# Probabilistic Methods

## Part I. Lovász Local Lemma

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# Introduction

Probabilistic methods

- prove the existence of combinatorial objects
- using probabilistic tools and arguments
  - First moment principles: linearity of expectation
  - Second moment inequalities
  - Lovász Local Lemma
  - Entropy Compression
  - Concentration inequalities
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# Outline of today's talk

- a warmup example
- hypergraph coloring problem
- statement of the Lovász Local Lemma
- application in hypergraph coloring
- application in acyclic graph coloring

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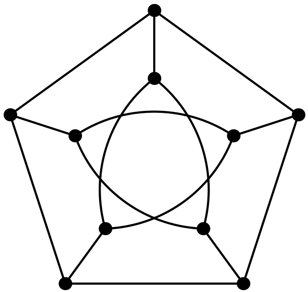

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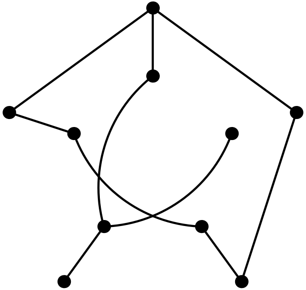

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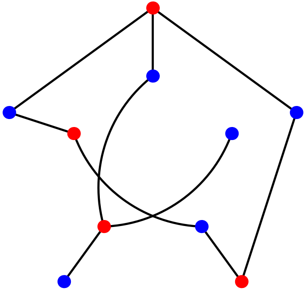

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The best we can hope for is $\sim \frac{m}{2}$ :

- a complete graph on $n$ vertices has $\binom{n}{2} \sim \frac{n^2}{2}$ edges
- a complete bipartite graph on $\lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor$ vertices has $\sim \frac{n^2}{4}$ edges

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$\mathbb{E}(\sum_{e \in E(G)} X_e) = \sum_{e \in E(G)} \mathbb{E}(X_e) = \frac{m}{2}$.

Therefore, there exists a coloring with at least $\frac{m}{2}$ bichromatic edges.

# Hypergraph coloring

A hypergraph $H = (V, E)$ is a couple of sets with

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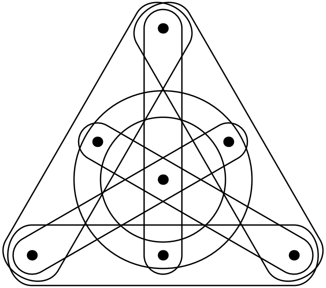

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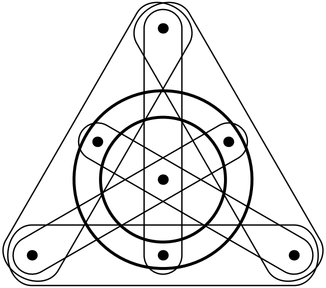

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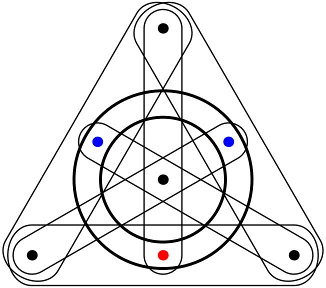

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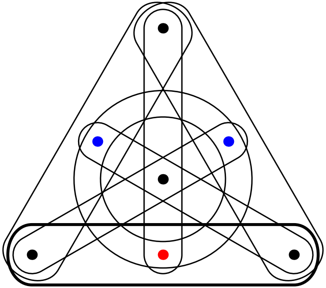

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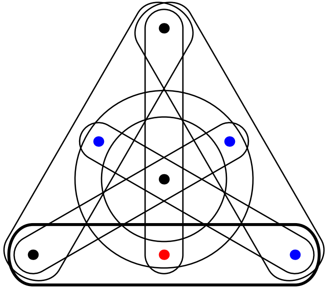

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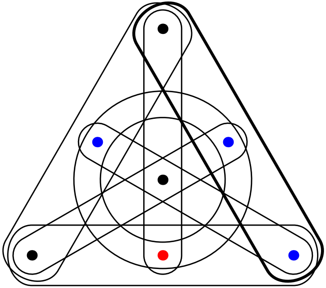

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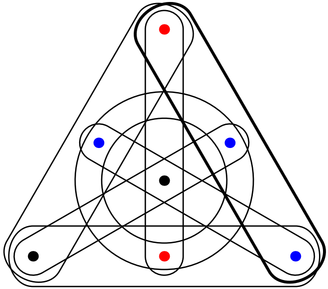

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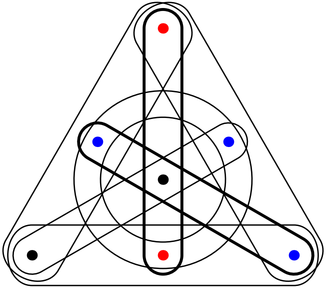

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If $\mathscr{A} := (A_e, e \in E)$ were independent, we would have

$$\mathbb{P}\left(\bigcap_{e \in E} \overline{A_e}\right) = \left(1 - \frac{1}{2^{k-1}}\right)^m > 0$$

# Mutually independent events

## Definition

*Let $A$ be an event and let $\mathscr{B}$ be a set of events in a probability space. We say that $A$ is <u>mutually independent</u> of $\mathscr{B}$ if*

$$\mathbb{P}\left(A \mid \bigcap_{B_i \in S} B_i\right) = \mathbb{P}(A)$$

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For example, in the context of random hypergraph coloring, $A_e$ is mutually independent of

$$\{A_{e'} : e \cap e' = \emptyset\}.$$

# Lovász Local Lemma

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## Theorem (Lovász Local Lemma, Symmetric version)

*Let* $\mathscr{A} = \{A_1, A_2, \ldots, A_n\}$ *be a set of events such that for each* $i = 1, 2 \ldots, n$

- $\mathbb{P}(A_i) \leq p$ *and*
- $\exists \mathscr{D}_i \subset \mathscr{A}$ *of size at most* $d$ *such that* $A_i$ *is mutually independent of* $\mathscr{A} \setminus \mathscr{D}_i$*.*

*If*

$$e \cdot p \cdot (d+1) \leq 1$$

*then*

$$\mathbb{P}\left(\bigcap_{i=1}^{n} \overline{A_i}\right) > 0.$$

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## Theorem (LLL)

*If* $\mathbb{P}(A_i) \le p$, $A_i$ *is mutually independent of* $\mathscr{A} \setminus \mathscr{D}_i$ *with* $|\mathscr{D}_i| \le d$, *and* $ep(d+1) \le 1$, *then* $\mathbb{P}\left(\bigcap_{i=1}^{n} \overline{A_i}\right) > 0$.

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In the context of random coloring of a $k$-regular $k$-uniform hypergraph, $p = \frac{1}{2^{k-1}}$ and each $A_e$ is mutually independent of all but at most $k^2$ other edges, so $d = k^2$.

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There exists a coloring without a monochromatic edge whenever

$$\frac{e}{2^{k-1}} \cdot k^2 \leq 1.$$

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### Theorem (Alon and Bregman 1988, Henning and Yeo 2013)

*Let $k \geq 4$. Then every $k$-regular $k$-uniform hypergraph is 2-colorable.*

# Acyclic graph coloring

## Definition

*Let $G = (V, E)$ be a graph. A coloring $\varphi : V(G) \to \{1, 2, \ldots, k\}$ is an acyclic coloring of $G$ if*

- $\varphi(u) \neq \varphi(v) \quad \forall uv \in E(G), \quad$ *($\varphi$ is a proper coloring)*
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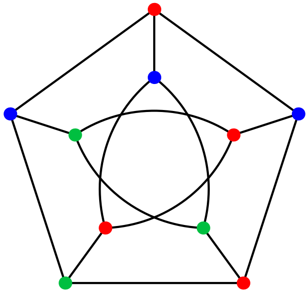

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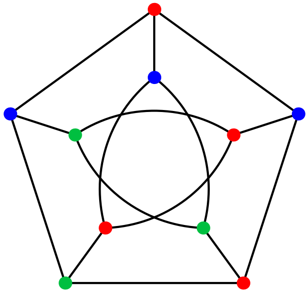

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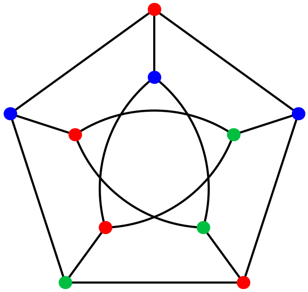

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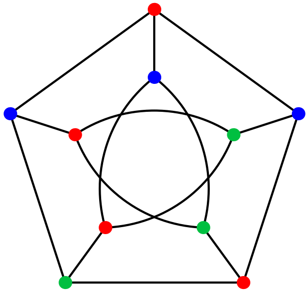

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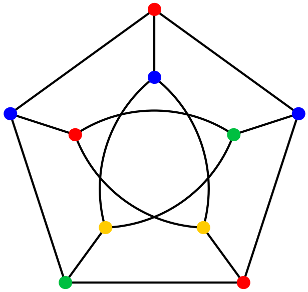

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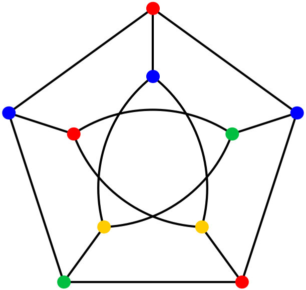

an acyclic coloring with 4 colors

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# Greedy bound

Can we bound $\chi_a(G)$ as a function of $\Delta(G)$, the maximum degree of $G$?

If we color every vertex with a color distinct from all the colors of its neighbors and the neighbors of its neighbors, surely we will not create any bichromatic cycle.

This is always possible provided we have at least

$$\Delta + \Delta(\Delta - 1) + 1 = \Delta^2 + 1$$

colors. Hence,

$$\chi_a(G) \leq \Delta^2 + 1$$

for every graph $G$.

# Using Lovász Local Lemma

Can we bound $\chi_a(G)$ as a function of $\Delta(G)$, the maximum degree of $G$?

## Theorem (Alon, McDiarmid, Reed 1991)

*Let $G$ be a graph with maximum degree $\Delta$. Then*

$$\chi_a(G) \leq 50\Delta^{4/3}.$$

*On the other hand, there are graphs for which*

$$\chi_a(G) = \Omega\left(\frac{\Delta^{4/3}}{(\log \Delta)^{1/3}}\right).$$

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Can we bound $\chi_a(G)$ as a function of $\Delta(G)$, the maximum degree of $G$?

## Theorem

*Let $G$ be a graph with maximum degree $\Delta$. Then*

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Let $C$ be a set of $K \geq 7\Delta^{3/2}$ colors.
Randomized procedure : For each vertex $v$, let $F(v)$ be the set of colors forbidden at $v$ – the colors of the neighbors already colored, and let $C(v) = C \setminus F(v)$ be the set of available colors at $v$. Clearly, $|F(v)| \leq \Delta$.

- Choose an integer $i \leq K - \Delta$ uniformly randomly and color $v$ with $i$-th available color.

This procedure gives a proper coloring of $G$.

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- Choose an integer $i \leq K - \Delta$ uniformly randomly and color $v$ with $i$-th available color.

Let $A_P$ be the event that a 4-vertex path $P = v_1 v_2 v_3 v_4$ gets only two colors.

$$\mathbb{P}(A_P) \leq \frac{1}{(K-\Delta)^2}.$$

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The dependency degree is (less than)

$$d < 4 \cdot 4 \cdot \Delta^3 = 16\Delta^3.$$

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## Theorem (LLL)

*If $\mathbb{P}(A_i) \leq p$, $A_i$ is mutually independent of $\mathscr{A} \setminus \mathscr{D}_i$ with $|\mathscr{D}_i| \leq d$, and $ep(d+1) \leq 1$, then $\mathbb{P}\left(\bigcap_{i=1}^{n} \overline{A_i}\right) > 0$.*

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$$p \leq \frac{1}{(K-\Delta)^2} \qquad \text{and} \qquad d < 4 \cdot 4 \cdot \Delta^3 = 16\Delta^3$$

and so

$$ep(d+1) \leq \frac{e(16\Delta^3)}{(7\Delta^{3/2} - \Delta)^2} < \frac{0.89\Delta}{(\Delta^{1/2} - \frac{1}{7})^2} < 1$$

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# Conclusion

LLL: If a set of bad events that are mostly mutually independent happen with low probability, then with positive probability none of them happen.

Applications in graphs, hypergraphs, coloring, transversals, satisfiability, combinatorics of words, etc.

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Applications in graphs, hypergraphs, coloring, transversals, satisfiability, combinatorics of words, etc.

Main inconvenience: Not algorithmic/non constructive, only proves existence.

Thank you for your attention!

# Frank Sinatra: Strangers in the night

Strangers in the night exchanging glances
Wondering in the night, what were the chances
we'd be sharing love before the night was through
Something in your eyes was so inviting
Something in your smile was so exciting
Something in my heart told me I must have you
Strangers in the night
Two lonely people we were strangers in the night
Up to the moment
When we said our first hello
Little did we know
Love was just a glance away
A warm embracing dance away, and
Ever since that night we've been together
Lovers at first sight, in love forever
It turned out so right
For strangers in the night
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# Alicia Keys: If I Ain't got you

Some people live for the fortune
Some people live just for the fame
Some people live for the power, yeah
Some people live just to play the game
Some people think that the physical
things define what's within
And I've been there before
That life's a bore
So full of the superficial
Some people want it all
But I don't want nothing at all
If it ain't you, baby
If I ain't got you, baby
Some people want diamond rings
Some just want everything
But everything means nothing
If I ain't got you, yeah
Some people search for a fountain
That promises forever young
Some people need three dozen roses
And that's the only way to prove you love
them
Hand me the world
On a silver platter
And what good would it be?
With no one to share
With no one who truly cares for me?
Some people want it all
But I don't want nothing at all
If it ain't you, baby
If I ain't got you, baby
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But everything means nothing
If I ain't got you, you, you
Some people want it all
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If it ain't you, baby
If I ain't got you, baby
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Some just want everything
But everything means nothing
If I ain't got you, yeah
If I ain't got you with me, baby
Oh, whoo-ooh
Said nothing in this whole wide world
don't mean a thing
If I ain't got you with me, baby

# Probabilistic Methods

## Part II. Entropy Compression

František Kardoš
LaBRI, Université de Bordeaux

ALEA Days, CIRM Marseille

14 March 2024

# Introduction

Entropy compression method

- analyze the performance of randomized algorithms
- prove that the algorithm eventually finds a solution

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$$\varphi : E(G) \to [1, k]$$

is acyclic if there is no bichromatic cycle.

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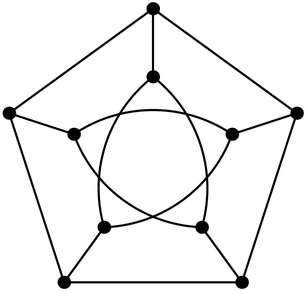

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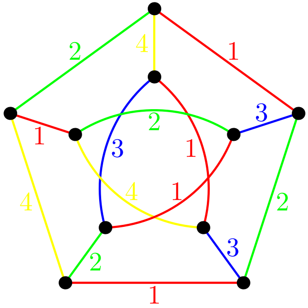

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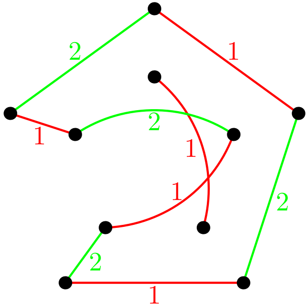

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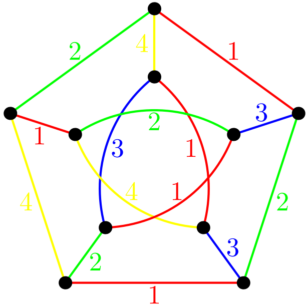

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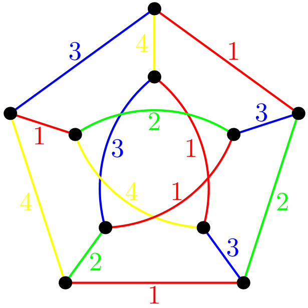

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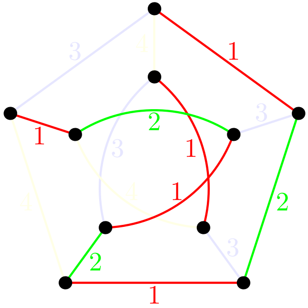

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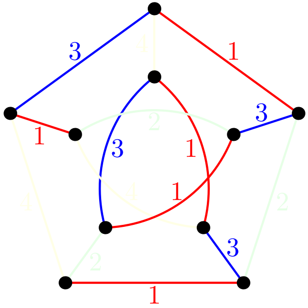

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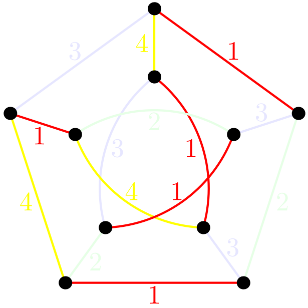

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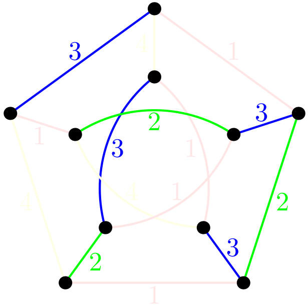

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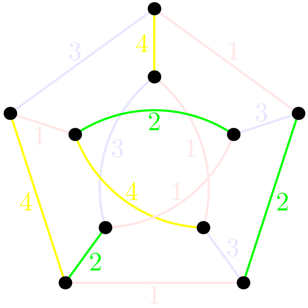

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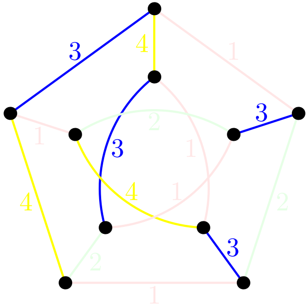

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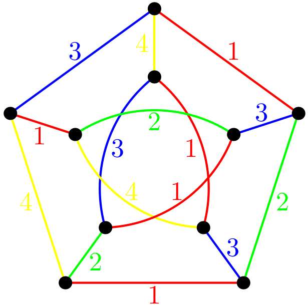

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For Petersen graph $P$ we have

$$\chi'_a(P) = \chi'(P) = 4.$$

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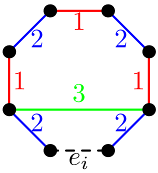

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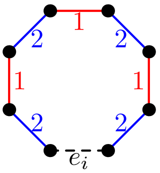

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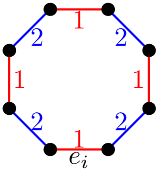

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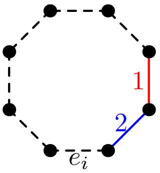

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Let's try to encode a run in a log file.

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- the new color for $e_i$
- (eventually) the path to uncolor.

# Entropy compression principle

Let the number of rounds $N$ be fixed.

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If we can prove that the number of possible combinations of $\{$final coloring $\times$ log file$\}$ is in $o(k^N)$, then we get a contradiction: a run that stops before round $N$ must exist.

# Log file: which edges are colored and which are not

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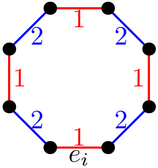

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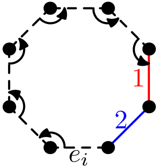
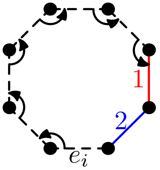


# Log file: what else?

Given

- the coloring after round $j$
- the information whether a path was uncolored or not
- (eventually) the uncolored path

we can determine the coloring before round $j$.

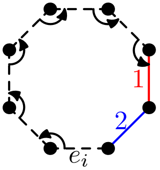


# Log file: what else?

Given

- the coloring after round $j$
- the information whether a path was uncolored or not
- (eventually) the uncolored path

we can determine the coloring before round $j$.




In particular, we can determine the color assigned to $e_i$.

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- the number of edges to uncolor, and
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Alternatively, log file can contain

- a series of booleans, indicating whether an edge is colored or uncolored; and
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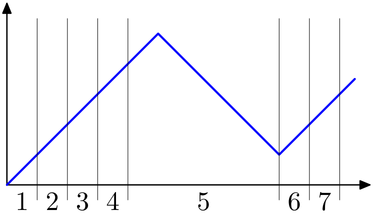

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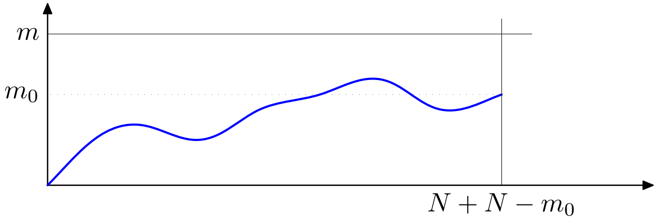

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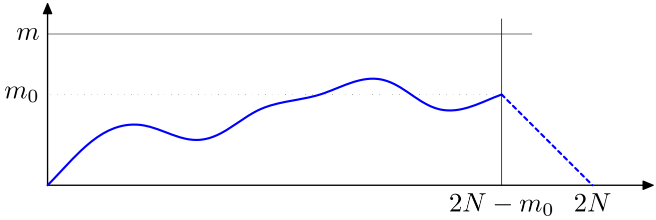

# Dyck words

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It is known that the number of Dyck words of length $2N$ is the $N$-th Catalan number

$$C_N = \frac{1}{N+1}\binom{2N}{N} \sim \frac{4^N}{N^{3/2}\sqrt{\pi}}$$

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# How many different colorings?

Repeat

- pick the first uncolored edge, say $e_i$
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As long as $K \geq 6\Delta$, the algorithm must find a valid coloring.

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Entropy compression: the history of a given process can be recorded in an efficient way – the amount of additional information that is recorded at each step of the process is (on average) less than the amount of new information randomly generated at each step.

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Thank you for your attention!