Journées ALEA, CIRM, 21st-25th March 2022

Scaling limits of random trees and graphs

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https://www.stats.ox.ac.uk/~goldschm/ALEAminicourse.html

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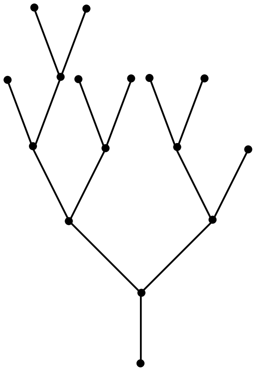
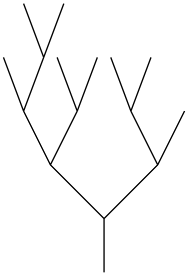
DEPARTMENT OF STATISTICS

1. INTRODUCTION: BINARY TREES



Binary trees

- Let $\mathbb{T}_n$ be the set of planted plane binary trees with n leaves.
- Note that every element of $\mathbb{T}_n$ has $n-1$ internal vertices and $2n-1$ edges.



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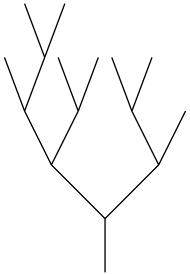
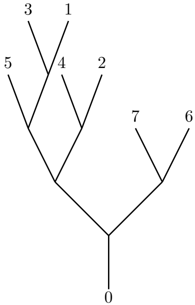
- Let $\mathbb{T}_n^*$ be the set of planted plane binary leaf-labelled trees with n labelled leaves.
- The root, labelled 0 is, by convention, not a leaf.





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$$|\mathbb{T}_n| = \frac{1}{n}\binom{2n-2}{n-1} \quad \text{(Catalan numbers)}, \quad |\mathbb{T}_n^*| = n!|\mathbb{T}_n|.$$

Uniform binary plane trees

$$|\mathbb{T}_n| = \frac{1}{n}\binom{2n-2}{n-1} \sim \frac{4^{n-1}}{n^{3/2}\sqrt{\pi}} \quad \text{as } n \to \infty.$$

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Rémy's algorithm recursively constructs a sequence $(T_n)_{n\geq 1}$ of trees such that T_n is uniform on $\mathbb{T}_n^*$ for each n.

Rémy's algorithm

- Start from a single edge with endpoints labelled 0 and 1.
- At step $n \geq 2$, pick an edge uniformly at random, divide it into two edges, insert a new vertex in the middle and attach to that vertex a new edge with a leaf labelled n at its other end, chosen to point in one of the two possible directions each with probability $1/2$.

Rémy's algorithm

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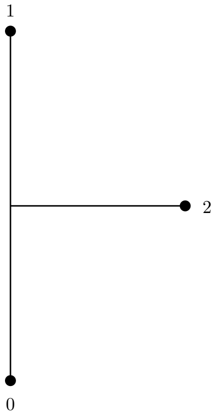

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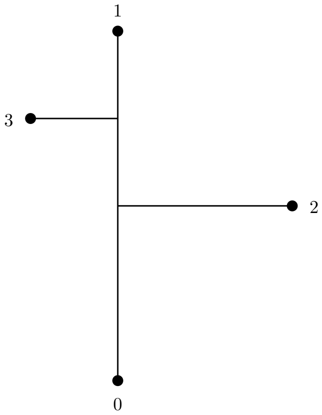

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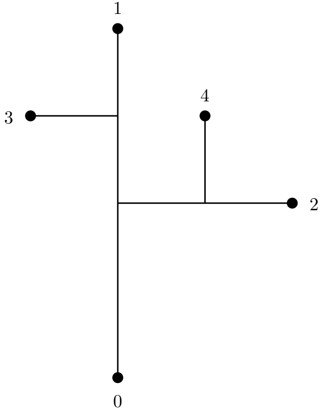

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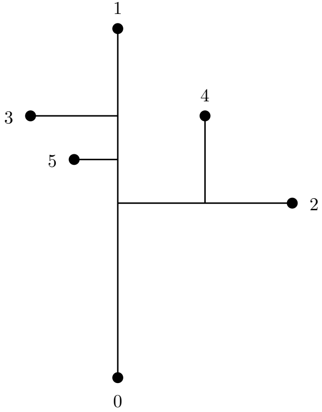

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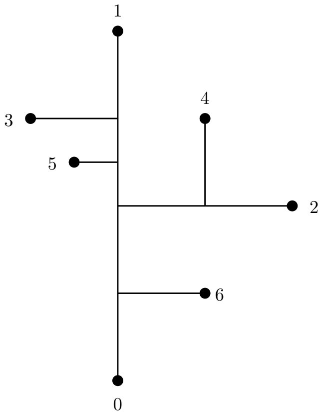

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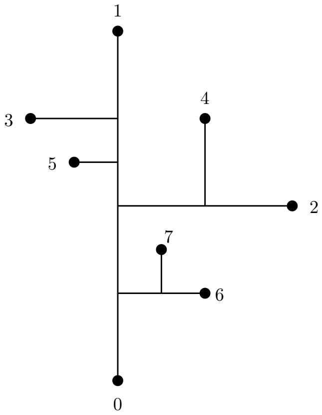
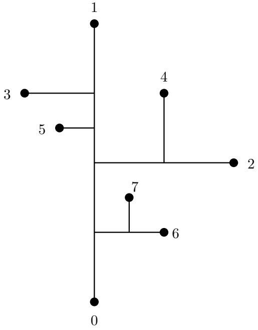
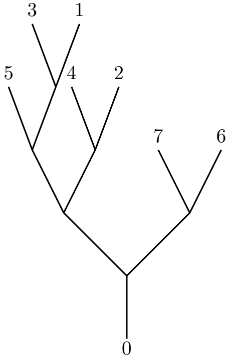


Rémy's algorithm

Claim: for each n, T_n is a uniform element of $\mathbb{T}_n^*$.





[J.-L. Rémy, **Un procédé itératif de dénombrement d'arbres binaires et son application à leur génération aléatoire**, *RAIRO. Informatique théorique* **19**:2 (1985), pp.179–195]

Taking limits

Vague question: what can we say about T_n as $n \to \infty$?

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Concrete first question: as $n \to \infty$, how does the distance between 0 and 1 behave?

An urn in Rémy's algorithm

The total number of edges present at step n is equal to $2n - 1$.

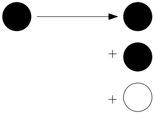
Consider the number of edges in the path between 0 and 1:

- If we add our new leaf somewhere along that path, it gets longer by 1.
- If we add our new leaf anywhere else, the length of the path remains the same.

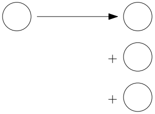
An urn in Rémy's algorithm

We have an urn process with two colours, say black and white, where each black ball represents an edge in the path between 0 and 1, and each white ball represents an edge elsewhere.

When we pick a black ball, we replace it in the urn together with one black and one white ball.



When we pick a white ball, we replace it in the urn together with two new white balls.



We start with a single black ball. At step n, we always have $2n - 1$ balls present.

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We have $B_1 = 1$.

For $n \geq 1$,

$$\mathbb{E}\left[B_{n+1} | \mathcal{F}_n\right] = \frac{B_n}{2n-1}(B_n + 1) + \frac{2n-1-B_n}{2n-1} B_n$$

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Define a sequence by $b_1 = 1$ and $b_{n+1} = \frac{2^{2n}(n!)^2}{(2n)!}$ for $n \geq 1$. Then

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Then we have that

$$\left(\frac{B_n}{b_n}\right)_{n \geq 1} \quad \text{is a non-negative martingale.}$$

Martingale limit

$(B_n/b_n)_{n\geq 1}$ is also bounded in L^2, hence uniformly integrable, and so it has an almost sure limit by the martingale convergence theorem.

Since

$$b_{n+1} = \frac{2^{2n}(n!)^2}{(2n)!} \sim \sqrt{\pi n},$$

we get that

$$\frac{B_n}{\sqrt{2n}} \to L \quad \text{a.s. as } n \to \infty$$

for some limit random variable L.

[P. Marchal, **A note on the fragmentation of the stable tree**, *Fifth Colloquium on Mathematics and Computer Science, DMTCS* (2008), pp.489–500]

Limiting distribution for the length

It also turns out (using a generating function argument) that the law of B_{n+1} is explicit:

$$\mathbb{P}\left(B_{n+1} = k\right) = \frac{k-1}{n} 2^{k-1} \frac{\binom{2n-k}{n-1}}{\binom{2n}{n}}$$

and so

$$\mathbb{P}\left(B_{n+1} = \lfloor x\sqrt{2n} \rfloor\right) \sim \frac{x}{\sqrt{2n}} e^{-x^2/2}, \quad x > 0.$$

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$$\mathbb{P}\left(B_{n+1} = \lfloor x\sqrt{2n} \rfloor\right) \sim \frac{x}{\sqrt{2n}} e^{-x^2/2}, \quad x > 0.$$

In other words, we get

$$\frac{B_n}{\sqrt{2n}} \to L \quad \text{a.s. as } n \to \infty,$$

where the limit L has the Rayleigh distribution, with density $xe^{-x^2/2}$ on $\mathbb{R}_+$.

[P. Flajolet, P. Dumas and V. Puyhaubert, **Some exactly solvable models of urn process theory**, *Fourth Colloquium on Mathematics and Computer Science: Algorithms, Trees, Combinatorics and Probabilities, DMTCS* (2006), pp.59–118]

Consequences

The distance between 0 and 1 varies as $\sqrt{2n}$, with a nice almost sure limit. What can we say about the distances between the other leaves as $n \to \infty$?

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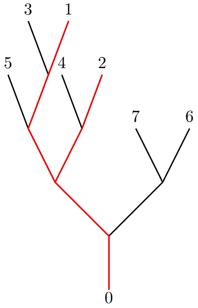
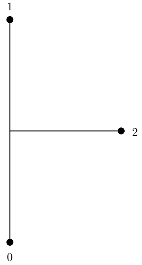
For example, let's think about the distance from 2 to the path between 0 and 1, and the position along that path at which it branches off.



More urns: self-similarity

At step 2 of Rémy's algorithm, we always have



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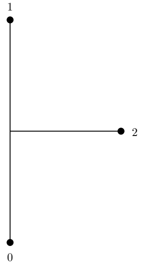

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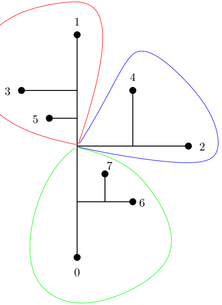
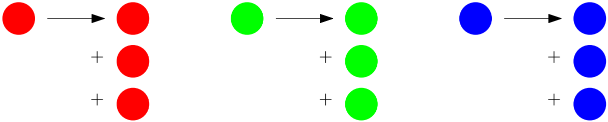
A useful consequence is that given the three sets of leaves, these three trees are themselves uniform binary plane trees.



More urns: self-similarity

The numbers of vertices in each of the three little trees evolve according to a variant of Pólya's urn with three colours, red, green and blue. We start with one ball of each colour. We pick a ball at random and replace it in the urn with **two** more of the same colour. Let R_n, G_n, B_n be the numbers of red, green and blue balls respectively at step n.



More urns: self-similarity

It is then standard that

$$\frac{1}{2n+3}(R_n, G_n, B_n) \to (\Delta_1, \Delta_2, \Delta_3) \quad \text{a.s. as } n \to \infty,$$

where $(\Delta_1, \Delta_2, \Delta_3) \sim \text{Dirichlet}(1/2, 1/2, 1/2)$.

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The Dirichlet distribution with parameters $\alpha_1, \alpha_2, \ldots, \alpha_k > 0$ has density

$$\frac{\Gamma(\sum_{i=1}^k \alpha_i)}{\prod_{i=1}^k \Gamma(\alpha_i)} x_1^{\alpha_1 - 1} \ldots x_k^{\alpha_k - 1}$$

with respect to Lebesgue measure on

$$\left\{ \mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}_+^k : \sum_{i=1}^k x_i = 1 \right\}.$$

More urns: self-similarity

The numbers of leaves in each of the three subtrees are given by

$$N_n^R = (R_n + 1)/2, \quad N_n^G = (G_n + 1)/2, \quad N_n^B = (B_n + 1)/2.$$

So we have

$$\frac{1}{n}(N_n^R, N_n^G, N_n^B) \to (\Delta_1, \Delta_2, \Delta_3) \quad \text{a.s.}$$

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Writing L_n^R, L_n^G, L_n^B for the lengths of the three paths at step n, we see that they look like small copies of the first urn model run for numbers of steps which are approximately $n\Delta_1$, $n\Delta_2$ and $n\Delta_3$.

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$$\frac{1}{\sqrt{2n}}(L_n^R, L_n^G, L_n^B) \to (\sqrt{\Delta_1}L_1, \sqrt{\Delta_2}L_2, \sqrt{\Delta_3}L_3) \quad \text{a.s.}$$

where L_1, L_2, L_3 are i.i.d. Rayleigh random variables, independent of $(\Delta_1, \Delta_2, \Delta_3)$.

Limiting subtree lengths

An elementary distributional calculation yields that

$$(\sqrt{\Delta_1}L_1, \sqrt{\Delta_2}L_2, \sqrt{\Delta_3}L_3) \stackrel{d}{=} \sqrt{\Gamma_2} \times \mathrm{Dir}(1,1,1),$$

where $\Gamma_2 \sim \mathrm{Gamma}(2, 1/2)$ and the two factors are independent.

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More generally, if we consider the subtree spanned by 0 and the leaves labelled $1, 2, \ldots, k$, we get $2k-1$ edges whose lengths are distributed as

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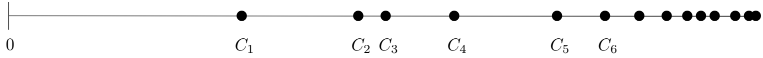
(Note that the $k = 1$ case fits into this pattern, since Rayleigh $\stackrel{d}{=} \sqrt{\Gamma_1}$.)

A limiting version of Rémy's algorithm: Aldous' line-breaking construction of the Brownian CRT

Take an inhomogeneous Poisson process on $\mathbb{R}_+$ of intensity t at t.



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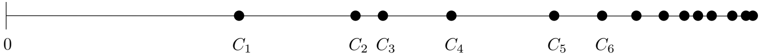

A useful way of constructing this is to let $E_1, E_2, \ldots$ be i.i.d. Exp$(1/2)$ and set $C_i = \sqrt{\sum_{j=1}^{i} E_j}$.

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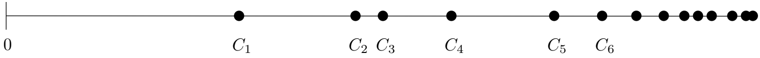

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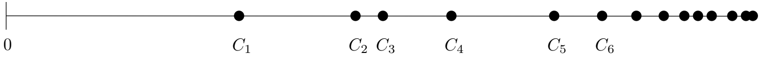

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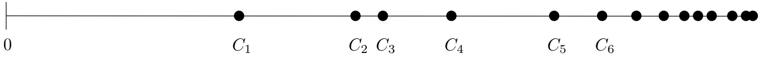

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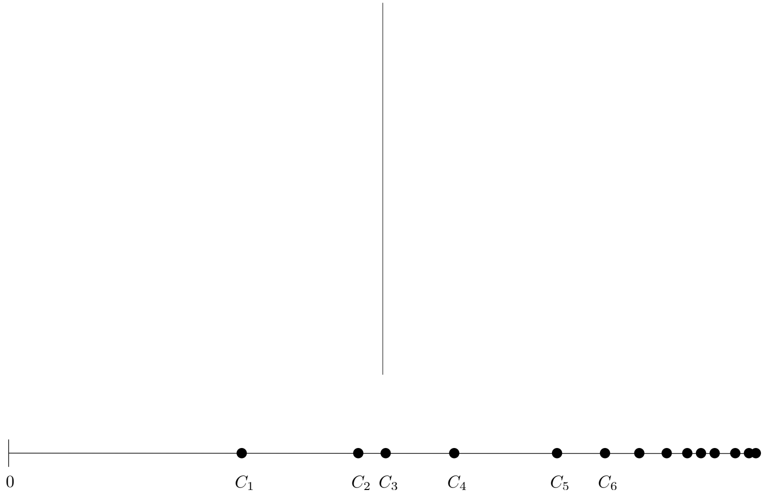
- Consider the line-segments $[0, C_1)$, $[C_1, C_2), \ldots$.
- Start from $[0, C_1)$ and proceed inductively.
- For $i \geq 2$, attach $[C_{i-1}, C_i)$ at a random point chosen uniformly over the existing tree.

Line-breaking construction



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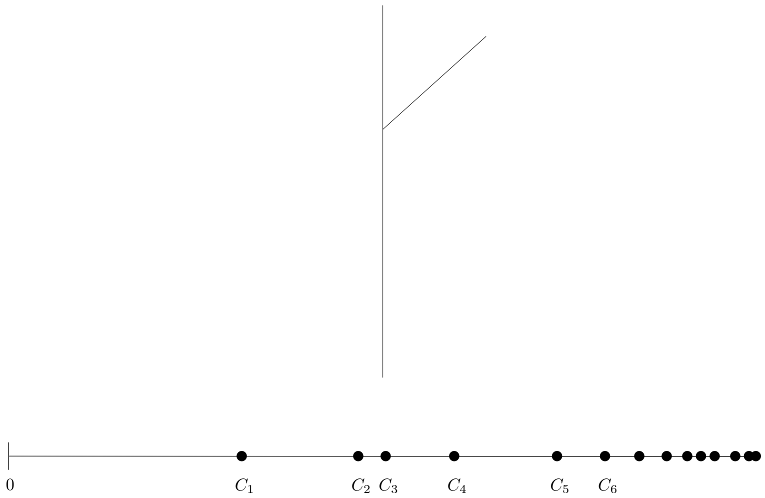

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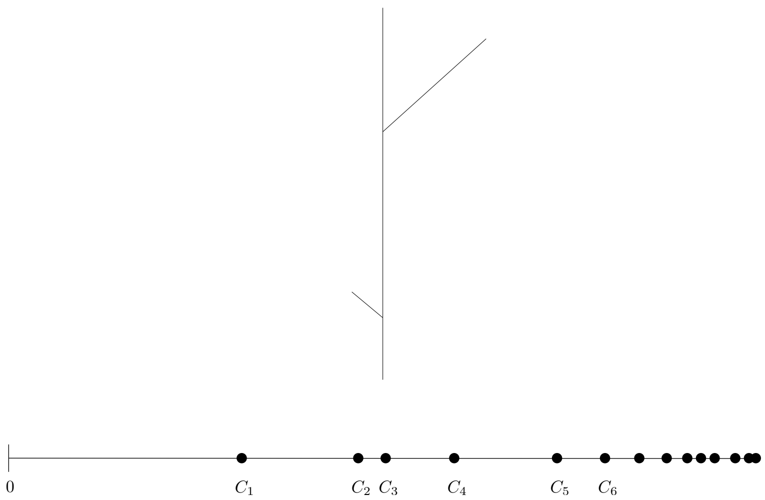

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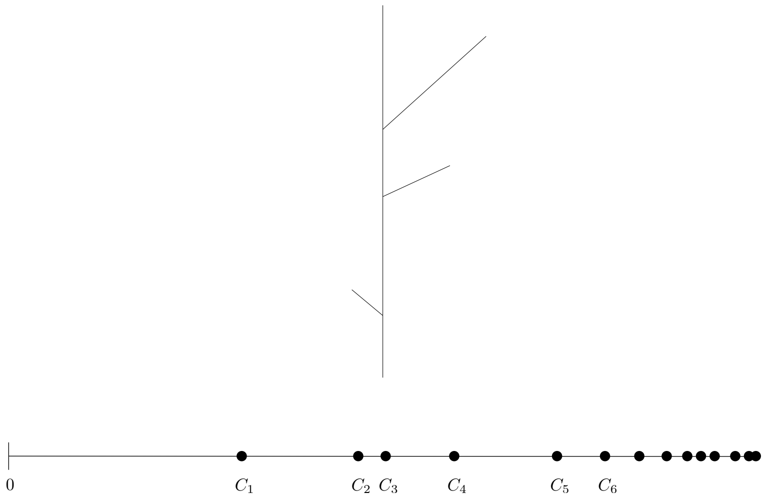

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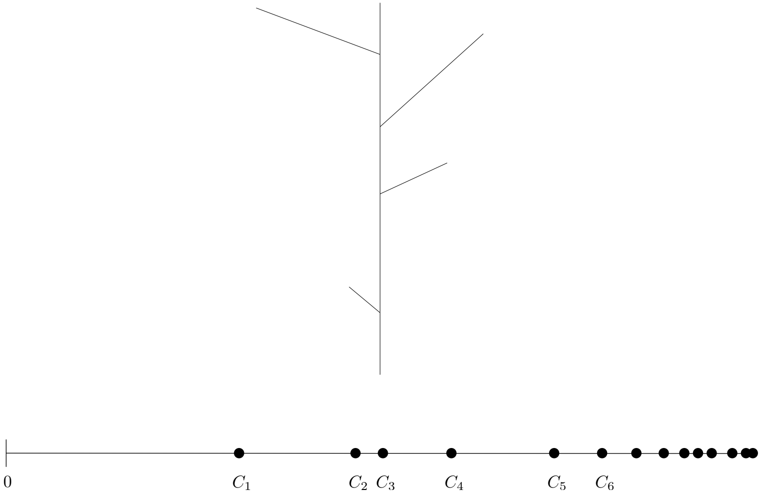

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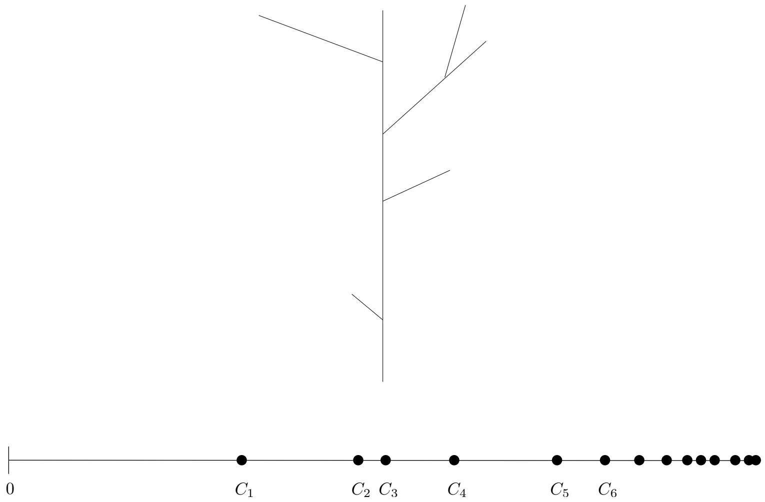

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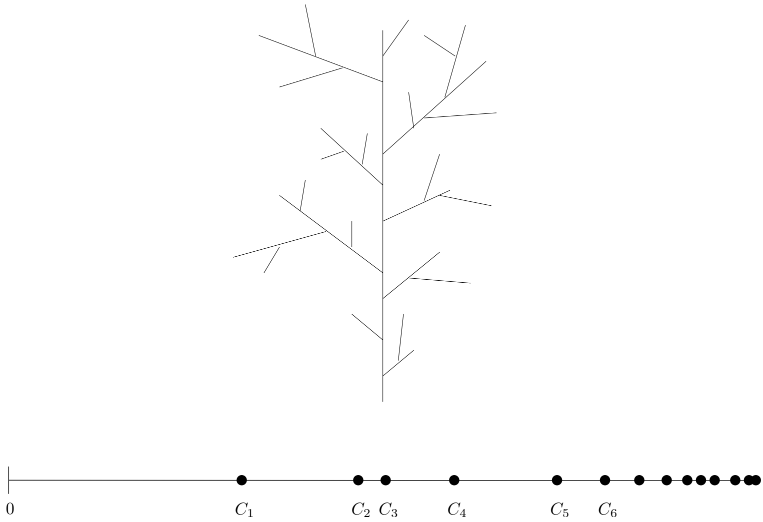

Why is this the right limit?

Claim: this gives the almost sure limit of the subtree spanned by 0 and the leaves 1, 2, . . . , k in the rescaled version of Rémy's algorithm.

- The tree at step $k \geq 1$ has total length

$$C_k = \sqrt{\sum_{i=1}^{k} E_i} \stackrel{d}{=} \sqrt{\Gamma_k},$$

  where $\Gamma_k \sim \text{Gamma}(k, 1/2)$.
- The combinatorics of the attachment mechanism are exactly the same as in Rémy's algorithm – so the underlying binary leaf-labelled tree has the right distribution.
- A calculation shows that the cut-points and attachment points split up the interval $[0, C_k)$ uniformly.

The line-breaking definition of the Brownian CRT

- Start from $[0, C_1)$ and proceed inductively.
- For $i \geq 1$, sample B_i uniformly from $[0, C_i)$ and attach $[C_i, C_{i+1})$ at the corresponding point of the tree constructed so far (this is a point chosen uniformly at random over the existing tree).

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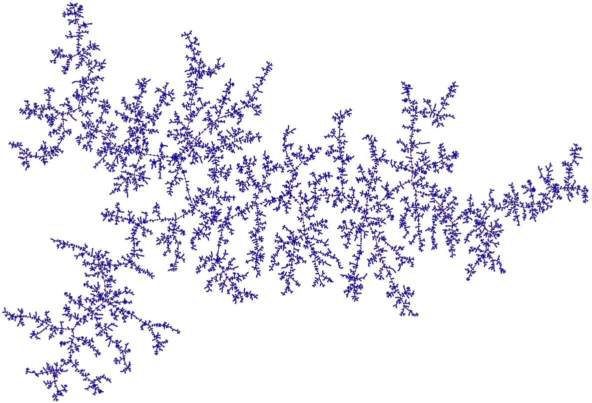
Now take the union of all the branches, thought of as a path metric space, and then take its completion.

This procedure gives (somewhat informally expressed) definition of the Brownian continuum random tree (CRT) which is a key object in this minicourse.

The line-breaking definition of the Brownian CRT



[Picture by Igor Kortchemski]

The scaling limit of the uniform binary plane tree

In the next section, we will make sense of the following statement.

Theorem. (Marchal (2003), Curien and Haas (2013))
As $n \to \infty$,

$$\frac{1}{\sqrt{2n}} T_n \to \mathcal{T} \quad \text{a.s.}$$

where $\mathcal{T}$ is the Brownian CRT.

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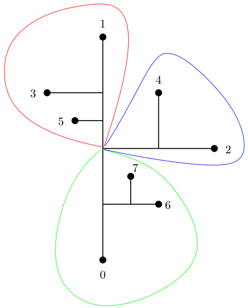
where $\mathcal{T}$ is the Brownian CRT.

We need to know what sort of objects we're really dealing with, and what is the topology in which the convergence occurs. (Also: why is $\mathcal{T}$ Brownian?!)

Before we do that, let's record an immediate consequence of the theorem.

Self-similarity of the Brownian CRT

Recall: we split our uniform binary plane tree into three little uniform binary plane trees of random sizes.

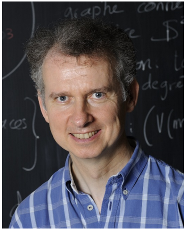


This property passes to the limit, and so the Brownian CRT can be split into three randomly rescaled Brownian CRTs. In particular, the Brownian CRT is a random fractal.

2. $\mathbb{R}$-TREES AND CONVERGENCE

Key reference:

Jean-François Le Gall, **Random trees and applications**, *Probability Surveys* **2** (2005) pp.245-311.



Continuous trees

We want a continuous notion of a tree. We don't really care about vertices: the important aspects are the shape of the tree and the distances. So it makes sense to think in terms of metric spaces.

$\mathbb{R}$-trees

Definition. A compact metric space $(\mathcal{T}, d)$ is an $\mathbb{R}$-tree if for all $x, y \in \mathcal{T}$,

- There exists a unique shortest path $[[x, y]]$ from x to y (of length $d(x, y)$).
- The only non-self-intersecting path from x to y is $[[x, y]]$.

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- The only non-self-intersecting path from x to y is $[[x, y]]$. (If g is a continuous injective map from $[0, 1]$ into $\mathcal{T}$, such that $g(0) = x$ and $g(1) = y$, then $g([0, 1]) = [[x, y]]$.)

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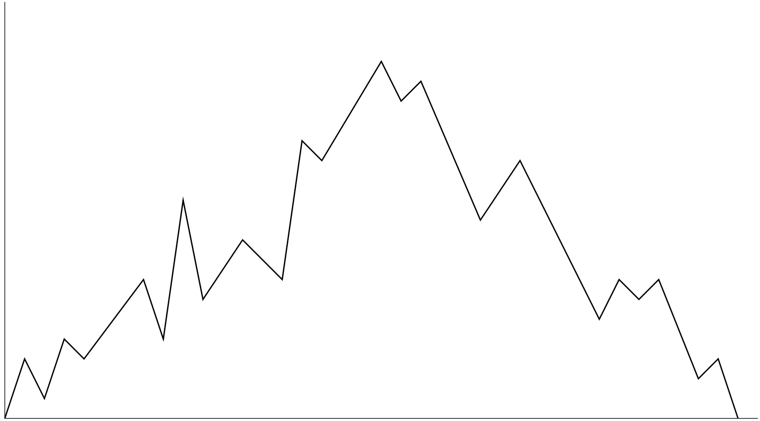
An element $v \in \mathcal{T}$ is called a vertex.
A rooted $\mathbb{R}$-tree has a distinguished vertex ρ called the root.
The height of a vertex v is its distance $d(\rho, v)$ from the root.
A leaf is a vertex v such that $v \notin [[\rho, w]]$ for any $w \neq v$.

Coding $\mathbb{R}$-trees

Let $h : [0,1] \to \mathbb{R}^+$ be an excursion, that is a continuous function such that $h(0) = h(1) = 0$ and $h(x) > 0$ for $x \in (0,1)$.



Coding $\mathbb{R}$-trees

Now put glue on the underside of the excursion and push the two sides together...

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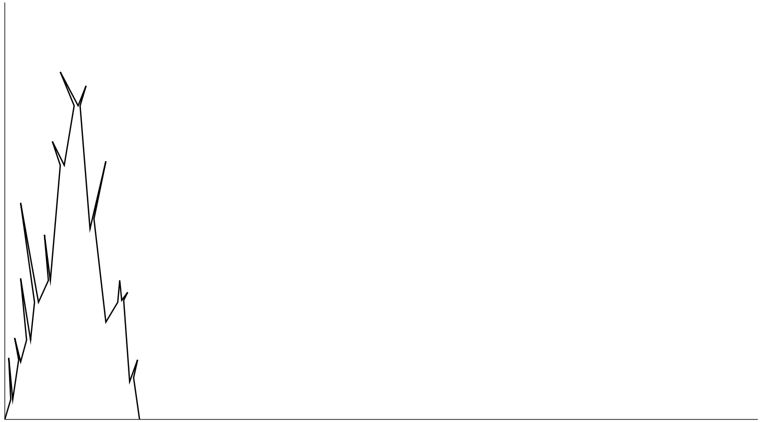

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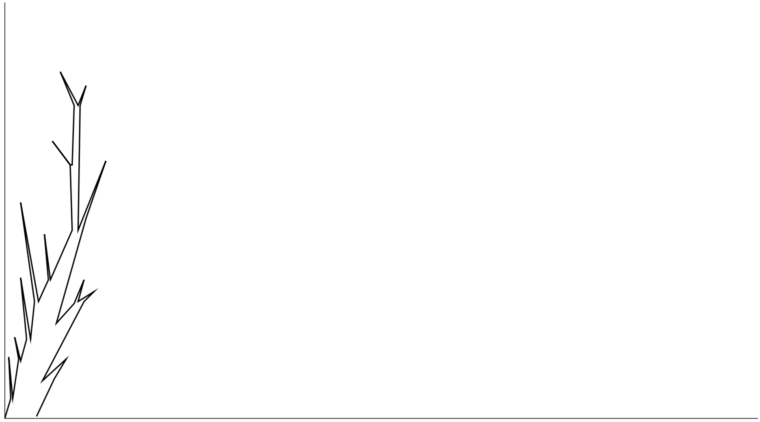

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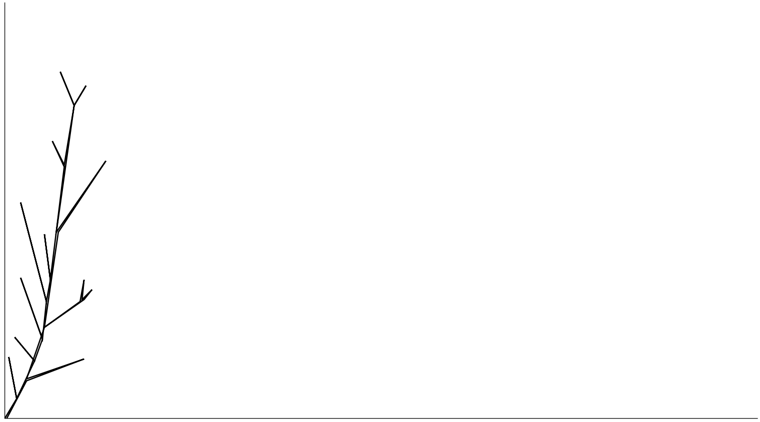
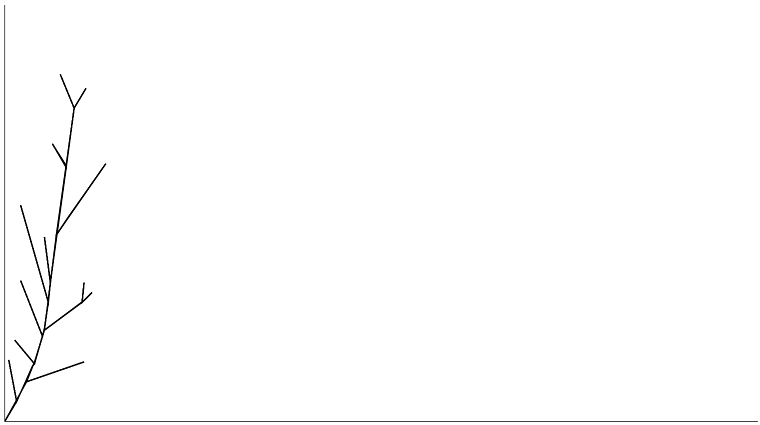


Coding ℝ-trees

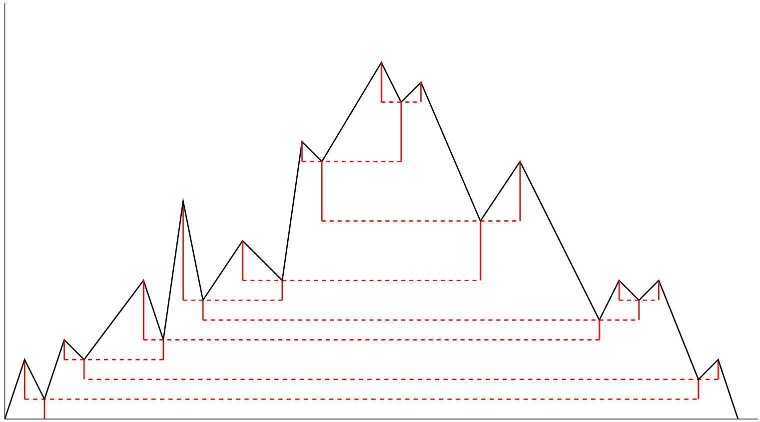
Now put glue on the underside of the excursion and push the two sides together to get a tree.



Coding $\mathbb{R}$-trees

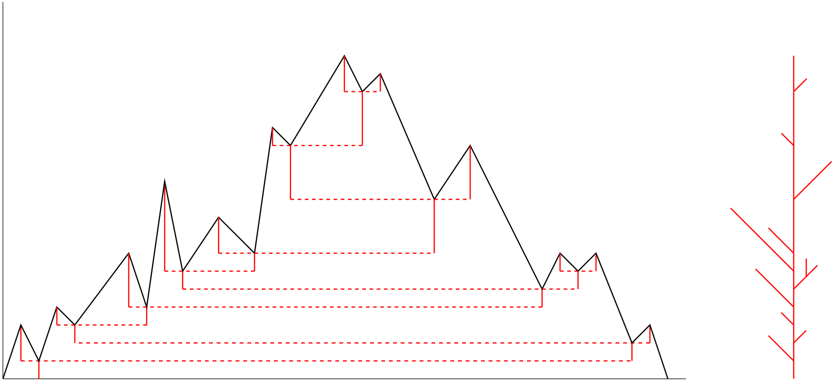
Formally, use h to define a distance:

$$d_h(x, y) = h(x) + h(y) - 2 \inf_{x \wedge y \leq z \leq x \vee y} h(z).$$



Coding $\mathbb{R}$-trees

Let $y \sim y'$ if $d_h(y, y') = 0$ and take the quotient $\mathcal{T}_h = [0,1]/\sim$.



Coding $\mathbb{R}$-trees

Theorem. For any excursion h, $(\mathcal{T}_h, d_h)$ is an $\mathbb{R}$-tree.

Write $\pi_h : [0,1] \to \mathcal{T}_h$ for the projection map.

We will often root $\mathcal{T}_h$ at $\rho = \pi_h(0) = \pi_h(1)$.

A natural measure

We will want to be able to sample random points in our trees. There is a natural "uniform" measure μ_h which is the push-forward of the Lebesgue measure on $[0, 1]$ onto $\mathcal{T}_h$.

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We will typically think of our continuous trees as triples $(\mathcal{T}_h, d_h, \mu_h)$.

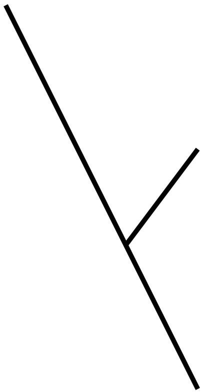
Topological considerations

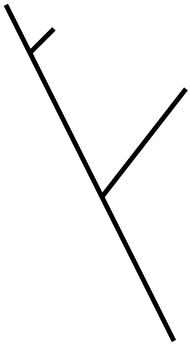
Let $\mathbb{M}$ be the space of compact metric spaces endowed with a Borel probability measure, up to measure-preserving isometry.

We will define a metric $\mathrm{d}_{\mathrm{GHP}}$, the Gromov-Hausdorff-Prokhorov distance on $\mathbb{M}$.

Topological considerations

Suppose that (X, d) and (X', d') are compact metric spaces.





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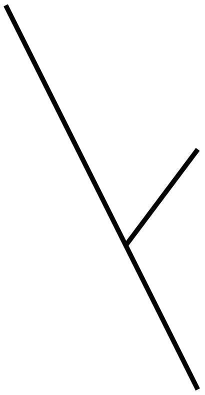
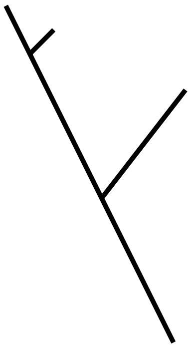
Suppose that (X, d) and (X', d') are compact metric spaces.

A correspondence R is a subset of $X \times X'$ such that for every $x \in X$, there exists $x' \in X'$ with $(x, x') \in R$ and vice versa.





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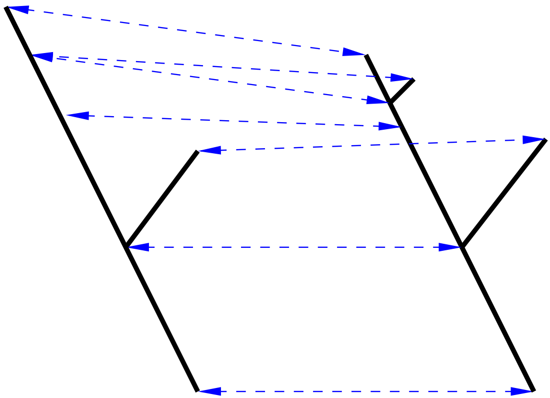
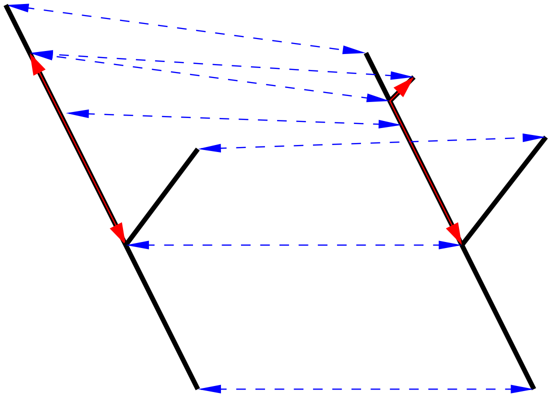


Topological considerations

The distortion of R is

$$\operatorname{dis}(R) = \sup\{|d(x,y) - d'(x',y')| : (x,x'),(y,y') \in R\}.$$



Topological considerations

Suppose that μ is a Borel probability measure on (X, d) and that μ' is a Borel probability measure on (X', d').

A measure ν on $X \times X'$ is a coupling of μ and μ' if $\nu(\cdot, X') = \mu(\cdot)$ and $\nu(X, \cdot) = \mu'(\cdot)$.

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Idea: find a correspondence and a coupling such that the correspondence has small distortion and the coupling "lines up" well with the correspondence i.e. if $(V, V') \sim \nu$ then $\mathbb{P}((V, V') \in R) = \nu(R)$ is close to 1.

Topological considerations

The Gromov-Hausdorff-Prokhorov distance between (X, d, μ) and (X', d', μ') is defined to be

$$\mathrm{d}_{\mathrm{GHP}}((X, d, \mu), (X', d', \mu')) = \frac{1}{2} \inf_{R,\nu} \max\{\mathrm{dis}(R), \nu(R^c)\}.$$

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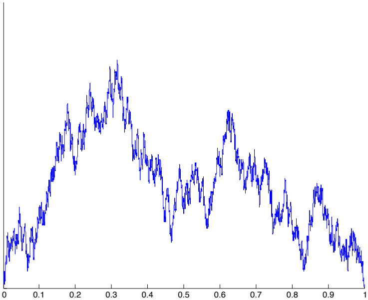
Theorem. $(\mathbb{M}, \mathrm{d_{GHP}})$ is a complete separable metric space.

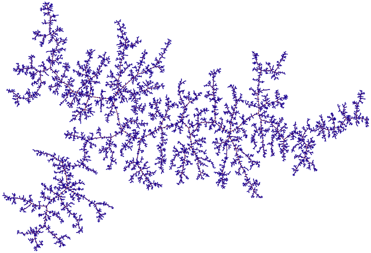
[S. Evans, J. Pitman and A. Winter, **Rayleigh processes, real trees, and root growth with re-grafting**, *Probability Theory and Related Fields* **134** (2006) pp.81-126.]

[R. Abraham, J.-F. Delmas and P. Hoscheit, **A note on the Gromov-Hausdorff-Prokhorov distance between (locally) compact metric measure spaces**, *Electronic Journal of Probability* **18** (2013), no. 14.]

The Brownian CRT

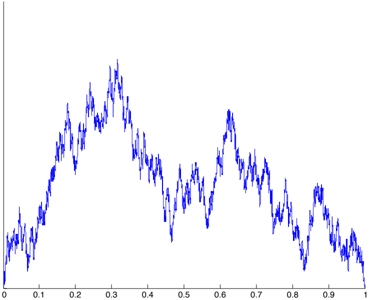
Definition. The Brownian continuum random tree is $(\mathcal{T}_{2e}, d_{2e}, \mu_{2e})$, where e is a standard Brownian excursion.

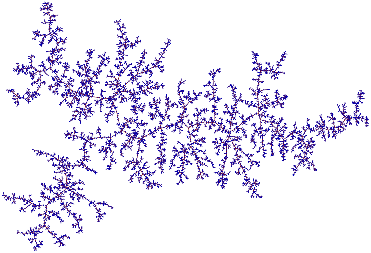




[Pictures by Igor Kortchemski]

A planar ordering



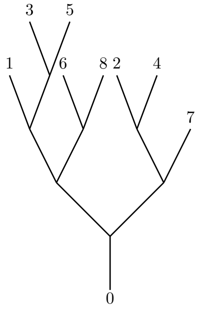


Observe that the excursion comes with more information than the the tree: if $s < t$ and $\pi_{2e}(s)$ and $\pi_{2e}(t)$ are leaves, it is natural to think of $\pi_{2e}(s)$ appearing to the left of $\pi_{2e}(t)$ (c.f. Rémy's algorithm).

Discrete trees as metric spaces

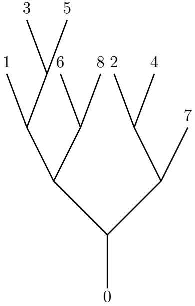
We want to think of $(T_n, n \geq 1)$ as metric spaces.

The vertices of T_n (labelled and unlabelled) come equipped with a natural metric: the graph distance d_n.



We sometimes write aT_n for the metric space (T_n, ad_n) given by the vertices of T_n with the graph distance scaled by a.

Uniform measure



We will endow T_n with μ_n, the measure which puts mass $1/(2n)$ on each of the $2n$ vertices.

Convergence

Theorem. As $n \to \infty$,

$$\left(T_n, \frac{d_n}{\sqrt{2n}}, \mu_n \right) \to (\mathcal{T}_{2e}, d_{2e}, \mu_{2e}) \quad \text{a.s.}$$

with respect to the Gromov-Hausdorff-Prokhorov topology.

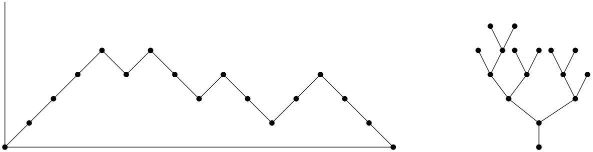
[P. Marchal, **Constructing a sequence of random walks strongly converging to Brownian motion**, *Discrete Mathematics and Theoretical Computer Science*, 2003, pp.181–190.]

[N. Curien & B. Haas, **The stable trees are nested**, *Probability Theory and Related Fields* **157**, 2013, pp.847–883.]

Binary trees and lattice excursions

There is a well-known bijection between planted binary plane trees with n leaves and lattice excursions with $2n$ steps.

Start every excursion with a $+1$ step. Now travel round the tree from left to right, recording a step whenever you see a vertex for the first time. The step is $+1$ if the vertex is not a leaf and -1 if the vertex is a leaf.



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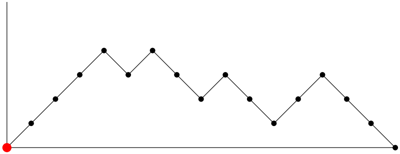

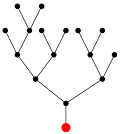

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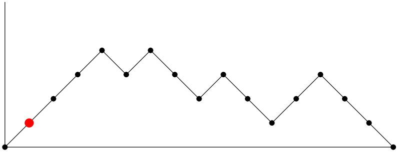

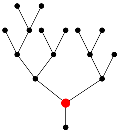

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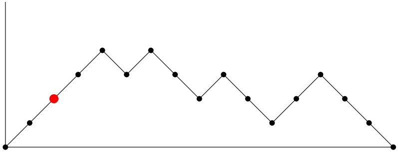

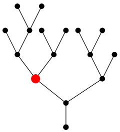

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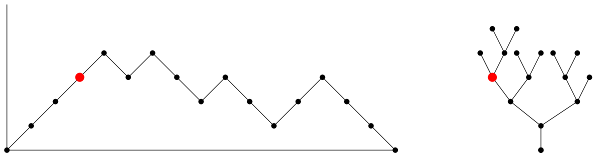

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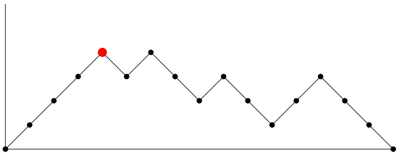

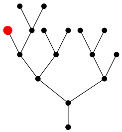

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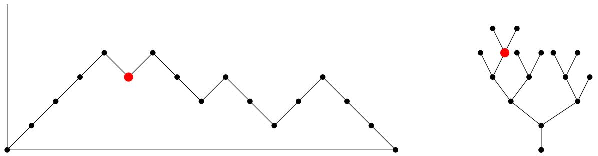

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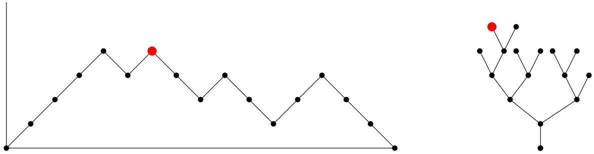

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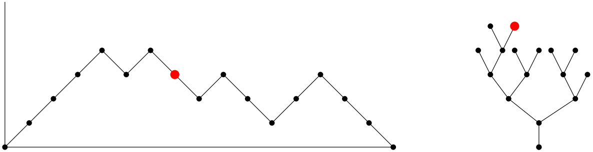

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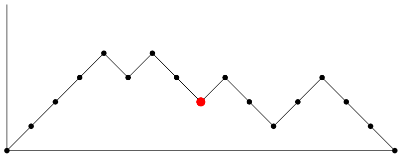

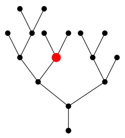

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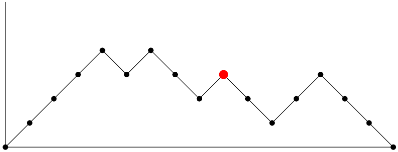

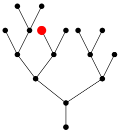

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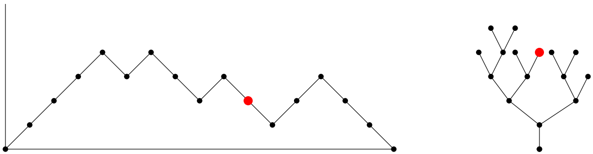

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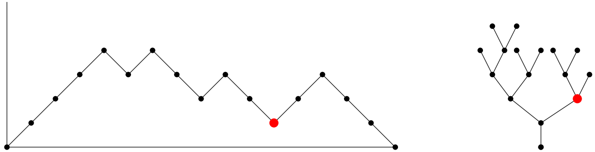

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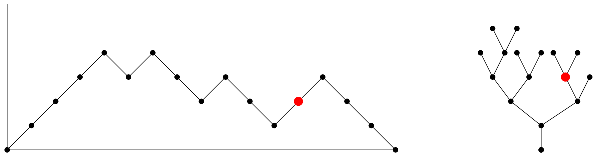

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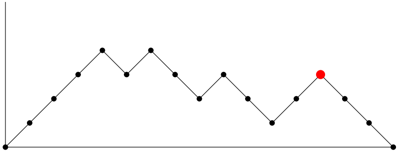

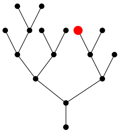

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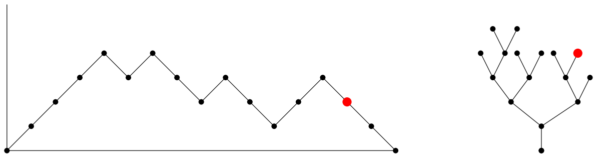

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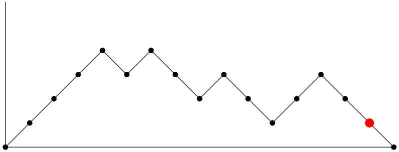

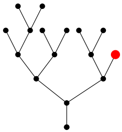

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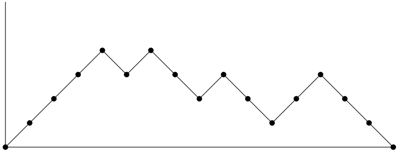

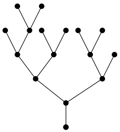

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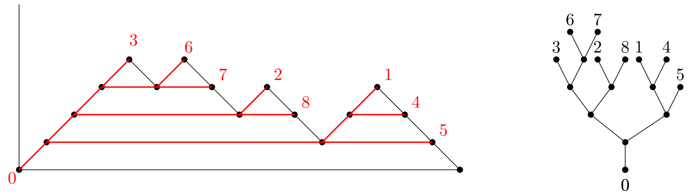

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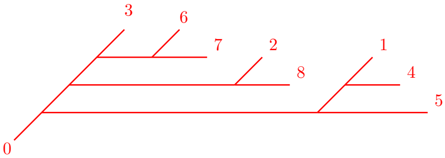

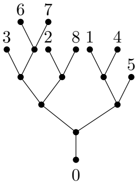

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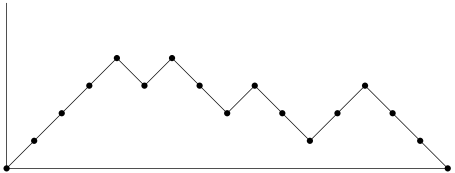
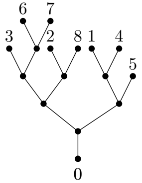
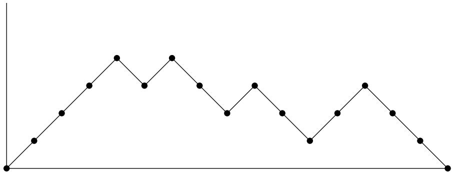
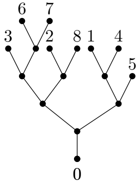




Since our trees are uniform, so are the lattice excursions. In other words, they are excursions of simple random walk away from 0.

Binary trees and lattice excursions





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Binary trees and lattice excursions

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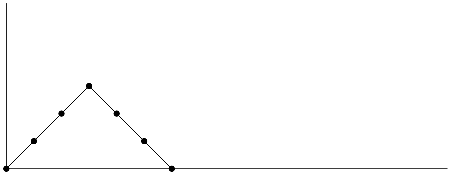

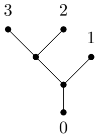

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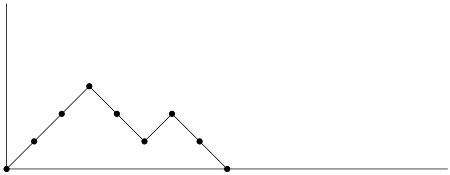

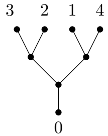

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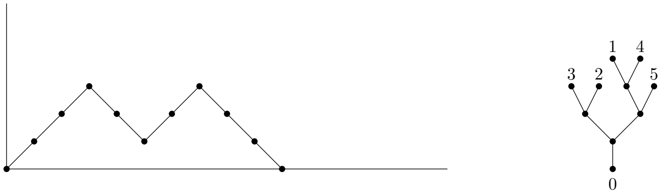

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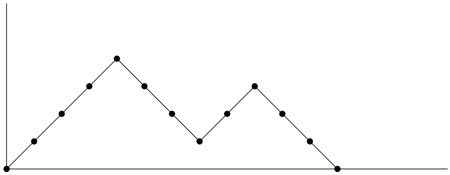

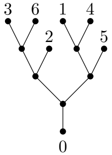

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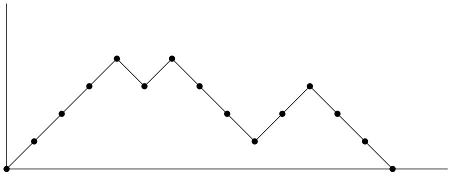

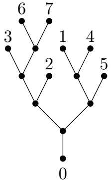

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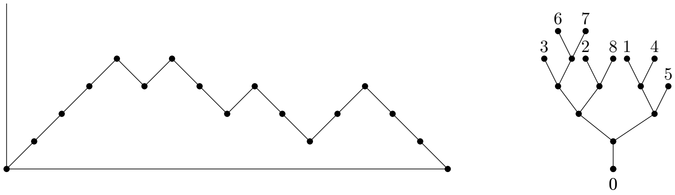


Binary trees and lattice excursions

Let $(E_n)_{n\geq 1}$ be the sequence of lattice excursions.

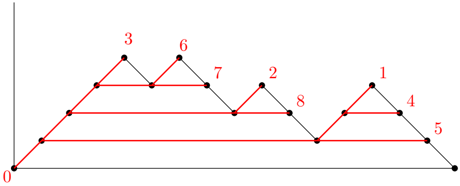
Theorem. (Marchal (2003))
As $n \to \infty$, we have

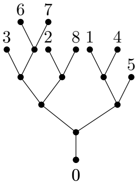
$$\frac{1}{\sqrt{2n}}(E_n(\lfloor 2nt \rfloor), 0 \leq t \leq 1) \to (e(t), 0 \leq t \leq 1)$$

uniformly on $[0, 1]$, almost surely.

Convergence of the trees

This is not quite enough to conclude that the trees converge in the GHP sense. The embedding of the tree in the excursion distorts distances.





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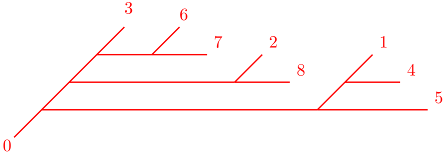

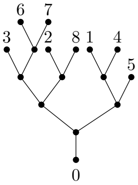

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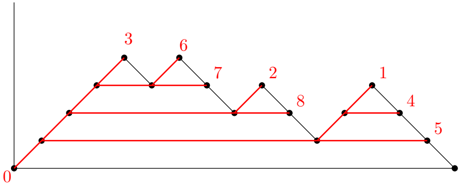

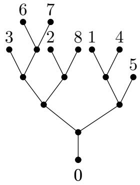

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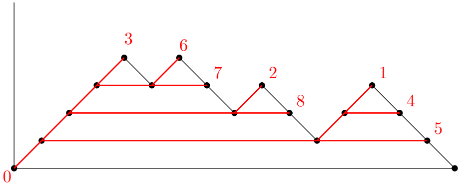

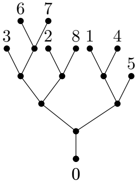

Write $H_n(k)$ for the distance from the root to the vertex visited at time k. Then

$$H_n(k) = \left| \left\{ 0 \leq i \leq k-1 : E_n(i) = \min_{i \leq j \leq k} E_n(k) \right\} \right|.$$

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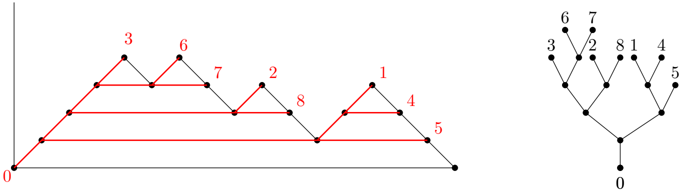

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It turns out that $H_n(k) \approx 2E_n(k)$.

Convergence of the trees

Theorem. As $n \to \infty$,

$$\frac{1}{\sqrt{2n}}(H_n(\lfloor 2nt \rfloor), 0 \le t \le 1) \to (2e(t), 0 \le t \le 1)$$

uniformly on $[0, 1]$, almost surely.

[J.-F. Marckert & A. Mokkadem, **The depth first processes of Galton-Watson trees converge to the same Brownian excursion**, *Annals of Probability*, **31**(3), pp.1655–1678, 2003.]

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Let's call the vertices be $v_0, v_1, \ldots, v_{2n-1}$ in the order we visit them, where v_0 is the root.

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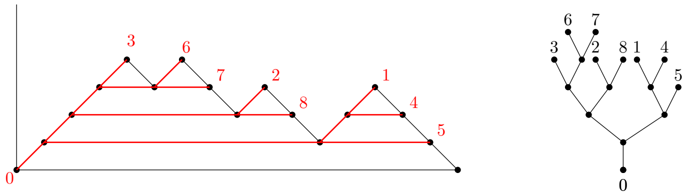
More generally, for $0 \leq i < j \leq 2n-1$, write $v_i \wedge v_j$ for the most recent common ancestor of v_i and v_j in the tree. Then

$$d_n(v_i, v_j) = d_n(v_0, v_i) + d_n(v_0, v_j) - 2d_n(v_0, v_i \wedge v_j).$$

Convergence of the trees

$$d_n(v_0, v_i \wedge v_j) = \begin{cases} \min_{i \le k \le j} H_n(k) - 1 & \text{if } v_i \text{ not an ancestor of } v_j \\ \min_{i \le k \le j} H_n(k) = H_n(i) & \text{if } v_i \text{ an ancestor of } v_j. \end{cases}$$



So

$$\left| d_n(v_0, v_i \wedge v_j) - \min_{i \le k \le j} H_n(k) \right| \le 1.$$

A correspondence

Define a correspondence R_n between $\{v_0, v_1, \ldots, v_{2n-1}\}$ and $[0,1]$ by declaring $(v_i, s) \in R_n$ if $i = \lfloor 2ns \rfloor$.

Endow $[0,1]$ with the pseudo-metric d_{2e}. We will bound $\mathrm{dis}(R_n)$.

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Suppose that $(v_i, s), (v_j, t) \in R_n$ with $s \leq t$. Then

$$\begin{aligned}
&|d_n(v_i, v_j) - d_{2e}(s,t)| \\
&\leq \left| \frac{1}{\sqrt{2n}} \left(H_n(\lfloor 2ns \rfloor) + H_n(\lfloor 2nt \rfloor) - 2 \min_{s \leq u \leq t} H_n(\lfloor 2nu \rfloor) \right) \right. \\
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The right-hand side converges to 0 uniformly in $s, t \in [0,1]$. So

$$\text{dis}(R_n) \to 0 \quad \text{a.s.}$$

A coupling

Recall that μ_n is the measure which puts mass $1/(2n)$ on each of the vertices $v_0, v_1, \ldots, v_{2n-1}$. Then we may couple μ_n and μ_{2e} by taking $U \sim \mathrm{U}[0,1]$ and taking ν to be the law of the pair

$$(v_{\lfloor 2nU \rfloor}, \pi_{2e}(U)).$$

This is precisely the natural coupling ν_n induced by the correspondence R_n, and so $\nu_n(R_n^c) = 0$.

GHP convergence

But then

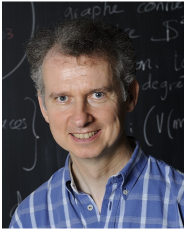
$$d_{\mathrm{GHP}}\left(\left(T_n, \frac{d_n}{\sqrt{2n}}, \mu_n\right), (\mathcal{T}_{2e}, d_{2e}, \mu_{2e})\right) \\ \leq \frac{1}{2}\max\{\mathrm{dis}(R_n), \nu_n(R_n^c)\} \to 0,$$

almost surely as $n \to \infty$. □

3. UNIVERSALITY

Key reference:

Jean-François Le Gall, **Random trees and applications**, *Probability Surveys* **2** (2005) pp.245-311.



Branching processes

A Bienaymé-Galton-Watson (BGW) branching process $(Z_n)_{n\geq 0}$ describes the size of a population which evolves as follows:

- Start with a single individual.
- This individual has a number of children distributed according to the offspring distribution p, where $p(k)$ gives the probability of k children, $k \geq 0$.
- Each child reproduces as an independent copy of the original individual.

Z_n gives the number of individuals in generation n (in particular, $Z_0 = 1$).

BGW trees

A BGW tree is the family tree arising from a BGW branching process. We will think of this as a rooted ordered tree.

BGW trees

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Consider the case where the offspring distribution p is critical i.e.

$$\sum_{k=1}^{\infty} kp(k) = 1.$$

This ensures, in particular, that the resulting tree, T, is finite.

Combinatorial trees in disguise

Let T be a BGW tree with offspring distribution p and total progeny N.

- If $p(0) = 1/2$ and $p(2) = 1/2$ then, conditional on $N = 2n - 1$, the tree is uniform on the set of rooted (unplanted!) plane binary trees with n leaves.
- If $p(k) = 2^{-k-1}$, $k \geq 0$ (i.e. Geometric(1/2) offspring distribution) then conditional on $N = n$, the tree is uniform on the set of plane trees with n vertices.
- If $p(k) = \frac{e^{-1}}{k!}$, $k \geq 0$ (i.e. Poisson(1) offspring distribution) then conditional on $N = n$, if we assign the vertices labels chosen uniformly at random from $\{1, 2, \ldots, n\}$ and then forget the ordering and the root, we obtain a labelled tree $\tilde{T}$ which is uniform on the set of possibilities.

The last example will be particularly important in Lecture 2.

A universal scaling limit

Let T_n be the family tree of a BGW process with critical offspring distribution of variance $\sigma^2 \in (0, \infty)$, conditioned to have total progeny n. Let d_n be the graph distance on T_n and let μ_n be the uniform measure on the vertices.

Theorem. (Aldous (1993), Le Gall (2005))
As $n \to \infty$,

$$\left(T_n, \frac{\sigma}{\sqrt{n}} d_n, \mu_n \right) \xrightarrow{d} (\mathcal{T}_{2e}, d_{2e}, \mu_{2e}),$$

where convergence is in the Gromov-Hausdorff-Prokhorov sense.

Two ways of encoding a tree

As we have seen, it is convenient to encode our trees in terms of discrete functions which are easier to manipulate.

We will do this is two different ways:

- the height function
- the depth-first walk (or Łukasiewicz path).

Height function

Suppose that our tree has n vertices. Let them be $v_0, v_1, \ldots, v_{n-1}$, listed in depth-first order.

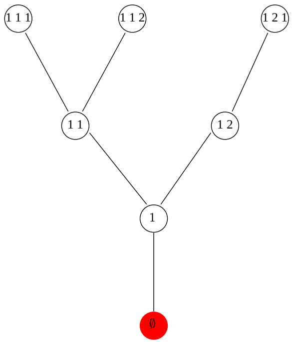
Height function

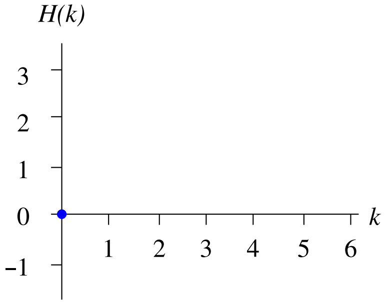
Suppose that our tree has n vertices. Let them be $v_0, v_1, \ldots, v_{n-1}$, listed in depth-first order.

Then the height function is defined by

$$H(k) = d_n(v_0, v_k), \quad 0 \leq k \leq n-1.$$

Height function





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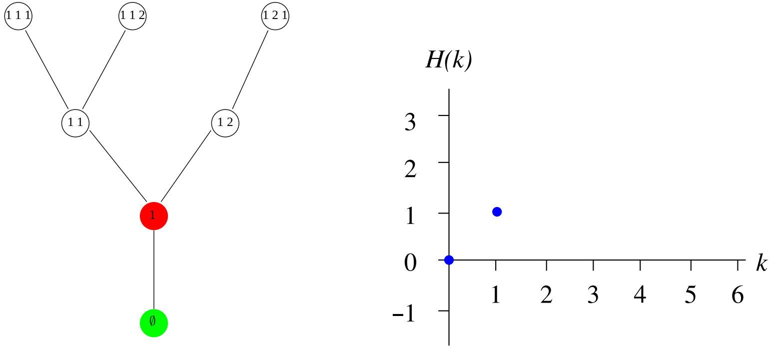

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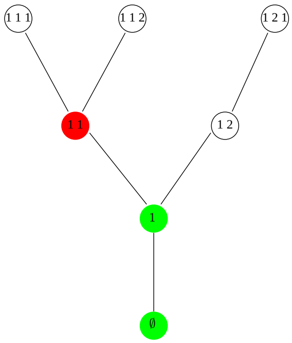
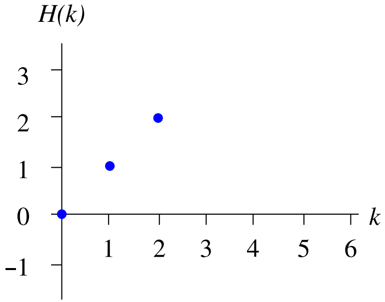
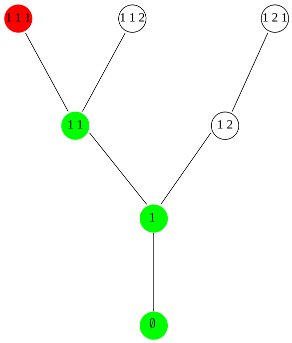
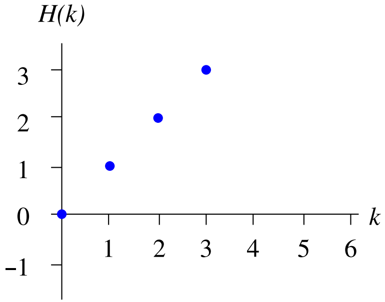




Height function





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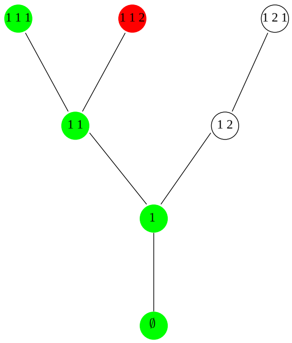
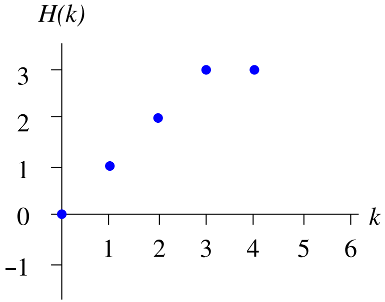
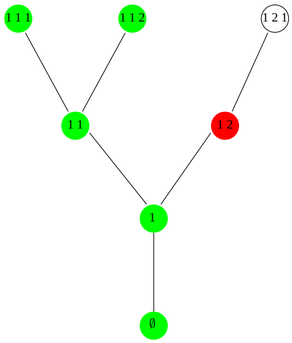
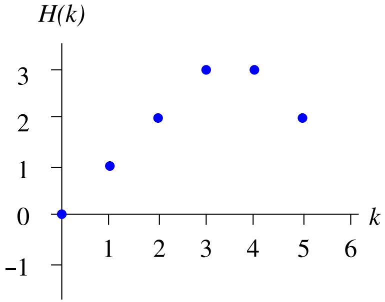




Height function





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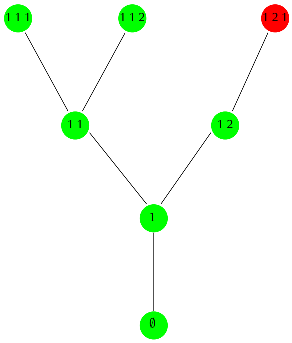

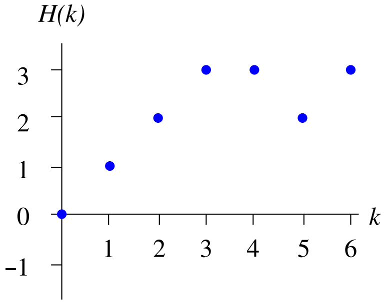

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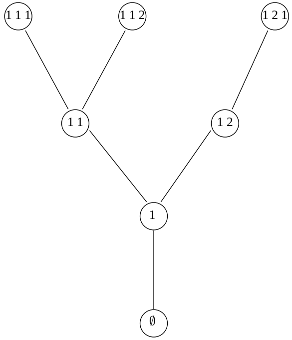

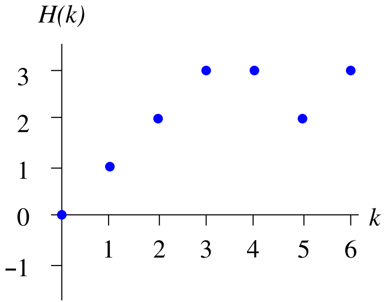

We can easily recover the tree from its height function.

Depth-first walk

Let $c(v)$ be the number of children of v, and that $v_0, v_1, \ldots, v_{n-1}$ is a list of the vertices in depth-first order.

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$$X(0) = 0,$$
$$X(i) = \sum_{j=0}^{i-1} (c(v_j) - 1), \text{ for } 1 \leq i \leq n.$$

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Define

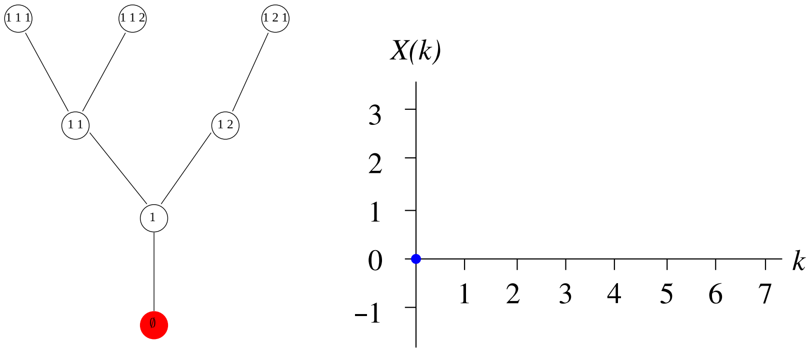
$$X(0) = 0,$$
$$X(i) = \sum_{j=0}^{i-1} (c(v_j) - 1), \text{ for } 1 \leq i \leq n.$$

In other words,

$$X(i+1) = X(i) + c(v_i) - 1, \quad 0 \leq i \leq n-1.$$

We can think of $X(i)$ as representing the number of vertices we have seen but not yet visited.

Depth-first walk



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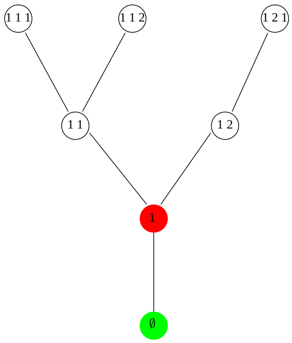

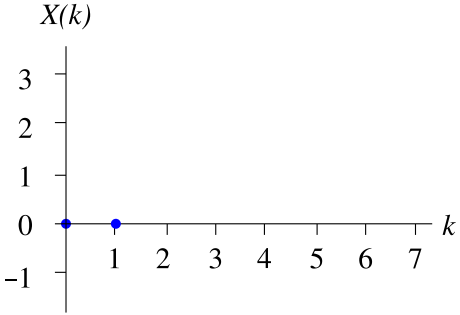

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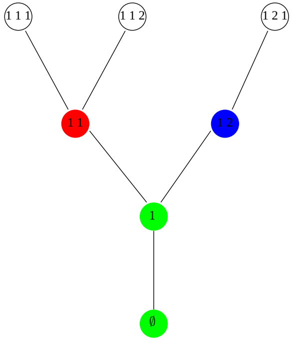

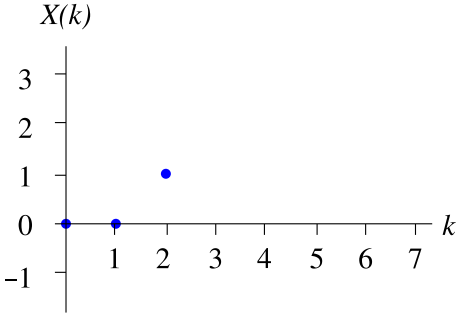

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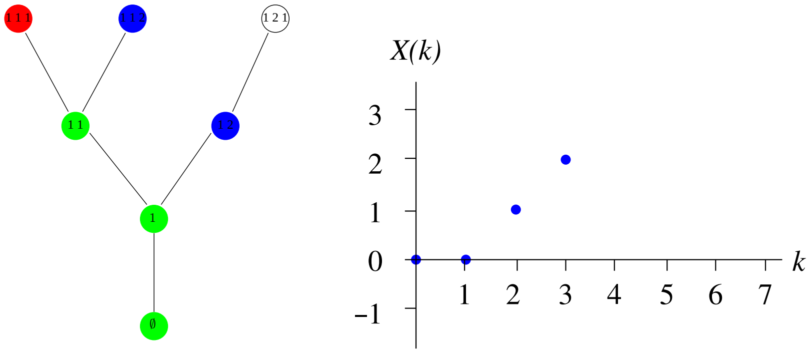

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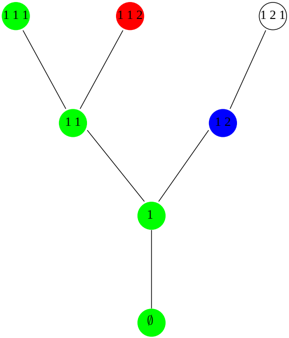

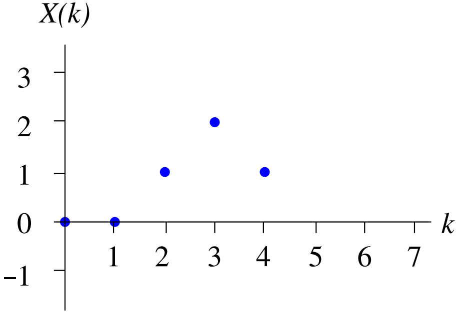

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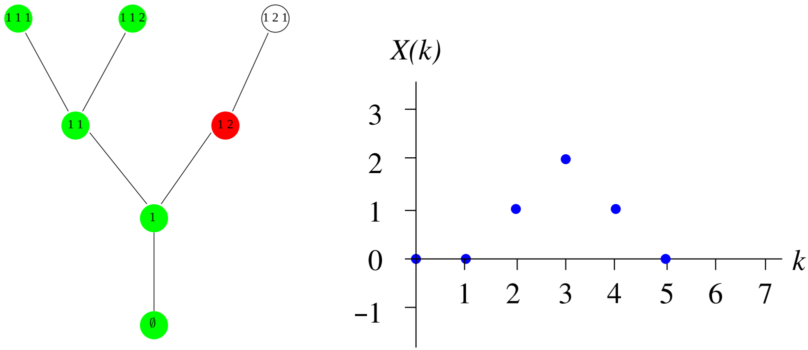

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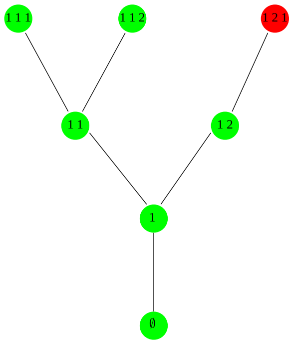

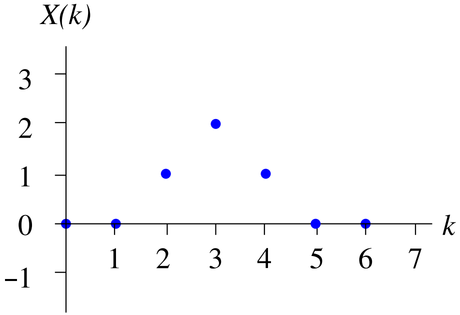

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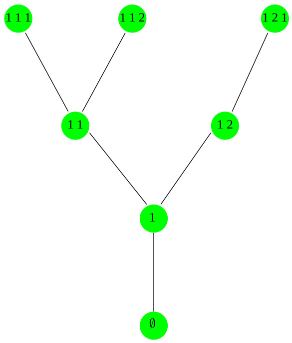

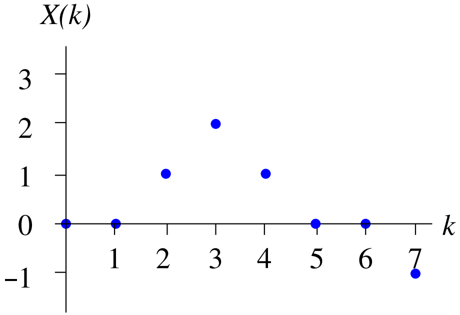

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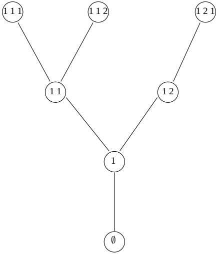

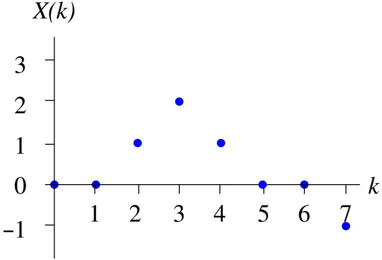

Proposition. For $0 \leq i \leq n-1$,

$$H(i) = \# \left\{ 0 \leq j \leq i-1 : X(j) = \min_{j \leq k \leq i} X(k) \right\}.$$

The depth-first walk of a BGW tree is a stopped random walk

Recall that p is a distribution on $\mathbb{Z}_+$ such that $\sum_{k=1}^{\infty} kp(k) = 1$.

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Proposition. Let $(R(k), k \geq 0)$ be a random walk with initial value 0 and step distribution $\nu(k) = p(k+1), k \geq -1$. Set

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Now suppose that T is a BGW tree with offspring distribution p and total progeny N. Then

$$(X(k), 0 \leq k \leq N) \stackrel{d}{=} (R(k), 0 \leq k \leq M).$$

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[Careful proof: see Le Gall (2005).]

BGW trees conditioned on their total progeny

Suppose now that we have offspring variance $\sigma^2 := \sum_{k=1}^{\infty}(k-1)^2 p(k) \in (0, \infty)$.

The depth-first walk X is a random walk with step mean 0 and variance σ^2, stopped at the first time it hits -1. The underlying random walk has a Brownian motion as its scaling limit, by Donsker's theorem.

The total progeny N is equal to $\inf\{k \geq 0 : X(k) = -1\}$. We want to condition on the event $\{N = n\}$.

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Standing assumption: $\mathbb{P}(N = n) > 0$ for all n sufficiently large.

BGW trees conditioned on their total progeny

Write $(X_n(k), 0 \leq k \leq n)$ for the depth-first walk conditioned on $\{N = n\}$. Then there is a conditional version of Donsker's theorem.

Theorem. As $n \to \infty$,

$$\frac{1}{\sigma\sqrt{n}}(X_n(\lfloor nt \rfloor), 0 \leq t \leq 1) \xrightarrow{d} (e(t), 0 \leq t \leq 1),$$

where $(e(t), 0 \leq t \leq 1)$ is a standard Brownian excursion.

[W.D. Kaigh, **An invariance principle for random walk conditioned by a late return to zero**, *Annals of Probability* **4**, 1976, pp.115-121.]

Height process

Let $(H_n(i), 0 \leq i \leq n)$ be the height process of a critical BGW tree with offspring variance $\sigma^2 \in (0, \infty)$, conditioned to have total progeny n, so that

$$H_n(i) = \# \left\{ 0 \leq j \leq i - 1 : X_n(j) = \min_{j \leq k \leq i} X_n(k) \right\}.$$

Theorem. As $n \to \infty$,

$$\frac{\sigma}{\sqrt{n}} \left(H_n(\lfloor nt \rfloor), 0 \leq t \leq 1 \right) \xrightarrow{d} 2 \left(e(t), 0 \leq t \leq 1 \right),$$

where $(e(t), 0 \leq t \leq 1)$ is a standard Brownian excursion.

Convergence to the Brownian CRT

The convergence

$$\left(T_n, \frac{\sigma}{\sqrt{n}} d_n, \mu_n\right) \xrightarrow{d} (\mathcal{T}_{2e}, d_{2e}, \mu_{2e}),$$

now follows by applying Skorokhod's theorem (in order to work on a probability space where the height process converges almost surely) and then using the same proof that we had in the case of binary trees.

Universality

The universality class of the Brownian CRT is, in fact, even larger. Some other examples of trees (and graphs!) with the Brownian CRT as their scaling limit are:

- uniform unordered unlabelled rooted trees
- uniform unordered unlabelled unrooted trees
- critical multi-type BGW trees
- random trees with a prescribed degree sequence satisfying certain conditions
- random dissections
- random graphs from subcritical classes.

4. THE CRITICAL ERDŐS–RÉNYI RANDOM GRAPH

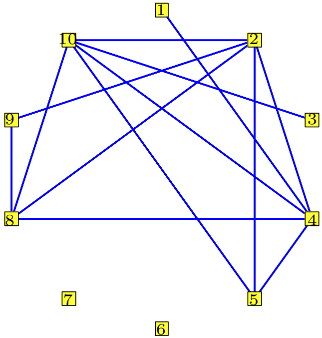
Key reference:

David Aldous, **Brownian excursions, critical random graphs and the multiplicative coalescent**, *Annals of Probability* **25**, 1997, pp.812–854.

The Erdős-Rényi random graph

Take n vertices labelled by $[n] := \{1, 2, \ldots, n\}$ and put an edge between any pair independently with probability p. Call the resulting model $G(n, p)$.

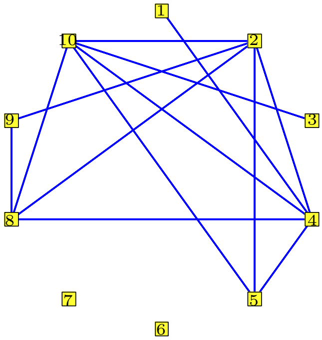
Example: $n = 10$, $p = 0.4$.



Connected components

We're going to be interested in the connected components of these graphs.

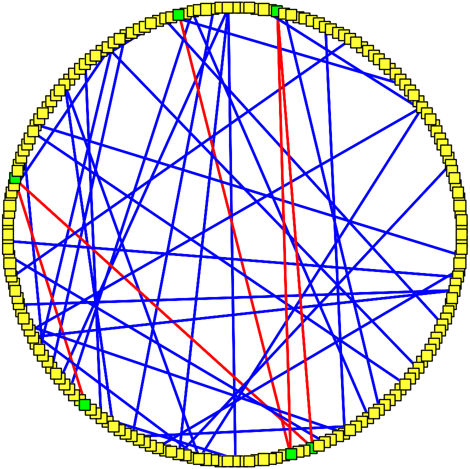
Below, there are three of them.



The phase transition

Let $p = c/n$ and consider the largest component (vertices in green, edges in red).

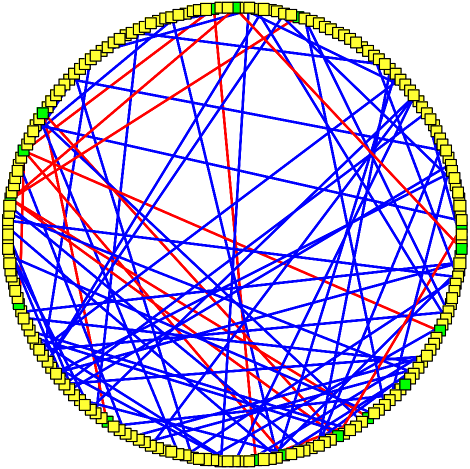
$n = 200$, $c = 0.4$



The phase transition

Let $p = c/n$ and consider the largest component (vertices in green, edges in red).

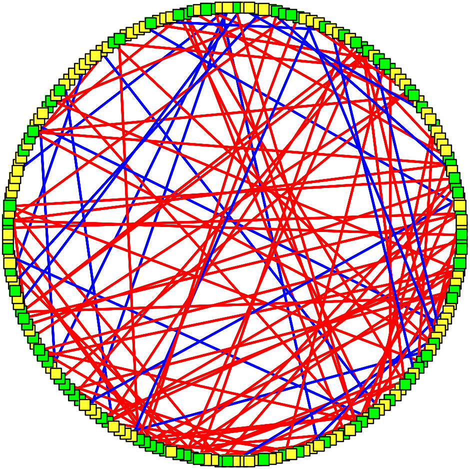
$n = 200$, $c = 0.8$



The phase transition

Let $p = c/n$ and consider the largest component (vertices in green, edges in red).

$n = 200$, $c = 1.2$



The phase transition (Erdős and Rényi (1960))

By the size of a component, we mean its number of vertices.

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Consider $p = c/n$.

- For $c < 1$, the largest connected component has size $O(\log n)$;
- for $c > 1$, the largest connected component has size $\Theta(n)$ (and the others are all $O(\log n)$).

(These statements hold with probability tending to 1 as $n \to \infty$.)

Heuristic picture of the phase transition

Vertex 1 has a Binomial$(n-1, c/n) \approx$ Poisson(c) number of neighbours, N.

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Consider now one of those neighbours. It has a Binomial$(n-N-1, c/n)$ number of neighbours we haven't seen before, which is still well-approximated by Poisson(c) as long as $N = o(n)$.

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Continuing in this way, we see that we can approximate the size of the component containing vertex 1 by the total progeny in a branching process with Poisson(c) offspring distribution (as long as the population doesn't get too large...).

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If $c \leq 1$, this branching process dies out with probability 1, which corresponds to getting only a small component containing vertex 1. A similar argument works for subsequent components.

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If, on the other hand, $c > 1$, there is a positive probability that the branching process will survive. The branching process approximation holds good until we explore the first component which does not die out; this component ends up being the giant.

The critical point of the phase transition

Recall: $p = c/n$.

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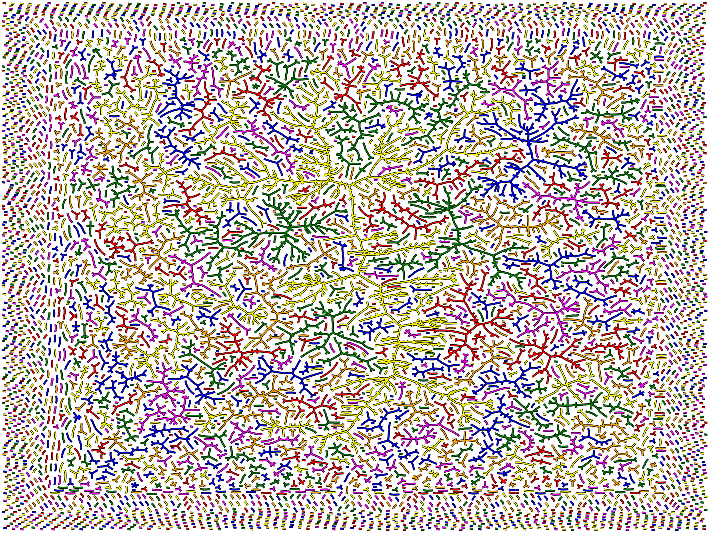
If $c = 1$, the largest component has size $\Theta(n^{2/3})$ and, indeed, there is a whole sequence of components of this order.

The critical random graph



[Picture by Nicolas Broutin]

The critical random graph

The critical window: $p = \frac{1}{n} + \frac{\lambda}{n^{4/3}}$, where $\lambda \in \mathbb{R}$. For such p, the largest components have size $\Theta(n^{2/3})$.

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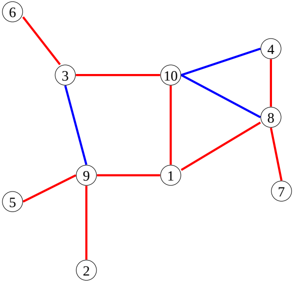
We will also be interested in the surplus of a component, the number of edges more than a tree that it has.

A component with surplus 3:



Convergence of the sizes and surpluses

Fix λ and let $C_1^n, C_2^n, \ldots$ be the sequence of component sizes of $G\left(n, \frac{1}{n} + \frac{\lambda}{n^{4/3}}\right)$ in decreasing order, and let $S_1^n, S_2^n, \ldots$ be the corresponding surpluses.

Write $\mathbf{C}^n = (C_1^n, C_2^n, \ldots)$ and $\mathbf{S}^n = (S_1^n, S_2^n, \ldots)$.

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Convergence for the first co-ordinate takes place in

$$\ell^2_{\searrow} := \left\{ \mathbf{x} = (x_1, x_2, \ldots) : x_1 \geq x_2 \geq \ldots \geq 0, \sum_{i=1}^{\infty} x_i^2 < \infty \right\}$$

with the usual ℓ^2-distance $\|\mathbf{x} - \mathbf{y}\|_2 = \sqrt{\sum_{i=1}^{\infty}(x_i - y_i)^2}$. For the second co-ordinate, convergence is in the distance

$$d(\mathbf{u}, \mathbf{v}) = 2^{-\inf\{i \geq 1 : u_i \neq v_i\}}$$

between integer sequences $\mathbf{u}$ and $\mathbf{v}$.

Limiting sizes and surpluses

Let $W_\lambda(t) = W(t) + \lambda t - \frac{t^2}{2}$, $t \geq 0$, where $(W(t), t \geq 0)$ is a standard Brownian motion.

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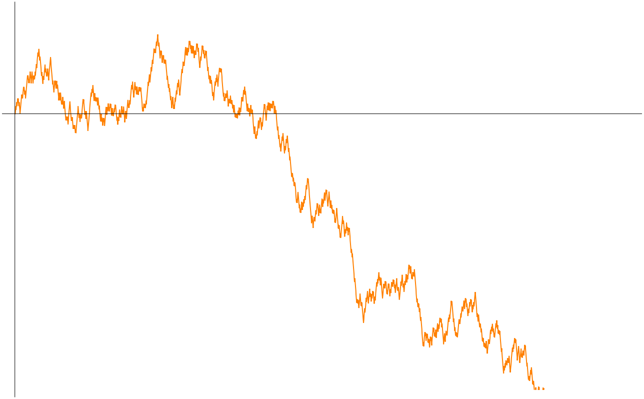

[Pictures by Louigi Addario-Berry]

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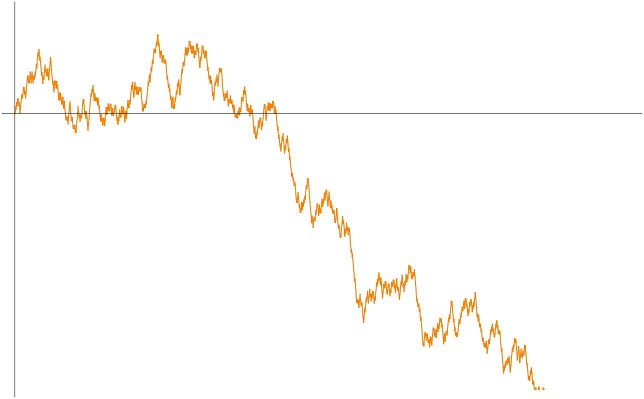

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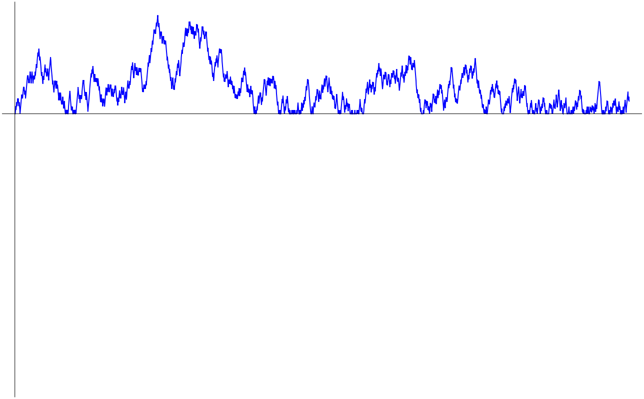

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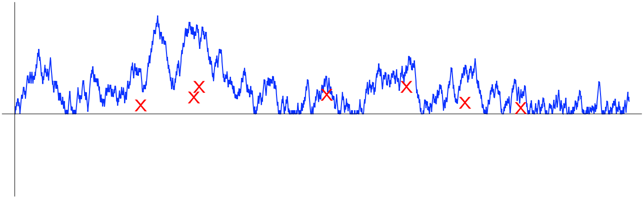
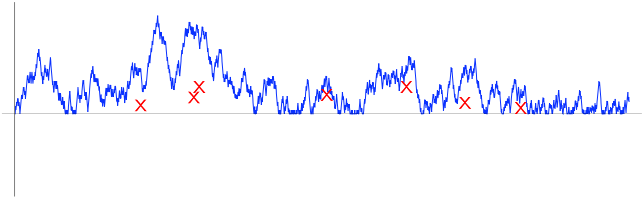


Decorate the picture with the points of a rate one Poisson process in the plane which fall above the x-axis and below the graph.



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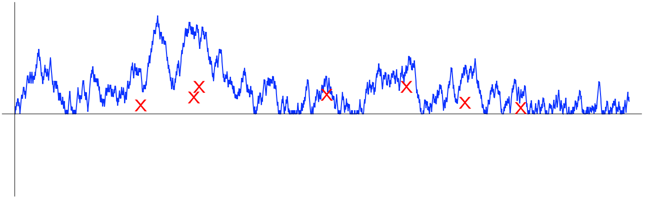
The path of B_λ can be split up into excursions above 0.



Decorate the picture with the points of a rate one Poisson process in the plane which fall above the x-axis and below the graph.

The path of B_λ can be split up into excursions above 0.

C is the sequence of lengths of these excursions, in decreasing order.

S is the sequence of numbers of points falling under those excursions.

Proof technique: depth-first exploration

As for our random trees, a key tool is a depth-first exploration.

For a rooted ordered tree, we defined the depth-first walk by $X(0) = 0$ and, for $1 \leq k \leq m$,

$$X(k) = \sum_{i=0}^{k-1} (c(v_i) - 1),$$

where $c(v)$ is the number of children of vertex v and $v_0, v_1, \ldots, v_{m-1}$ are the vertices in depth-first order.

We need to adapt this idea to the setting of graphs with multiple components, which are not a priori ordered or rooted.

Depth-first exploration

We root each component at its lowest-labelled vertex, and also use the vertex labels to provide a canonical ordering of the neighbours of a vertex.

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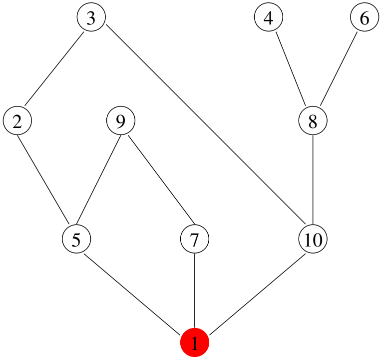
So we start by exploring from the vertex 1, and there's no need to stop when we hit the end of the first component: we can just keep going by starting again from the next lowest-labelled vertex that we have not yet explored.

$X(k)$ will then be the number of vertices seen but not visited at step k minus the number of components already fully explored.

Depth-first exploration: an example

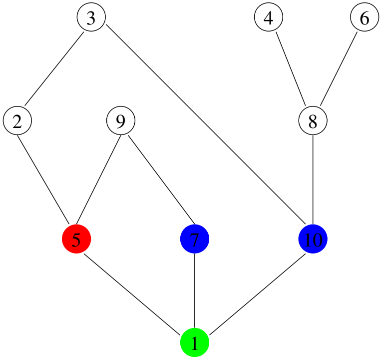
Step 0



Current: 1   Seen: none   Visited: none $X(0) = 0$.

Depth-first exploration: an example

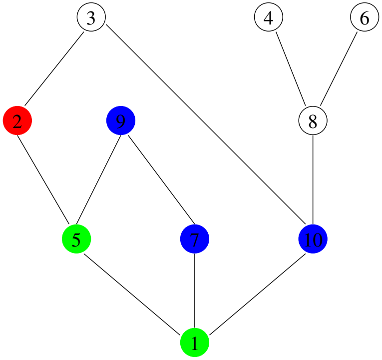
Step 1



Current: 5 Seen: 7, 10 Visited: 1 $X(1) = 2$.

Depth-first exploration: an example

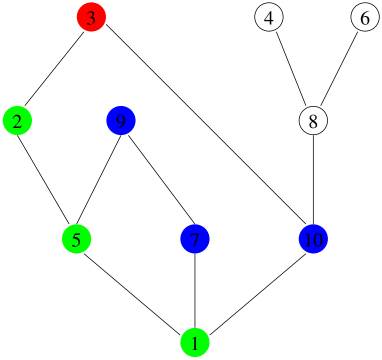
Step 2



Current: 2 Seen: 9, 7, 10 Visited: 1, 5 $X(2) = 3$.

Depth-first exploration: an example

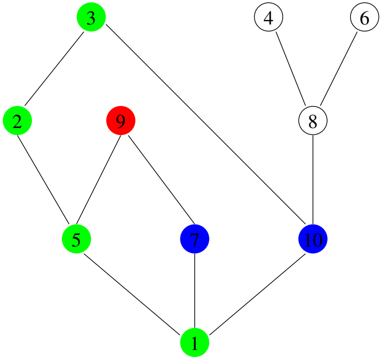
Step 3



Current: 3   Seen: 9, 7, 10   Visited: 1, 5, 2 $X(3) = 3$.

Depth-first exploration: an example

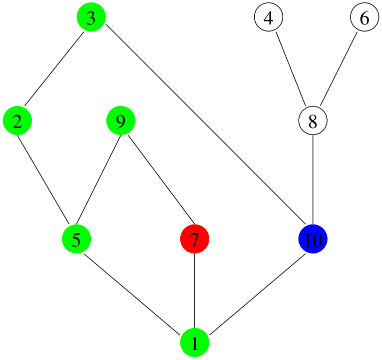
Step 4



Current: 9 Seen: 7, 10 Visited: 1, 5, 2, 3 $X(4) = 2$.

Depth-first exploration: an example

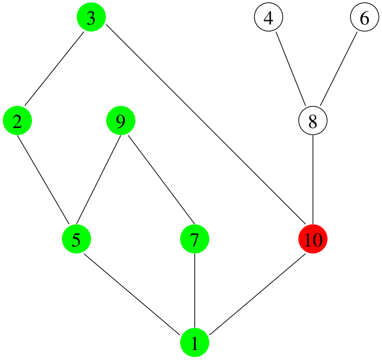
Step 5



Current: 7 Seen: 10 Visited: 1, 5, 2, 3, 9 $X(5) = 1$.

Depth-first exploration: an example

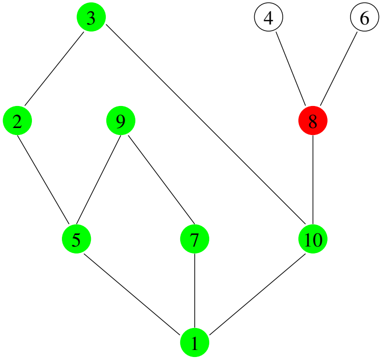
Step 6



Current: 10   Seen: none   Visited: 1, 5, 2, 3, 9, 7 $X(6) = 0$.

Depth-first exploration: an example

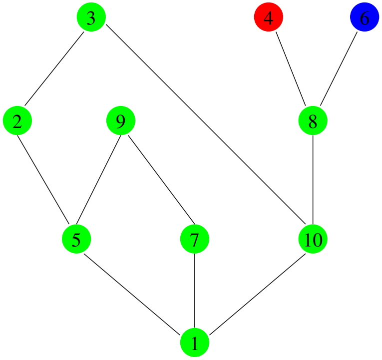
Step 7



Current: 8   Seen: none   Visited: 1, 5, 2, 3, 9, 7, 10 $X(7) = 0$.

Depth-first exploration: an example

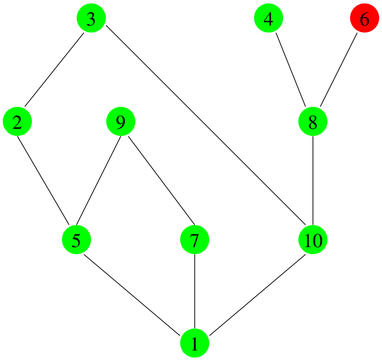
Step 8



Current: 4   Seen: 6   Visited: 1, 5, 2, 3, 9, 7, 10, 8 $X(8) = 1$.

Depth-first exploration: an example

Step 9

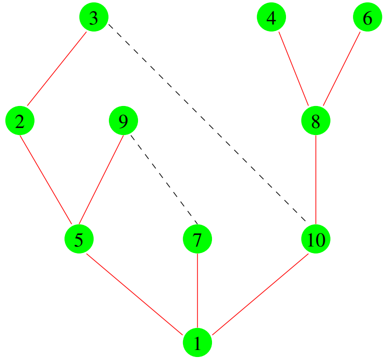


Current: 6   Seen: none   Visited: 1, 5, 2, 3, 9, 7, 10, 8, 4

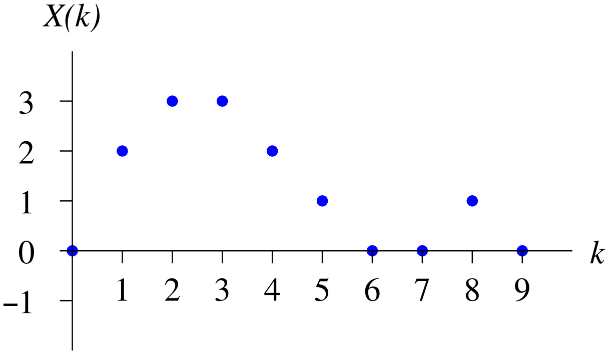
$X(9) = 0.$

Depth-first exploration: an example

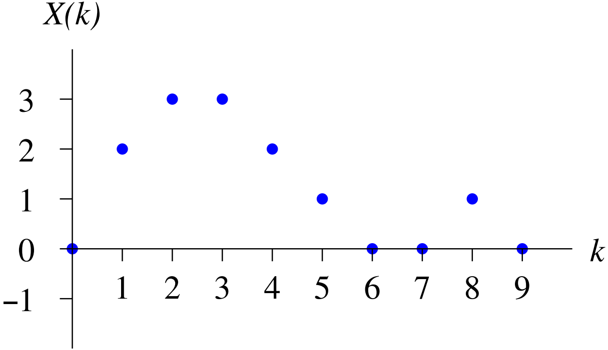
We explored the graph as if the dashed edges weren't there:



Depth-first walk



Depth-first walk



If there are several components, $T(k) = \inf\{i \geq 0 : X(i) = -k\}$ marks the beginning of the $(k+1)$th component. So the component sizes are $\{T(k+1) - T(k), k \geq 0\}$. This sequence can clearly be reconstructed from the path of $(X(i), i \geq 0)$.

Convergence of the depth-first walk

Let X_λ^n be the depth-first walk associated with $G\left(n, \frac{1}{n} + \frac{\lambda}{n^{4/3}}\right)$.

Theorem. (Aldous (1997)) As $n \to \infty$,

$$(n^{-1/3} X_\lambda^n(\lfloor n^{2/3} t \rfloor), t \geq 0) \xrightarrow{d} (W_\lambda(t), t \geq 0),$$

uniformly on compact time-intervals.

Sketch of proof

X^n is a (time-inhomogeneous) Markov process. We need to understand its step distribution.

At time i,

- v_i is the current vertex;
- i vertices are dead;
- $X^n(i)$ vertices are alive;
- we want to know $c(v_i)$, the number of children of v_i.

We have not yet looked at the possible edges from v_i to any of the other $n - i - X^n(i)$ unexplored vertices in the graph. Each of these is present with probability $\frac{1}{n} + \frac{\lambda}{n^{4/3}}$ independently. So, given $X^n(i)$,

$$c(v_i) \sim \text{Bin}\left(n - i - X^n(i), \frac{1}{n} + \frac{\lambda}{n^{4/3}}\right).$$

As long as $X^n(i) = o(n)$ and $i = O(n^{2/3})$,

$$(n - i - X^n(i)) \left(\frac{1}{n} + \frac{\lambda}{n^{4/3}} \right) \approx 1 + \frac{\lambda}{n^{1/3}} - \frac{i}{n} + o(n^{-1/3}),$$

and so we approximately have

$$X^n(i+1) - X^n(i) \sim \text{Poisson} \left(1 + \frac{\lambda}{n^{1/3}} - \frac{i}{n} \right) - 1.$$

So X^n is close to being a random walk with a deterministic (but time-dependent) drift. Let

$$M^n(i) = X^n(i) - \sum_{j=0}^{i-1} \left(\frac{\lambda}{n^{1/3}} - \frac{i}{n} \right) \approx X^n(i) - \frac{\lambda i}{n^{1/3}} + \frac{i^2}{2n}.$$

$$X^n(i+1) - X^n(i) \sim \text{Poisson}\left(1 + \frac{\lambda}{n^{1/3}} - \frac{i}{n}\right) - 1$$

and so if

$$M^n(i) \approx X^n(i) - \frac{\lambda i}{n^{1/3}} + \frac{i^2}{2n}$$

then $(M^n(i), i \geq 0)$ is approximately a martingale.

Plug in $i = \lfloor tn^{2/3} \rfloor$:

$$n^{-1/3} M^n(\lfloor tn^{2/3} \rfloor) \approx n^{-1/3} X^n(\lfloor tn^{2/3} \rfloor) - \lambda t + \frac{t^2}{2}.$$

Since the Poisson distribution here has variance ≈ 1 for all i, we can apply the martingale functional CLT (a more general version of Donsker's theorem) to obtain

$$\left(n^{-1/3} X^n(\lfloor tn^{2/3} \rfloor) - \lambda t + \frac{t^2}{2}, t \geq 0\right) \xrightarrow{d} (W(t), t \geq 0).$$

Question

So we now understand the limiting sizes and surpluses of components of the critical random graph.

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But what do the limiting components look like?

They are no longer (in general) trees. Again, the vertex-labels are irrelevant: we are really interested in what shapes and distances look like in the limit. So we will give a metric space answer, and convergence will be in the Gromov-Hausdorff-Prokhorov distance.

Approach

Consider the components one by one.

Simple but important fact: a component of $G(n, p)$ conditioned to have a particular set of m vertices and s surplus edges is a uniform connected graph on those m vertices with $m + s - 1$ edges.

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So, given the vertex-sets of the components and their surpluses, we can just sample uniform connected graphs in order to get back the whole graph.

4. CONNECTED GRAPHS

Joint work with Louigi Addario-Berry (McGill) and Nicolas Broutin (Sorbonne Université).

[L. Addario-Berry, N. Broutin & C. Goldschmidt, **The continuum limit of critical random graphs**, *Probability Theory and Related Fields* **152**(3-4), 2012, pp.367–406.]

[L. Addario-Berry, N. Broutin & C. Goldschmidt, **Critical random graphs: limiting constructions and distributional properties**, *Electronic Journal of Probability* **15**, 2010, paper no. 25, pp.741–775.]

Uniform connected graph with fixed surplus

Fix $k \geq 0$ and let G_n^k be a uniform connected graph with vertices labelled by $1, 2, \ldots, n$ and $n + k - 1$ edges (so that G_n^k has surplus k).

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Theorem. (Addario-Berry, Broutin & G. (2012))
There exists a random compact metric measure space $(\mathcal{G}^k, d^k, \mu^k)$ such that

$$\left(G_n^k, \frac{d_n^k}{\sqrt{n}}, \mu_n^k\right) \xrightarrow{d} (\mathcal{G}^k, d^k, \mu^k)$$

as $n \to \infty$.

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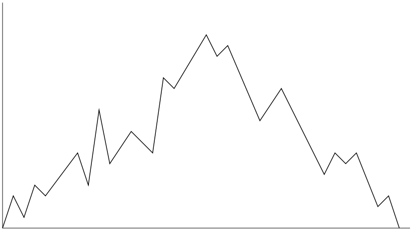
as $n \to \infty$.

We can give an explicit description for the scaling limit.

Scaling limit

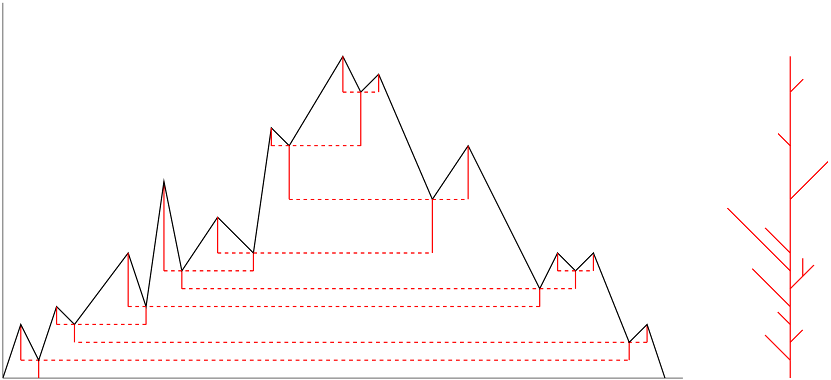
Let e be a standard Brownian excursion. Define a random excursion $\tilde{e}^k : [0,1] \to \mathbb{R}_+$ via a change of measure as follows. For any suitable test-function $f : \mathcal{C}([0,1], \mathbb{R}_+) \to \mathbb{R}$,

$$\mathbb{E}\left[f(\tilde{e}^k(t), 0 \le t \le 1)\right] = \frac{\mathbb{E}\left[f(e(t), 0 \le t \le 1)\left(\int_0^1 e(u)du\right)^k\right]}{\mathbb{E}\left[\left(\int_0^1 e(u)du\right)^k\right]}$$



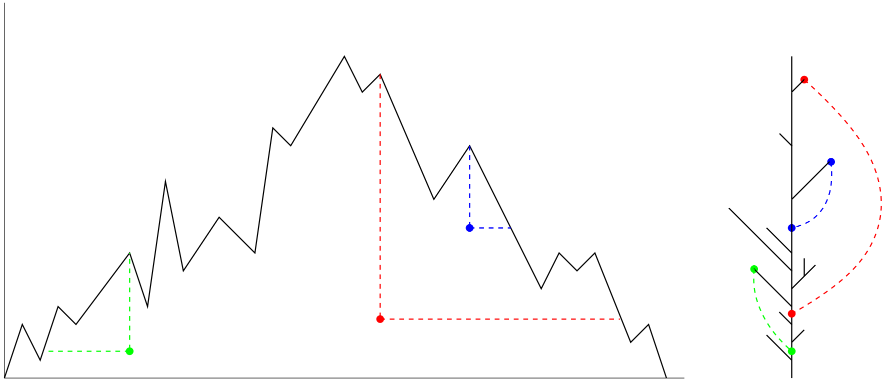
Scaling limit

Use $2\tilde{e}^k$ to encode a continuum random tree $(\tilde{\mathcal{T}}^k, \tilde{d}^k, \tilde{\mu}^k)$.



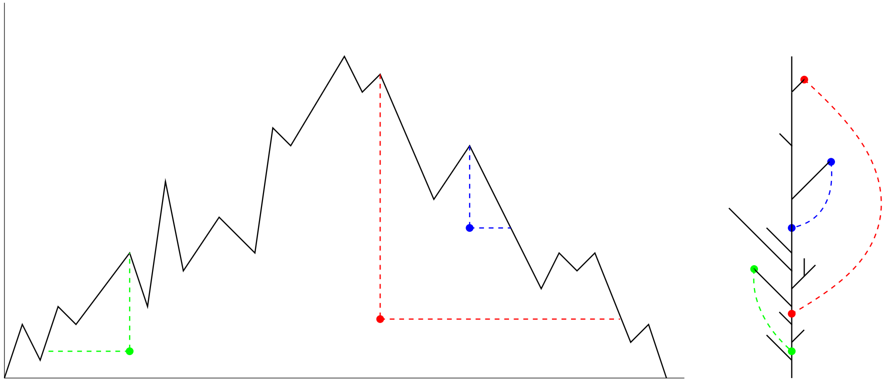
Scaling limit

Sample k independent uniform marks in the area under the curve.
Each mark picks out two points of the tree.

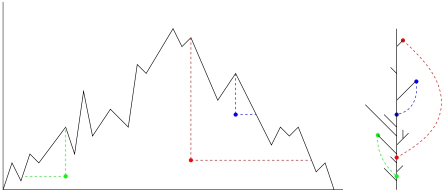


Scaling limit

Sample k independent uniform marks in the area under the curve.
Each mark picks out two points of the tree. Identify them.

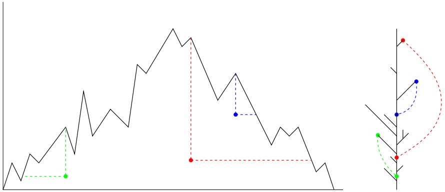


Vertex identifications



Write π^k for the usual projection $[0, 1] \to \tilde{\mathcal{T}}^k$.

Vertex identifications



Write π^k for the usual projection $[0,1] \to \tilde{\mathcal{T}}^k$.

We have marks $(x_1, y_1), \ldots, (x_k, y_k)$ which are uniform in the area under the excursion. For $1 \leq i \leq k$, let

$$t_i = \inf\{t \geq x_i : 2\tilde{e}^k(t) = y_i\}.$$

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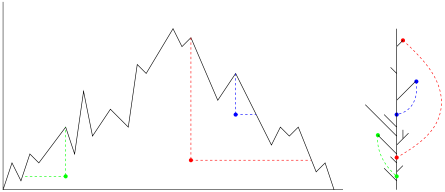

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Define an equivalence relation $\sim$ this time on $\tilde{\mathcal{T}}^k$ by declaring $\pi^k(x_i) \sim \pi^k(t_i)$ for each $1 \leq i \leq k$ and let $\mathcal{G}^k = \tilde{\mathcal{T}}^k / \sim$.

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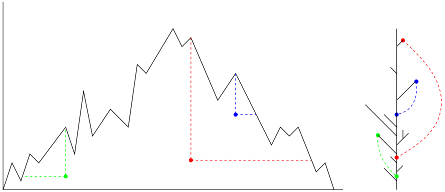

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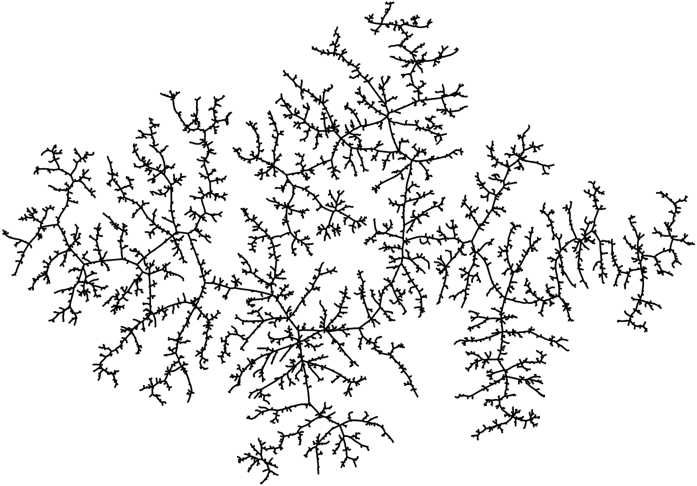
Let d^k be the metric and μ^k the measure induced in the obvious way from $\tilde{d}^k$ and $\tilde{\mu}^k$ respectively.

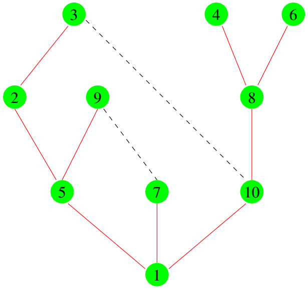
Scaling limit $(\mathcal{G}^k, d^k, \mu^k)$ for $k = 4$



[Picture by Nicolas Broutin]

Proof technique: back to the depth-first exploration

In a depth-first exploration of a connected graph G, we effectively explore a spanning tree; the dashed surplus edges make no difference.



Call the spanning tree the depth-first tree associated with the graph G, and write $T(G)$. X is also the depth-first walk of T.

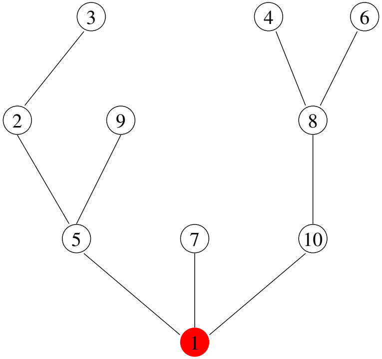
Permitted edges

Look at things the other way round: for a given tree T, which connected graphs G have depth-first tree $T(G) = T$?

In other words, where can we put surplus edges so that they don't change T?

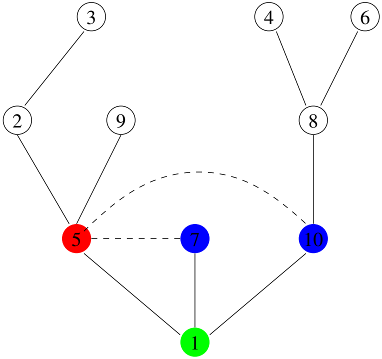
Call such edges permitted.

Depth-first walk and permitted edges



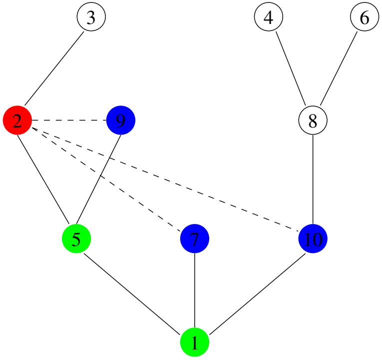
Step 0: $X(0) = 0$.

Depth-first walk and permitted edges



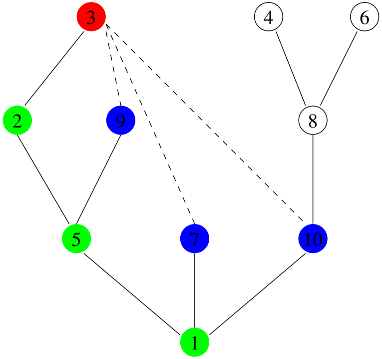
Step 1: $X(1) = 2$.

Depth-first walk and permitted edges



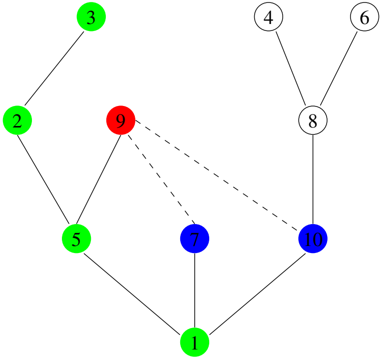
Step 2: $X(2) = 3$.

Depth-first walk and permitted edges



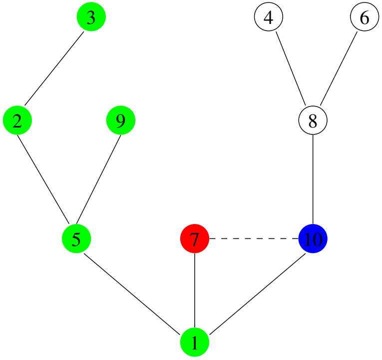
Step 3: $X(3) = 3$.

Depth-first walk and permitted edges



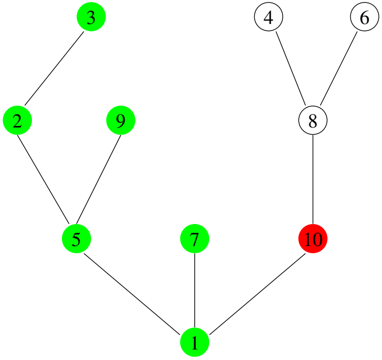
Step 4: $X(4) = 2$.

Depth-first walk and permitted edges



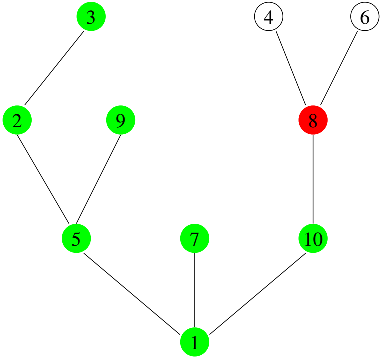
Step 5: $X(5) = 1$.

Depth-first walk and permitted edges



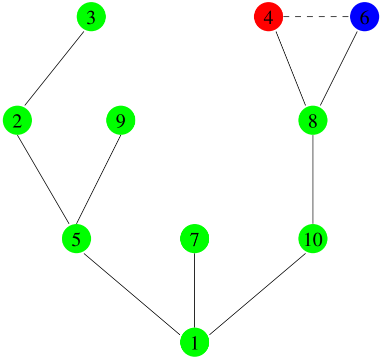
Step 6: $X(6) = 0$.

Depth-first walk and permitted edges



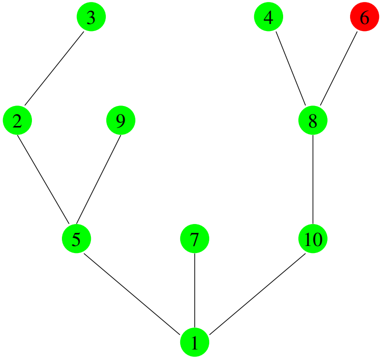
Step 7: $X(7) = 0$.

Depth-first walk and permitted edges



Step 8: $X(8) = 1$.

Depth-first walk and permitted edges



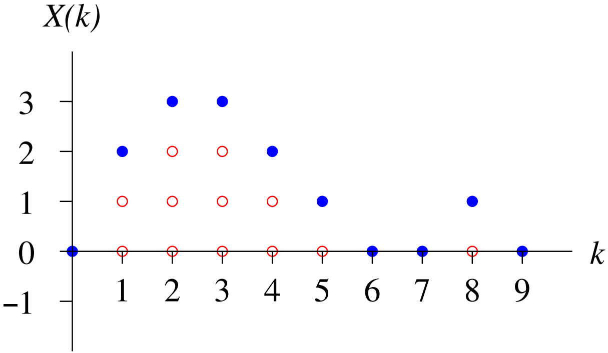
Step 10: $X(9) = 0$.

Area

At step $k \geq 0$, there are $X(k)$ permitted edges. So the total number is

$$a(T) = \sum_{k=0}^{n-1} X(k).$$

We call this the area of T.



Classifying graphs by depth-first tree

Let $\mathbb{G}_T$ be the set of graphs G such that $T(G) = T$. It follows that $|\mathbb{G}_T| = 2^{a(T)}$, since each permitted edge may either be included or not.

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Let $\mathbb{T}_{[n]}$ be the set of trees with label-set $[n] = \{1, 2, \ldots, n\}$. Then

$$\left\{\mathbb{G}_T : T \in \mathbb{T}_{[n]}\right\}$$

is a partition of the set of connected graphs on $[n]$.

Recipe for creating a uniform connected graph

Then we can create a uniform connected graph G_n^k as follows.

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- Choose a uniform k-set from among the $a(\tilde{T}_n^k)$ permitted edges and add them to the tree.

Taking limits

We essentially need to show

- the tree $\tilde{T}_n^k$ converges to an $\mathbb{R}$-tree coded by the excursion $\tilde{e}^k$;
- the locations of the surplus edges converge to the locations in the limiting picture.

Taking limits for the tree

Write $\tilde{X}_n^k$ for the depth-first walk associated with $\tilde{T}_n^k$. Then

$$a\left(\tilde{T}_n^k\right) = \sum_{i=0}^{n-1} \tilde{X}_n^k(i) = \int_0^n \tilde{X}_n^k(\lfloor s \rfloor) ds = n^{3/2} \int_0^1 n^{-1/2} \tilde{X}_n^k(\lfloor nu \rfloor) du,$$

by changing variable in the integral.

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by changing variable in the integral.

If T_n is a uniform random tree on $[n]$ and X_n is its depth-first walk, then

$$(n^{-1/2} X_n(\lfloor nt \rfloor), 0 \le t \le 1) \xrightarrow{d} (e(t), 0 \le t \le 1).$$

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So by the continuous mapping theorem,

$$\int_0^1 n^{-1/2} X_n(\lfloor nu \rfloor) du \xrightarrow{d} \int_0^1 e(u) du.$$

Taking limits for the tree

Use the change of measure to get from $\tilde{X}_n^k$ to X_n: for any bounded continuous function f,

$$\mathbb{E}\left[f\left(n^{-1/2}\tilde{X}_n^k(\lfloor nt \rfloor), 0 \le t \le 1\right)\right] = \frac{\mathbb{E}\left[f\left(n^{-1/2}X_n(\lfloor nt \rfloor), 0 \le t \le 1\right)\binom{n^{3/2}\int_0^1 n^{-1/2}X_n(\lfloor nu \rfloor)du}{k}\right]}{\mathbb{E}\left[\binom{n^{3/2}\int_0^1 n^{-1/2}X_n(\lfloor nu \rfloor)du}{k}\right]}$$

Taking limits for the tree

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We have

$$n^{-3k/2}\binom{n^{3/2}\int_0^1 n^{-1/2}X_n(\lfloor nu\rfloor)du}{k} \xrightarrow{d} \frac{\left(\int_0^1 e(s)ds\right)^k}{k!} \quad \text{as } n \to \infty.$$

Taking limits for the tree

It turns out that we also have uniform integrability, so we obtain

$$\begin{aligned}\mathbb{E}\left[f\left(n^{-1/2}\tilde{X}_n^k(nt), 0 \leq t \leq 1\right)\right] \to & \frac{\mathbb{E}\left[f(e)\left(\int_0^1 e(u)du\right)^k\right]}{\mathbb{E}\left[\left(\int_0^1 e(u)du\right)^k\right]} \\ & = \mathbb{E}\left[f(\tilde{e}^k)\right].\end{aligned}$$

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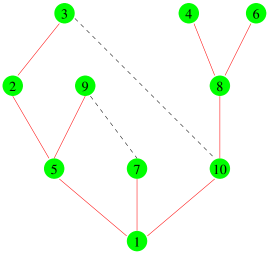
$$= \mathbb{E}\left[f(\tilde{e}^k)\right].$$

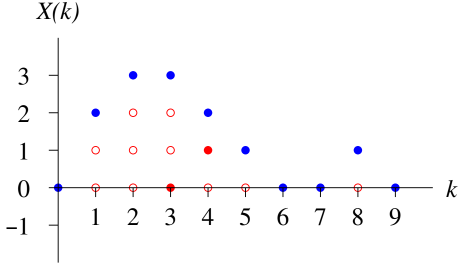
This (after converting to the height process) entails that

$$\left(\tilde{T}_n^k, \frac{d_n^k}{\sqrt{n}}, \mu_n^k\right) \xrightarrow{d} (\tilde{\mathcal{T}}^k, \tilde{d}_k, \tilde{\mu}_k).$$

Taking limits for the surplus edges

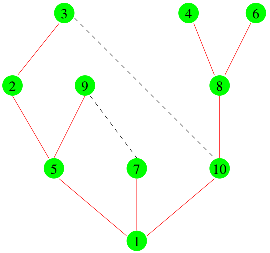
The permitted edges are in bijective correspondence with the integer points under the graph of the depth-first walk.

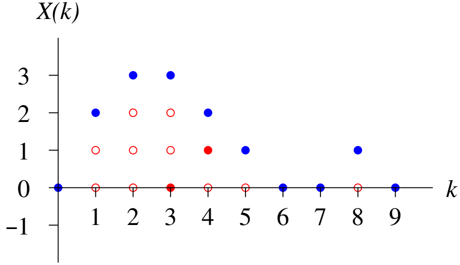




Taking limits for the surplus edges

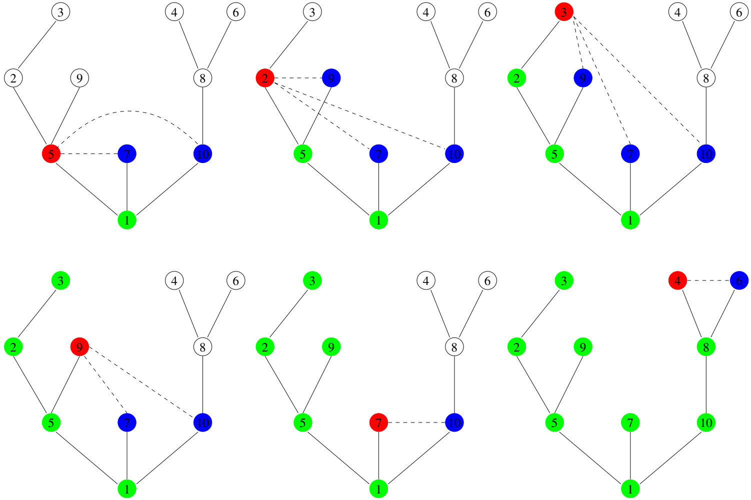
The permitted edges are in bijective correspondence with the integer points under the graph of the depth-first walk. Since we pick a uniform *k*-set from among these points, in the limit what we see is just *k* points picked independently and uniformly from the area under the limit curve.





Taking limits for the surplus edges

Surplus edges almost go to ancestors... In fact, they always go to younger children of ancestors of the current vertex.



Taking limits for the surplus edges

When we rescale, the distance between a vertex and one of its children vanishes and so, in the limit, surplus "edges" do go to ancestors of the current vertex (i.e. vertices on the path down to the root).

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Taking limits for the surplus edges

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Taking care over the details, this completes the proof.

Back to the critical Erdős-Rényi random graph

Let $p = 1/n + \lambda n^{-4/3}$ for fixed $\lambda \in \mathbb{R}$. Recall that $C_1^n, C_2^n, \ldots$ are the sizes of the components of $G(n, p)$ listed in decreasing order of size and $S_1^n, S_2^n, \ldots$ are the associated surpluses.

Theorem. (Aldous (1997)) As $n \to \infty$,

$$(n^{-2/3}\mathbf{C}^n, \mathbf{S}^n) \xrightarrow{d} (\mathbf{C}, \mathbf{S}).$$

Back to the critical Erdős-Rényi random graph

Let $\mathcal{C}_1^n, \mathcal{C}_2^n, \ldots$ be the components of $G(n,p)$ listed in decreasing order of size. Let $d_1^n, d_2^n, \ldots$ be the graph distances and $\chi_1^n, \chi_2^n, \ldots$ be the counting measures respectively, so that $\chi_i^n(\mathcal{C}_i^n) = C_i^n$.

Theorem. (Addario-Berry, Broutin & G. (2012))
Jointly with the convergence $(n^{-2/3}\mathbf{C}^n, \mathbf{S}^n) \xrightarrow{d} (\mathbf{C}, \mathbf{S})$, as $n \to \infty$,

$$\left(\left(\mathcal{C}_1^n, \frac{d_1^n}{n^{1/3}}, \frac{1}{n^{2/3}}\chi_1^n\right), \left(\mathcal{C}_2^n, \frac{d_2^n}{n^{1/3}}, \frac{1}{n^{2/3}}\chi_2^n\right), \ldots\right)$$

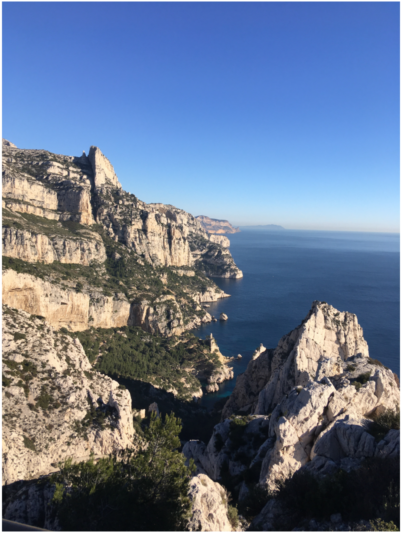
$$\xrightarrow{d} ((\mathcal{C}_1, d_1, \chi_1), (\mathcal{C}_2, d_2, \chi_2), \ldots)$$

in an ℓ_4 version of GHP, where $\chi_i(\mathcal{C}_i) = C_i$ for each $i \geq 1$, and conditionally on $(\mathbf{C}, \mathbf{S})$,

$$(\mathcal{C}_i, d_i, \chi_i) \stackrel{d}{=} (\mathcal{G}^{S_i}, \sqrt{C_i} \cdot d^{S_i}, C_i \cdot \mu^{S_i}),$$

independently for different $i \geq 1$.

5. PERSPECTIVES



Universality

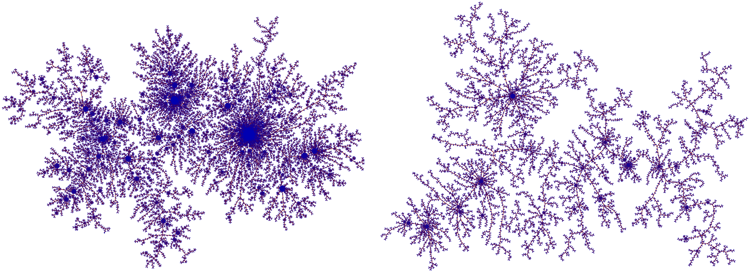
Just as for the Brownian continuum random tree, there are several critical random graph models with the same scaling limit as the critical Erdős-Rényi random graph. Examples are random graphs generated according to the configuration model, and rank-one inhomogeneous random graphs (under various conditions). In each case we must assume that the empirical degree distribution has finite third moment.

[S. Bhamidi, N. Broutin, S. Sen & X. Wang, **Scaling limits of random graph models at criticality: Universality and the basin of attraction of the Erdős-Rényi random graph**, `arXiv:1411.3417`.]

[S. Bhamidi & S. Sen, **Geometry of the vacant set left by random walk on random graphs, Wright's constants, and critical random graphs with prescribed degrees**, *Random Structures and Algorithms*, **56**, 2020, pp.676-721]

Stable trees

The BGW trees we considered were all critical and had finite offspring variance. If we assume instead that the offspring distribution is in the domain of attraction of a stable law of index $\alpha \in (1, 2)$, we obtain the so-called α-stable trees as scaling limits.

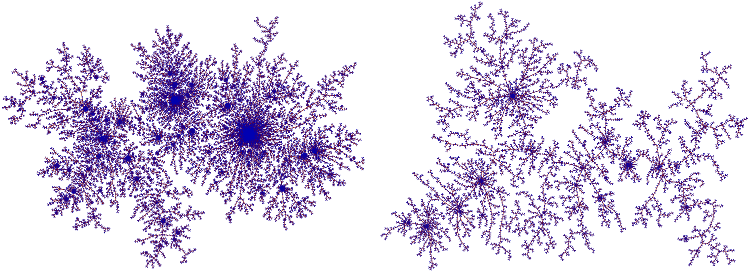


[Pictures by Igor Kortchemski]

[T. Duquesne & J.-F. Le Gall, **Random trees, Lévy processes and spatial branching processes**, *Astérisque* **281**, 2002, vi+147.]

[T. Duquesne, **A limit theorem for the contour process of conditioned Galton-Watson trees**, *Annals of Probability*, **31**, 2003, pp.996?1027.]

Stable trees



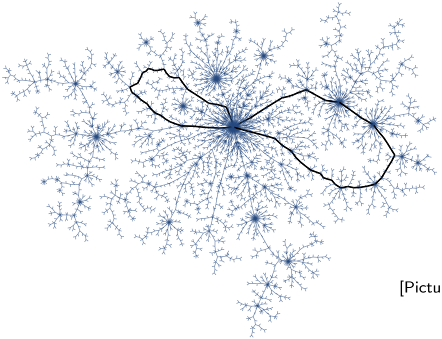
There is an analogue of Rémy's algorithm due to Marchal (2008) and there is also a (more complicated) line-breaking construction.

[P. Marchal, **A note on the fragmentation of a stable tree**, *Fifth Colloquium on Mathematics and Computer Science (DMTCS)*, 2008, pp.489–500.]

[C. Goldschmidt & B. Haas, **A line-breaking construction of the stable trees**, *Electronic Journal of Probability* **20**, 2015, Paper no. 16, pp.1–24.]

Stable graphs

The natural graph model whose scaling limit involves the stable trees is the configuration model with i.i.d. degrees having appropriate power-law tail behaviour.



[Picture by Delphin Sénizergues]

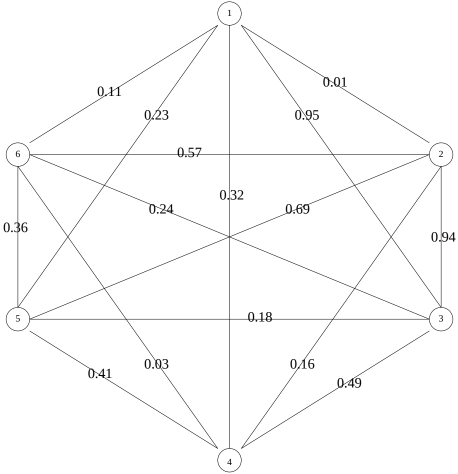
[G. Conchon-Kerjan & C. Goldschmidt, **The stable graph: the metric space scaling limit of a critical random graph with i.i.d. power-law degrees**, `arXiv:2002.04954`]

[C. Goldschmidt, B. Haas and D. Sénizergues, **Stable graphs: distributions and line-breaking construction**, `arXiv:1811.06940`]

[A. Joseph, **The component sizes of a critical random graph with given degree sequence**, *Annals of Applied Probability* **24**(6), 2014, pp.2560–2594.]

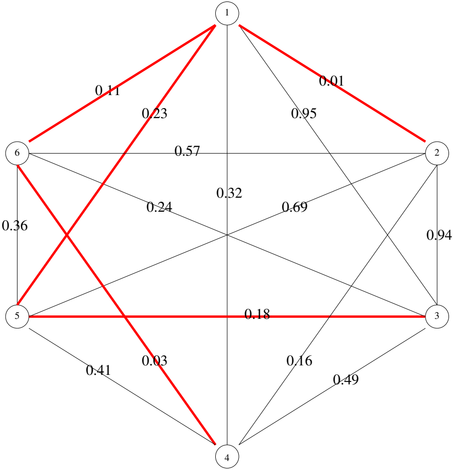
The scaling limit of the minimum spanning tree of the complete graph

Consider the complete graph on n vertices with independent edge-weights which are uniformly distributed on $[0, 1]$.



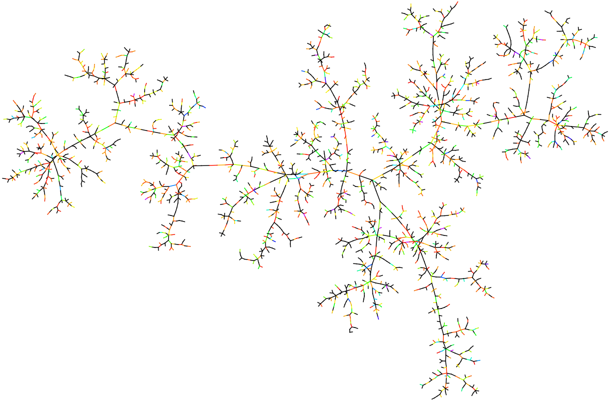
The scaling limit of the minimum spanning tree of the complete graph

Find the minimum spanning tree (MST).



The scaling limit of the minimum spanning tree of the complete graph

Question. Does the MST of the complete graph on n vertices possess a scaling limit?



[Picture by Louigi Addario-Berry]

The scaling limit of the minimum spanning tree of the complete graph

Let M_n be the MST of the complete graph on n vertices, let d_n be its graph distance, and μ_n its uniform measure.

Theorem. (Addario-Berry, Broutin, G. & Miermont (2017))
There exists a random compact measured real tree $(\mathcal{M}, d, \mu)$ such that

$$\left(M_n, \frac{d_n}{n^{1/3}}, \mu_n\right) \xrightarrow{d} (\mathcal{M}, d, \mu)$$

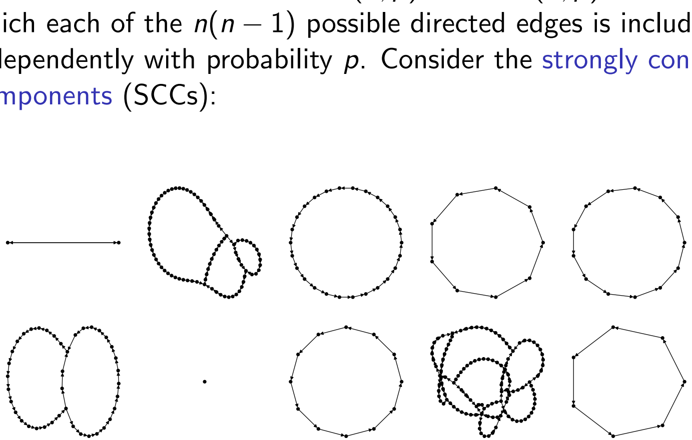
as $n \to \infty$, in GHP. $\mathcal{M}$ is binary and has Minkowski dimension 3 almost surely.

The key to understanding this result is a connection between the Erdős-Rényi random graph and Kruskal's algorithm for constructing the MST.

[L. Addario-Berry, N. Broutin, C. Goldschmidt & G. Miermont, **The scaling limit of the minimum-spanning tree of the complete graph**, *Annals of Probability* **45**(5), 2017, pp.3075–3144.]

Random directed graphs

Consider the directed version $D(n,p)$ of the $G(n,p)$ model, in which each of the $n(n-1)$ possible directed edges is included independently with probability p. Consider the strongly connected components (SCCs):



Random directed graphs

This model also undergoes a phase transition from at $p = 1/n$: below we have only microscopic SCCs; above, there is a unique giant SCC.

Using similar methods to those deployed in the undirected setting, we can prove that there is a scaling limit for $D(n, \frac{1}{n} + \lambda n^{-4/3})$ with components having sizes (and lengths!) on the order of $n^{1/3}$.

[C. Goldschmidt & R. Stephenson, **The scaling limit of a critical random directed graph**, `arXiv:1905.05397`]

Thank you!