Chemins dans le quart de plan II

Kilian Raschel

Universität Bielefeld

Journées ALÉA 2011

1. Introduction and main results
   - Introduction
   - Results
2. Proofs
   - Explicit expression of the counting generating functions
     - Reduction to boundary value problems
     - Conformal gluing and uniformization
   - Nature of the counting generating functions
3. Conclusion

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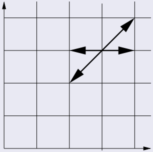
Counting the numbers of walks confined to the quarter plane

Let $\mathcal{S}$ be a step set



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- with increments in $\mathcal{S}$;

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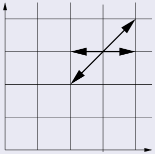

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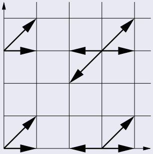

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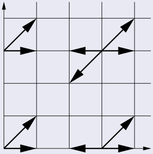

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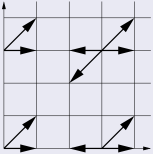

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$$Q(x,y;t)=\sum_{i,j,n\geqslant 0} q(i,j;n)x^i y^j t^n.$$

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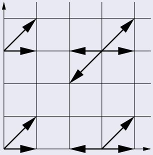

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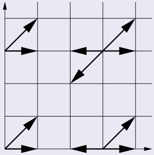

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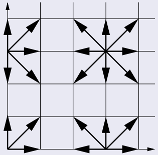
- $Q(x,y;t)$: explicit expression;
- $Q(x,y;t)$: dependence on $\mathcal{S}$, e.g., its nature (rational, algebraic, (non-)holonomic).

Class of the walks with small steps

$\mathcal{S} \subset \{-1,0,1\}^2 \setminus \{(0,0)\}$.



There are 2^8 such problems.

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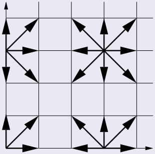

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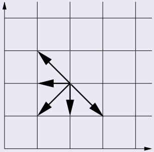
Some of these 2^8 models are:



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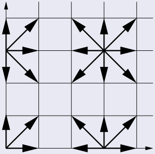

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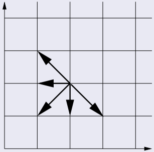

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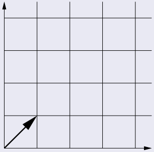

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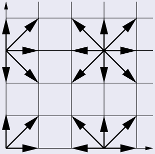

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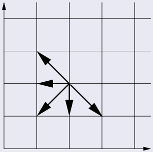

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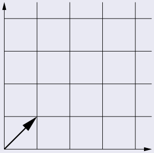

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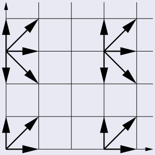

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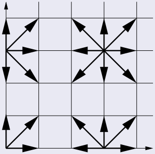

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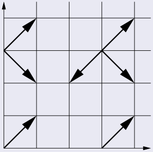

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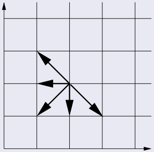

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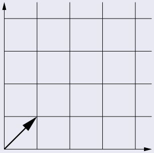

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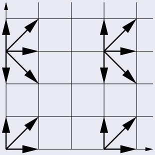

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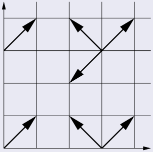

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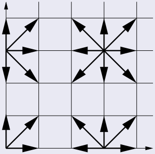

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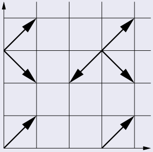

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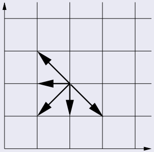

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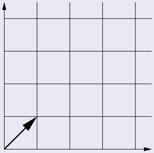

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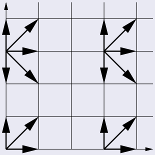

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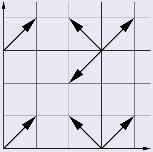

symmetrical.

Finally, it remains 79 problems! [BMM]

The functional equation

The functional equation for $Q(x,y;t)$:

$$K(x,y;t)Q(x,y;t) =$$
$$K(x,0;t)Q(x,0;t) + K(0,y;t)Q(0,y;t) - K(0,0;t)Q(0,0;t) - xy.$$

The functional equation

The kernel:

$$K(x,y;t) = xyt\left[\sum_{(k,\ell)\in\mathcal{S}} x^k y^\ell - 1/t\right].$$

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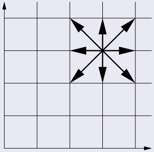
The group of the walk



$$\sum_{(k,\ell)\in\mathcal{S}} x^k y^\ell = \begin{cases} B_{-1}(y)x^{-1} + B_0(y) + B_{+1}(y)x^{+1} \\ A_{-1}(x)y^{-1} + A_0(x) + A_{+1}(x)y^{+1} \end{cases}$$

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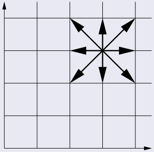

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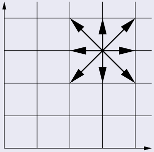
$$\Psi(x,y) = \left(x, \frac{A_{-1}(x)}{A_{+1}(x)}\frac{1}{y}\right), \qquad \Phi(x,y) = \left(\frac{B_{-1}(y)}{B_{+1}(y)}\frac{1}{x}, y\right)$$

The group of the walk



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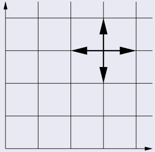
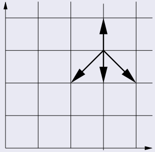
and thus by any element of the group

$$\langle \Psi, \Phi \rangle.$$

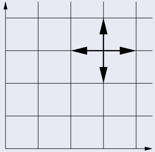
Examples

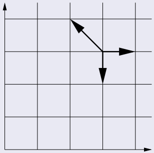


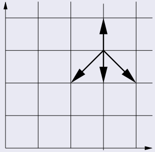


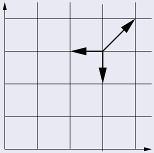
Order 4;

Examples



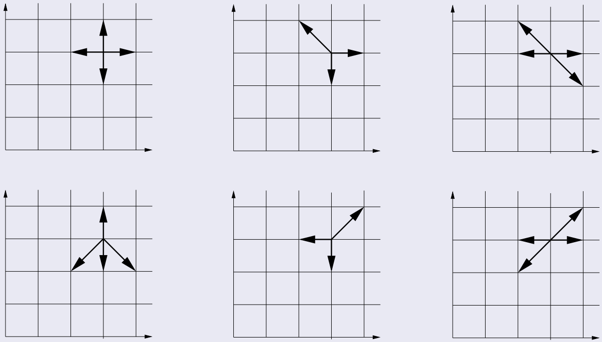






Order 4; order 6;

Examples



Order 4; order 6; order 8;

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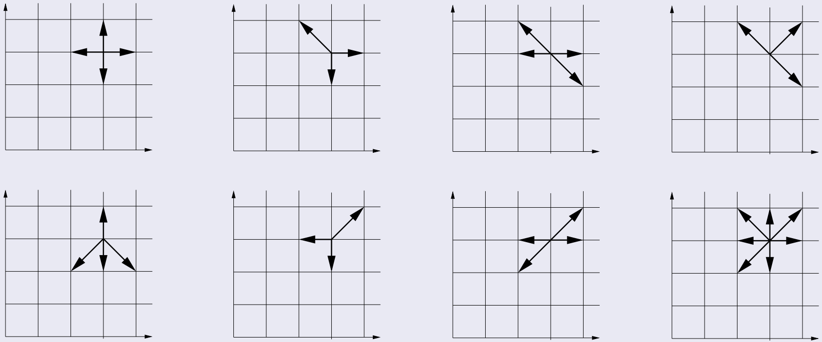

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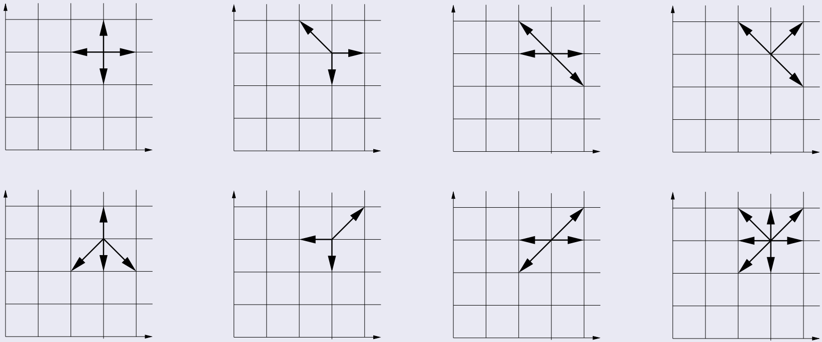
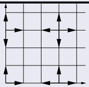
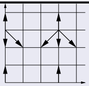
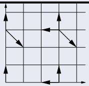
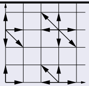
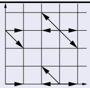
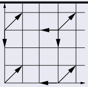
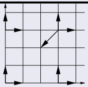
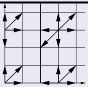
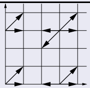


Order 4; order 6; order 8; order ∞.

Classification of the 79 models [BMM]

- For 23 walks, $\langle \Psi, \Phi \rangle$ is finite;
- For 56 walks, $\langle \Psi, \Phi \rangle$ is infinite.

Existing results for the 23 finite group cases

Group	Covariance	Walks	$Q(x, y; t)$
4	$= 0$	  and 14 others	holonomic [BMM]
6	< 0	 	holonomic [BMM]
8	< 0		holonomic [BMM]
6	> 0	  	algebraic [BMM]
8	> 0		algebraic [BK]

Existing results for the 56 infinite group cases

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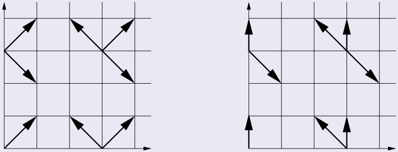

Expression & nature (non-holonomic) of $Q(x, y; t)$ in [MR].

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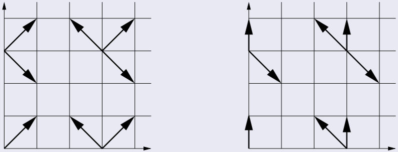

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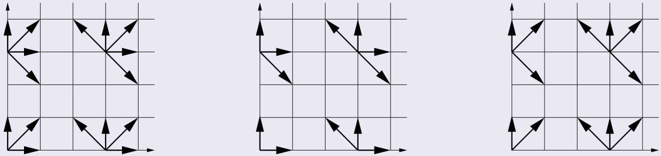

In progress.

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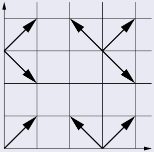
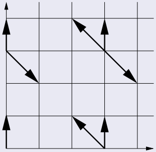
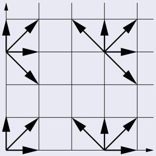
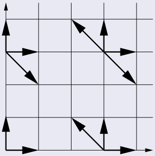
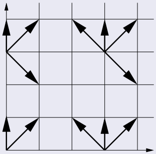



Expression & nature (non-holonomic) of $Q(x, y; t)$ in [MR].

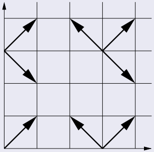
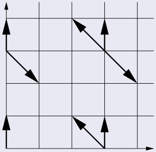




In progress.

- 51 non-singular walks?

Existing results for the 56 infinite group cases

- $5 = 2 + 3$ singular walks:

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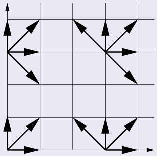
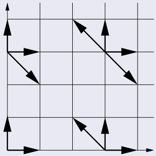
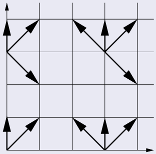

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where

- $Y_0(x;t)$ and $Y_1(x;t)$ are the two roots of the kernel $y \mapsto K(x,y;t)$,
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- It remains to find w!

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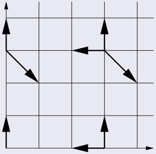

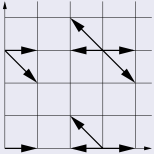

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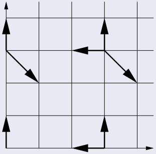

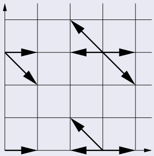

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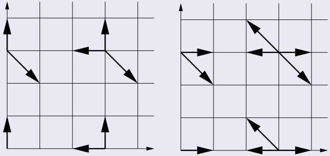

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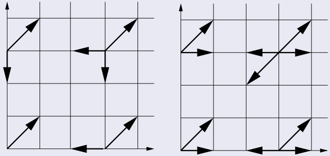

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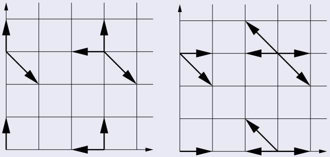

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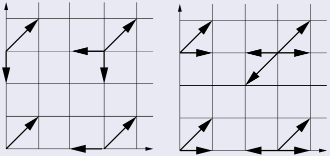

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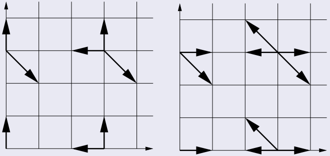

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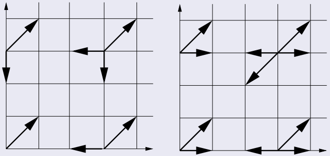

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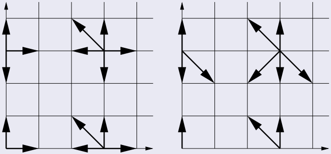

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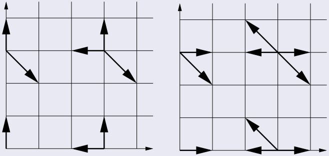

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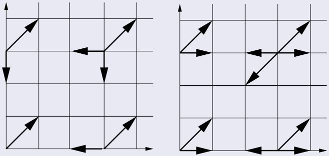

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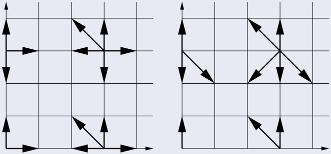

- w explicit [$\wp$-Weierstrass functions]

Results (4/4)

- Comparison between the nature of Q and that of w & $\widetilde{w}$:

Group	Covariance	$Q(x,y;t)$	$w(x;t)$ & $\widetilde{w}(y;t)$
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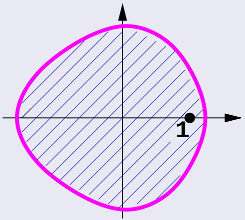
A 3-steps method

- Boundary value problems



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A 3-steps method

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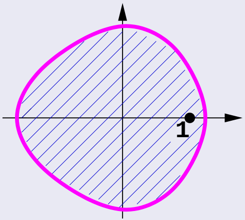

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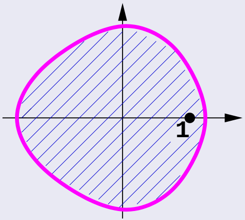

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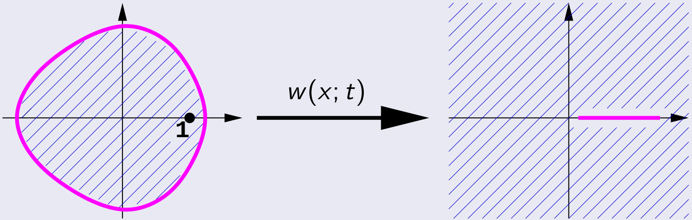

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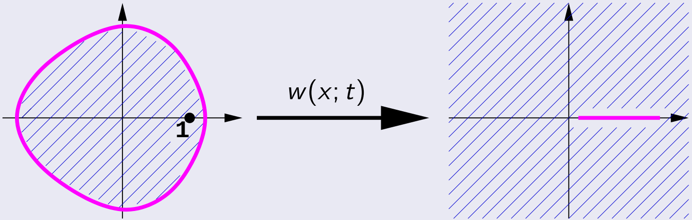

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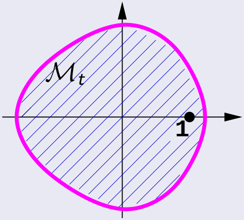
Boundary value problem of Riemann-Carleman type

There exists a curve $\mathcal{M}_t$, symmetrical w.r.t. the horizontal axis,



such that: $\forall u \in \mathcal{M}_t$,

$$K(u,0;t)\,Q(u,0;t) - K(\overline{u},0;t)\,Q(\overline{u},0;t) = uX_0^{-1}(u;t) - \overline{u}X_0^{-1}(\overline{u};t),$$

X_0 being a root of the kernel $x \mapsto K(x,y;t) = xyt\left[\sum_{(k,\ell)\in\mathcal{S}} x^k y^\ell - 1/t\right]$.

How to obtain this Riemann-Carleman problem? (1/2)

The functional equation:

$$\begin{aligned} K(x,y;t)Q(x,y;t) &= K(x,0;t)Q(x,0;t) \\ &+ K(0,y;t)Q(0,y;t) - K(0,0;t)Q(0,0;t) - xy. \end{aligned}$$

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Roots of the kernel:

$$K(x,y;t) = xyt\left[\sum_{(k,\ell)\in\mathcal{S}} x^k y^\ell - 1/t\right] = 0 \Longleftrightarrow x = X_0(y;t) \text{ or } X_1(y;t).$$

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A new functional equation:

$$0 = K(X_\ell(y;t),0;t)Q(X_\ell(y;t),0;t) + K(0,y;t)Q(0,y;t) - K(0,0;t)Q(0,0;t) - X_\ell(y;t)y.$$

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We get:

$$\begin{aligned} K(X_0(y;t),0;t)Q(X_0(y;t),0;t) - K(X_1(y;t),0;t)Q(X_1(y;t),0;t) \\ = X_0(y;t)y - X_1(y;t)y. \end{aligned}$$

How to obtain this Riemann-Carleman problem? (2/2)

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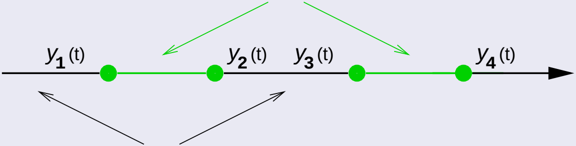

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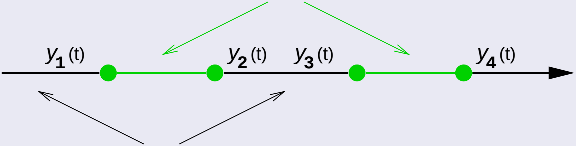

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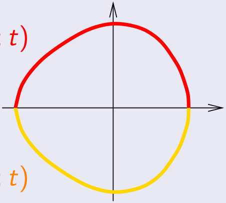
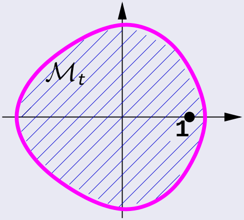
$X_0([y_1(t), y_2(t)];t)$



$X_1([y_1(t), y_2(t)];t)$

Boundary value problem of Riemann-Carleman type

There exists a curve $\mathcal{M}_t$, symmetrical w.r.t. the horizontal axis,

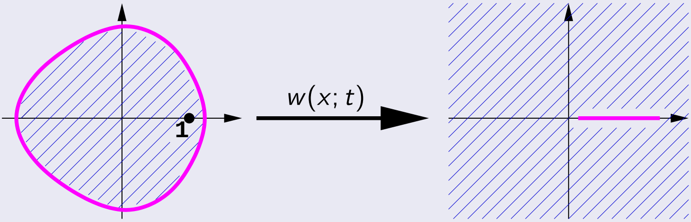


such that: $\forall u \in \mathcal{M}_t$,

$$K(u,0;t)\,Q(u,0;t) - K(\overline{u},0;t)\,Q(\overline{u},0;t) = uX_0^{-1}(u;t) - \overline{u}X_0^{-1}(\overline{u};t),$$

X_0 being a root of the kernel $x \mapsto K(x,y;t) = xyt\left[\sum_{(k,\ell)\in\mathcal{S}} x^k y^\ell - 1/t\right]$.

Conformal gluing function



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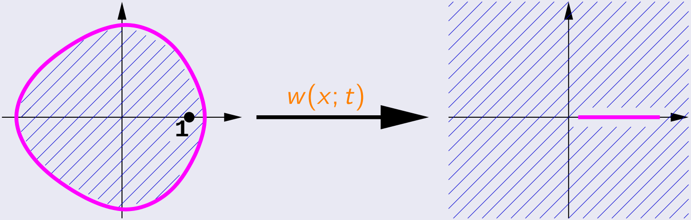

Resolution of this boundary value problem of Riemann-Carleman type

$$K(x,0;t)Q(x,0;t)-K(0,0;t)Q(0,0;t)=$$
$$\frac{1}{2\pi\imath}\int_{\mathcal{M}_t} uX_0^{-1}(u;t)\left[\frac{\partial_u w(u;t)}{w(u;t)-w(x;t)}-\frac{\partial_u w(u;t)}{w(u;t)-w(0;t)}\right]\mathrm{d}u.$$

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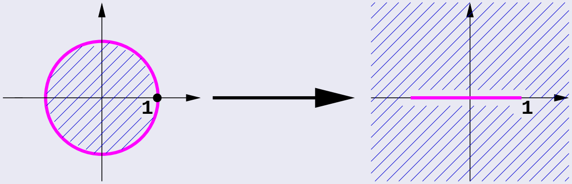
Example: the unit circle



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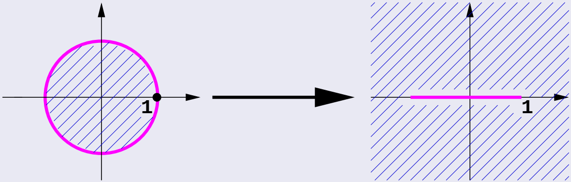

$w(x) = \frac{1}{2}\left(x + \frac{1}{x}\right)$ is a good CGF

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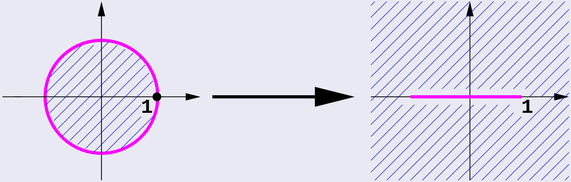

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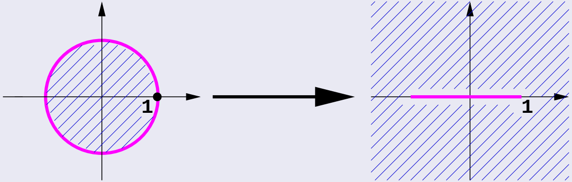

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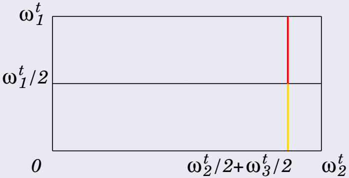
Main idea: transforming the curve $\mathcal{M}_t$ into a simple curve

In our case:



Different formulations for $\mathcal{K}_t$

$$\mathcal{K}_t = \left\{(x, y) \in \mathbb{C}^2 : \sum_{(k,\ell)\in\mathcal{S}} x^k y^\ell - 1/t = 0\right\}$$

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Riemann surface of the square root of a third degree polynomial

Let $g_2^3 - 27g_3^2 \neq 0$ and $\mathcal{L} = \{(u,v) \in \mathbb{C}^2 : v^2 = 4u^3 - g_2 u - g_3\}$.

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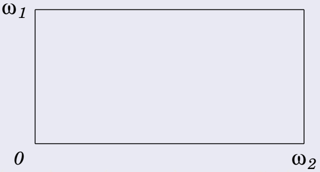

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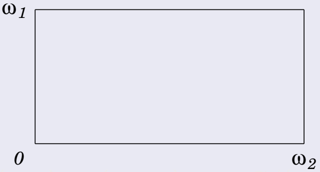

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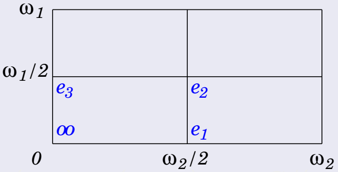

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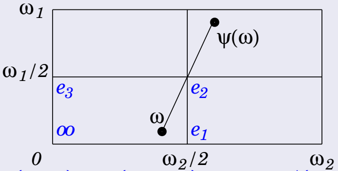
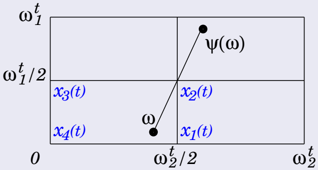


- $\mathcal{L}$ is stable by $(u,v) \mapsto (u,-v) \iff \mathbb{C}/(\omega_1\mathbb{Z}+\omega_2\mathbb{Z})$ is stable by $\psi(\omega) = -\omega + [\omega_1 + \omega_2]$.

A symmetric view point



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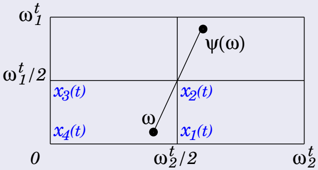

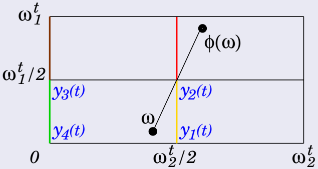

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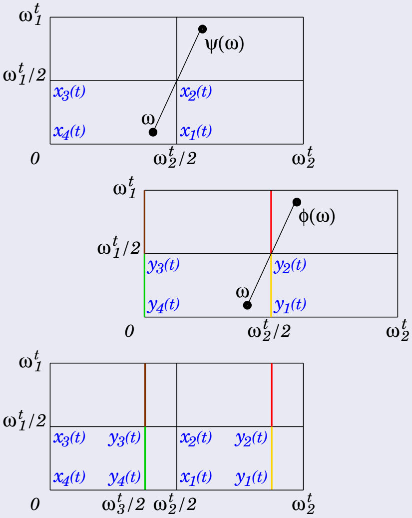

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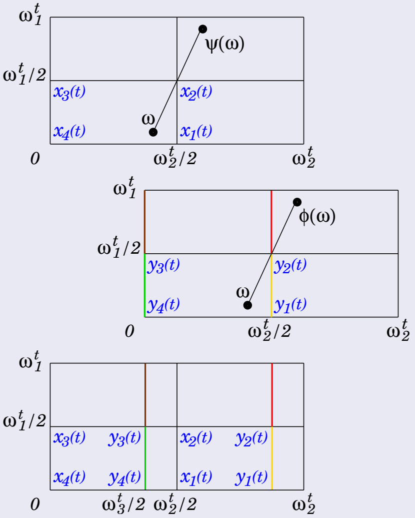

$$\psi(\omega) = -\omega + [\omega_1^t + \omega_2^t]$$

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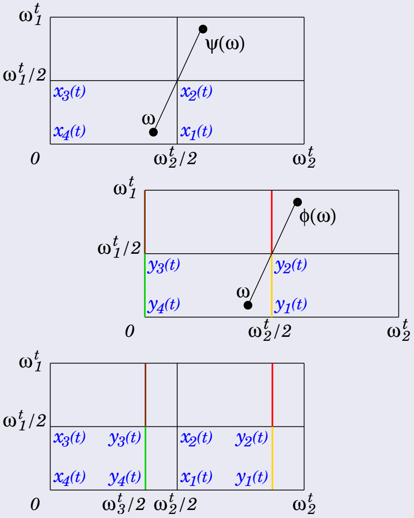

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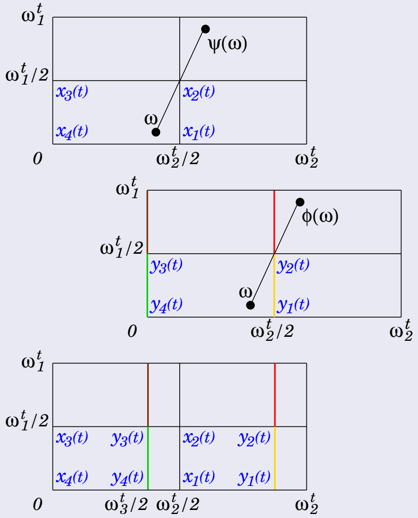

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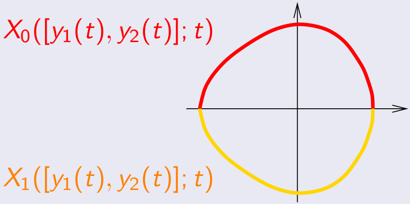
Conformal gluing function



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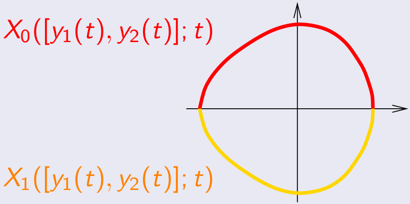

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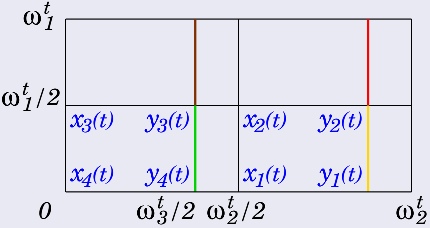

$$w(x(\omega);t)$$
$$\|$$
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Expression of the CGFs w & $\widetilde{w}$

We have

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- $\log_0$ is holonomic, with vanishing differential equation $xy' - 1 = 0$;
- For all $\ell \in \mathbb{Z}$, $\log_\ell$ is also holonomic, with the same equation:
    - Expression of $\log_\ell$ in terms of $\log_0$;

An enlightening example: the logarithm function

- The function
$$\log_0(x) = \log(|x|) + \imath \arg(x)$$
is holomorphic for $x \in \mathbb{C} \setminus \mathbb{R}_-$;
- $\log_0$ doesn't admit a direct meromorphic continuation through $\mathbb{R}_-$;
- $\log_0$ has a meromorphic continuation along a path going through $\mathbb{R}_-$, say $\log_1$;
- $\log_1$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}_-$ and
$$\log_1 = \log_0 + 2\pi\imath;$$
- Similarly we can define $\log_\ell$ for any $\ell \in \mathbb{Z}$, and
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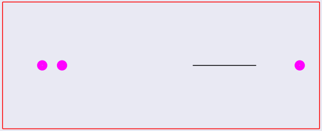
  - Expression of $\log_\ell$ in terms of $\log_0$;
  - Reasoning via a meromorphic continuation along a path.

Our reasoning

- The branches of $Q(x, 0; t)$:



$Q_0(x, 0; t)$

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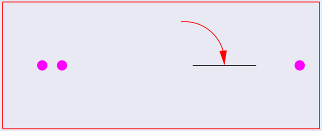

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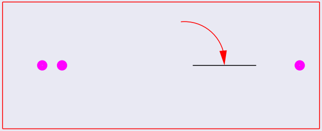

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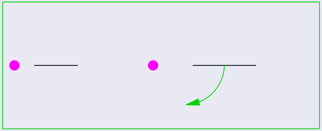

$Q_1(x,0;t)$

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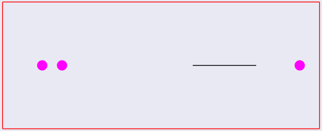

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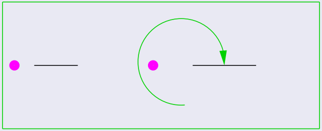

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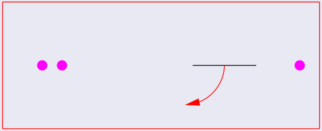

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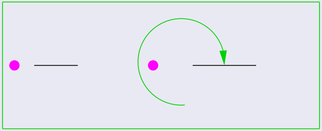

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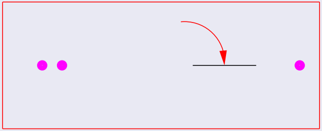

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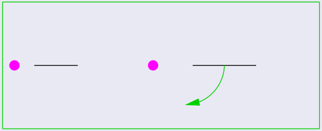

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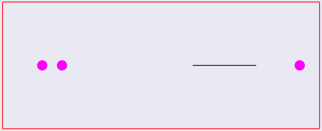

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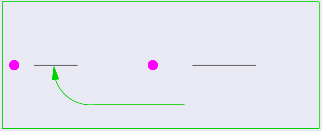

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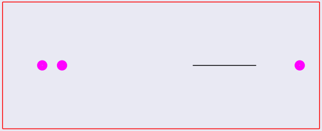

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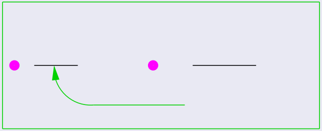

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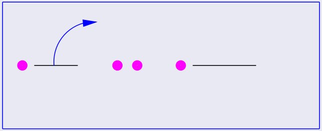

$Q_2(x, 0; t)$

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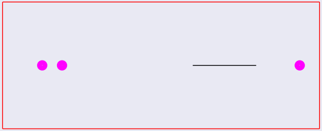

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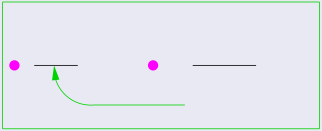

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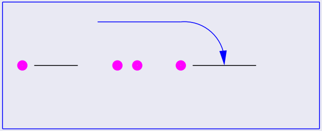

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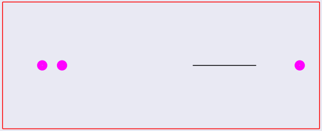

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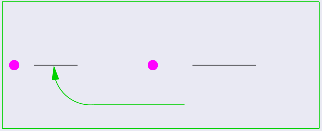

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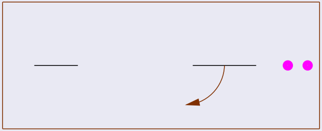

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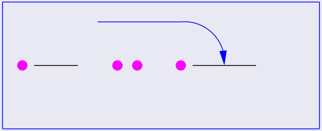

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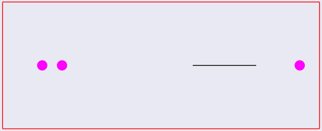

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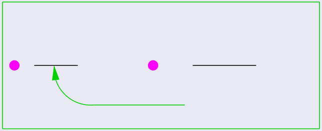

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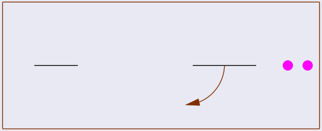

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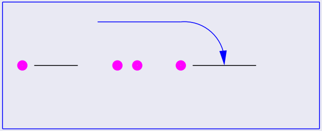
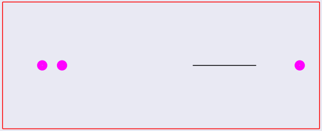


$Q_2(x,0;t)$

- There are infinitely many poles.

Our reasoning

- The branches of $Q(x,0;t)$:



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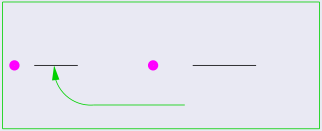

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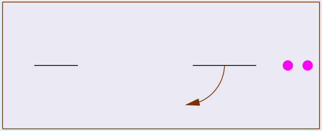

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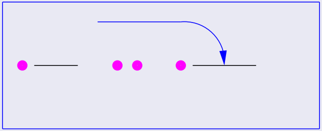
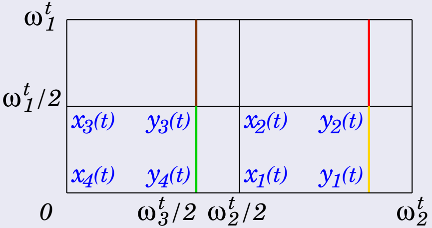


$Q_2(x,0;t)$

- There are infinitely many poles.
- If $Q(x,0;t)$ satifies a differential equation, all its branches satisfy the same equation.

The universal covering



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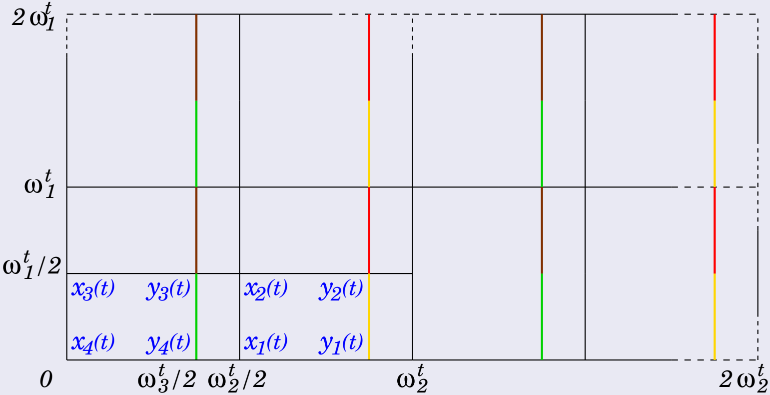

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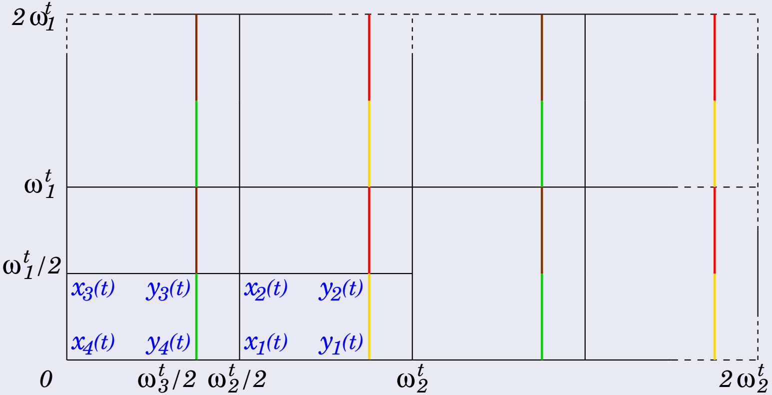

A functional equation for $q_x(\omega) = Q(x(\omega), 0; t)$ on the universal covering

We have: $q_x(\omega + \omega_3^t) = q_x(\omega) + xy(\omega + \omega_3^t) - xy(-\omega)$.

Consequence of the functional equation for $q_x(\omega) = Q(x(\omega), 0; t)$

Remember: $q_x(\omega + \omega_3^t) = q_x(\omega) + xy(\omega + \omega_3^t) - xy(-\omega)$.

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- Known results in the finite group case;

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Proof of the functional equation on the universal covering

$$KQ(x, y; t) = KQ(x, 0; t) + KQ(0, y; t) - KQ(0, 0; t) - xy.$$

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$$KQ(x, y; t) = KQ(x, 0; t) + KQ(0, y; t) - KQ(0, 0; t) - xy.$$

If $K(x, y; t) = 0$,

$$0 = KQ(x, 0; t) + KQ(0, y; t) - KQ(0, 0; t) - xy,$$
$$0 = KQ(\Phi(x, 0); t) + KQ(0, y; t) - KQ(0, 0; t) - \Phi(xy).$$

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Making the difference gives

$$KQ(\Phi(x, 0); t) - KQ(x, 0; t) = \Phi(xy) - xy.$$

Consequence of the functional equation for $q_x(\omega) = Q(x(\omega), 0; t)$

Remember: $q_x(\omega + \omega_3^t) = q_x(\omega) + xy(\omega + \omega_3^t) - xy(-\omega)$.

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Remember:

$$\Psi \circ \Phi \longleftrightarrow \omega \mapsto \omega - \omega_3^t, \qquad \Phi \longleftrightarrow \omega \mapsto -\omega + [\omega_2^t + \omega_3^t].$$

1 Introduction and main results
- Introduction
- Results

2 Proofs
- Explicit expression of the counting generating functions
  - Reduction to boundary value problems
  - Conformal gluing and uniformization
- Nature of the counting generating functions

3 Conclusion

Perspectives

- Non-holonomy of the counting generating functions in the variable z;

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Thanks for your attention!