$$

□ Indeed, by Fatou's lemma,

$$Q[W_\infty^\theta]\le\limsup_n Q[W_n^\theta]=0\ ,$$

yielding the first statement. For the second one, use Markov inequality

$$Q(W_n\ge\exp\{-\epsilon a_n\})\le c\exp\{-a_n(1-\epsilon\theta)\}$$

with $\epsilon=\theta^{-1}(1-\delta)$, and apply the Borel-Cantelli lemma. ∎

To prove Theorem 1.5.1, we will check (1.5.64) with

$$a_n=\begin{cases} c_1 n^{1/3} & \text{if } d=1\\ c_2\sqrt{\ln n} & \text{if } d=2\end{cases} \tag{1.5.66}$$

where $c_1,c_2\in(0,\infty)$ are some constants. In this respect, we first prove an auxiliary lemma.

Lemma 1.5.3 *For $\theta\in[0,1]$ and $\Lambda\subset\mathbb{Z}^d$,*

$$Q\left[W_{n-1}^\theta I_n\right]\ge\frac{1}{|\Lambda|}Q\left[Z_{n-1}^\theta\right]-\frac{2}{|\Lambda|}P(\omega_n\notin\Lambda)^\theta. \tag{1.5.67}$$

□ Repeating the argument in [48, page 453], we see that

$$\begin{aligned}
I_n &\geq \sum_{z\in\Lambda} \mu_{n-1}(\omega_n = z)^2 \\
&\geq \frac{1}{|\Lambda|}\mu_{n-1}(\omega_n \in \Lambda)^2 && (\text{Cauchy}-\text{Schwarz}) \\
&= \frac{1}{|\Lambda|}\left(1-\mu_{n-1}(\omega_n \notin \Lambda)\right)^2 \\
&\geq \frac{1}{|\Lambda|}\left(1-2\mu_{n-1}(\omega_n \notin \Lambda)\right) \\
&\geq \frac{1}{|\Lambda|}\left(1-2\mu_{n-1}(\omega_n \notin \Lambda)^\theta\right) && (\text{Convexity}, \theta\in(0,1))
\end{aligned}$$

Note also that

$$\begin{aligned}
Q\left[W_{n-1}^\theta \mu_{n-1}(\omega_n \notin \Lambda)^\theta\right] &\leq Q\left[W_{n-1}\mu_{n-1}(\omega_n \notin \Lambda)\right]^\theta \\
&= P(\omega_n \notin \Lambda)^\theta.
\end{aligned}$$

We therefore see that

$$\begin{aligned}
Q\left[W_{n-1}^\theta I_n\right] &\geq \frac{1}{|\Lambda|}Q\left[W_{n-1}^\theta\right] - \frac{2}{|\Lambda|}Q\left[W_{n-1}^\theta \mu_{n-1}(\omega_n \notin \Lambda)^\theta\right] \\
&\geq \frac{1}{|\Lambda|}Q\left[W_{n-1}^\theta\right] - \frac{2}{|\Lambda|}P(\omega_n \notin \Lambda)^\theta.
\end{aligned}$$

■

□ second proof of Theorem 1.5.1. Assume now that $\theta \in (0,1)$, and define a function $f : (-1,\infty) \to [0,\infty)$ by

$$f(u) = 1 + \theta u - (1+u)^\theta.$$

It is then clear that there are constants $c_1, c_2 \in (0,\infty)$ such that

$$\frac{c_1 u^2}{2+u} \leq f(u) \leq c_2 u^2 \quad \text{for all } u \in (-1,\infty). \tag{1.5.68}$$

Using (1.4.41), (1.5.68) and (1.4.54) in this order, we see that

$$\begin{aligned}
Q^{\mathcal{G}_{n-1}}\Delta(W_n^\theta) &= W_{n-1}^\theta Q^{\mathcal{G}_{n-1}}\left((1+U_n)^\theta - 1\right) \\
&= -W_{n-1}^\theta Q^{\mathcal{G}_{n-1}} f(U_n) \\
&\leq -c_3 W_{n-1}^\theta I_n.
\end{aligned}$$

We therefore have by (1.5.67) that

$$QW_n^\theta \leq \left(1 - \frac{c_3}{|\Lambda|}\right) Q\left[W_{n-1}^\theta\right] + \frac{2c_3}{|\Lambda|}P(\omega_n \notin \Lambda)^\theta. \tag{1.5.69}$$

Now, we proceed to the optimal choice for Λ, which will depend on the dimension.

- For $d=1$, set $\Lambda = (-n^{2/3}, n^{2/3}]$. Then, by Cramer theorem,

$$P(\omega_n \notin \Lambda) = P\left(\left|\frac{\omega_n}{n^{1/2}}\right| \geq n^{1/6}\right) \leq 2\exp(-\frac{n^{1/3}}{2}),$$

so that (1.5.69) reads,

$$QW_n^\theta \le \left(1 - \frac{c_3}{2n^{2/3}}\right) Q\left[W_{n-1}^\theta\right] + 4c_3 \exp(-\frac{n^{1/3}}{2}). \qquad (1.5.70)$$

Now, one can conclude (1.5.64) with $a_n = c_1 n^{1/3}$, using Gronwall's Lemma.

Lemma 1.5.4 *Let $u(t), t \ge 0$, be positive and absolutely continuous such that*

$$u'(t) \le -b(t)u(t) + a(t)$$

with a, b measurable and locally bounded. Then, for $B(t) = \int_0^t b(s)ds$,

$$u(t) \le u(0)e^{-B(t)} + \int_0^t a(s)e^{B(s)-B(t)}ds \qquad (1.5.71)$$

(By assumption, $(ue^B)' \le ae^B$, yielding by integration $u(t)e^{B(t)} - u(0) \le \int_0^t a(s)e^{B(s)}ds$. This implies the result.)

In view of (1.5.70), we use Gronwall's Lemma with piecewise constant functions $b(t) = c_3/(2n^{2/3})$ and $a(t) = 4c_3 e^{n^{1/3}/2}$ for $t \in (n-1, n]$. We easily check that each term in the right-hand side of (1.5.71) is eventually bounded by $\exp -cn^{1/3}$ for some $c > 0$.

- For $d = 2$, we set
$$\Lambda = (-n^{1/2}\ln^{1/4} n, n^{1/2}\ln^{1/4} n]^2$$
to get (1.5.64) with $a_n = c_2\sqrt{\ln n}$ in a similar way as above.

■

For further use, we keep in mind the above estimate, although that the exponent is not optimal, as we will see in Theorem 1.5.6.

Lemma 1.5.5 *Assume $d = 1$ and $\beta \ne 0$. Then,*

$$QW_n^\theta \le c_4 e^{-c_5 n^{1/3}} \qquad \text{for } d = 1. \qquad (1.5.72)$$

1.5.3 Localization in dimension 1

The next result shows that, in dimension $d = 1$, the polymer is always localized.

Theorem 1.5.6 *Assume the dimension is $d = 1$.*

$$\beta_c = 0$$

Equivalently, for all non degenerate Q and all $\beta \ne 0$, $p(\beta) < \lambda(\beta)$.

In this section we use the notation of section 1.3. In addition to (1.3.22) we introduce the notation ($k < n$, $x, y \in \mathbb{Z}^d$),

$$W_{k,n}^x(y) = P^x\Big(\exp\{\beta\sum_{j=1}^{n-k}\eta(k+j,\omega_j) - (n-k)\lambda(\beta)\}1_{\omega_{n-k}=y}\Big), \qquad (1.5.73)$$

with P^x the law of the simple random walk starting from x at time 0. In the sequel, $W_n(x)$ will stand for $W^0_{0,n}(x)$. The Markov property of the simple random walk yields

$$W_n = \sum_{x,y\in\mathbb{Z}^d} W_k(x) W^x_{k,n}(y). \tag{1.5.74}$$

We start with a lemma.

Lemma 1.5.7 *In any dimension $d \geq 1$ we have the following inequality*

$$p(\beta) - \lambda(\beta) \leq \inf_{m\geq 1, \theta\in(0,1]} \frac{1}{m\theta} \ln Q \sum_x W_m(x)^\theta \ . \tag{1.5.75}$$

Remark: Observe that for $m = 1$, one recovers the bound (1.2.16): since

$$QW_1(x)^\theta = (2d)^{-1} \exp\{\lambda(\theta\beta) - \theta\lambda(\beta)\} \ ,$$

the right-hand side of (1.5.75) with $m = 1$ is equal to

$$\inf_{\theta\in(0,1]} \frac{1}{\theta}[(1-\theta)\ln(2d) + \lambda(\theta\beta) - \theta\lambda(\beta)] = \inf_{\theta\in(0,1]} \frac{\ln(2d) + \lambda(\theta\beta)}{\theta} - \lambda(\beta) - \ln(2d) \ ,$$

and we recover (1.2.16). Hence, the lemma improves on this upper bound we found in section 1.2.2.

□ Let $\theta \in (0,1)$ and m be a positive integer. By using the subadditive estimate

$$\forall u, v > 0, \qquad (u+v)^\theta < u^\theta + v^\theta, \tag{1.5.76}$$

we have for all $n \geq 1$,

$$\begin{aligned}
Q\frac{1}{n}\ln W_{nm} &= Q\frac{1}{\theta n}\ln W^\theta_{nm} \\
&\overset{(1.5.74)}{=} Q\frac{1}{\theta n}\ln\left(\sum_{x_1,\ldots,x_n} W_m(x_1)\ldots W^{x_{n-1}}_{(n-1)m,nm}(x_n)\right)^\theta \\
&\overset{(1.5.76)}{\leq} Q\frac{1}{\theta n}\ln\sum_{x_1,\ldots,x_n} W_m(x_1)^\theta\ldots W^{x_{n-1}}_{(n-1)m,nm}(x_n)^\theta \\
&\overset{(\text{Jensen})}{\leq} \frac{1}{\theta n}\ln Q\sum_{x_1,\ldots,x_n} W_m(x_1)^\theta\ldots W^{x_{n-1}}_{(n-1)m,nm}(x_n)^\theta \\
&= \frac{1}{\theta n}\ln\sum_{x_1,\ldots,x_{n-1}} Q\left[W_m(x_1)^\theta\ldots W^{x_{n-2}}_{(n-2)m,nm}(x_{n-1})^\theta\right] Q\left[\sum_{x_n} W^{x_{n-1}}_{(n-1)m,nm}(x_n)^\theta\right] \\
&\overset{(\text{stationarity})}{=} \frac{1}{\theta n}\ln\left(Q\sum_x W_m(x)^\theta\right)^n \\
&= \frac{1}{\theta}\ln Q\sum_x W_m(x)^\theta
\end{aligned}$$

The proof is complete by taking the limit as $n \to \infty$ and then by taking the infimum over all $\theta \in]0,1]$ and $m \geq 1$. ■

□ Proof of Theorem 1.5.6: Let $d = 1$, $\theta \in (0,1)$ and $\beta > 0$. By Lemma 1.5.5, there exists a $c(\theta) > 0$ such that

$$\forall m \geq 1 \qquad Q(W_m^\theta) \leq e^{-c(\theta)m^{\frac{1}{3}}}.$$

Let us fix an integer $m \geq 1$ and define L_m to be the set of points visited by the simple random walk at time m:

$$L_m \stackrel{def}{=} \{x \in \mathbb{Z}^d; P(w_m = x) > 0\}.$$

Then,

$$\begin{aligned} Q(\sum_{x \in L_m} (W_m(x))^\theta) &\leq \mid L_m \mid Q(W_m^\theta) \\ &\leq \mid L_m \mid e^{-c(\theta)m^{\frac{1}{3}}} \\ &\longrightarrow 0\,, \qquad m \to \infty, \end{aligned}$$

where we have used the fact that $\mid L_m \mid = O(m)$. In particular, there exists $m \geq 1$ such that

$$Q(\sum_{x \in L_m} (W_m(x))^\theta) < 1.$$

We have $\ln Q(\sum_{x \in L_m} (W_m(x))^\theta) < 0$ and so by lemma 1.5.7 $p(\beta) < 0$. ■

Remark 1.5.8 *(Case $d = 2$) Little was known in dimension 2 until recently. With computations similar to those in the cavity method in spin-glass models, Carmona and Hu [12] for a gaussian environment, proved that, for all $\beta \neq 0$, there is a constant $c \in (0, \infty)$ such that*

$$\overline{\lim_{n \nearrow \infty}} I_n \geq c, \quad Q\text{-}a.s. \tag{1.5.77}$$

This was extended tothe general case, see [16, Proposition 1.4 (b)].

Lacoin proved recently that $\beta_c = 0$ [45], hence the lim sup *can be replaced with a* lim inf *in the Cesaro sense.*

Chapter 2

Oriented ρ-percolation

In this chapter we show how the directed polymer model relates to the oriented ρ-percolation model. We will obtain estimates on the number of open paths in an ρ-percolation model in dimension $1+d$, or, equivalenly, the number of ρ-open path in an oriented percolation model.

2.1 Model of ρ-percolation

We briefly introduce the model and outline our approach.

2.1.1 Orientation

Consider the graph $\mathbb{N} \times \mathbb{Z}^d$, and fix some parameter $p \in (0,1)$. To each site of this graph except the origin, assign a variable taking value 1 with probability p and 0 with probability $1-p$, independently of the other sites. An oriented (sometimes also called *semi-oriented*) path of length n is a sequence $(0,x_0),(1,x_1),(2,x_2),\ldots,(n,x_n)$, where $x_0=0$ and x_i,x_{i+1} are neighbours in $\mathbb{Z}^d$, $i=0,\ldots,n-1$: viewing the first coordinate as time, one can think of such path as a path of the d-dimensional simple random walk. Fix another parameter $\rho \in [0,1]$; the concept of ρ-percolation was introduced by Menshikov and Zuev in [53], as the occurence of an infinite length path with asymptotic density of 1s larger of equal to ρ. As in classical percolation [32], this event obeys a zero-one law [53], it has probability 1 or 0 according to p larger or smaller than some critical threshold, which was later studied by Kesten and Su [41] in the asymptotics of large dimension.

In the present chapter, we focus on paths of finite length n, in the limit $n \to \infty$. An oriented path of length n is called ρ-open, if the proportion of 1s in it is at least ρ. From standard percolation theory [32] it is known that for large p there are 1-open oriented paths with nonvanishing probability, and from [53] that for any p one can find ρ larger than p such that, almost surely, there are ρ-open oriented paths for large n. in the literature. Here, we will address the slightly different – but related – question of how many such paths of length n are there in typical situation ?

We expect that the number of different ρ-open paths of length n behaves like $e^{n\alpha(\rho)(1+o(1))}$, for some deterministic exponent $\alpha(\rho)$, depending on p and d. We will prove this statement, and also that the function $\alpha(\cdot)$ is the negative convex conjugate of the free energy of directed polymers in random environment. This will allow us to obtain, when $d \geq 3$, the explicit expression for $\alpha(\rho)$ in a certain range of values for ρ depending on the parameters p and d. The reason for this remarkable fact is the existence of the weak-disorder region in the polymer model (cf. Theorem 1.3.1), this reflects here into a parameter region where the number of paths is of the same order as its expected value.

At this point the reader may be tempted to use first and second moment methods to estimate the number of paths, and this is performed in [42]. The first moment is easily computed, and serves as an upper bound in complete generality. The second moment is more difficult to analyse. However, it can be checked that in large dimension and for density close to the parameter p of the Bernoulli, the ratio second-to-first-squared remains bounded in the limit of an infinitely long path. This means that, under these circumstances, the upper bound gives the right order of magnitude with a positive probability. However, (i) this method does not tell us anything on α for general parameters, (ii) it fails to keep track of the correlation between counts for different values of the density.

Our strategy is quite different. We will study the *moment generating function* of the number of paths, which is not surprising in such a combinatorial problem. The point is that the moment generating function is simply the partition function of the directed polymer in random, Bernoulli environment. From the existence and known properties of the free energy, we will derive the existence of α and its expression in thermodynamics terms.

Moreover, in a more restricted range of values for ρ, we even obtain an equivalent for the number of paths which achieves exactly a given density of 1s. This is clearly a very sharp estimate, that we obtain by using the power of complex analysis, and convergence of the renormalized moment generating function in the sense of analytic functions. Certainly a naive moments method cannot lead to such an equivalent.

2.1.2 The model

We start to define formally the model. Let $\eta(t,x), t = 1,2,\ldots, x \in \mathbb{Z}^d$ be a sequence of independent identically distributed Bernoulli random variables, with common parameter $p \in (0,1)$, $Q(\eta(t,x)=1) = p = 1 - Q(\eta(t,x)=0)$. We denote by $(\Omega_\eta, \mathcal{G}, Q)$ the probability space where this sequence is defined. The vertex (t,x) is *open* if $\eta(t,x)=1$ and *closed* in the opposite case $\eta(t,x)=0$. A nearest neighbour path ω in $\mathbb{Z}^d$ of length n $(1 \le n \le \infty)$ is a sequence $\omega = (\omega_t; t=0,\ldots,n), \omega_t \in \mathbb{Z}^d, \omega_0 = 0, \|\omega_t - \omega_{t-1}\|_1 = 1$ for $t = 1,\ldots,n$. We denote by $\mathcal{P}_n$ the set of such paths ω, and by $\mathcal{P}_\infty$ the set of infinite length nearest neighbour paths. For $\omega \in \mathcal{P}_n$, let

$$H_n(\omega) = \sum_{t=1}^{n} \eta(t,\omega_t) \tag{2.1.1}$$

be the number of open vertices along the path ω.

In oriented percolation, one is concerned with the event that there exists an infinite open path ω, i.e.

$$\texttt{Perc} = \left\{\text{there exists } \omega \in \mathcal{P}_\infty : \eta(t,\omega_t) = 1 \text{ for all } t \ge 1\right\}.$$

It is well known [27, 32] that there exists $\vec{p}_c(d) \in (0,1)$, called the critical percolation threshold, such that

$$Q(\texttt{Perc}) \begin{cases} > 0 & \text{if} \quad p > \vec{p}_c(d), \\ = 0 & \text{if} \quad p < \vec{p}_c(d). \end{cases} \tag{2.1.2}$$

For $\rho \in (p,1]$, Menshikov and Zuev [53] introduced ρ-percolation as the event that there exists an infinite path ω with asymptotic proportion at least ρ of open sites,

$$\rho\text{-}\texttt{Perc} = \left\{\text{there exists } \omega \in \mathcal{P}_\infty : \liminf_{n\to\infty} H_n(\omega)/n \ge \rho\right\}.$$

They showed that there also exists a threshold $\vec{p}_c(\rho,d)$ such that (2.1.2) holds with ρ-Perc instead of Perc. In fact, by tail triviality, the probability of ρ-Perc is equal to 0 or 1, the former holding when $p < \vec{p}_c(\rho,d)$ and the latter when $p > \vec{p}_c(\rho,d)$. Very little has been proved for ρ-percolation. The asymptotics of $\vec{p}_c(\rho,d)$ for large d are obtained in [41] at first order, showing that $d^{1/\rho}\vec{p}_c(\rho, d)$ has a limit as $d \to \infty$, and that the limit is different from the analogous quantity for d-ary trees. As mentioned in this reference, the equality $\vec{p}_c(1, d) = \vec{p}_c(d)$ follows from Theorem 5 of [46].

In this paper we are interested in the number of oriented paths of length n which have exactly k open vertices ($k \in \{0, \dots, n\}$),

$$Q_n(k) = \text{Card}\big\{\omega \in \mathcal{P}_n : H_n(\omega) = k\big\} \tag{2.1.3}$$

(Card$A$ denotes the cardinality of A) and the related quantity given, for $\rho \in [0, 1]$, by

$$R_n(\rho) = \begin{cases} \text{Card}\big\{\omega \in \mathcal{P}_n : H_n(\omega) \geq n\rho\big\}\ , & \rho \geq p, \\ \text{Card}\big\{\omega \in \mathcal{P}_n : H_n(\omega) \leq n\rho\big\}\ , & \rho < p. \end{cases} \tag{2.1.4}$$

Note that $Q_n(k), R_n(\rho)$ are random variables, that $R_n(\rho) = \sum_{k \geq n\rho} Q_n(k)$ when $\rho \geq p$, and that $\texttt{Perc} = \bigcap_n \{Q_n(n) \geq 1\} = \bigcap_n \{R_n(1) \geq 1\}$. Since one expects that, typically, most paths will have energy H_n close to np, $R_n(\rho)$ represents the tails of the distribution Q_n.

A word of warning: the reader will take a special care to distinguish the environmental measure Q and the counts $Q_n(\cdot)$, the Bernoulli percolation parameter p and the free energy $p(\beta)$, and finally, the partition function $Z^\eta_{n,\beta}$ of the polymer model and the unnormalized quantity Z_n defined below.

2.2 Rate of growth for the number of ρ-open paths

We start to relate these quantities to the model of directed polymers in random environment. The generating function of H_n is defined by the first equality below, and the second one comes from the definition (1.1.4) of the partition function:

$$Z_n = \sum_{\omega \in \mathcal{P}_n} \exp\{\beta H_n(\omega)\} = (2d)^n Z^\eta_{n,\beta}\ .$$

In chapter 1.2, we have seen that

$$\phi(\beta) = \lim_{n \to \infty} \frac{1}{n} Q \ln Z_n = p(\beta) + \ln(2d) \tag{2.2.5}$$

exists in $\mathbb{R}$, and that the event $\Omega_0(\beta)$ defined by

$$\Omega_0(\beta) = \Big\{\lim_{n \to \infty} \frac{1}{n} \ln Z_n = \phi(\beta)\Big\} \tag{2.2.6}$$

has full measure, $Q(\Omega_0(\beta)) = 1$, see theorem 1.2.1. The function ϕ is a non-decreasing and convex function of β. Consider the event $\Omega_0 = \bigcap_{\beta \in \mathbb{Q}} \Omega_0(\beta)$: we have $Q(\Omega_0) = 1$, and observe, for further purpose, that on this event the convergence (2.2.6) holds for *all* real number β by convexity.

The Legendre conjugate

$$\phi^*(\rho) = \sup\{\beta\rho - \phi(\beta); \beta \in \mathbb{R}\}\ , \tag{2.2.7}$$

is a convex, lower semi-continuous function from $[0, 1]$ to $\mathbb{R} \cup \{+\infty\}$, such that $\phi^*(\rho) \geq \phi^*(p) = -\ln(2d)$. Legendre convex duality is better understood by taking a glance at the graphical construction, e.g. figures 2.2.1 and 2.2.2 in [20]; here, on Figure 2.1 we illustrate how the functions ϕ and ϕ^* typically look in our situation.

We now define two deterministic quantities,

$$\rho^+ = \lim_{n\to\infty} \max_{\omega\in\mathcal{P}_n} \frac{H_n(\omega)}{n} \,, \quad \rho^- = \lim_{n\to\infty} \min_{\omega\in\mathcal{P}_n} \frac{H_n(\omega)}{n} \,, \qquad Q\text{-a.s.} \tag{2.2.8}$$

which are called *time constants* in last passage percolation and first passage percolation problems. Their existence can be obtained by specifying a direction for the ending point ω_n, which allows using subadditive arguments [40], and then summing over the possible directions. We give here a short, different proof for existence, which is in the spirit of this book. Since

$$\exp\{\beta \max_{\omega\in\mathcal{P}_n} H_n(\omega)\} \le Z_n \le (2d)^n \exp\{\beta \max_{\omega\in\mathcal{P}_n} H_n(\omega)\},$$

we have

$$\frac{1}{n\beta}\ln Z_n - \frac{1}{\beta}\ln(2d) \le \max_{\omega\in\mathcal{P}_n} \frac{H_n(\omega)}{n} \le \frac{1}{n\beta}\ln Z_n \,.$$

Taking the limits $n\to\infty$ and then $\beta\to+\infty$, we see that the quantity ρ^+ from (2.2.8) is well-defined as an a.s. limit and in L^1, and is in fact equal to the slope

$$\rho^+ = \lim_{\beta\to+\infty} \phi(\beta)/\beta \tag{2.2.9}$$

of the asymptotic direction of ϕ at $+\infty$, which exists by convexity. In particular, ρ^+ is deterministic. Similarly, we have

$$\rho^- = \lim_{\beta\to-\infty} \phi(\beta)/\beta \,.$$

From standard properties of convex duality, the range of the derivative

$$(d/d\beta)(1/n)\ln Z_n(\beta)$$

converges almost surely to $[\rho^-,\rho^+]$ in Hausdorff distance, and $\phi^*(\rho) < +\infty$ if and only if $\rho\in[\rho^-,\rho^+]$. For such ρ, we have $\phi^*(\rho)\le 0$.

Our first result is the existence of the rate of growth of $R_n(\rho)$, together with the identification of the rate.

Theorem 2.2.1 (Rate of growth) *For all $\rho\in[0,1]$ with $\rho\ne\rho^+,\rho^-$, the following limit*

$$\alpha(\rho) = \lim_{n\to\infty} \frac{1}{n}\ln R_n(\rho) \tag{2.2.10}$$

exists Q-a.s. (possibly assuming the value $-\infty$), and is given by

$$\alpha(\rho) = -\phi^*(\rho) \,.$$

Clearly, α is concave, with values in $[0,\ln(2d)]\cup\{-\infty\}$ and $\alpha(p)=\ln(2d)$. On (ρ^-,ρ^+), the functions α and ϕ^* are finite, and they are infinite on $(-\infty,\rho^-)\cup(\rho^+,+\infty)$. Note that, for all $\rho\in(\rho^-,\rho^+)$ and almost every η,

$$R_n(\rho) = \exp n[\alpha(\rho)+o(1)] \,, \quad \text{as } n\to\infty \,,$$

or, equivalently,

$$R_n(\rho)^{1/n} \longrightarrow \exp\alpha(\rho) \,, \quad \text{as } n\to\infty \,.$$

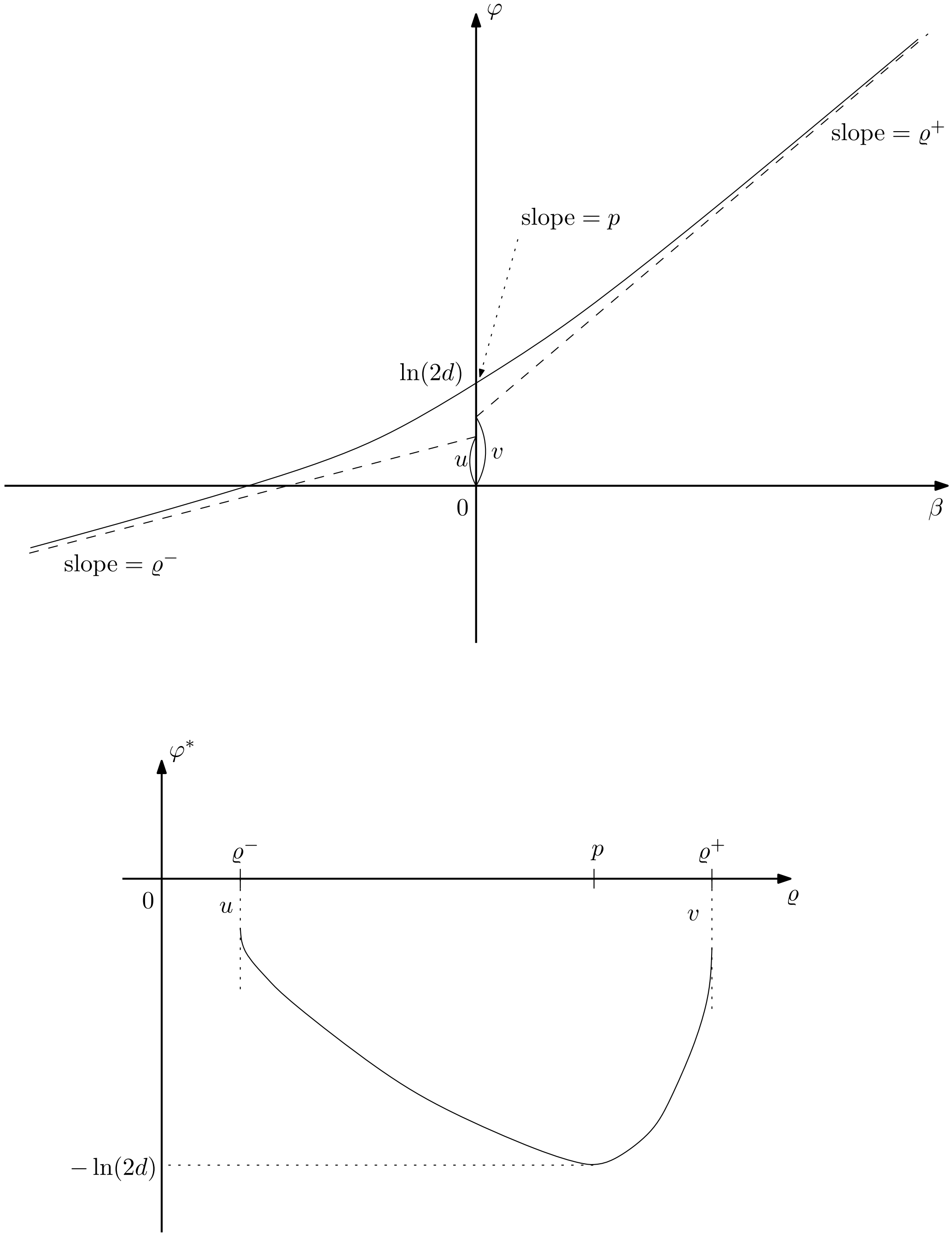

Figure 2.1: The function ϕ and its Legendre transform ϕ^*

Remark 2.2.2

By convexity the function α is continuous on (ρ^-,ρ^+). For now, it is not clear whether the limit $\alpha(\rho^+ - 0) = \lim_{\rho\nearrow\rho^+}\alpha(\rho)$ should be equal to 0 in the case $p \le \vec{p}_c(d)$, or should be positive on the contrary. In the case $p > \vec{p}_c(d)$, it is possible to show by subadditive arguments that, conditionally on percolation, the limit $\alpha(1)$ in (2.2.10) exists and is positive, but it is not clear whether α is continuous at 1.

Since $\eta(t,x)$ are Bernoulli variables, we have $QZ_n = \exp\{n\hat\lambda(\beta)\}$ with

$$\hat\lambda(\beta) = \lambda(\beta) + \ln(2d)\ , \qquad \lambda(\beta) = \ln\Big[1 + p(e^\beta - 1)\Big]\ .$$

A direct computation shows that the Legendre conjugate $\hat\lambda^*(\rho) = \sup\{\beta\rho - \hat\lambda(\beta); \beta \in \mathbb{R}\}$ of $\hat\lambda$ is equal to

$$\begin{aligned}\hat\lambda^*(\rho) &= -\ln(2d) + \rho\ln\frac{\rho}{p} + (1-\rho)\ln\frac{1-\rho}{1-p} \\ &= \rho\ln\frac{\rho}{2dp} + (1-\rho)\ln\frac{1-\rho}{2d(1-p)}.\end{aligned} \tag{2.2.11}$$

We now summarize what we know, from the results for the polymer model, on the growth rate α and its relations with $\hat\lambda^*$. (We recall that both functions depend on the Bernoulli parameter p, but we don't write explicitely the dependence.) Note that these two functions coincide at $\rho = p$ and take the value $\ln(2d)$.

Theorem 2.2.3 *Let $p \in (0,1)$.*

1. *We have* the annealed bound*: For all ρ,*

$$\alpha(\rho) \le -\hat\lambda^*(\rho)\ . \tag{2.2.12}$$

2. *The function $\alpha(\rho) + \hat\lambda^*(\rho)$ is nonincreasing for $\rho \in [p,\rho^+)$ and is nondecreasing for $\rho \in (\rho^-, p]$.*

3. *The set*

$$\mathcal{V}(p) = \{\rho \in (0,1) : \alpha(\rho) = -\hat\lambda^*(\rho)\} \tag{2.2.13}$$

is an interval containing p (here, "interval" is understood in broad sense, i.e., it can reduce to the single point $\{p\}$).

4. *In dimension $d=1$ and $d=2$, $\mathcal{V}(p) = \{p\}$, i.e. the inequality in (2.2.12) is strict for all $\rho \ne p$.*

5. *In dimension $d \ge 3$, $\mathcal{V}(p)$ contains a neighborhood of p.*

6. *Let $d \ge 3$, and π_d be the probability for the d-dimensional simple random walk to ever return to the starting point. When $p > \pi_d$, then $[p,1) \subset \mathcal{V}(p)$, so that the equality holds in (2.2.12) for all $\rho \in [p,1)$. Similarly, when $p < 1-\pi_d$, then $(0,p] \subset \mathcal{V}(p)$, so that the equality holds for all $\rho \in (0,p]$.*

7. *If $p < (1/2d)$, then $\sup\mathcal{V}(p) < 1$. Similarly, if $p > 1-(1/2d)$, we have $\inf\mathcal{V}(p) > 0$.*

□ Proof of theorem 2.2.3.

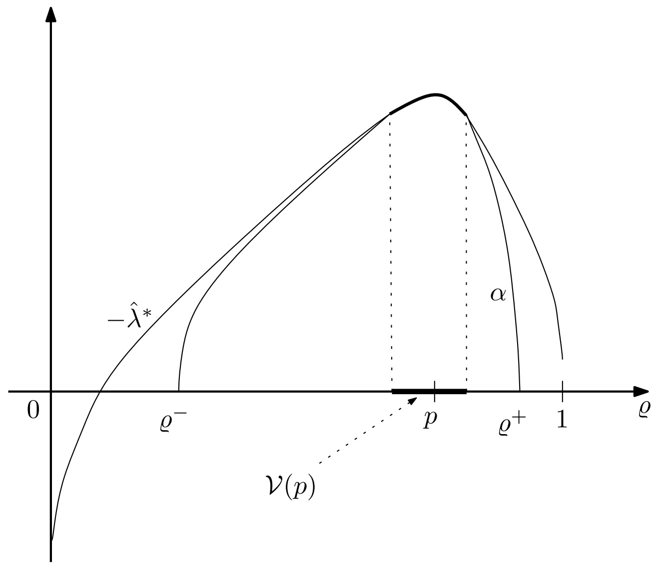

Figure 2.2: Typical behaviour of the function α when $d \geq 3$

1. By the annealed bound (1.2.11),

$$Q \ln Z_n \leq n\hat{\lambda}(\beta) \ .$$

   Then, $\phi(\beta) \leq \hat{\lambda}(\beta)$, which implies $\phi^*(\rho) \geq \hat{\lambda}^*(\rho)$ from the definition of Legendre transform. The inequality now follows from $\alpha \leq -\phi^*$.

2. Set $\phi_n(\beta) = n^{-1}Q \ln Z_n(\beta)$. From the monotonicity proposition 1.2.10 we have

$$\hat{\lambda}'(\beta) \geq \phi_n'(\beta)$$

   for all $\beta \geq 0$. Hence, for $\rho \geq p$, the reciprocal functions are such that

$$(\hat{\lambda}')^{-1}(\rho) \leq (\phi_n')^{-1}(\rho) \ .$$

   Since $(\hat{\lambda}')^{-1} = (\hat{\lambda}^*)'$ and $(\phi_n')^{-1} = (\phi_n^*)'$, we have

$$(\hat{\lambda}^*)'(\rho) \leq (\phi^*)'(\rho)$$

   for all $\rho \geq p$ where ϕ^* is differentiable. Since $\alpha = -\phi^*$ for $\rho \neq \rho^+$, this proves the first half of the desired statement. The other half is similar.

3. From the monotonicity proposition 1.2.10, the "low temperature" region

$$\mathcal{W}(p) = \{\beta \in \mathbb{R} : \phi(\beta) = \hat{\lambda}(\beta)\}$$

   is an interval containing 0. Let $\beta \in \mathcal{W}(p)$, and $\rho = \lambda'(\beta) = \hat{\lambda}'(\beta)$. From Theorem 2.3 (a) in [16] it is known that $\beta \in \mathcal{W}(p)$ implies $\phi^*(\rho) \leq 0$. Then, the supremum defining $\hat{\lambda}^*(\rho)$ is achieved at β, which implies the first equality in

$$-\hat{\lambda}^*(\rho) = -[\beta\rho - \hat{\lambda}(\beta)] = -[\beta\rho - \phi(\beta)] = -\phi^*(\rho) = \alpha(\rho) \ ,$$

   where the second equality holds for $\beta \in \mathcal{W}(p)$, the third one because of $\phi'(\beta) = \hat{\lambda}'(\beta) = \rho$, and the last one because $\phi^*(\rho) \leq 0$.

Let now $\beta \notin \mathcal{W}(p)$, and $\rho = \lambda'(\beta)$. Then,

$$-\hat{\lambda}^*(\rho) = -[\beta\rho - \hat{\lambda}(\beta)] > -[\beta\rho - \phi(\beta)] \geq -\phi^*(\rho) \geq \alpha(\rho) .$$

Observe that λ' is a diffeomorphism from $\mathbb{R}$ to $(0,1)$. From this we can identify the set $\mathcal{V}(p)$ defined by (2.2.13),

$$\mathcal{V}(p) = \{\lambda'(\beta); \beta \in \mathcal{W}(p)\} , \tag{2.2.14}$$

which is an interval containing p.

4. When $d = 1, 2$, it is known that $\mathcal{W}(p) = \{0\}$, see theorem 1.5.6 [and Lacoin [45]]. Hence, $\mathcal{V}(p)$ reduces to $\{p\}$.

5. When $d \geq 3$, from celebrated results of Imbrie and Spencer [35], Bolthausen [7], it is known that $\mathcal{W}(p)$ contains a neighborhood of 0. In view of (2.2.14), $\mathcal{V}(p)$ is in its turn a neighborhood of p.

6. This is a consequence of [17, example 2.1.1], which shows for instance that, if $p > \pi_d$, then $\mathcal{W}(p) \supset \mathbb{R}^+$. Indeed, in view of (2.2.14), this implies that $\mathcal{V}(p)$ contains $[p, 1)$, and α is still equal to $-\hat{\lambda}^*$ at $\rho = 1$ by upper semi-continuity of both functions. The case of $p < 1 - \pi_d$ is similar.

7. This is a consequence of [17, example 2.2.1], which shows for instance that, if $p < (1/2d)$, then $\mathcal{W}(p)$ is bounded from above. The other case is similar.

■

We end with an interesting property of the rate α.

Theorem 2.2.4 *The functions ϕ^* and α are differentiable in the interior of their domains.*

□ *Proof of Theorem 2.2.4:* One can prove that ϕ is strictly convex. By a classical property of Legendre duality, it implies the differentiability of ϕ^*. ■

2.3 Proof of the logarithmic asymptotics and large deviations for the energy

Recall the notation P for the simple random walk on $\mathbb{Z}^d$ starting from 0, and recall that $Z_n = (2d)^n Z^\eta_{n,\beta}$. denote by $\nu_n = \nu_n^\eta$ the law of $(1/n)H_n$ under P: ν_n is the probability measure on $\mathbb{R}$ which concentrates on $n^{-1} \times \{0, 1, \ldots, n\}$, given by $\nu_n(\{\rho\}) := P(H_n(S) = n\rho)$, is such that

$$\nu_n(\{\rho\}) = \frac{Q_n(n\rho)}{(2d)^n} \quad \text{if} \quad n\rho \in \{0, 1, \ldots, n\}. \tag{2.3.15}$$

To prove Theorem 2.2.1, all what we need is an almost sure large deviation principle for ν_n, see Proposition 2.3.1 below. Recall first the event $\Omega_0 = \bigcap_{\beta \in \mathbb{Q}} \Omega_0(\beta)$ defined below (2.2.6), where the convergence (2.2.6) holds for any real number β.

Proposition 2.3.1 *The function*

$$I(\rho) = \ln(2d) + \phi^*(\rho) \in [0, \ln(2d)] \cup \{+\infty\}$$

is lower semi-continuous and convex on $[0,1]$. Moreover, for all $\eta \in \Omega_0$ the sequence $(\nu_n, n \geq 1)$ obeys a large deviation principle with rate function I. That is, *(i) for any closed* $F \subset [0,1]$, *we have*

$$\limsup_{n\to\infty} n^{-1} \ln \nu_n(F) \leq -\inf_{\rho\in F} I(\rho),$$

(ii) for any open (in the induced topology on $[0,1]$*)* $G \subset [0,1]$, *we have*

$$\liminf_{n\to\infty} n^{-1} \ln \nu_n(G) \geq -\inf_{\rho\in G} I(\rho).$$

We first finish the proof of Theorem 2.2.1, and then prove the above proposition.

□ *Proof of Theorem 2.2.1.* Assume that $\rho \in [p, \rho^+)$ and $\eta \in \Omega_0$. Applying (i) of Proposition 2.3.1 with $F = [\rho, 1]$ and using (2.3.15) together with the fact that $\rho \geq p$, we see that the limit in (2.2.10) is not larger than $\ln(2d) - I(\rho) = \alpha(\rho)$. Applying (ii) of Proposition 2.3.1 with $G = (\rho+\varepsilon, 1]$ $(\varepsilon > 0)$ and using the fact that $\rho \geq p$, we see that the limit is at least $\alpha(\rho+\varepsilon)$. Since $\rho < \rho^+$, this quantity tends to $\alpha(\rho)$ as $\varepsilon \searrow 0$. This proves (2.2.10) for $\rho \in [p, \rho^+)$. The case $\rho \in (\rho^-, p)$ is completely similar. Finally, when $\rho > \rho^+$ (the case $\rho < \rho^-$ is similar) we have $I(\rho) = \infty$ and then $R_n(\rho) = 0$ for large n, proving (2.2.10) in this case. ■

□ *Proof of Proposition 2.3.1.* The properties of I are clear from the definition. Fix $\eta \in \Omega_0$. In view of (2.2.5) and (2.2.6), the Laplace transforms of $\nu_n(\cdot) = P(n^{-1}H_n = \cdot)$ have logarithmic asymptotics:

$$\lim_{n\to\infty} \frac{1}{n} \ln P(\exp\{\beta H_n(S)\}) = \phi(\beta) - \ln(2d)$$

for all real β. From the Gärtner-Ellis theorem (Theorem 2.3.6 in [20]), the full statement (i) in Proposition 2.3.1 follows, and we obtain for open $G \subset [0,1]$ that

$$\liminf_{n\to\infty} \frac{1}{n} \ln \nu_n(G) \geq -\inf\{I(\rho); \rho \in G \cap \mathcal{E}\}\ , \tag{2.3.16}$$

where

$$\mathcal{E} = \Big\{\rho \in [0,1] : \exists\beta\ \forall r \neq \rho,\ \beta\rho - \phi^*(\rho) > \beta r - \phi^*(r)\Big\}$$

is the set of exposed points of ϕ^* from (2.2.7). Its complement is the set of all points ρ such that ϕ^* is linear in a neighborhood of ρ. We will improve (2.3.16) into (ii) of Proposition 2.3.1 with a subadditivity argument. We start by showing that ϕ is differentiable at 0 with that $\phi'(0) = p$. Indeed, using Jensen inequality twice, we have

$$\frac{1}{n} \ln P[e^{Q\beta(H_n - np)}] \leq Q\frac{1}{n} \ln P[e^{\beta(H_n - np)}] \leq \frac{1}{n} \ln QP[e^{\beta(H_n - np)}]$$

Computing the extreme terms and taking the limit $n \to \infty$ for the middle one, we get

$$0 \leq \phi(\beta) - \beta p \leq \lambda(\beta) - \beta p\ ,$$

which shows that $\phi'(0) = p$ since $\lambda'(0) = p$. This implies that $p \in \mathcal{E}$ and that $\mathcal{E}$ is a neighborhood of p. Let $\rho \in (\rho^-, \rho^+) \cap G$ be a non-exposed point of ϕ^*. For definiteness, we assume $\rho > p$. Let

$$\rho_1 = \sup\{\rho' \in \mathcal{E}; \rho' < \rho\}\ , \quad \rho_2 = \inf\{\rho' \in \mathcal{E}; \rho' > \rho\}\ .$$

Recall that ϕ is strictly convex. This implies that the function ϕ^* cannot have a linear piece that goes up to ρ^+, cf. Figure 2.1. Then, $p < \rho_1 < \rho < \rho_2 < \rho^+$, and $\rho_1, \rho_2 \in \mathcal{E}$. Let $\gamma \in (0,1)$ such that $\rho = \gamma\rho_1 + (1-\gamma)\rho_2$. Since the interval (ρ_1, ρ_2) consists of non-exposed points, we have $I(\rho) = \gamma I(\rho_1) + (1-\gamma) I(\rho_2)$. Since G is open and contains ρ, we can find $\varepsilon > 0$ and $k, \ell \in \mathbb{N}^*$ such that

$$|u - \rho_1| < \varepsilon, |v - \rho_2| < \varepsilon \Longrightarrow \frac{ku + \ell v}{k + \ell} \in G^\varepsilon$$

with G^ε the set of $r \in G$ at distance at least ε from the outside of G. The key fact is

$$\begin{aligned}
&\mathrm{Card}\Big\{\omega \in \mathcal{P}_{n(k+\ell)} : \frac{H_{n(k+\ell)}(\omega)}{n(k+\ell)} \in G^\varepsilon\Big\} \\
\geq\ & \sum_{x \in \mathbb{Z}^d} \mathrm{Card}\Big\{\omega \in \mathcal{P}_{n(k+\ell)} : \frac{H_{nk}(\omega)}{nk} \in (\rho_1 - \varepsilon, \rho_1 + \varepsilon), S_{nk} = x\Big\} \\
&\times \mathrm{Card}\Big\{\omega \in \mathcal{P}_{n(k+\ell)} : S_{nk} = x, \\
&\qquad \frac{H_{n(k+\ell)}(\omega) - H_{nk}(\omega)}{n\ell} \in (\rho_2 - \varepsilon, \rho_2 + \varepsilon)\Big\} \\
\geq\ & \mathrm{Card}\Big\{\omega \in \mathcal{P}_{n(k+\ell)} : \frac{H_{nk}(\omega)}{nk} \in (\rho_1 - \varepsilon, \rho_1 + \varepsilon)\Big\} \\
&\times \min_{\|x\|_1 \leq nk} \mathrm{Card}\Big\{\omega \in \mathcal{P}_{n(k+\ell)} : S_{nk} = x, \\
&\qquad \frac{H_{n(k+\ell)}(\omega) - H_{nk}(\omega)}{n\ell} \in (\rho_2 - \varepsilon, \rho_2 + \varepsilon)\Big\} \\
=\ & \mathrm{Card}\Big\{\omega \in \mathcal{P}_{nk} : \frac{H_{nk}(\omega)}{nk} \in (\rho_1 - \varepsilon, \rho_1 + \varepsilon)\Big\} \\
&\times \min_{\|x\|_1 \leq nk} \mathrm{Card}\Big\{\omega \in \mathcal{P}_{n\ell} : \frac{H_{n\ell}^{(nk,x)}(\omega)}{n\ell} \in (\rho_2 - \varepsilon, \rho_2 + \varepsilon)\Big\}
\end{aligned}$$

with $H_{n\ell}^{(nk,x)}(\omega) = \sum_{t=1}^n \eta(t+nk, \omega_t + x)$ the Hamiltonian in the time-space shifted environment. Similarly, we denote by $\nu_{n\ell}^{(nk,x)}$ the measure $\nu_{n\ell}^{(nk,x)}(\cdot) = P(H_{n\ell}^{(nk,x)} \in \cdot)$. The above display implies that

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n(k+\ell)} \ln \nu_{n(k+\ell)}(G^\varepsilon) \\
\geq\ & \frac{k}{k+\ell} \liminf_{n\to\infty} \frac{1}{nk} \ln \nu_{nk}\big((\rho_1 - \varepsilon, \rho_1 + \varepsilon)\big) \\
&+ \frac{\ell}{k+\ell} \liminf_{n\to\infty} \frac{1}{n\ell} \min_{\|x\|_1 \leq nk} \ln \nu_{n\ell}^{(nk,x)}\big((\rho_2 - \varepsilon, \rho_2 + \varepsilon)\big)
\end{aligned}$$

It is straightforward to check that

$$\liminf_{n\to\infty} \frac{1}{n(k+\ell)} \ln \nu_{n(k+\ell)}(G^\varepsilon) \leq \liminf_{n\to\infty} \frac{1}{n} \ln \nu_n(G) ,$$

and it is not difficult to see that

$$\liminf_{n\to\infty} \frac{1}{n\ell} \min_{\|x\|_1 \leq nk} \ln \nu_{n\ell}^{(nk,x)}\big((\rho_2 - \varepsilon, \rho_2 + \varepsilon)\big) \geq -I(\rho_2) , \quad Q\text{-a.s.} \tag{2.3.17}$$

We postpone the proof of (2.3.17) for the moment. Hence, the key inequality implies

$$\begin{aligned}
\liminf_{n\to\infty} \frac{1}{n} \ln \nu_n(G) &\geq -\frac{k}{k+\ell} I(\rho_1 + \varepsilon) - \frac{\ell}{k+\ell} I(\rho_2 + \varepsilon) , \\
\liminf_{n\to\infty} \frac{1}{n} \ln \nu_n(G) &\geq -\big[\gamma I(\rho_1) + (1-\gamma) I(\rho_2)\big] = -I(\rho),
\end{aligned}$$

letting $\varepsilon \searrow 0$ and $k/(k+\ell) \to \gamma$. This yields statement (ii) in Proposition 2.3.1.

Now, let us prove (2.3.17). By a standard concentration inequality (e.g., Theorem 4.2 in [18]), we have

$$Q(|\ln Z_n - Q\ln Z_n| \geq u) \leq 2\exp\Big\{-\frac{u^2}{4\beta^2 n}\Big\}.$$

Therefore we have, Q-a.s. as $n \to \infty$,

$$\max_{\|x\|_1 \leq m \leq n} \Big|\frac{1}{n}\ln Z_n^{(m,x)}(\beta) - \phi(\beta)\Big| \to 0\,, \quad \beta \in \mathbb{R}\,,$$

with $Z_n^{(m,x)}$ the partition function associated to $H_n^{(m,x)}$. Since ρ_2 is an exposed point for ϕ^*, (2.3.17) follows from the Gärtner-Ellis theorem. ∎

Let us comment on the above proof. We could improve (2.3.16) into the full lower bound (ii) in Proposition 2.3.1 with a subadditivity argument, implying convexity of the rate function. If we knew that $(\rho^-, \rho^+) \subset \mathcal{E}$ – or, equivalently, that ϕ is differentiable –, we could directly conclude without this extra argument. We tried to prove it, but we could not. We state it as a conjecture:

Conjecture 2.3.2 *The functions $p(\beta)$ and $\phi(\beta)$ are everywhere differentiable.*

2.4 Sharp asymptotics

In some part of the parameter region – implying weak disorder –, we can improve on our estimates.

2.4.1 An equivalent

We will obtain much sharper results for large dimension and ρ's not too far from p. The reason is that the partition function Z_n behaves smoothly as $n \nearrow \infty$. The almost-sure limit

$$W_\infty(\beta) = \lim_{n\to\infty} Z_n(\beta) e^{-n\hat{\lambda}(\beta)}$$

exists for all β, since the sequence is a positive $(\mathcal{G}_n)_n$-martingale, where $\mathcal{G}_n = \sigma\{\eta(t,x); t \leq n, x \in \mathbb{Z}^d\}$. So, let us now concentrate on the case of large dimension, $d \geq 3$. When β belongs to some neighborhood of the origin (known as the weak disorder region), the limit W_∞ is strictly positive a.s. In a smaller neighborhood of the origin, the limit can be expressed as a (random) perturbation series in L^2 [65]. Moreover, the convergence holds in much stronger sense, namely, in the sense of analytic functions [19]. We will use strong tools from complex analysis, as it is classically done to obtain limit theorems for sums of random variables [**?**].

Theorem 2.4.1 *Assume $d \geq 3$. There exist a neighborhood U_3 of p in $\mathbb{R}$ and an event Ω_2 with full probability such that for every sequence k_n with $k_n/n \to \rho \in U_3$ and all $\eta \in \Omega_2$,*

$$Q_n(k_n) = \sqrt{\frac{-\alpha''(\rho)}{2\pi n}}\, W_\infty(\beta(\rho))\, \exp\Big\{n\alpha\Big(\frac{k_n}{n}\Big)\Big\}\,(1+o(1))$$

where $o(1)$ tends to 0 as $n \to \infty$, and $\beta(\rho) = \ln\frac{(1-p)\rho}{p(1-\rho)}$. The neighborhood U_3 is contained in $\mathcal{V}(p)$, hence we have $\alpha = -\hat{\lambda}^*$ with $\hat{\lambda}^*$ given by (2.2.11).

We note that the leading order is deterministic, but the prefactor is random (as W_∞), depending on the particular realization of the Bernoulli field. This theorem is a corollary of a more refined result (Theorem 2.4.3), which can be found in Section 2.4. This will be proved by complex analysis arguments, considering the Fourier transform of H_n under some (polymer) measure. Fourier methods are quite strong, they are used in a different spirit in [6] to obtain sharp results on the polymer path itself for small β. The disadvantage is that we have to restrict the parameter domain. It would be tempting to use only real variable techniques as in the Ornstein-Zernike theory for the Bernoulli bond percolation [10], but we take another, shorter route.

Bibliographic note: The model is also interesting with real-valued $\eta(t,x)$ with general distribution. This is motivated by first-passage time percolation. The results given here at the exponential order, have been generalized in [11] to the case of variables with exponential moments. (The reader will find in this paper, a different, nice proof.) For the case of the Gaussian law, we mention [44] on the so-called REM conjecture: it is proved that the local statistics of $(H_n(\omega); \omega \in \mathcal{P}_n)$ approach that of a Poisson point process, provided that one focuses on values distant from the mean QH_n by at most $o(n^{1-\varepsilon})$.

We can interpret our last result in this spirit. In our case, $(H_n(\omega); \omega \in \mathcal{P}_n)$ spreads on the lattice, and natural local statistics of the energy levels are the ratios $Q_n(k_n)/QQ_n(k_n)$. For $d \geq 3$ and $k_n \sim n\rho \in U_3$,

$$Q_n(k_n)/QQ_n(k_n) \simeq W_\infty(\beta(\rho))$$

since $QW_\infty(\beta) = 1$. We emphasize that here the energy level k_n is of order n (far from the bulk), and that the limit is not universal but depends on the lattice and the law of the environment η.

2.4.2 Analytic martingales

Assume $d \geq 3$. Let U_0 be the open set in the complex plane given by $U_0 = \{\beta \in \mathbb{C} : |\operatorname{Im}\beta| < \pi\}$. Then, U_0 is a neighborhood of the real axis, and $\lambda(\beta) = \log Q[\exp\{\beta\eta(t,x)\}]$ is an analytic function on U_0. Define, for $n \geq 0$ and $\beta \in U_0$,

$$W_n(\beta) = P\Big[\exp\Big(\beta\sum_{t=1}^{n}\eta(t,S_t) - n\lambda(\beta)\Big)\Big]\,. \tag{2.4.18}$$

Then, for all $\beta \in U_0$, the sequence $(W_n(\beta), n \geq 0)$ is a $(\mathcal{G}_n)_n$-martingale with complex values, where $\mathcal{G}_n = \sigma\{\eta(t,x); t \leq n, x \in \mathbb{Z}^d\}$. At the same time, for each n and η, $W_n(\beta)$ is an analytic function of $\beta \in U_0$.

Define the real subset

$$U_1 = \big\{\beta \in \mathbb{R} \ : \ \lambda(2\beta) - 2\lambda(\beta) < -\ln\pi_d\big\}\,, \tag{2.4.19}$$

which is an open interval (β_1^-, β_1^+) containing 0 $(-\infty \leq \beta_1^- < 0 < \beta_1^+ \leq +\infty)$. The following is established in [19]:

Proposition 2.4.2 *Define U_2 to be the connected component of the set*

$$\Big\{\beta \in U_0 \ : \ \lambda(2\operatorname{Re}\beta) - 2\operatorname{Re}\lambda(\beta) < -\ln\pi_d\Big\}$$

which contains the origin. Then, U_2 is a complex neighborhood of U_1. Furthermore, there exists an event Ω_1 with $Q(\Omega_1) = 1$ such that,

$$W_n(\beta) \to W_\infty(\beta) \text{ as } n \to \infty, \quad \textit{for all } \eta \in \Omega_1, \beta \in U_2\,,$$

where the convergence is locally uniform. In particular, the limit $W_\infty(\beta)$ is holomorphic in U_2, and all derivatives of W_n converge locally uniformly to the corresponding ones of W_∞. Finally, $W_\infty(\beta) > 0$ for all $\beta \in U_1$, Q-a.s.

For the sake of completeness we repeat the proof here.

Proof of Proposition 2.4.2: Since $\overline{(e^z)} = e^{\overline{z}}$ and $\overline{Q[f]} = Q[\overline{f}]$, we have $\overline{\lambda(\beta)} = \lambda(\overline{\beta})$, and

$$\begin{aligned}
Q\Big[|W_n(\beta)|^2\Big] &= Q\Big[P[\exp\{\beta H_n(S) - n\lambda(\beta)\}]P[\exp\{\overline{\beta} H_n(\tilde S) - n\overline{\lambda(\beta)}\}]\Big] \\
&= P^{\otimes 2}\Big[Q\Big[\exp\{\beta H_n(S) + \overline{\beta} H_n(\tilde S) - 2n \operatorname{Re}\lambda(\beta)\}\Big]\Big] \\
&= P^{\otimes 2}\Big[\exp\Big\{[\lambda(2\operatorname{Re}\beta) - 2\operatorname{Re}\lambda(\beta)]\sum_{t=1}^{n} \mathbf{1}\{S_t = \tilde S_t\}\Big\}\Big] \\
&\le P^{\otimes 2}\Big[\exp\Big\{[\lambda(2\operatorname{Re}\beta) - 2\operatorname{Re}\lambda(\beta)]\sum_{t=1}^{\infty} \mathbf{1}\{S_t = \tilde S_t\}\Big\}\Big] && (2.4.20) \\
&< \infty && (2.4.21)
\end{aligned}$$

if $\beta \in U_2$. Indeed, the random variable $\sum_{t=1}^\infty \mathbf{1}\{S_t = \tilde S_t\}$ (which is the number of meetings between two independent d-dimensional simple random walks) is geometrically distributed with parameter π_d.

For any real $\beta \in U_2$, the positive martingale $W_n(\beta)$ is bounded in L^2, hence it converges almost surely and in L^2-norm to a non-negative limit $W_\infty(\beta)$. Moreover, the event $\{W_\infty(\beta) = 0\}$ is a tail event, so it has probability 0 or 1. Since $QW_\infty(\beta) = 1$, we have necessarily $W_\infty(\beta) > 0$, Q-a.s.

We need a stronger convergence result. Fix a point $\beta \in U_2$ and a radius $r > 0$ such that the closed disk $D(\beta, r) \subset U_2$. Choosing $R > r$ such that $D(\beta, R) \subset U_2$, we obtain by Cauchy's integral formula for all $\beta' \in D(\beta, r)$,

$$W_n(\beta') = \frac{1}{2i\pi}\int_{\partial D(\beta,R)} \frac{W_n(z)}{z-\beta'}dz = \int_0^1 \frac{W_n(\beta + Re^{2i\pi u})Re^{2i\pi u}}{(\beta + Re^{2i\pi u}) - \beta'}du\ ,$$

hence

$$X_n := \sup\{|W_n(\beta')|; \beta' \in D(\beta, r)\} \le R\int_0^1 \frac{|W_n(\beta + Re^{2i\pi u})|}{R-r}du\ .$$

Letting $C = (R/(R-r))^2$, we obtain by the Schwarz inequality

$$\begin{aligned}
(Q[X_n])^2 &\le CQ[\int_0^1 |W_n(\beta + Re^{2i\pi u})|^2 du] \\
&\le C\sup\{Q[|W_n(\beta'')|^2]; n \ge 1, \beta'' \in D(\beta, R)\} \\
&< \infty
\end{aligned}$$

in view of (2.4.21). Notice now that X_n, a supremum of positive submartingales, is itself a positive submartingale. Since $\sup Q[X_n] < \infty$, X_n converges Q-a.s. to a finite limit X_∞. Finally,

$$\sup\{|W_n(\beta')|; \beta' \in D(\beta, r), n \ge 1\} < \infty \quad Q\text{-a.s.},$$

and W_n is uniformly bounded on compact subsets of U_2 on a set of environments of full probability. On this set, $(W_n, n \ge 0)$ is a normal sequence [63] which has a unique limit on the real axis: since U_2 is connected, the full sequence converges to some limit W_∞, which is holomorphic on U_2, and, as mentioned above, positive on the real axis. ■

We do not know that $W_\infty(\beta) \neq 0$ for general $\beta \in U_2$, only for $\beta \in U_1$. Therefore, for all $\eta \in \Omega_1$, we fix another complex neighborhood U_3 of U_1, included in U_2 and depending on η, such that W_∞ and W_n (for n large) belongs to $\mathbb{C} \setminus \mathbb{R}_-$. Recall that

$$Z_n(\beta) = W_n(\beta) \exp\{n\hat{\lambda}(\beta)\} \tag{2.4.22}$$

by definition.

It is sometimes convenient to consider, for real β, the β-tilted law

$$\nu_{n,\beta}(k) = Z_n(\beta)^{-1} e^{\beta k} Q_n(k) \ , \qquad k \in \{0, 1, \dots, n\} \ ,$$

which is a probability measure on the integers $0, 1, \dots, n$. Its mean is equal to $(d/d\beta) \ln Z_n(\beta)$, and its variance is

$$D_{n,\beta} = \frac{d^2}{d\beta^2} \ln Z_n(\beta) \ . \tag{2.4.23}$$

These quantities depend also on η, and $D_{n,\beta} > 0$ as soon as the Bernoulli configuration $(\eta(t,x), t \leq n, \|x\|_1 \leq n, \|x\|_1 = n \mod 2)$ is not identically 0 or 1 on each “hyperplane” $t = k$, $k = 1, \dots, n$. This happens eventually with probability 1, so we will not worry about degeneracy of the variance $D_{n,\beta}$. By positivity of the variance, for all u in the range of $(d/d\beta) \ln Z_n(\cdot)$ there exists unique $\beta = \beta_n(u) \in \mathbb{R}$ such that

$$\frac{d}{d\beta} \ln Z_n(\beta_n(u)) = u \ . \tag{2.4.24}$$

Observe that the function β_n is itself random. Define for $\beta \in \mathbb{R}, k \in \mathbb{N}$,

$$I_n(k) = \sup\{\beta k - \ln Z_n(\beta); \beta \in \mathbb{R}\} - n \ln(2d) \ . \tag{2.4.25}$$

(We will see in the proof of Theorem 2.4.1 below, that $I_n(k) \sim nI(k/n)$ with I as in Proposition 2.3.1.) For k in the range of $(d/d\beta) \ln Z_n(\cdot)$, we have

$$I_n(k) = \beta_n(k)k - \ln Z_n(\beta_n(k)) - n \ln(2d) \ . \tag{2.4.26}$$

Recall (β_1^-, β_1^+) defined in (2.4.19).

Theorem 2.4.3 *There exist an event Ω_2 with $Q(\Omega_2) = 1$ and a real neighborhood U_4 of 0, $U_4 \subset (\beta_1^-, \beta_1^+)$, with the following property. Let $k_n \in \{0, 1, \dots, n\}$ be a sequence such that $\beta_n(k_n)$ remains in a compact subset K of U_4, and let $\hat{D}_n = D_{n,\beta_n(k_n)}$. Then, for all $\eta \in \Omega_2$,*

$$Q_n(k_n) = \frac{1}{\sqrt{2\pi \hat{D}_n}} \exp\{-I_n(k_n) + n \ln(2d)\} \times \big(1 + o(1)\big),$$

where $o(1) \to 0$ as $n \to \infty$.

Proof of Theorem 2.4.3. Suppose that β is a real number. Note that the Fourier transform of the tilted measure is

$$\sum_{k=0}^{n} e^{iku} \nu_{n,\beta}(k) = \frac{Z_n(\beta + iu)}{Z_n(\beta)} \ .$$

From the usual inversion formula for Fourier series we have

$$Q_n(k_n) = Z_n(\beta) e^{-\beta k_n} \times \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{Z_n(\beta + iu)}{Z_n(\beta)} e^{-ik_n u} \, du \ .$$

Taking $\beta = \beta_n(k_n)$ and using (2.4.26) this becomes

$$Q_n(k_n) = e^{-I_n(k_n)+n\ln(2d)} \times \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{Z_n\Big(\beta_n(k_n)+iu\Big)}{Z_n\Big(\beta_n(k_n)\Big)} e^{-ik_n u}\, du\ . \tag{2.4.27}$$

For the moment, K is any compact subset of (β_1^-, β_1^+). From the Taylor expansion of Z_n at $\beta = \beta_n(k_n)$ and (2.4.24), we have

$$\log Z_n\Big(\beta_n(k_n)+iu\Big) = \log Z_n\Big(\beta_n(k_n)\Big) + iuk_n - \frac{u^2}{2}\hat{D}_n + \mathrm{Rest}_n\ ,$$

where the remainder can be estimated by the Cauchy integral formula,

$$|\mathrm{Rest}_n| \le |u|^3 \delta_K^{-3} \max\{|\log Z_n(\beta')|; \beta' \in D(\beta'', \delta_K), \beta'' \in K\}$$

for all $|u| \le \delta_K$, with $\delta_K > 0$ equal to half of the distance from K to the complement of U_3. From Proposition 2.4.2 and the definition of U_3, the above maximum is less that $C_K n$ for all $n \ge 1$, with C_K random but finite and independent of n.

Moreover, in view of Proposition 2.4.2 and (2.4.22,2.4.23), we see that

$$\hat{D}_n = n\lambda''(\beta_n(k_n)) + W_n''(\beta_n(k_n)) \tag{2.4.28}$$

is such that $C_K' n \le \hat{D}_n \le C_K'' n$ for some positive constants C_K', C_K''.

We split the integral in (2.4.27) according to $|u| \le \varepsilon_n := (\ln n/n)^{1/2}$ or not, and the first contribution is

$$\begin{aligned}
&\int_{|u|\le\varepsilon_n} \frac{Z_n\Big(\beta_n(k_n)+iu\Big)}{Z_n\Big(\beta_n(k_n)\Big)} e^{-ik_n u}\, du \\
&= \int_{|u|\le\varepsilon_n} \exp\Big\{-\frac{u^2}{2}\hat{D}_n\Big\} du(1+o(1)) \\
&= \frac{1}{\sqrt{\hat{D}_n}} \int_{|u|\le\varepsilon_n \hat{D}_n^{1/2}} \exp\Big\{-\frac{u^2}{2}\Big\} du(1+o(1)) \\
&= \frac{1}{\sqrt{2\pi\hat{D}_n}}(1+o(1))
\end{aligned} \tag{2.4.29}$$

since $\varepsilon_n \hat{D}_n^{1/2} \to \infty$ by (2.4.28).

Finally, to show that the other contribution is negligible, we need the following fact:

Lemma 2.4.4 *There exist an event Ω_3 with $Q(\Omega_3) = 1$, an integer random variable n_0, a neighborhood U_5 of 0 in $\mathbb{R}$, and $\kappa > 0$ such that $n_0(\eta) < \infty$ for $\eta \in \Omega_3$ and*

$$\Big|\frac{Z_n(\beta+iu)}{Z_n(\beta)}\Big| \le \exp\{-\kappa n u^2\} + \exp\{-\kappa n\}$$

for $\eta \in \Omega_3, \beta \in U_5, u \in [-\pi, \pi]$, and $n \ge n_0(\eta)$.

With the lemma to hand, for η, β, u as above, we bound

$$\int_{\varepsilon_n < u \le \pi} \frac{Z_n\big(\beta_n(k_n)+iu\big)}{Z_n\big(\beta_n(k_n)\big)} e^{-ik_n u}\, du = o\big(\hat{D}_n^{-1/2}\big)$$

where we have used $n = \mathcal{O}(\hat{D}_n)$ of (2.4.28). Combined with (2.4.29) and (2.4.27) this estimate yields the proof of the theorem, with $\Omega_2 = \Omega_1 \cap \Omega_3$, and $U_4 = U_5 \cap U_3$. ■

We turn to the proof of Lemma 2.4.4, which states that the distribution $\nu_{n,\beta}$ does not concentrate on a sublattice of $\mathbb{Z}$, and is not too close from such a distribution. In our proof we take advantage of some (conditional) independance in the variables $\eta(t, S_t)$ under $\nu_{n,\beta}$. This is reminiscent of a construction of [24] for central limit theorem and equivalence of ensembles for Gibbs random fields.

Proof of Lemma 2.4.4: To simplify notations, for real β, we abreviate $\mu_n = \mu^{\eta}_{n,\beta}$ the Gibbs polymer measure, and write E_{μ_n} for its expectation. Introduce

$$\mathcal{I}(x, y) = \{z \in \mathbb{Z}^d : \|x - z\|_1 = \|z - y\|_1 = 1\} , \quad x, y \in \mathbb{Z}^d \tag{2.4.30}$$

the set of lattice points which are next to both x and y, and the event

$$M(\eta, t, y, z) = \Big\{ \text{Card}\{\eta(t, x); x \in \mathcal{I}(y, z)\} = 2 \Big\} .$$

The reason for introducing $M(\eta, t, y, z)$ is that on this event, a path ω conditioned on $\omega_{t-1} = y, \omega_{t+1} = z$, has the option to pick up a $\eta(t, \omega_t)$ value that can be either 0 or 1, bringing therefore some amount of randomness. This event plays a key role here. Note for further purpose that

$$Q\Big(M(\eta, t, y, z)\Big) = 1 - \Big(q^{\text{Card}\mathcal{I}(y,z)} + (1 - q)^{\text{Card}\mathcal{I}(y,z)}\Big) =: \bar{q}(y - z) . \tag{2.4.31}$$

We have

$$\begin{aligned}
\left|\frac{Z_{2n}(\beta + iu)}{Z_{2n}(\beta)}\right| &= \left|E_{\mu_{2n}} e^{iuH_{2n}}\right| \\
&= \left|E_{\mu_{2n}} E_{\mu_{2n}} \left[e^{iuH_{2n}} \mid \Sigma^e\right]\right| \\
&\le E_{\mu_{2n}} \left|E_{\mu_{2n}} \left[e^{iuH_{2n}} \mid \Sigma^e\right]\right| \\
&= E_{\mu_{2n}} \prod_{t=1}^{n} \left|E_{\mu_{2n}} \left[e^{iu\eta(2t-1, S_{2t-1})} \mid \Sigma^e\right]\right|
\end{aligned}$$

by conditional independence of $\Sigma_1, \Sigma_3, \ldots, \Sigma_{2n-1}$ under μ_{2n} given $\Sigma^e = \sigma(\Sigma_{2k}, k \ge 0)$. Recall the notation $\mathcal{I}$ from (2.4.30) and denote by

$$m_\ell = \text{Card}\big\{x \in \mathcal{I}(S_{2t-2}, S_{2t}) : \eta(2t - 1, x) = \ell\big\} , \qquad \ell = 0, 1, \ldots ,$$

the number of sites which can be reached by the walk at time $2t - 1$ and where $\eta(\cdot)$ equals to 0 and 1 respectively ($m_1 + m_0 \le 2d$). Then, for $m_0, m_1 \ge 1$,

$$\begin{aligned}
\left|E_{\mu_{2n}} \left[e^{iu\eta(2t+1, S_{2t+1})} \mid \Sigma^e\right]\right| &= \left|\frac{m_1 e^{\beta + iu} + m_0}{m_1 e^{\beta} + m_0}\right| \\
&\le \exp\{-Cu^2\} , \qquad |u| \le \pi ,
\end{aligned}$$

where the constant C is uniform for $\beta \in K$, and $1 \le m_0, m_1 \le 2d$. We obtain

$$\left|E_{\mu_{2n}} e^{iuH_{2n}}\right| \le E_{\mu_{2n}} \exp\Big\{ - Cu^2 \sum_{t=1}^{n} \mathbf{1}\{M(\eta, 2t - 1, \Sigma_{2t-2}, \Sigma_{2t})\}\Big\} .$$

So far our arguments do not require β to be small. From this point, we will use a perturbation argument. Since $\mu_{2n} = \mu_{2n,\beta}^{\eta}$ is equal to P for $\beta = 0$, we study the term on the right-hand side for the simple random walk measure P instead of the polymer measure μ_{2n}, and estimate the error from this change of measure. This procedure is rather weak, we believe that the result of the lemma holds for a much larger range of β, but we we do not know how to control the term in the right-hand side in a different way.

For $\varepsilon > 0$ we split the last expectation according to the sum being larger or smaller than $n\varepsilon$,

$$\begin{aligned} \left| E_{\mu_{2n}} e^{iuH_{2n}} \right| &\leq e^{-C\varepsilon u^2} + \mu_{2n}\Big(\sum_{t=1}^{n} \mathbf{1}\{M(\eta, 2t-1, \Sigma_{2t-2}, \Sigma_{2t})\} \leq n\varepsilon \Big) \\ &\leq e^{-C\varepsilon u^2} + e^{2n\beta} P\Big(\sum_{t=1}^{n} \mathbf{1}\{M(\eta, 2t-1, \Sigma_{2t-2}, \Sigma_{2t})\} \leq n\varepsilon \Big) \end{aligned} \tag{2.4.32}$$

by the obvious inequalities $0 \leq H_{2n} \leq 2n$. For $\gamma \in (0, 1]$, note that

$$\begin{aligned} Q \exp\Big\{ -\gamma \mathbf{1}\{M(\eta, 2t-1, S_{2t-2}, S_{2t})\} \Big\} &= e^{-\gamma} q(S_{2t-2} - S_{2t}) \\ &\quad + [1 - q(S_{2t-2} - S_{2t})] \ , \end{aligned}$$

with q defined in (2.4.31). Then, there exists some $C_1 > 0$ such that

$$\sup_{\substack{x : P(\Sigma_2 = x) > 0, \\ \|x\|_\infty \leq 1}} \big(e^{-\gamma} q(x) + [1 - q(x)] \big) \leq \exp\{-C_1 \gamma\} \ , \qquad \gamma \in (0, 1].$$

Hence,

$$\begin{aligned} & QP \exp\Big\{ -\gamma \sum_{t=1}^{n} \mathbf{1}\{M(\eta, 2t-1, \Sigma_{2t-2}, \Sigma_{2t})\} \Big\} \\ &= P \exp\Big\{ -C_1 \gamma \sum_{t=1}^{n} \mathbf{1}\{\|\Sigma_{2t-2} - \Sigma_{2t}\|_\infty \leq 1\} \Big\} \\ &= \Big(P \exp\Big\{ -C_1 \gamma \mathbf{1}\{\|\Sigma_2\|_\infty \leq 1\} \Big\} \Big)^n \\ &= \Big(\frac{(2d-1)e^{-C_1\gamma} + 1}{2d} \Big)^n \\ &\leq e^{-nC_2\gamma} \end{aligned}$$

with $C_2 > 0$. Now, we choose $\varepsilon = C_2/2, \gamma = 1$, and we get

$$\begin{aligned} & QP \Big(\sum_{t=1}^{n} \mathbf{1}\{M(\eta, 2t-1, \Sigma_{2t-2}, \Sigma_{2t})\} \leq n\varepsilon \Big) \\ &\leq e^{n\gamma\varepsilon} QP \exp\Big\{ -\gamma \sum_{t=1}^{n} \mathbf{1}\{M(\eta, 2t-1, \Sigma_{2t-2}, \Sigma_{2t})\} \Big\} \\ &\leq e^{-nC_2/2} \ , \end{aligned}$$

and then

$$Q\Big(P\big(\sum_{t=1}^{n} \mathbf{1}\{M(\eta, 2t-1, \Sigma_{2t-2}, \Sigma_{2t})\} \leq n\varepsilon \big) \geq e^{-nC_2/4} \Big) \leq e^{-nC_2/4} \ .$$

By Borel-Cantelli lemma, the set Ω_3 of all environments such that

$$P\Big(\sum_{t=1}^{n}\mathbf{1}\{M(\eta,2t-1,\Sigma_{2t-2},\Sigma_{2t})\}\leq n\varepsilon\Big)\leq e^{-nC_2/4}\qquad\text{eventually},$$

is of full measure. We define n_0 as the first integer (if exists) from which the previous bound is fulfilled, and $U_5=(-C_2/4,C_2/4)$. From (2.4.32) we easily check that Lemma 2.4.4 holds true with $\kappa=\min(C\varepsilon,C_2/2)$. ■

Proof of Theorem 2.4.1: The theorem is a corollary of Theorem 2.4.3, where Ω_2 and U_3 are introduced. In particular we know that $\alpha=-\eta^*$ in U_3. Note that $\beta(\rho)$ is the maximizer in the definition of $\lambda^*(\rho)$ as a Legendre transform. Since $k_n/n\to\rho$, we have that $\beta_n(k_n)\to\beta(\rho)$. By (2.4.28), $\hat{D}_n\sim n\lambda''(\beta(\rho))$, and by Legendre duality,

$$(\lambda^*)'\circ\lambda'=\mathrm{Id}\ ,$$

and so $\lambda''(\beta(\rho))=1/(\lambda^*)''(\rho)$. The only quantity left to be studied is $I_n(k_n)$. Combining (2.4.25, 2.4.22) and performing the change of variable $\beta=\beta(k_n/n)+v$, we have

$$\begin{aligned}
I_n(k_n) &= \sup\{\beta k_n-n\hat{\lambda}(\beta)-\ln W_n(\beta);\beta\in\mathbb{R}\}\\
&= \sup\Big\{(\beta(k_n/n)+v)k_n-n\hat{\lambda}(\beta(k_n/n)+v)\\
&\qquad\qquad -\ln W_n(\beta(k_n/n)+v);v\in\mathbb{R}\Big\}\\
&= \sup\Big\{n\Big[\hat{\lambda}(\beta(k_n/n))-\hat{\lambda}(\beta(k_n/n)+v)+\hat{\lambda}'(\beta(k_n/n))v\Big]\\
&\qquad\qquad -\ln W_n(\beta(k_n/n)+v);v\in\mathbb{R}\Big\}+n\hat{\lambda}^*(k_n/n)\\
&= n\hat{\lambda}^*(k_n/n)-\ln W_n(\beta(k_n/n))\\
&\quad +\sup\Big\{n\Big[\hat{\lambda}(\beta(k_n/n))-\hat{\lambda}(\beta(k_n/n)+v)+\hat{\lambda}'(\beta(k_n/n))v\Big]\\
&\qquad\qquad -\ln W_n(\beta(k_n/n)+v)+\ln W_n(\beta(k_n/n));v\in\mathbb{R}\Big\}\\
&= n\hat{\lambda}^*(k_n/n)-\ln W_n(\beta(k_n/n))+o(1)\\
&= n\hat{\lambda}^*(k_n/n)-\ln W_n(\beta(\rho))+o(1)
\end{aligned}$$

by strict convexity of $\hat{\lambda}$ and the fact that $|\ln[W_n(\beta+v)/W_n(\beta)]|\leq|v|$. ■

Chapter 3

Appendix

3.1 Crash course on Gibbs measures

Let $\mathcal{P}(\Omega)$ denote the set of probability measures on $(\Omega, \mathcal{A})$.

Definition 3.1.1 *Let $\mu, \nu \in \mathcal{P}(\Omega)$. The (relative) entropy* [1] *$H(\nu|\mu)$ of ν with respect to μ is*

$$H(\nu|\mu) = \begin{cases} \nu(\ln \frac{d\nu}{d\mu}) & \text{if } \nu \ll \mu \text{ and } \ln \frac{d\nu}{d\mu} \in L^1(\nu) \\ +\infty & \text{otherwise} \end{cases} \tag{3.1.1}$$

In the first case above, we have $\frac{d\nu}{d\mu} \ln \frac{d\nu}{d\mu} \in L^1(\mu)$ and

$$H(\nu|\mu) = \mu\left(\Phi(\frac{d\nu}{d\mu})\right), \quad \Phi(t) = t \ln t,\ \Phi(0) = 0.$$

The function $\Phi : [0, +\infty) \to \mathbb{R}$ is strictly convex, with $\Phi'(t) = 1 + \ln t, \Phi''(t) = 1/t, t \in (0, +\infty)$, it is shown in figure 3.1.

We state some elementary properties.

Proposition 3.1.2 *(i) $H(\nu|\mu) \in [0, +\infty]$,* and *$H(\nu|\mu) = 0 \iff \nu = \mu$*
(ii) $H(\nu|\mu)$ is a convex function of μ and a convex function of ν.

□ (i) Without loss of generality, we can assume $\nu \ll \mu$. By Jensen's inequality,

$$H(\nu|\mu) = \mu\left(\Phi(\frac{d\nu}{d\mu})\right) \geq \Phi\left(\mu(\frac{d\nu}{d\mu})\right) = \Phi\left(\nu(\Omega)\right) = \Phi(1) = 0$$

By strict convexity, the equality holds if and only if $\frac{d\nu}{d\mu}$ is μ-a.s. constant, i.e., if and only if $\mu = \nu$.
(ii) Let $\lambda \in (0,1), \mu, \mu', \nu, \nu'$. Without loss of generality, we can assume $\nu \ll \mu$, $\nu \ll \mu'$ with ν-integrability of logarithms of derivatives, and the same assumptions for ν'. By convexity of Φ,

$$\begin{aligned} H(\lambda\nu + (1-\lambda)\nu'|\mu) &= \mu\left(\Phi(\lambda\frac{d\nu}{d\mu} + (1-\lambda)\frac{d\nu'}{d\mu})\right) \\ &\leq \mu\left(\lambda\Phi(\frac{d\nu}{d\mu}) + (1-\lambda)\Phi(\frac{d\nu'}{d\mu})\right) \\ &= \lambda H(\nu|\mu) + (1-\lambda)H(\nu'|\mu) \end{aligned}$$

[1] The relative entropy is also called Kullback or Kullback-Leibler information in statistics, and information gain in information theory.

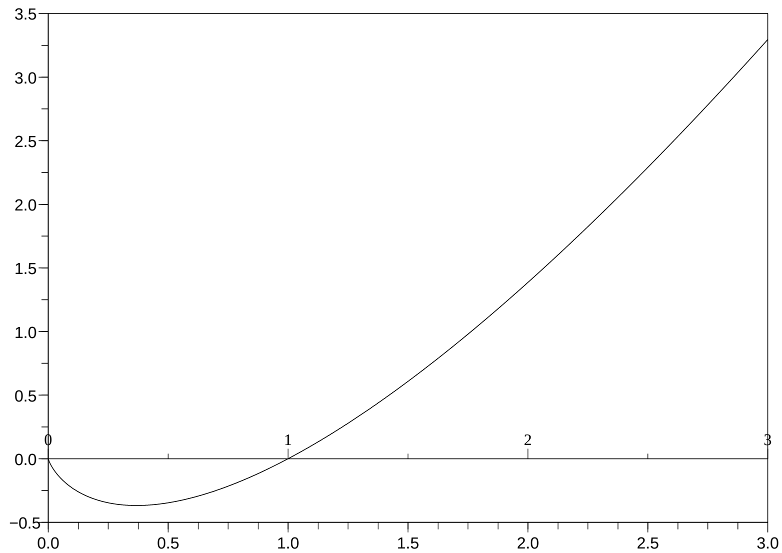

Figure 3.1: The function $\Phi : x \mapsto x \ln x$ on $[0, +\infty)$. The minimum is achieved at $x = e^{-1}$ and is equal to $-e^{-1}$. The tangent at $x = 0$ is vertical.

By concavity of the logarithm,

$$\ln \frac{d\nu}{\lambda d\mu + (1-\lambda) d\mu'} = -\ln \frac{\lambda d\mu + (1-\lambda) d\mu'}{d\nu} \leq -\lambda \ln \frac{d\mu}{d\nu} - (1-\lambda) \ln \frac{d\mu'}{d\nu}$$

with $\frac{d\mu}{d\nu} = (\frac{d\nu}{d\mu})^{-1}$. ■

Remark 3.1.3 *The relative entropy $H(\mu|\nu)$ is a popular method of measuring the similarity between two probability distributions μ, ν. In spite of its properties 3.1.2, it is not a distance because it is not symmetric and it does not satisfy the triangle inequality. To settle the first issue, one may consider the symmetrized version*

$$D_{KL}(\mu|\nu) = \frac{1}{2}\big(H(\mu|\nu) + H(\nu|\mu)\big),$$

which is called the Kullback-Leibler divergence.

Relative entropy compares to distances between probability measures: Recall the total variation norm of a finite, signed measure m on $(\Omega, \mathcal{A})$,

$$\|m\|_{TV} = \sup\{m(f); f : A \to \mathbb{R} \text{ measurable}, \|f\|_\infty \leq 1\}.$$

Proposition 3.1.4 (Gibbs variational formula) *For $f : \Omega \to \mathbb{R}$ measurable with $\mu(e^f) < \infty$ and $\mu(|f|e^f) < \infty$,*

$$\ln \mu(e^f) = \sup\{\nu(f) - H(\nu|\mu); \nu \in \mathcal{P}(\Omega), \nu(|f|) < \infty\}$$

This formula is an energy/entropy balance: it shows the competition between the energy gain $\nu(f)$ and the entropy cost $H(\nu|\mu)$.

□ Let $\nu \ll \mu$, and A be the event $A = \{\frac{d\nu}{d\mu} > 0\}$. It holds

$$\begin{aligned}
\mu(e^f) &\geq \mu(e^f \; ; \; A) \\
&= \nu(e^f \times \frac{1}{\frac{d\nu}{d\mu}} \; ; \; A) \\
&= \nu(e^f \times \frac{1}{\frac{d\nu}{d\mu}}) \\
&= \nu(\exp\{f - \ln \frac{d\nu}{d\mu}\}) \\
&\geq \exp\{\nu(f) - \nu(\ln \frac{d\nu}{d\mu})\} \qquad (3.1.2)
\end{aligned}$$

by using Jensen's inequality in (3.1.2). Hence, the left-hand side is not smaller than the right-hand side. To show the equality, note that the equality is achieved with the probability measure ν given by $d\nu = e^f/\mu(e^f)d\mu$:

$$\nu(f) - H(\nu|\mu) = \frac{\mu(fe^f)}{\mu(e^f)} - \frac{\mu(e^f[f - \ln \mu(e^f)])}{\mu(e^f)} = \ln \mu(e^f) \qquad (3.1.3)$$

■

Corollary 3.1.5 (Gibbs variational principle) *The supremum in the Gibbs variational formula 3.1.4 is unique and achieved for ν with $d\nu = e^f/\mu(e^f)d\mu$.*

□ What remains to prove is the uniqueness of the minimizer that we have found is (3.1.3). To reach equality in Jensen's inequality in (3.1.2), it is needed that $f - \ln \frac{d\nu}{d\mu}$ is ν-a.s. constant. Hence, any maximizer is of the form $d\nu = Z^{-1}e^f d\mu$, with $Z = \mu(e^f)$ since ν has mass 1. ■

Exercise 3.1.6 *Prove Property 3.1.2 using the above variational formula.*

Let $f : \Omega \to \mathbb{R}$, some function, not μ-a.s. constant. Define, for $\beta \in \mathbb{R}$,

$$\theta(\beta) = \ln \mu(e^{\beta f}) \in (-\infty, +\infty] \qquad (3.1.4)$$

Then, $\theta : \mathbb{R} \to (-\infty, +\infty]$ is convex. Indeed, for $\lambda \in (0, 1)$ and $\beta, \beta' \in \mathbb{R}$, we have, from Hölder inequality with $p = \lambda^{-1}, q = (1 - \lambda)^{-1}$

$$\begin{aligned}
\exp \theta[\lambda\beta + (1 - \lambda)\beta'] &= \mu(\exp\{\lambda\beta f + (1 - \lambda)\beta' f\}) \\
&\leq \mu(\exp \beta f)^\lambda \times \mu(\exp \beta' f)^{1-\lambda} \qquad \text{(Hölder)} \\
&= \exp\{\theta[\lambda\beta] + \theta[(1 - \lambda)\beta']\}
\end{aligned}$$

Hence, the domain $Dom(\theta) := \{\beta \in \mathbb{R} : \theta(\beta) < +\infty\}$ is an interval containing 0. The function θ is the logarithmic Laplace transform of the random variable f. In statistical mechanics, Ω is the the set of all configurations ω of the system under consideration; the measure μ describe the ideal state of the system, i.e., when no interaction is present; the function $-f$ has the meaning of an energy (interaction energy or internal energy); and θ is called the free energy at temperature inverse β.[2]

We will assume

[2] In physics, $\beta = 1/(kT)$ with T the temperature and $k = 1.3810^{-23}$ joules per absolute degrees.

Assumption 3.1.7 *$Dom(\theta)$ has positive length.*

Definition 3.1.8 (Gibbs Measures) *For $\beta \in Dom(\theta)$, the (finite volume) Gibbs measure at temperature inverse β is the probability measure $\gamma_\beta \in \mathcal{P}(\Omega)$ given by*

$$d\gamma_\beta = Z^{-1} e^{\beta f} d\mu \ , \qquad Z = Z(\beta) = \mu(e^{\beta f})$$

On the interior of $Dom(\theta)$, the function θ is infinitely differentiable, with

$$\theta'(\beta) = \gamma_\beta(f) \ ,$$

$$\theta''(\beta) = \mathrm{Var}_{\gamma_\beta}(f) = \gamma_\beta(f^2) - \gamma_\beta(f)^2 \ .$$

In particular, θ is strictly convex on its domain. Moreover, the set

$$I := \theta'(Dom(\theta)) = \{\theta'(\beta); \beta \in Dom(\theta), \theta'(\beta) \text{ exists}\}$$

is an interval, and is non empty under assumption 3.1.7. By definition,

$$a \in I \iff \exists \beta : \gamma_\beta(f) = a \ .$$

The Gibbs variational formula writes

$$\ln \mu(e^{\beta f}) = \sup_{a \in \mathbb{R}} \{\beta a - \inf_{\nu \in \mathcal{P}(\Omega), \nu(f) = a} H(\nu|\mu)\} \ .$$

From the Gibbs variational principle, it follows that for all $a \in I$, the infimum

$$\mathcal{I}(a) = \inf\{H(\nu|\mu); \nu \in \mathcal{P}(\Omega), \nu(f) = a\}, \tag{3.1.5}$$

is achieved at the unique point $\nu = \gamma_\beta$, where β is the solution of $\theta'(\beta) = a$. This is the **minimum entropy principle**. It indicates that Gibbs measures are natural objects, since it states that Gibbs measures are the less informative probability measures satisfying linear constraints.

Note that the function $\mathcal{I}$ defined in (3.1.5) is convex.

We now give an identity for the entropy – more precisely, the Kullback-Leibler divergence – of two Gibbs measures. Let $\beta_1, \beta_2 \in Dom(\theta), \beta_1 \neq \beta_2$. By strict convexity of θ, we have $(\beta_1 - \beta_2)\big(\theta'(\beta_1) - \theta'(\beta_2)\big) > 0$. In fact we have

$$(\beta_1 - \beta_2)\big(\theta'(\beta_1) - \theta'(\beta_2)\big) = H(\gamma_{\beta_1}|\gamma_{\beta_2}) + H(\gamma_{\beta_2}|\gamma_{\beta_1}). \tag{3.1.6}$$

Indeed,

$$\begin{aligned} H(\gamma_{\beta_1}|\gamma_{\beta_2}) &= \gamma_{\beta_1}\left[\ln \frac{d\gamma_{\beta_1}}{d\gamma_{\beta_2}}\right] \\ &= \gamma_{\beta_1}\left[(\beta_1 - \beta_2) f - \big(\theta(\beta_1) - \theta(\beta_2)\big)\right] \\ &= (\beta_1 - \beta_2)\gamma_{\beta_1}[f] - \big(\theta(\beta_1) - \theta(\beta_2)\big) \ . \end{aligned}$$

Therefore,

$$\begin{aligned} H(\gamma_{\beta_1}|\gamma_{\beta_2}) + H(\gamma_{\beta_2}|\gamma_{\beta_1}) &= (\beta_1 - \beta_2)\left(\gamma_{\beta_1}[f] - \gamma_{\beta_2}[f]\right) \\ &= (\beta_1 - \beta_2)\big(\theta'(\beta_1) - \theta'(\beta_2)\big) \ . \end{aligned}$$

3.2 Superadditive lemma

A real sequence $(u_n; n \geq 1)$ is called superadditive if

$$u_{n+m} \geq u_n + u_m \ , \qquad n, m \geq 1 \ .$$

The following result is standard, e.g., [22, lemma 3.1.3].

Lemma 3.2.1 (Superadditive lemma) *If $(u_n; n \geq 1)$ is a superadditive sequence, then*

$$\lim_{n\to\infty} \frac{u_n}{n} = \sup_{m\geq 1} \frac{u_m}{m} \in \mathbb{R} \cup \{+\infty\} \ .$$

□ Fix $m \geq 1$, and let $M = \min\{u_\ell; \ell = 1, \ldots, m-1\} \wedge 0$. Then, decomposing any integer n as $n = km + \ell$ with $k = \lfloor \frac{n}{m} \rfloor$ and $\ell \in \{0, \ldots, m-1\}$, we have by superadditivivity, and setting $u_0 = 0$,

$$\frac{u_n}{n} \geq k\frac{u_m}{m} + \frac{u_\ell}{n} \geq k\frac{u_m}{m} + \frac{M}{n}$$

Hence,

$$\liminf_{n\to\infty} \frac{u_n}{n} \geq \frac{u_m}{m} \ , \qquad m \geq 1 \ .$$

This shows one inequality. Since $\limsup_{n\to\infty} n^{-1}u_n \leq \sup_{m\geq 1} m^{-1}u_m$ is trivial, the lemma is proved. ■

3.3 Concentration inequalities

Loosely, concentration inequalities state that a function of many independent random variables, which does not depend much on any of them, strongly concentrated around its typical value. This theory has been recently renewed by technical works, of Talagrand in particular. We will take the straightforward approach by martingales, which yields weaker results but of the same nature.

We start with the famous Azuma's lemma, see e.g. [1]. Maurey (1979) popularized its use by deriving an isoperimetric inequality for the symmetric group, showing its potential to study normed spaces (see Milman-Scechtman (1986). This was the first step in opening a wide field of applications. This lemma is quite useful.

Lemma 3.3.1 (Azuma's lemma) *Let $M_k, 0 \leq k \leq n$ be martingale starting from $M_0 = 0$ such that*

$$|M_k - M_{k-1}| \leq 1,$$

for $k = 1, \ldots, n$. Then, for $\theta \in \mathbb{R}$, we have

$$\mathbf{E}(e^{\theta M_n}) \leq \exp n\theta^2/2 \ ,$$

and, for all $r \geq 0$,

$$\mathbf{P}(M_n \geq nr) \leq e^{-nr^2/2} \ .$$

□ By assumption $\Delta M_{k-1} \stackrel{\text{def}}{=} M_k - M_{k-1}$ has absolute value smaller than 1. From the barycentric relation

$$\Delta M_k = \frac{1 + \Delta M_k}{2} \cdot 1 + \frac{1 - \Delta M_k}{2} \cdot (-1) \ ,$$

it follows by convexity of the function $u \mapsto \exp \theta u$ for $\theta \in \mathbb{R}$, that

$$e^{\theta \Delta M_k} \leq \frac{1+\Delta M_k}{2}\, e^{\theta} + \frac{1-\Delta M_k}{2}\, e^{-\theta}\ ,$$

and by the martingale property,

$$\mathbf{E}^{\mathcal{H}_k} e^{\theta \Delta M_k} \leq \cosh \theta \leq e^{\theta^2/2}\ .$$

The last inequality comes from the identities $\cosh\theta := (e^{\theta}+e^{-\theta})/2$, $(\cosh\theta)' = \tanh\theta$, and the elementary fact $|\tanh\theta| \leq |\theta|, \theta \in \mathbb{R}$. Hence, for $\theta \in \mathbb{R}$,

$$\begin{aligned}
\mathbf{E}(e^{\theta M_n}) &= \mathbf{E}\mathbf{E}^{\mathcal{H}_{n-1}} \prod_{k=0}^{n-1} e^{\theta \Delta M_k} \\
&= \mathbf{E}\big(\prod_{k=0}^{n-2} e^{\theta \Delta M_k}\big) \mathbf{E}^{\mathcal{H}_{n-1}} e^{\theta \Delta M_{n-1}} \\
&\leq \mathbf{E} \prod_{k=0}^{n-2} e^{\theta \Delta M_k} e^{\theta^2/2} \\
&\leq \exp n\theta^2/2
\end{aligned}$$

by induction. This is the first claim. From this and Markov inequality we obtain for $\theta \geq 0$,

$$\begin{aligned}
e^{n\theta r}\mathbf{P}(M_n \geq nr) &\leq \mathbf{E}\left(e^{\theta M_n}; \{M_n \geq nr\}\right) \qquad (\theta \geq 0) \\
&\leq \mathbf{E}(e^{\theta M_n}) \\
&\leq \exp n\theta^2/2.
\end{aligned}$$

Hence,

$$\mathbf{P}(M_n \geq nr) \leq \exp -n \sup_{\theta \geq 0}(r\theta - \theta^2/2),$$

which yields the result by taking the optimal θ ($\theta = r$ is non negative when $r \geq 0$). ■

Corollary 3.3.2 *Let $(X_k)_{k \leq n}$ a sequence of independent random variables in E, and $Y_n = f_n(X_1, \ldots, X_n)$ with $f_n : E^n \to \mathbb{R}$ a measurable function. Assume that for $k = 1, \ldots n$, all $x_1, \ldots, x_n, y_k$ in E,*

$$|f_n(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n) - f_n(x_1, \ldots, x_{k-1}, y_k, x_{k+1}, \ldots, x_n)| \leq 1$$

Then, we have the subgaussian estimates: $\mathbf{E}(\exp\theta[Y_n - \mathbf{E}Y_n]) \leq \exp n\theta^2/2$ for all real θ, and, for all $r \geq 0$,

$$\mathbf{P}(|Y_n - \mathbf{E}Y_n| \geq nr) \leq 2e^{-nr^2/2}\ .$$

□ Let $\mathcal{H}_k = \sigma(X_1, \ldots X_k)$, and $M_k = \mathbf{E}^{\mathcal{H}_k}(Y_n) - \mathbf{E}(Y_n)$. Then, $(M_k, k \leq n)$ is a $(\mathcal{H}_k, k \leq n)$-martingale, with $M_n = Y_n - \mathbf{E}Y_n$. Moreover, by independence,

$$\begin{aligned}
\Delta M_{k-1} &\stackrel{\text{def}}{=} M_k - M_{k-1} \\
&= \mathbf{E}^{\mathcal{H}_k}(Y_n) - \mathbf{E}^{\mathcal{H}_{k-1}}(Y_n) \\
&= \mathbf{E}^{\mathcal{H}_k}[f_n(X_1, \ldots, X_{k-1}, X_k, X_{k+1}, \ldots, X_n) \\
&\qquad -f_n(X_1, \ldots, X_{k-1}, Z_k, X_{k+1}, \ldots, X_n)]
\end{aligned}$$

with Z_k an independent copy of X_k. By assumption, we have

$$|\Delta M_k| \leq 1, \quad k = 0, \dots n-1.$$

Then, the corollary directly follows from Azuma's lemma 3.3.1. ■

Exercise 3.3.3 *Assume that the random variables η's are bounded, $|\eta(t,x)| \leq K$ a.s. for some finite $K > 0$. Using Azuma's lemma, show that*

$$Q\Big(\exp\{\theta[p^\eta_{n,\beta} - Qp^\eta_{n,\beta}]\}\Big) \leq \exp\{C_1\theta^2/n\} \ ,$$

and deduce that

$$\begin{aligned} \mathrm{Var}_Q(p^\eta_{n,\beta}) &\leq C_2/n; , \\ Q\Big(|p^\eta_{n,\beta} - Qp^\eta_{n,\beta}|\Big) &\leq C/\sqrt{n} \ . \end{aligned}$$

What is the smallest constant $C = C(\beta, K)$ that you can get this way? [Answer: $C_1 = 2\beta^2K^2$.]

Asking bounded oscillation in all variables is quite strong an assumption. In the following concentration inequality we relax it into a control for the exponential moment of the oscillation.

Lemma 3.3.4 [Concentration] *Suppose that $X \in L^1(Q)$ is $\mathcal{G}_n$-measurable for some n and that there exist $\delta \in (0,\infty)$, $A \in (0,\infty)$, $X_1, ..., X_n \in L^1(Q)$ such that*

$$Q^{\mathcal{G}_{j-1}}[X_j] = Q^{\mathcal{G}_j}[X_j], \ \ \textit{and} \ \ Q^{\mathcal{G}_{j-1}}[\exp(\delta|X - X_j|)] \leq A \tag{3.3.7}$$

for all $j = 1, ..., n$. Then, with $B = 2\sqrt{6}A^2/\delta^2$,

$$Q(|X - Q[X]| \geq \varepsilon n) \leq 2\exp(-\varepsilon^2 n/(4B)) \quad \textit{for all } \varepsilon \in [0, B\delta]. \tag{3.3.8}$$

□ We consider a sequence $D_j = Q^{\mathcal{G}_j}[X] - Q^{\mathcal{G}_{j-1}}[X]$. We first observe that

$$Q^{\mathcal{G}_{j-1}}[e^{\delta|D_j|}] \leq A^2 \ , \qquad j = 1, \dots n. \tag{3.3.9}$$

Indeed, since $Q^{\mathcal{G}_{j-1}}[X_j] = Q^{\mathcal{G}_j}[X_j]$, we have

$$\begin{aligned} |D_j| &\leq |Q^{\mathcal{G}_j}[X - X_j]| + |Q^{\mathcal{G}_{j-1}}[X - X_j]| \\ &\leq Q^{\mathcal{G}_j}[Y_j] + Q^{\mathcal{G}_{j-1}}[Y_j], \ \text{ with } Y_j = |X - X_j|. \end{aligned}$$

It follows from Jensen inequality that

$$e^{\delta Q^{\mathcal{G}_{j-1}}[Y_j]} \leq Q^{\mathcal{G}_{j-1}}[e^{\delta Y_j}] \leq A.$$

Similarly,

$$Q^{\mathcal{G}_{j-1}}[e^{\delta Q^{\mathcal{G}_j}[Y_j]}] \leq Q^{\mathcal{G}_{j-1}}[Q^{\mathcal{G}_j}[e^{\delta Y_j}]] = Q^{\mathcal{G}_{j-1}}[e^{\delta Y_j}] \leq A.$$

These imply (3.3.9) by writing

$$Q^{\mathcal{G}_{j-1}}[e^{\delta|D_j|}] \leq e^{\delta Q^{\mathcal{G}_{j-1}}[Y_j]} Q^{\mathcal{G}_{j-1}}[e^{\delta Q^{\mathcal{G}_j}[Y_j]}] \leq A^2.$$

We now infer from (3.3.9) that

$$Q^{\mathcal{G}_{j-1}}[e^{\theta D_j}] \leq e^{B\theta^2} \ , \qquad \theta \in [-\delta/2, \delta/2], \quad j = 1, \dots, n. \tag{3.3.10}$$

First observe that

$$\tfrac{1}{4!}Q^{\mathcal{G}_{j-1}}[|D_j|^4] = \tfrac{1}{4!}Q^{\mathcal{G}_{j-1}}[\delta^4|D_j|^4]/\delta^4 \le Q^{\mathcal{G}_{j-1}}[e^{\delta|D_j|}]/\delta^4 \le A^2/\delta^4$$

and hence that

$$Q^{\mathcal{G}_{j-1}}[|D_j|^2 e^{\delta|D_j|/2}] \le Q^{\mathcal{G}_{j-1}}[|D_j|^4]^{1/2} Q^{\mathcal{G}_{j-1}}[e^{\delta|D_j|}]^{1/2} \le 2\sqrt{6}A^2/\delta^2 = B.$$

Since $e^x \le 1 + x + |x|^2 e^{|x|}/2$ for all $x \in \mathbb{R}$, we obtain (3.3.10) by estimating

$$Q^{\mathcal{G}_{j-1}}[e^{\theta D_j}] \le 1 + \theta^2 B/2 \le \exp(B\theta^2)$$

Finally, since $X - Q[X] = D_n + ... + D_1$, it follows from the last estimate that

$$Q[\exp(\theta(X - Q[X]))] \le \exp(B\theta^2 n)\ , \qquad \theta \in [-\delta/2, \delta/2] \tag{3.3.11}$$

via a simple iterative procedure. To see (3.3.8), we take $\theta = \frac{\varepsilon}{2B} \le \frac{\delta}{2}$. Then, by Chebychev's inequality and (3.3.11),

$$\begin{aligned} Q(|X - Q[X]| \ge \varepsilon n) &= Q(\theta|X - Q[X]| \ge \theta\varepsilon n) \\ &\le 2\exp((B\theta^2 - \theta\varepsilon)n) = 2\exp(-\varepsilon^2 n/(4B)). \end{aligned}$$

■

3.4 Harris correlation inequality

Recall that a function $f : \mathbb{R}^k \to \mathbb{R}$ is increasing if $f(x) \le f(y)$ whenever it $x_i \le y_i\ \forall i \le k$. This is equivalent to f being coordinatewise increasing, i.e., $f(x) \le f(y)$ for all y such that $y_i = x_i$ for all $i \ne j$ and $x_j < y_j$.

Definition 3.4.1 *A family $X = (X_i; 1 \le i \le k)$ of real random variables defined on the same probability space are called* **positively associated** *if for any $f, g : \mathbb{R}^k \to \mathbb{R}$ bounded increasing functions,*

$$\mathbf{E}[f(X)g(X)] \ge [\mathbf{E}f(X)][\mathbf{E}g(X)]\ . \tag{3.4.12}$$

The inequality (3.4.12) is called the Fortuyn-Kasteleyn-Ginibre (FKG) inequality. The inequality simply means that increasing functions are positively correlated. The following example is crucial in the applications, it is due to Harris.

Proposition 3.4.2 (FKG-Harris inequality) *A family of independent, real random variables is positively associated.*

□ We prove Proposition 3.4.2 by induction on k. For $k = 1$, consider an independent copy Y_1 of X_1, and write

$$\begin{aligned} \mathrm{Cov}(f(X_1), g(X_1)) &= \mathbf{E}[f(X_1)g(X_1)] - [\mathbf{E}f(X_1)][\mathbf{E}g(X_1)] \\ &= \frac{1}{2}\mathbf{E}\left[[f(X_1) - f(Y_1)][g(X_1) - g(Y_1)]\right]\ , \end{aligned}$$

where the integrand is a.s. non-negative by monotonicity of f, g. Let now $k > 1$, and write the standard decomposition

$$\mathrm{Cov}(f(X), g(X)) = \mathrm{Cov}(\mathbf{E}[f(X)|X_1], \mathbf{E}[g(X)|X_1]) + \mathbf{E}\mathrm{Cov}(f(X), g(X)|X_1)\ ,$$

with $\mathrm{Cov}(U,V|X_1)$ the covariance of U,V given X_1. We show that both terms are non-negative. For f non-decreasing, $\mathbf{E}[f(X)|X_1]$ is itself a non-decreasing function of X_1, and the first term will be non-negative according to the case $k=1$. Indeed, by independence, $\mathbf{E}[f(X)|X_1] = \psi(X_1)$ is given by $\psi(x_1) = \mathbf{E}f(x_1, X_2, \ldots, X_k)$, and then,

$$\psi(x_1) - \psi(x_1') = \mathbf{E}[f(x_1, X_2, \ldots, X_k) - (x_1', X_2, \ldots, X_k)] \geq 0 \quad \text{if} \quad x_1 \geq x_1' \ .$$

Again by independence, the second term is equal to $\mathbf{E}\Phi(X_1)$, where

$$\Phi(x_1) = \mathrm{Cov}(F_{x_1}(X_2, \ldots, X_k), G_{x_1}(X_2, \ldots, X_k)) \ ,$$

with F_{x_1}, G_{x_1} the partial functions f,g at x_1, e.g., $F_{x_1}(x_2, \ldots, x_k) = f((x_1, \ldots, x_k)$. Clearly F_{x_1}, G_{x_1} are non-decreasing for all x_1, and the the conditional covariance $\mathrm{Cov}(f(X), g(X)|X_1)$ is a.s. non-negative by the induction assumption. ■

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