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**Abstract:** This paper studies double imaginary characteristic roots in the case of time-delay systems with two delays as parameters. We aim to identify the direction in which double roots cross the imaginary axis, when the delay parameters change. To determine what happens when parameters are under a small perturbation, we present two methods: the algebraic and geometric approaches. Taking a theoretical example, we show that, even if the two methods are conceptually different, the provided results are consistent.

*Keywords:* Characteristic roots; Spectral analysis; Time delay; Stability analysis; Distributed Parameter Systems.

## 1. INTRODUCTION AND PROBLEM STATEMENT

The stability of dynamical systems with time-delays has been a problem of great interest in the last few decades. It has been shown that the presence of delays is responsible for rather complex behaviours, and it might induce stability or instability, function of the domain in which the delays take value. For systems with one delay as parameter, methods of identifying all the stable delay intervals are given in Lee and Hsu (1969) and Walton and Marshall (1987). As for systems with two delays as parameters, a rich collection of stability charts (the parameter regions where the system is stable) are presented in Stépán (1989).

In this paper, we study time-delay systems with two delays, and focus our attention on how the stability charts look like in a neighbourhood of a double (non semi-simple) root of the characteristic equation. We present two methods, the algebraic approach in Section 2 and the geometric approach in Section 3, and briefly compare them in Section 4. The algebraic approach we present involves the computation of eigenvectors corresponding to the double root. Geometric approach was introduced by Gu et al. (2005, 2015) and is based on the continuity of characteristic roots as functions of parameters (the two delays). Both methods aim to judge characteristic roots direction of crossing (the imaginary axis) when parameters change.

Consider a linear time-delay system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau_1) + A_2 x(t-\tau_2), \qquad (1)$$

where $x \in \mathbb{R}^n$ is the state vector, the constant delays $\tau_1$ and $\tau_2$ are real and positive, with $A_0, A_1, A_2 \in \mathbb{R}^{n\times n}$. The *characteristic matrix* of the system (1) is given by

$$M(s,\tau) = sI - A_0 - A_1 e^{-s\tau_1} - A_2 e^{-s\tau_2}, \qquad (2)$$

where $\tau = (\tau_1\ \tau_2)^T$, $I$ is the $n \times n$ identity matrix, and $s$ is the Laplace variable. Similar to the finite-dimensional case, the *characteristic equation* of system (1) is given by

$$\det M(s,\tau) = 0. \qquad (3)$$

The roots of (3) are called *characteristic roots* of system (1). Thus, the finite-dimensional nonlinear eigenvalue problem of system (1) can be written as

$$M(s,\tau)u = 0, \qquad (4)$$

where the vector $u \in \mathbb{C}^n \setminus \{0\}$ is called a *right eigenvector* corresponding to the characteristic root $s$. In a similar manner, we can also construct a *left eigenvector* $v^T \in \mathbb{C}^n \setminus \{0\}$ corresponding to the eigenvalue $s$, satisfying $v^T M(s,\tau) = 0$. Furthermore, the multiplicity of an eigenvalue as a root of the characteristic equation (3) is called *algebraic multiplicity.* An eigenvalue is called *simple* if its algebraic multiplicity is equal to one. Otherwise, the eigenvalue is called *multiple.* Note that, since matrices $A_0$, $A_1$ and $A_2$ are real, eigenvalues are real or appear in complex conjugate pairs. If in the case of a simple eigenvalue there is a single corresponding eigenvector (up to a scaling factor), a multiple eigenvalue can have one or several corresponding eigenvectors. The maximal number of linearly independent eigenvectors corresponding to a multiple eigenvalue is called *geometric multiplicity* of the eigenvalue. In other words, the geometric multiplicity is equal to the dimension of the null space of the characteristic matrix (2). The geometric multiplicity is less than or equal to the algebraic multiplicity. A multiple eigenvalue is called *semi-simple* if the algebraic and geometric multiplicities are equal, otherwise the eigenvalue is called *non semi-simple.* If there is a single eigenvector corresponding to the eigenvalue, then the eigenvalue is called *nonderogatory.* For instance, the nonderogatory case of zero eigenvalue, $s = 0$, with algebraic multiplicity two and geometric multiplicity one is known as Bogdanov-Taken singularity, and it has been extensively studied in the literature, especially in models with time delays (see Bogdanov (1975), Bogdanov (1981), Boussaada et al. (2014), Campbell and Yuan (2008), Faria (2003), Takens (1974)).

In this paper, we suppose that real matrices $A_0, A_1, A_2$ are such that the *characteristic function* $p(s,\tau) := \det M(s,\tau)$ (also called *quasi-polynomial*) of system (1) has the form

$$p(s,\tau) = p_0(s) + p_1(s)e^{-s\tau_1} + p_2(s)e^{-s\tau_2}, \qquad (5)$$

where $p_k(s)$, $k \in \{0,1,2\}$ are polynomials of $s$ with real coefficients.

*Remark 1.* For $n = 2$, consider

$$A_0 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

$$A_1 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, A_2 = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}.$$

If at least one of the conditions

$$\begin{aligned} &\text{(C1)}: \quad \{b_{11} = 0, b_{12} = 0, c_{11} = 0, c_{12} = 0\} \\ &\text{(C2)}: \quad \{b_{11} = 0, b_{21} = 0, c_{11} = 0, c_{21} = 0\} \\ &\text{(C3)}: \quad \{b_{12} = 0, b_{22} = 0, c_{12} = 0, c_{22} = 0\} \\ &\text{(C4)}: \quad \{b_{21} = 0, b_{22} = 0, c_{21} = 0, c_{22} = 0\} \end{aligned}$$

are satisfied, the characteristic function of system (1) can be written of the form (5). Note that conditions (C1)–(C4) are satisfied when matrices $A_1$ and $A_2$ are of rank 1. $\square$

*Remark 2.* (An equivalent eigenvalue problem). The eigenvalue problem (4) corresponding to the functional differential equation (1) is nonlinear and finite-dimensional. By considering (as, for instance, in Hale and Lunel (1993)) a linear operator $\mathcal{A} = A_0x(t) + A_1x(t-\tau_1) + A_2x(t-\tau_2)$, with the domain

$$\mathcal{D}(\mathcal{A}) = \left\{ \phi \,\middle|\, \phi, \frac{d\phi}{d\theta} \in \mathcal{C}\left([-\max\{\tau_1,\tau_2\},0], \mathbb{R}^n\right), \right.$$
$$\left. \phi(0) = A_0\phi(0) + A_1\phi(-\tau_1) + A_2\phi(-\tau_2) \right\},$$

such that $\mathcal{A}\phi = \frac{d\phi}{d\theta}$, where $\mathcal{C}\left([-\max\{\tau_1,\tau_2\},0], \mathbb{R}^n\right)$ is the space of continuous functions from $[-\max\{\tau_1,\tau_2\},0]$ to $\mathbb{R}^n$, we can rewrite system (1) as an abstract ordinary differential equation $\dot{x}_t = \mathcal{A}x_t$, where $x_t(\phi) \in \mathcal{C}\left([-\max\{\tau_1,\tau_2\},0], \mathbb{R}^n\right)$ is the function segment defined by $x_t(\phi)(\theta) = x(\phi)(t+\theta), \quad \theta \in [-\max\{\tau_1,\tau_2\},0]$. Note that the characteristic roots are the eigenvalues of $\mathcal{A}$. This allows us to write an infinite-dimensional and linear eigenvalue problem,equivalent to (4), $(sI - \mathcal{A})u = 0$, with $u \in \mathcal{C}\left([-\max\{\tau_1,\tau_2\},0], \mathbb{C}^n\right)$. In addition, the corresponding *eigenfunctions* of eigenvalue $s$ take the form $ue^{s\theta}$, with $\theta \in [-\max\{\tau_1,\tau_2\},0]$. However, the algebraic approach described in Section 2 is based on the finite-dimensional and nonlinear form of eigenvalue problem (4). $\square$

This paper studies the behaviour of double imaginary characteristic roots (double roots $s = \pm i\omega \neq 0$ of characteristic function (5)) under a small deviation of delay parameters ($\tau_1$ and $\tau_2$) and compares the geometric approach and the algebraic approach in the case of time-delay systems. By definition (see for instance Campbell and Yuan (2008); Gu et al. (2015)), we say that system (5) has a double root $s_0 = \pm i\omega_0$ at $\omega = \omega_0$ for $\tau_0 = (\tau_{10}\ \tau_{20})^T$ if and only if

$$p(s_0,\tau_0) = \left.\frac{\partial p}{\partial s}\right|_{\substack{s=s_0\\ \tau=\tau_0}} = 0 \quad \text{and} \quad \left.\frac{\partial^2 p}{\partial s^2}\right|_{\substack{s=s_0\\ \tau=\tau_0}} \neq 0.$$

In the sequel, we present the two approaches and illustrate their implications by considering the following example.

*Example 3.* (*Case study*) Let the characteristic matrix be

$$M(s,\tau) = \begin{pmatrix} s + 2e^{-s\tau_1}\cos(1) - 1 & 1 \\ -1 - e^{-s\tau_2} + 2e^{-s\tau_1}\sin(1) & s - 1 \end{pmatrix}. \qquad (6)$$

We compute the characteristic function corresponding to the characteristic matrix (6), and obtain

$$\begin{aligned} p(s,\tau) &= s^2 - 2s + 2 \\ &+ \left[(2\cos 1)\, s - 2\,(\cos 1 + \sin 1)\right] e^{-\tau_1 s} + e^{-\tau_2 s}. \end{aligned} \qquad (7)$$

For $\tau = \tau_0 = (\tau_{10}, \tau_{20})^T = (1,2)$, the characteristic function (7) has a double imaginary root $s = s_0 = \pm i\omega_0$ at $\omega_0 = 1$. $\square$

## 2. ALGEBRAIC APPROACH

The eigenvalue $s_0 = \pm i\omega_0 = \pm i$ in Example 3 is double and non semi-simple. This implies that there exist $u_0$ and $v_0$, a right, respectively left eigenvectors, and $u_1$ and $v_1$ a right, respectively left generalized eigenvectors, such that conditions

$$M_0 u_0 = 0 \qquad (8)$$
$$M_0 u_1 + M_1^0 u_0 = 0 \qquad (9)$$
$$v_0^T M_0 = 0 \qquad (10)$$
$$v_1^T M_0 + v_0^T M_1^0 = 0 \qquad (11)$$

are simultaneously satisfied (see Campbell and Yuan (2008), Chapter 7 in Hale and Lunel (1993)), where

$$M_0 = M(s_0,\tau_0), \qquad (12)$$
$$M_1^0 = \left.\frac{\partial M(s,\tau)}{\partial s}\right|_{\substack{s=s_0\\ \tau=\tau_0}}. \qquad (13)$$

*Proposition 4.* Eigenvectors $u_0$, $u_1$, $v_0$ and $v_1$ satisfy

$$v_0^T M_1^0 u_0 = 0 \qquad (14)$$
$$v_1^T M_1^0 u_0 = v_0^T M_1^0 u_1 \neq 0. \qquad (15)$$

**Proof.** Multiply equation (9) by $v_0^T$ on the left and obtain (14) in view of (10). Multiply equation (9) by $v_1^T$ on the left and obtain $v_1^T M_0 u_1 = -v_1^T M_1^0 u_0$. Multiply equation (11) by $u_1$ on the right and obtain $v_1^T M_0 u_1 = -v_0^T M_1^0 u_1$. Thus, we have (15) because $v_1^T M_1^0 \neq 0$ and $M_1^0 u_1 \neq 0$. $\square$

We consider a simple case, where the delay parameters are under a small perturbation $\varepsilon > 0$:

$$\tau_1(\varepsilon) = \tau_{10} + \varepsilon\tau_{11}, \qquad (16)$$
$$\tau_2(\varepsilon) = \tau_{20} + \varepsilon\tau_{21}. \qquad (17)$$

As $\tau(\varepsilon)$ is smooth, we can write a Taylor series expansion of the characteristic matrix (2). Moreover, provided that the eigenvalue is non semi-simple, we can write a Puiseux series expansion of it and of the corresponding right eigenvector (see, for instance, Kato (1995)):

$$s(\varepsilon) = s_0 + \varepsilon^{\frac{1}{2}} s_1 + \varepsilon s_2 + \dots \tag{18}$$

$$u_0(\varepsilon) = u_0 + \varepsilon^{\frac{1}{2}} w_1 + \dots \tag{19}$$

We can now replace equations (16), (17), (18), and (19) in the eigenvalue problem (4) and write

$$M(s(\varepsilon), \tau(\varepsilon))u(\varepsilon) = 0.$$

Note that we use series expansion of exponential functions, as $\varepsilon \to 0$ (for instance, we write $e^{-\varepsilon s_0 \tau_{10}} = 1 - \varepsilon s_0 \tau_{11} + \frac{\varepsilon^2 s_0^2 \tau_{11}^2}{2}$). We collect the terms of equal powers of $\varepsilon$, more precisely we collect only the first three orders ($\varepsilon^0$, $\varepsilon^{\frac{1}{2}}$, $\varepsilon^1$) and obtain

$$M_0 u_0 = 0,$$

$$M_0 w_1 + s_1 M_1^0 u_0 = 0, \tag{20}$$

$$s_1 M_1^0 w_1 + \left( \frac{s_1^2}{2} M_2^0 + s_2 M_1^0 + M_1^1 \right) u_0 = 0, \tag{21}$$

where the matrices $M_0$, $M_1^0$, $M_2^0$, and $M_1^1$ are given by

$$M_0 = s_0 I - A_0 - A_1 e^{-s_0 \tau_{10}} - A_2 e^{-s_0 \tau_{20}}, \tag{22}$$

$$M_1^0 = I + \tau_{10} A_1 e^{-s_0 \tau_{10}} + \tau_{20} A_2 e^{-s_0 \tau_{20}}, \tag{23}$$

$$M_2^0 = -\tau_{10}^2 A_1 e^{-s_0 \tau_{10}} - \tau_{20}^2 A_2 e^{-s_0 \tau_{20}}, \tag{24}$$

$$M_1^1 = \tau_{11} s_0 A_1 e^{-s_0 \tau_{10}} + \tau_{21} s_0 A_2 e^{-s_0 \tau_{20}}. \tag{25}$$

Notice that equations (22) and (23) are explicit expressions of (12) and (13), respectively. For the sake of uniqueness of $u_0$ and without any loss of generality, we can use the normalization

$$v_1^T M_1^0 u_0 = 1 = v_0^T M_1^0 u_1. \tag{26}$$

Divide equation (20) by $s_1 \neq 0$ and obtain $\frac{1}{s_1} M_0 w_1 = M_0 u_1$ in view of (9). Since $M_0 \neq 0$, we can write $w_1$ as

$$w_1 = s_1 u_1. \tag{27}$$

Replace (27) in equality (21) and multiply by $v_0^T$ on the left. We obtain

$$s_1^2 v_0^T M_1^0 u_1 + \frac{s_1^2}{2} v_0^T M_2^0 u_0 + s_2 v_0^T M_1^0 u_0 + v_0^T M_1^1 u_0 = 0. \tag{28}$$

Now we use the second equality from the normalization (26), and condition (14) from Proposition 4 in equation (28), and write the expression of $s_1$:

$$s_1 = \pm \sqrt{-\frac{v_0^T M_1^1 u_0}{1 + \frac{1}{2} v_0^T M_2^0 u_0}}. \tag{29}$$

Given the continuity properties of the spectrum (Chapter 1 in Michiels and Niculescu (2014)) and the Puiseux series expansion (18), it is easy to see that under a small perturbation of delay parameters (16)–(17) the double characteristic root $s_0 = i\omega_0$ splits up into two simple characteristic roots, which will move to the right or left half-plane function of the sign of $\mathrm{Re}(s(\epsilon))$. These results can be summarized in the following proposition and remarks.

*Proposition 5.* Assume that a double and non semi-simple characteristic root of system (5) is located on the imaginary axis, but not at the origin. Then, under a small perturbation of delay parameters of the form (16)–(17) this double characteristic root splits up into two simple roots, and each one of them will move towards stability (instability) if $\mathrm{Re}(s(\varepsilon)) < 0$ ($\mathrm{Re}(s(\varepsilon)) > 0$). □

*Remark 6.* Provided that $\mathrm{Re}(s_0) = 0$, we might suppose, in general, that the sign of $\mathrm{Re}(s(\varepsilon))$ is given by the sign of $\mathrm{Re}(s_1)$ as defined in equation (29), when $\varepsilon$ is very small. However, knowing the sign of $\mathrm{Re}(s_1)$ is not enough to conclude over a global tendency of the double root to move towards stability or instability, as we shall illustrate in the sequel. This is why we take a further step, multiply equation (21) with $v_1^T$ on the left, and obtain the following expression of $s_2$,

$$s_2 = \frac{v_0^T M_1^1 u_0}{1 + \frac{1}{2} v_0^T M_2^0 u_0} \left( v_1^T M_1^0 u_1 + \frac{1}{2} v_1^T M_2^0 u_0 \right) - v_1^T M_1^1 u_0, \tag{30}$$

in view of normalization (26). Other terms of $s(\varepsilon)$ might be found in a similar way. □

*Remark 7.* In order to approximate the value of $s(\varepsilon)$, function of $\varepsilon$, $\tau_{11}$, and $\tau_{21}$, we use $s(\varepsilon)$ defined as in equation (18), where $s_1$ is given by (29), $s_2$ is given by (30), with $M_1^0$, $M_2^0$, and $M_1^1$ given by (23)–(25), $u_0$ and $v_0^T$ are normalized right, respectively left eigenvectors, satisfying (26), and $u_1$ and $v_1$ are generalized right, respectively left eigenvectors, satisfying equations (8)–(11). □

*Remark 8.* Note that the double non semi-simple root will split up into two simple roots. The plus sign from equation (29) corresponds to one of these simple roots, and the minus sign corresponds to the other simple root. □

*Remark 9.* Roughly speaking, the double root will split up into two simple roots that will follow one of the two tendencies: either one root moves towards a half-plane, and the other root towards the other half-plane, or both roots moves towards the same half-plane. This means that we have two types of qualitative behaviour when a perturbation $\varepsilon$ arises. We illustrate both situations on Example 3. □

In the sequel, we consider the characteristic matrix (6) in Example 3. A right eigenvector $u_0$ satisfying (8) is of the form

$$u_0 = \begin{pmatrix} \alpha_1 + i\alpha_2 \\ -(\cos(2) - i(-1 + \sin(2)))\,(\alpha_1 + i\alpha_2) \end{pmatrix},$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$. We compute the generalized right eigenvector $u_1 = (\beta_1 + i\beta_2 \ u_{12})^T$, verifying (9), with $\beta_1, \beta_2 \in \mathbb{R}$ and

$$u_{12} = (\cos(2) - i\sin(2))\alpha_1 + (i\cos(2) + \sin(2))\alpha_2 - - (i + \cos(2) - i\sin(2))\,(\beta_1 + i\beta_2),$$

In the same manner we write the left eigenvector $v_0^T$ satisfying equation (10)

$$v_0 = \begin{pmatrix} (1 - i)\,(\gamma_1 + i\gamma_2) \\ \gamma_1 + i\gamma_2 \end{pmatrix},$$

and the left generalized eigenvector $v_1^T$ such that condition (11) holds:

$$v_1 = \begin{pmatrix} -\gamma_1 - i\gamma_2 + (1 - i)\delta_1 + (1 + i)\delta_2 \\ \delta_1 + i\delta_2 \end{pmatrix},$$

with $\gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$. The normalization condition (26) leads to the following constraint on $\alpha_1$, $\alpha_2$, $\gamma_1$, and $\gamma_2$:

$$\alpha_1 = 0, \qquad \alpha_2 = -\frac{1}{\gamma_2\,(\cos 2 + \sin 2 \tan 2)},$$

$$\gamma_1 = -\gamma_2 \tan 2, \qquad \gamma_2 \neq 0.$$

Therefore, the normalized eigenvectors satisfy

$$
\begin{aligned}
u_0 &= \frac{1}{\gamma_2}\begin{pmatrix} -i\cos 2 \\ \cos 2(-1 + i\cos 2 + \sin 2) \end{pmatrix}, \\
u_{12} &= \frac{\cos 2(-i\cos 2 - \sin 2)}{\gamma_2} - \\
&\quad - (\cos 2 - i(-1 + \sin 2))\,(\beta_1 + i\beta_2), \\
v_0 &= \gamma_2 \begin{pmatrix} (1+i)(1 + i\tan 2) \\ -\tan 2 + i \end{pmatrix}, \\
v_1 &= \begin{pmatrix} (-i + \tan 2)\gamma_2 + (1-i)\,(\delta_1 + i\delta_2) \\ \delta_1 + i\delta_2 \end{pmatrix}.
\end{aligned}
$$

Now we write $M_2^0$ and $M_1^1$ of the form

$$
\begin{aligned}
M_2^0 &= \frac{\partial^2}{\partial s^2} M(s,\tau)\Big|_{\substack{s=s_0\\ \tau=\tau_0}}, \\
M_1^1 &= \tau_{11}\frac{\partial}{\partial \tau_1} M(s,\tau)\Big|_{\substack{s=s_0\\ \tau=\tau_0}} + \tau_{21}\frac{\partial}{\partial \tau_2} M(s,\tau)\Big|_{\substack{s=s_0\\ \tau=\tau_0}},
\end{aligned}
$$

and recover the explicit formulae (24) and (25), respectively. More precisely, we obtain

$$
\begin{aligned}
M_2^0 &= \begin{pmatrix} 2e^{-i}\cos 1 & 0 \\ -i - (4-i)e^{-2i} & 0 \end{pmatrix}, \\
M_1^1 &= \begin{pmatrix} -2ie^{-i}\cos 1\tau_{11} & 0 \\ e^{-2i}\left(i\tau_{21} - \left(-1 + e^{2i}\right)\tau_{11}\right) & 0 \end{pmatrix}.
\end{aligned}
$$

We compute $s_1$ function of $\tau_{11}$ and $\tau_{21}$, using equation (29):

$$
s_1 = \pm\sqrt{\frac{(2i + (4+2i)e^{2i})\,\tau_{11} - 2i\tau_{21}}{-1 + (1-2i)e^{2i}}}.
$$

Using formula (30), we write $s_2$ function of $\tau_{11}$ and $\tau_{21}$

$$
\begin{aligned}
s_2 = {} & \frac{\left(\frac{3}{80} + \frac{i}{80}\right)\sec 2(\cos 1 - i\sin 1)}{(\cos 1 - (1-i)\sin 1)((1+i)\cos 1 + \sin 1)} \\
& [(1+i)((115 + 82i)\cos 1 + (87 + 14i)\cos 3 \\
& + (61 - 2i)\cos 5 + (9 + 18i)\cos 7 - (73 - 194i)\sin 1 \\
& - (53 + 70i)\sin 3 - (25 - 70i)\sin 5 - (45 + 18i)\sin 7)\tau_{11} \\
& + 2(\cos 1 - i\sin 1)((42 - 40i) - (21 - 4i)\cos 2 \\
& + (22 - 40i)\cos 4 - (3 - 4i)\cos 6 + (4 + 28i)\sin 2 \\
& + (4 + 4i)\sin 4 - (4 - 24i)\sin 6)\tau_{21}].
\end{aligned}
$$

Now that we have concrete expressions of $s_1$ and $s_2$ function of $\tau_{11}$ and $\tau_{21}$ for Example 3, we proceed with giving values to $\tau_{11}$, $\tau_{21}$, and $\varepsilon$ in order to illustrate Proposition 5. Suppose $\gamma_2 = \beta_1 = \beta_2 = \gamma_1 = \gamma_2 = 1$. We consider two simple cases, as follows, and compute the value of $s(\varepsilon)$ for each case:

| **case** | $\tau_{11}$ | $\tau_{21}$ | $\varepsilon$ | $s(\varepsilon)$ |
|---|---|---|---|---|
| **(a)** | 0 | 1 | 0.001 | $0.0076423 + 0.963458i$ |
| | | | | $0.0000623462 + 1.02993i$ |
| **(b)** | 0 | -1 | 0.001 | $0.0293821 + 1.0071i$ |
| | | | | $-0.0370867 + 0.999517i$ |

We notice that if we fix $\tau_1$ and increase $\tau_2$, then the double root splits up into two simple roots moving towards instability (with positive real part), as depicted in Figure 1, left. On the other hand, if we fix $\tau_1$ and decrease $\tau_2$, then the two simple roots will move one towards instability, and the other one towards stability, as illustrated in Figure 1, right. Figure 1 was obtained by using QPmR algorithm

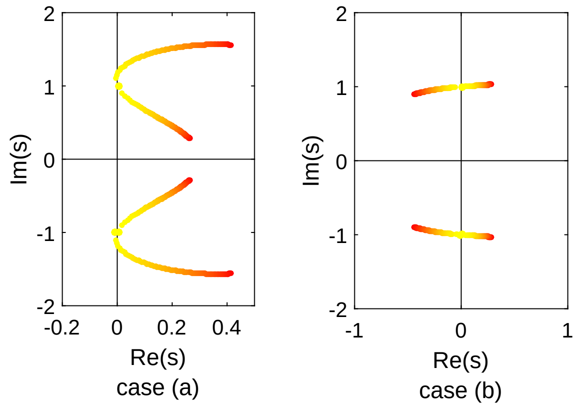


Fig. 1. Double charcteristic root behaviour for Example 3. Yellow points (lying on the imaginary axis) correspond to $\tau_{10} = 1$ and $\tau_{20} = 2$. Case (a): fix $\tau_1 = 1$ and increase $\tau_2$ from 2 to 3. Case (b): fix $\tau_1 = 1$ and decrease $\tau_2$ from 2 to 1.9.

developed by Vyhlidal and Zitek (2009). Yellow points $s = \pm i$ correspond to $\varepsilon = 0$, $\tau_1 = 1$ and $\tau_2 = 2$. Fix $\tau_1 = 1$ (i.e. $\tau_{11} = 0$). For case (a), the characteristic roots of the quasi-polynomial are computed by using QPmR algorithm, for each value of $\tau_2$, from 2 to 3 using a step of $10^{-3}$. In Figure 1 left, characteristic roots $s(\varepsilon)$ are represented by coloured points (yellow points corresponding to $\varepsilon = 0$). We remark that these points become red as $\tau_2$ increases, i.e. as perturbation $\varepsilon$ increases. We can see that double root $s = \pm i$ splits up into two simple roots that will move towards the right half-plane. This is coherent to our computation of $s(\varepsilon)$ of the form (18), using equations (29) and (30). For case (b), we decrease $\tau_2$ from 2 (yellow points) to 1.9 (red points) using a step of $10^{-3}$, and we notice in Figure 1 right that the double root splits up into two simple roots going in opposite directions, one to the left half-plane, and the other one to the right half-plane. This is also consistent with our prediction based on computing $s(\varepsilon)$ as in (18).

Thus, we identify two type of tendencies when a double non semi-simple root lying on the imaginary axis is subject to a perturbation: either one root will go into a half-plane, and the other root into the other half-plane, or both roots move towards the same half-plane.

## 3. GEOMETRIC APPROACH

In order to use this approach as described in Gu et al. (2015), the standing assumption

$$
D = \det\begin{pmatrix} \mathrm{Re}\left(\frac{\partial p}{\partial \tau_1}\right) & \mathrm{Re}\left(\frac{\partial p}{\partial \tau_2}\right) \\ \mathrm{Im}\left(\frac{\partial p}{\partial \tau_1}\right) & \mathrm{Im}\left(\frac{\partial p}{\partial \tau_2}\right) \end{pmatrix}_{\substack{s=s_0\\ \tau=\tau_0}} \neq 0 \qquad (31)
$$

has to be satisfied. It has been proven that there is a bijection between the complex plane (where the quasi-polynomial (5) characteristic roots lie) and parameter space ($\tau_1 - \tau_2$ space). More precisely, the parameter space is divided by the stability crossing curve $\mathcal{T}$ (defined as in Gu et al. (2005)) into an S-sector and a G-sector, as depicted in Figure 2.

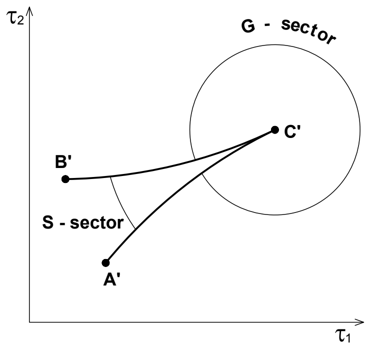


Fig. 2. $\tau_1 - \tau_2$ parameter space is divided by the stability crossing curve into a small sector, called S-sector, and a large sector, named G-sector.

The point C′ in Figure 2 corresponds to the double root $s_0 = \pm i\omega_0$ on the complex plane. This means that the coordinates of $\tau_1$ and $\tau_2$ corresponding to C′ are nothing else than $\tau_{10}$ and $\tau_{20}$, for which the quasi-polynomial (3) has a double root at $s_0 = \pm i\omega_0$. Moreover, for $\tau_1$ and $\tau_2$ taking values on the curve A′C′B′ (on the stability crossing curve $\mathcal{T}$), the quasi-polynomial (3) has simple characteristic roots $s = \pm i\omega$. Note that the stability crossing curve contains both positive and negative local stability crossing curves (A′C′ and B′C′). For instance, if A′C′ is the positive (negative) local stability crossing curve, and $(\tau_1, \tau_2)$ takes values on this curve, then the characteristic equation (3) has at least one imaginary characteristic root $s = i\omega$, with $\omega > \omega_0$ ($\omega < \omega_0$). In addition, the stability crossing curve $\mathcal{T}$ is a separation curve that divides $\tau_1 - \tau_2$ parameter space into regions, such that the number of characteristic roots on the right half complex plane remain constant as the parameters vary within each such region.

Theorem 7 in Gu et al. (2015) confirms what it has been discussed in the previous section, that when a perturbation occurs on the parameters $\tau_1$ and $\tau_2$ the characteristic roots move according to a specific pattern: either both roots move toward the same half-plane (corresponding to the case where the pair $(\tau_1, \tau_2)$ moves towards S-sector in the parameter space), or one root moves into the right half-plane and the other one into the left half-plane (corresponding to the case where the parameter set $(\tau_1, \tau_2)$ moves towards G-sector in the parameter space). However, judging towards which half-plane the two characteristic roots move when $(\tau_1, \tau_2)$ is in S-sector additionally requires knowing the sign of a parameter $\kappa$, defined as follows.

$$\kappa = \operatorname{Re}\left[\frac{\partial^2 p}{\partial s^2}\left(-\frac{\partial^3 p}{\partial s^3} + 3\frac{\partial^2 p}{\partial \tau_1 \partial s}\frac{\partial^2 \tau_1}{\partial u^2} + 3\frac{\partial^2 p}{\partial \tau_2 \partial s}\frac{\partial^2 \tau_2}{\partial u^2}\right)\right]_{\substack{s=s_0\\ \tau_1=\tau_{10}\\ \tau_2=\tau_{20}\\ \gamma=i}},$$

where $u$ represents the perturbation on $s$. We can evaluate $\frac{\partial^2 \tau_1}{\partial u^2}$ and $\frac{\partial^2 \tau_2}{\partial u^2}$, by using

$$\begin{bmatrix}\frac{\partial^2 \tau_1}{\partial u^2}\\ \frac{\partial^2 \tau_2}{\partial u^2}\end{bmatrix} = -\left[\begin{pmatrix}\operatorname{Re}\left(\frac{\partial p}{\partial \tau_1}\right) & \operatorname{Re}\left(\frac{\partial p}{\partial \tau_2}\right)\\ \operatorname{Im}\left(\frac{\partial p}{\partial \tau_1}\right) & \operatorname{Im}\left(\frac{\partial p}{\partial \tau_2}\right)\end{pmatrix}^{-1}\begin{matrix}\operatorname{Re}\left(\frac{\partial^2 p}{\partial s^2}\gamma^2\right)\\ \operatorname{Im}\left(\frac{\partial^2 p}{\partial s^2}\gamma^2\right)\end{matrix}\right]_{\substack{s=s_0\\ \tau_1=\tau_{10}\\ \tau_2=\tau_{20}}}, \tag{32}$$

with $\gamma = \pm i$. Notice that the inverse matrix in equation (32) exists due to hypotheses (31). Thus, the case where $(\tau_1, \tau_2)$ pair moves towards S-sector can be summarized in the following criterion.

*Criterion 10.* If $(\tau_1, \tau_2)$ is in the S-sector in a sufficiently small neighbourhood of $(\tau_{10}, \tau_{20})$, then the two characteristic roots in the neighborhood of $s_0$ are both in the left (right) half-plane if $\kappa < 0$ ($\kappa > 0$). □

In the sequel, we use the geometric approach and apply Criterion 10 on Example 3.

To begin with, we verify whether the standing assumption (31) is satisfied, and compute $D \simeq 1.74159 \neq 0$. The parameter space is depicted in Figure 3. The stability crossing curve has been computed as in Gu et al. (2005).

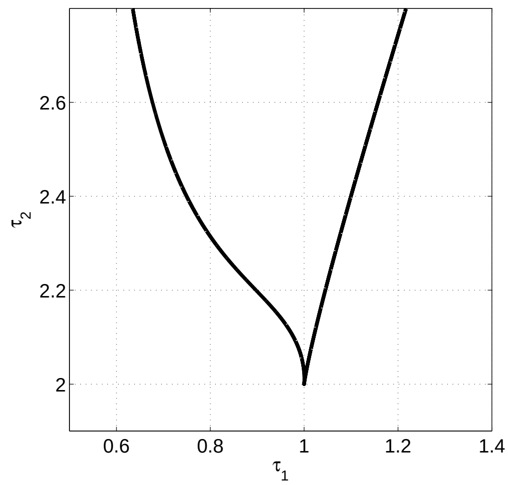


Fig. 3. $\tau_1 - \tau_2$ parameter space for Example 3.

We note that the neighbourhood of $(\tau_{10}, \tau_{20}) = (1, 2)$ is divided by the stability crossing curve $\mathcal{T}$ into an S-sector and a G-sector. The cusp at $(1, 2)$ corresponds to $s_0 = \pm i\omega_0$.

Provided that $D > 0$, we compute $\kappa \simeq 30.7082 > 0$. Therefore, according to Criterion 10, both imaginary roots move to the right half-plane as $(\tau_1, \tau_2)$ moves to S-sector. According to Theorem 7 from Gu et al. (2015), as $(\tau_1, \tau_2)$ moves to G-sector, one of the two imaginary roots moves to the right half-plane, the other moves to the left half-plane. In other words, as $(\tau_1, \tau_2)$ moves from the S-sector to the G-sector through the $(1, 2)$, one root moves from the right half-plane to the left half-plane passing through the point $i$ on the imaginary axis, another root on the right half-plane moves to touch the imaginary axis at $i$ then return to the right half-plane.

## 4. CONCLUDING DISCUSSION ON THE TWO METHODS

In this paper we have presented two methods that can be used to analyse the behaviour of a double non semi-simple characteristic root under a small perturbation of the parameters. More precisely, we have studied the case where the parameters are two delays of a system written of the form (1), for which the characteristic equation is given by (3).

The two methods are rather different and we denote them by *algebraic* and *geometric approach*, respectively.

However, both methods suggest that characteristic roots follow the same pattern when a perturbation occurs under the parameters.

The former method is based on eigenvectors computation. Even if in this paper we have only discussed the non semi-simple double roots case, a similar study can be made for double semi-simple characteristic roots. The latter method involves the computation of partial derivatives of the quasi-polynomial (3) with respect to the delays, up to the third order. Using the geometric approach implies that condition (31) must be satisfied. Nonetheless, one difference with respect to the method presented in Section 2 is that the geometric approach (and Criterion 10) presented in this paper applies to both semi-simple and non semi-simple characteristic roots. In other words, the main limitation when using the algebraic approach is that we have to separately treat semi-simple and non semi-simple double characteristic roots, and the geometric approach main limitation is that assumption (31) has to be satisfied.

Both approaches show that there are typically two type of situations when a deviation on the delays occurs in the parameter space: either the two imaginary roots move to the same half-plane of the complex plane, or one of the imaginary roots moves to the left half-plane and the other one moves to the right half-plane. This is rather surprising in the case of double non semi-simple characteristic roots: the trajectory of such a root when a perturbation $\varepsilon$ occurs is described by the equation (18), where $s_0$ lies on the imaginary axis, and $s_1$ is given by equation (29). Note that the double root splits up into two simple roots, the positive real part $s_1$ corresponding to a root, and the $s_1$ with negative real part corresponding to the other simple root. Some publications in the literature suggest that this is why non semi-simple double roots always splits up into two simple roots following only one type of tendency, which is to move towards different half-planes of the complex plane. The main message of this paper is that we have to pay attention to the quantification of the *small neighbourhood*, which remains an open problem at least for the algebraic approach. We can briefly illustrate this on Example 3, as follows.

We consider a point in S-sector, $(\tau_1, \tau_2) = (1.1, 2.6)$, as depicted in Figure 3. We have seen in Section 3 that in this case both imaginary roots move to the right half-plane. We can verify that we obtain the same result by using algebraic approach, by choosing $\varepsilon = 0.1$, $\tau_{11} = 1$ and $\tau_{21} = 6$ in equations (16)–(17). Thus, we compute $s(\varepsilon)$ written of the form (18), yielding the value of the two roots $s = 0.975521 - 0.254494i$ and $s = 0.826199 - 1.58361i$. As both roots have positive real part, they are both in the right half-plane. Of course, we can always find an extremely small value for $\varepsilon$ such that we end up with two simple roots lying on different sides of the imaginary axis, but this choice rises a natural question regarding the relevance of considering such a value for the control community. We conclude by saying that when choosing values for $\varepsilon$, $\tau_{11}$ and $\tau_{21}$ we have to decide upon what a *small neighbourhood* means, which remains an open question.

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