Probabilistic methods: Exercices

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March 14, 2024, ALEA Days

1 Proof of Lovász Local Lemma

Let us recall the symmetric version of LLL:

Theorem 1 (Lovász Local Lemma) *Let $\mathscr{A} = \{A_1, A_2, \ldots, A_n\}$ be a set of events such that for each $i = 1, 2 \ldots, n$*

- $\mathbb{P}(A_i) \leq p$ *and*
- $\exists \mathscr{D}_i \subset \mathscr{A}$ *of size at most d such that A_i is mutually independent of $\mathscr{A} \setminus \mathscr{D}_i$.*

If $e \cdot p \cdot (d+1) \leq 1$, then

$$\mathbb{P}\left(\bigcap_{i=1}^{n} \overline{A_i}\right) > 0.$$

We will derive this theorem from the following more general statement:

Theorem 2 (LLL, general version) *Let $\mathscr{A} = \{A_1, A_2, \ldots, A_n\}$ be a set of events such that for each $i = 1, 2 \ldots, n$*

- $\exists \mathscr{D}_i \subset \mathscr{A}$ *such that A_i is mutually independent of $\mathscr{A} \setminus \mathscr{D}_i$,*
- $\exists x_i \in [0; 1[$ *such that*
$$\mathbb{P}(A_i) \leq x_i \cdot \prod_{A_j \in \mathscr{D}_i} (1 - x_j).$$

Then

$$\mathbb{P}\left(\bigcap_{i=1}^{n} \overline{A_i}\right) \geq \prod_{i=1}^{n} (1 - x_i) > 0.$$

Exercice 1 *Derive the symmetric version of LLL from the general version.*

Hint: Try $x_i = \frac{1}{d+1}$.

Exercice 2 *Prove the general version of LLL.*

To achieve this, rewrite the probalities as telescopic products and use and prove the following claim:
For every $i = 1, \ldots, n$ and for every set $S \subseteq [n] \setminus \{i\}$,

$$\mathbb{P}\left(A_i \mid \bigcap_{j \in S} \overline{A_j}\right) \leq x_i.$$

Hint: Prove the claim by induction on $|S|$, separate dependent and independent events.

2 Formula satisfiability

A k-CNS formula is a conjunction of clauses, where each clause is a disjunction of k literals, with a literal either a boolean variable or a negation of one; we require that each variable appears at most once (including its negation) in each clause.

A k-CNS formula is *satisfiable* if there exists an assignement of boolean values to variables such that every clause has at least one true literal.

Exercice 3 *Use linearity of expectation to prove that an unsatisfiable k-CNS formula has at least 2^k clauses.*

Find an unsatisfiable formula with exactly 2^k clauses.

Exercice 4 *Find a function $f(k)$ such that if for every clause C, there are at most $f(k)$ other clauses sharing at least one variable with C, then the whole formula admits a satisfying assignment.*

A k-CNS formula is *linear* if any two clauses share at most one variable.

Exercice 5 *Find a function $g(k)$ (faster than 2^k) such that every linear k-CNS formula with at most $g(k)$ clauses is satisfiable.*

More reading: H. Gebauer, R.A. Moser, D. Scheder, E. Welzl, The Lovász Local Lemma and Satisfiability, in S. Albers, H. Alt, S. Näher (eds.), Efficient Algorithms, Lecture Notes in Computer Science 5760, Springer 2009.

3 Independent transversals in graphs

Let G be a graph with maximum degree Δ and let $V(G) = V_1 \cup V_2 \cup \ldots \cup V_r$ be a partition of its vertex set into r pairwise disjoint subsets called *parts* of G. An *independent transversal* of G with respect to the partition $\{V_i\}_{i \in [r]}$ is an independent set of vertices which contains exactly one vertex from each part V_i.

The best known ratio so far between the part size and maximum degree is due to Haxell:

Theorem 3 (Haxell 2001) *Let G be a graph with maximum degre at most Δ, whose vertex set is partitioned into parts of size at least 2Δ. Then G admits an independent transversal.*

Exercice 6 *Prove a similar result for parts of size at least $c \cdot \Delta$ (for a suitable constant c) using LLL.*

Exercice 7 *Prove a similar result for parts of size at least $c \cdot \Delta$ (for a suitable constant c) using Entropy Compression.*