Square-tiled surfaces and metric maps
ALEA Conference 2024

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Part I:
Square-tiled surfaces (Origamis)

Square-tiled surface

Definition

A **square-tiled surface** *(or* **origami***) is an oriented compact connected surface obtained by gluing a finite number of isometric squares along parallel sides by translation (right $\leftrightarrow$ left, up $\leftrightarrow$ down).*

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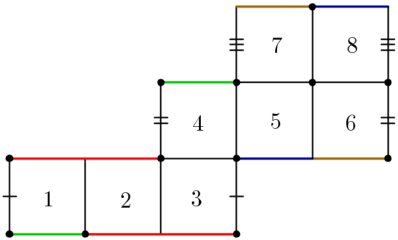
Labelled origami: squares are numbered



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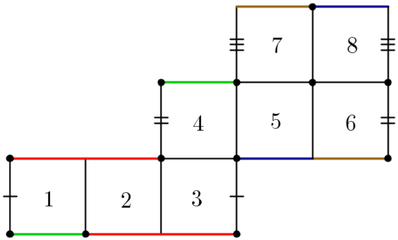


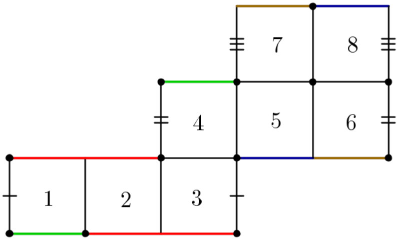
$h = (1, 2, 3)(4, 5, 6)(7, 8)$
$v = (1, 2, 3, 4)(5, 7, 6, 8)$

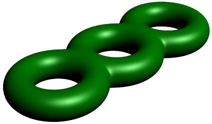
Equivalent definition

A labelled origami with N squares is a pair of permutations $(h, v) \in S_N \times S_N$ acting transitively on $\{1, \ldots, N\}$.

Geometry of square-tiled surfaces

- topology (genus)





Geometry of square-tiled surfaces

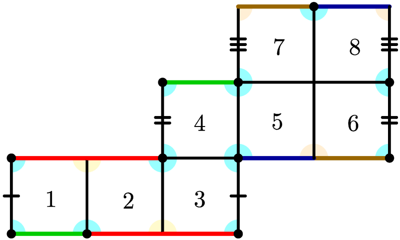
- topology (genus)
- flat metric with conical singularities (coming from the euclidean metric on $\mathbb{R}^2$)

Degre k_i of a singularity: number of extra turns.

Euler-Poincaré

$$2g - 2 = \sum_i k_i.$$

$k_1 + 1, \ldots, k_n + 1$ is the cycle type of $v^{-1}h^{-1}vh$.



$g = 3, \; k = 4$
$v^{-1}h^{-1}vh = (2, 7, 3, 4, 6)$

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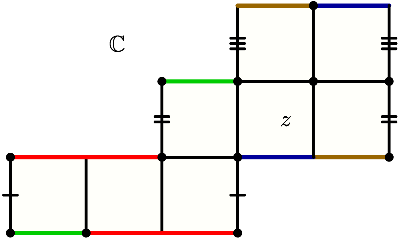

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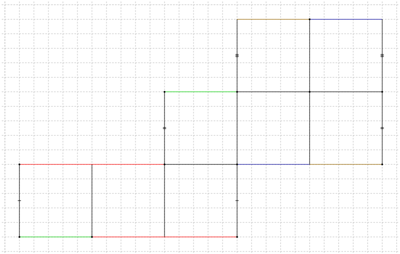

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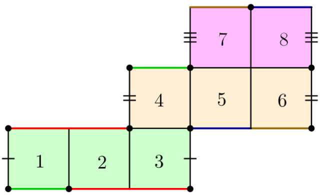
It is an example of translation surface (see Part II).

Cylinders

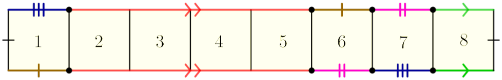
Definition

A **cylinder** *is a maximal collection of parallel closed geodesics*

- **3 cylinders** SQT with 8 squares, genus 3, one singularity of degree 4



- **1 cylinder** SQT with 8 squares, genus 3, one singularity of degree 4.

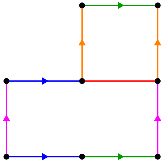


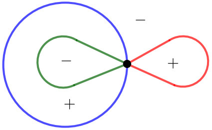
Cylinders

Definition

A **cylinder** *is a maximal collection of parallel closed geodesics*

The reunion of the boundaries of all cylinders is the reunion all horizontal segments emerging from the singularities, so it determines a (possibly disconnected) bipartite map with n vertices of valencies $2(k_i + 1)$. This map has an integer metric (each side has an integer length).





Results and conjectures – Teaser for Part II

Fact: (see Part II)

$$\left|\{\text{SQT of sing. type } (k_1, \dots, k_n) \text{ with } \leq N \text{ squares}\}\right| \sim cN^d \text{ as } N \to \infty$$

where $d = \sum(k_i + 1) + 1 = \text{Nber of edges} + 1 = 2g + n - 1.$

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Let c_1 be the corresponding constant for 1-cylinder surfaces, and $c_{1,1}$ the corresponding constant for SQT with 1 horizontal cylinder AND 1 vertical cylinder.

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$$\frac{c_1}{c} = \frac{c_{1,1}}{c_1} \sim \frac{1}{d} \quad \text{as } d \to \infty.$$

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[Delecroix-G-Zograf-Zorich], Combination of results of [DGZZ] and [Chen-Möller-Zagier], [Aggarwal], [Sauvaget]...

Results and conjectures – Teaser for Part II

Conjecture

The distribution of k-cylinder SQTs of type $(k_1, \ldots, k_n)$ converges to the distribution of the (unsigned) Stirling numbers of the first kind $c(k, d)$, as $d \to \infty$.

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Interpretation :

- Asymptotically, no more constraints on the permutations.
- Compare to distribution of vertices for random bipartite maps with d edges.
- Expect a strong convergence result: mod-Poisson convergence of parameter $\log(d)$ (Kowalski-Nikeghbali), as in Hwang result for the number of cycles of uniformly random permutations of S_d.

Half-translation SQTs

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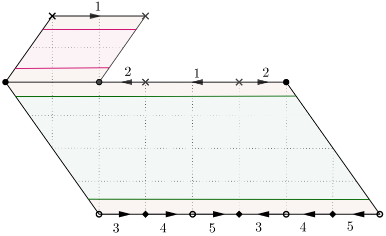
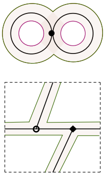
⇝ Flat metric with conical singularities of angle $(k+2)\pi$ (degree k: number of extra half-turns), with $k \geq -1$.

⇝ Maps with vertices of valency $k_i + 2 \geq 1$, equipped with an integer metric.





Large genus asymptotics: half-translation case

Here we assume that the half-translation SQts have degrees $k_i = 1$. We let $d = \sum(k_i/2+1) = 6g-6$.

Theorem (Delecroix-G-Zograf-Zorich)

- *Separating 1-cylinder SQTs*

$$\frac{c(sep)}{c(nonsep)} \sim \sqrt{\frac{2}{3\pi g}} \cdot \frac{1}{4^g} \quad as\ g \to \infty.$$

- *Proportion of 1-cylinder surfaces*

$$\frac{cyl_1}{\mathrm{Vol}} \sim \sqrt{\frac{\pi}{4d}} \quad as\ g \to \infty.$$

- *Heights*
  The probability that all the heights are bounded by m tends to $\sqrt{\frac{m}{m+1}}$ *a* $g \to \infty$.

Large genus asymptotics: half-translation case

Theorem (Delecroix-G-Zograf-Zorich)

- *Global separation:*
  All singularities of a SQT are located on the same horizontal layer with probability that tends to 1 when g tends to infinity.
- *Distribution of number of cylinders*
  It converges in a strong sense to the Poisson distribution of parameter $\lambda_d = \log(d)/2$ *[convergence mod-Poisson of parameter* λ_d *and limiting function* $t\Gamma(3/2)/\Gamma(1+t/2)$*].*

Conjecture: it holds for all SQT (with a bounded number of singularities of degree -1).

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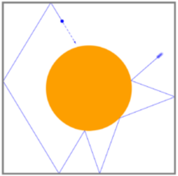
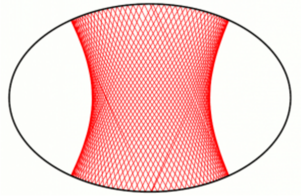
Compare to the distribution of faces of random maps [Bodini-Courtiel-Dovgal-Hwang] and [Budzinski-Curien-Petri].

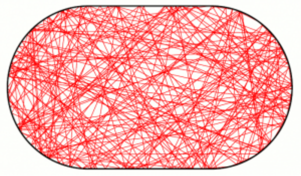
Part II:
Motivations:
Billiards and flat surfaces

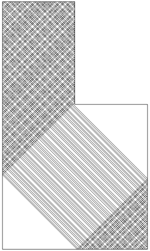
Billiards

Differents types of billiards...





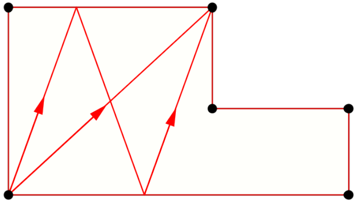




...different mathematics.

Rational polygonal billiards

Example of recent results: Right-angled billiards



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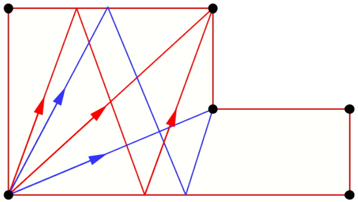

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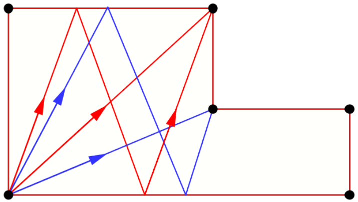
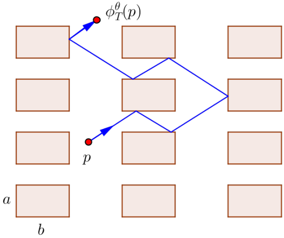


There are (asymptotically) 4 times more blue trajectories than red trajectories (Athreya-Eskin-Zorich, 2012).

Rational polygonal billiards

Example of recent results: Windtree models



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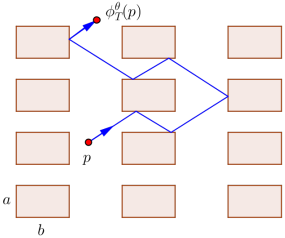
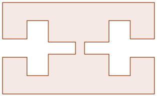


Diffusion rate: 2/3
(Delecroix-Hubert-Lelièvre,
2011)

Rational polygonal billiards

Example of recent results: Windtree models

with an obstacle



Diffusion rate:
$(2m)!!/(2m+1)!!$
(Delecroix-Zorich, 2015)

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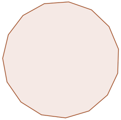
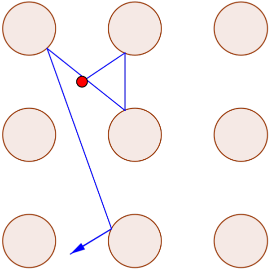
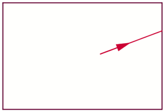


Diffusion rate: ??



Diffusion rate: 1/2

From billiards to flat surfaces



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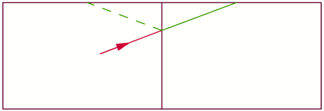

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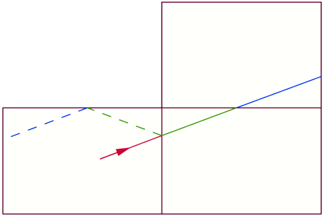

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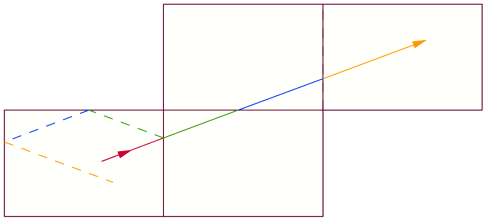

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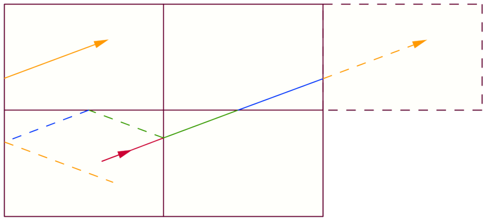

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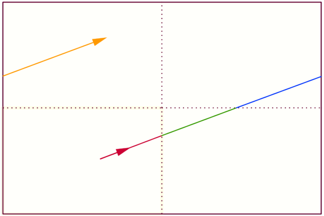

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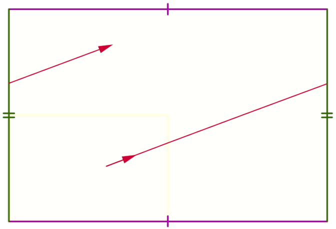

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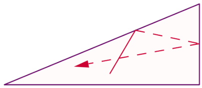

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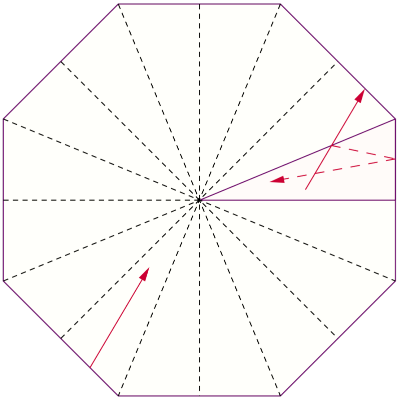

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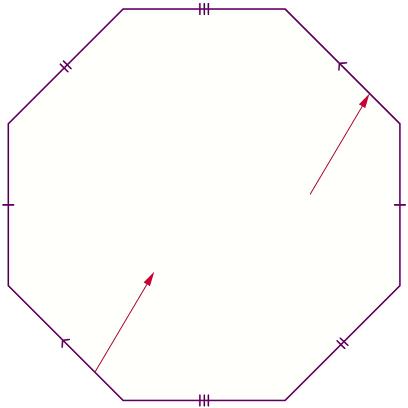

Flat surfaces

Flat surfaces

Translation surface

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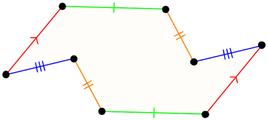

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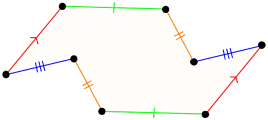
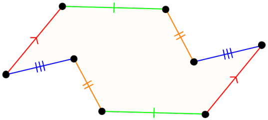


Flat metric
conical angles $(k+1) \cdot 2\pi$

Flat surfaces

Translation surface



Flat metric
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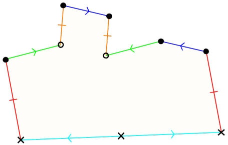
Half-translation surface



Flat metric
Conical angles $(k+2) \cdot \pi$

Moduli space of translation surfaces:

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Strata :

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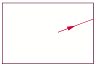
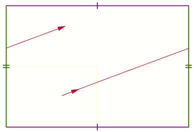
The familly of (independant) sides of the polygonal pattern for a surface S, viewed as vectors (or complex numbers), forms a system of local coordinates for the stratum $\mathcal{H}(\underline{k})$ around S.

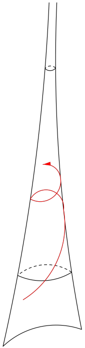
Lebesgue measure in these coordinates gives rise to a well globally defined measure with good features:

- the measure is finite (after renormalization)
- it is $SL(2, \mathbb{R})$-invariant.

Etude des billards: renormalisation

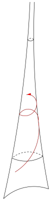






billiard flow rational polygon	*unfolding* $\longrightarrow$	linear flow flat surface	*renormalization* $\longrightarrow$	geodesic flow $SL(2,\mathbb{R})-$ orbit

Lyapunov exponents
(mean eigenvalues of the monodromy)



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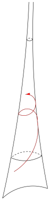

Siegel-Veech constants
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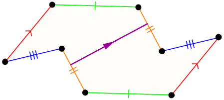

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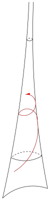

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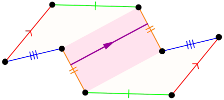

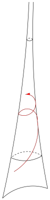

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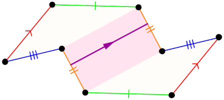

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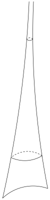

Volumes of strata

Lyapunov exponents
(mean eigenvalues of the monodromy)

[EKZ]

Siegel-Veech constants
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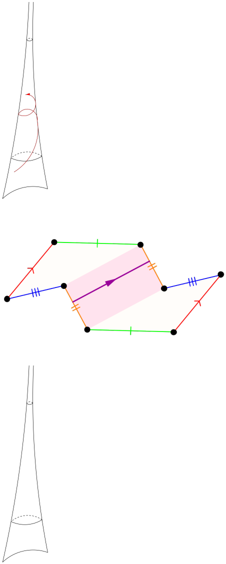
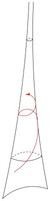
Volumes of strata





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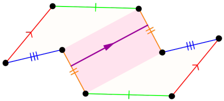

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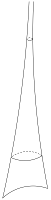
[EMZ] [MZ] [G]



Volumes of strata

Volumes of strata and SQTs

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Idea: To evaluate the Masur-Veech volume of strata, it suffices to count the number of "integer" points in a large radius "ball" (rather hyperboloid here).

$$\mathrm{Vol}\,\mathcal{H}(k_1, \ldots, k_n) = \lim_{N\to\infty} \frac{2d}{N^d} \big| \{ SQT \text{ of type } (k_1, \ldots, k_n) \text{ of area } \leq N \} \big|.$$

Volumes of strata and SQTs

Computation of the volume of $\mathcal{H}(2)$.

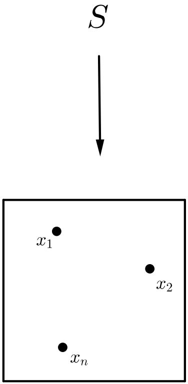
Some magic

Stratum	1 *cyl*	2 *cyl*	3 *cyl*	*Vol*
$\mathcal{H}(2,2)$	$\frac{1}{12}\zeta(5)$	$-\frac{1}{12}\zeta(5)$		$\frac{1}{3}\zeta(4) = \frac{\pi^4}{270}$
		$\frac{1}{6}\zeta(2)\zeta(3)$	$-\frac{1}{6}\zeta(2)\zeta(3)$	
			$\frac{1}{3}\zeta(4)$	
$\mathcal{Q}(3,1^5)$	$40\zeta(4)$	$50\zeta(4)$		$90\zeta(4) = \pi^4$
$\mathcal{Q}(4,1^2)$	$9\zeta(3)$	$8\zeta(2) - 9\zeta(3)$		$8\zeta(2) = \frac{4\pi^2}{3}$
$\mathcal{Q}(4,3^2)$	$\frac{11}{2}\zeta(5)$	$-\frac{11}{2}\zeta(5)$		$12\zeta(4) = \frac{2\pi^4}{15}$
		$+3\zeta(2)\zeta(3)$	$-3\zeta(2)\zeta(3)$	
		$+\frac{16}{3}\zeta(4)$	$+\frac{20}{3}\zeta(4)$	

Part III:
More on volumes, counting metric maps, . . .

Historical methods to compute volumes

The first complete results for the computation of volumes of strata of translation surfaces, are due to Eskin-Okounkov.



Set of "integer points" in the stratum: ramified covers of the torus, over n points $x_1, \ldots, x_n$, with profile $(k_i + 1, 1, \ldots, 1)$ over x_i.

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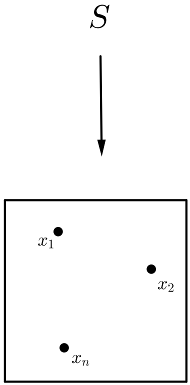

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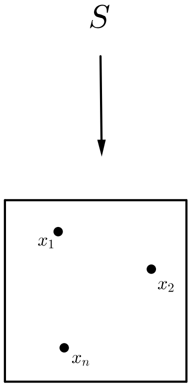

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Let $\mathcal{N}_N(\underline{k})$ be the number of such covers of degree N (counted with automorphims), and define the generating series $Z(q) = \sum_N \mathcal{N}_N(\underline{k}) q^N$.

Theorem (Bloch-Okounkov, Eskin-Okounkov, Eskin-Okounkov-Pandharipande)

This generating series is a quasimodular form of mixed weight (explicit in terms of the k_i*'s).*

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Why this result is useful to compute volume?

- the asymptotics of $\sum_{n=1}^{N} \mathcal{N}_n(\underline{k})$ as $N \to \infty$ is related to the asymptotics of $Z(q)$ as $q \to 1$
- The ring of quasimodular forms is "small": we can compute the series $Z(q)$ knowing just the first coefficients.
- The modularity property relates the asymptotics of $Z(q)$ as $q \to 1$ to the asymptotics of $Z(q)$ as $q \to 0$.

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This results implies that $\mathrm{Vol} \in \pi^{2g}\mathbb{Q}$.

Methods using counting of metric maps

In his proof of Witten conjecture, Kontsevich in 1992 uses the counting of (integer) metric maps, of "general" type (valencies of the vertices are supposed ≥ 3, but not fixed).
One result that we can extract from his work is the following :

Theorem (Kontsevich, Norbury)

$$N_{g,n}(b_1, \ldots, b_n) = \sum_{Maps_{g,n}} \left| \{\textit{integer metrics, faces of length } b_1, \ldots, b_n\} \right|$$

is a symmetric quasi-polynomial in b_i^2 *, whose higher order term form a polynomial that satisfies some nice recursions.*

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This result allows to compute the volume of "principal" strata (all $k_i = 1$) of half-translation surfaces (+ asymptotic results on number of cylinders) [DGZZ].

Recent advances

The conjecture of Eskin-Zorich in the case of translation surfaces

Theorem

$$\mathrm{Vol}\,\mathcal{H}(k_1,\ldots,k_n) \sim \frac{4}{(k_1+1)(k_2+1)\ldots(k_n+1)} \quad \textit{as } d \to \infty$$

was proved recently by several results

- [Chen-Möller-Zagier] in some case : pushing forward the arguments and techniques of Eskin-Okounkov
- [Aggarwal]: pure combinatorics !
- [Sauvaget] and [Chen-Möller-Sauvaget-Zagier]: algebraic interpretation of volumes and (simple) quadratic recursion.

Recent advances

The conjecture of ADGZZ in the case of half-translation surfaces

Theorem

$$\mathrm{Vol}\,\mathcal{Q}(k_1,\ldots,k_n) \sim \frac{4}{\pi}\prod_i \frac{2^{k_i+2}}{k_i+2} \quad \text{as } d \to \infty$$

was proved recently in the case of all $k_i = 1$ by several results

- [Aggarwal] asymptotics of intersection numbers (appearing in Kontsevich polynomials) via Virasoro contraints
- [Chen-Möller-Sauvaget]: algebraic interpretation of volumes

Aggarwal's results on Kontsevich polynomials allow to get precise asymptotics for the distribution of cylinders.
Open problem: other strata? large number of $k_i = -1$?