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Abstract: The interest in using delayed design is emphasized in recent research on the stabilization of finite/infinite dimensional dynamical systems. In particular a property called multiplicity-induced-dominancy (or MID for short) allowing for a reduced-complexity delayed-controller design showed its efficiency in fast damping of harmful oscillations. This contribution is concerned with oil-well drilling systems torsional vibrations, which constitute an important source of economic losses; drill bit wear, pipes disconnection, borehole disruption and prolonged drilling time, among other consequences. Such torsional vibrations are assumed to be governed by a wave equation with weak damping term. The MID-based design is further exploited to quench the torsional vibrations along the rotary drilling system. The proposed control law guarantees the existence of robustness margins with respect to delays and parameters uncertainties. Numerical investigation of the performance as well as the robustness of the corresponding control law will be presented.

1. INTRODUCTION

In this paper we consider the stabilization of torsional oscillations for a drilling system. Our approach is based on a multiplicity-induced-dominancy (MID) property allowing the design of a reduced-complexity delayed-controller (adjusted from Boussaada et al. (2019a)). The resulting closed-loop system is proved to be robust to uncertainties on the parameters and to delays in the loop.

The dynamical behavior of drill-strings is complex as many dynamic phenomena are involved, as vibrations, bending and twisting quasi-static motion, bit-rock interactions Kapitaniak et al. (2015); Spanos et al. (2003). In particular, the drill-string interaction with the borehole gives rise to a wide variety of non-desired oscillations Dunayevsky et al. (1998); Jansen (1995); Saldivar et al. (2011) which can be classified depending on the direction they appear. Among them, the longitudinal (or axial) oscillations may cause a bit-bouncing effect that is characterized by a repetitive loss of contact of the bit with the bottom of the well Aarsnes and Aamo (2016), Depouhon and Detournay (2014), Germay et al. (2009); Zhou and Krstic (2016). Torsional vibrations can appear due to downhole conditions (such as significant drag, tight or formation characteristics Nandakumar and Wiercigroch (2013) for instance) or due to side forces induced by Coulomb friction Aarsnes and Shor (2018). They are known as *stick-slip* and are characterized by a series of *stick* (the bit stop rotating) and *slip* (a sudden release of energy) Kamel and Yigit (2014). These self-excited undesirable vibrations may cause fatigue of the equipment or a deterioration of the performance of the process implying a reduction of the Rate of Penetration (ROP). They can also lead to premature failing of the bit or provoke premature wear and tear of drilling equipment resulting in fatigue and induced failures such as pipe wash-out and twist-off Mason et al. (1996). Thus, they may cause catastrophic damages and at least wear to expensive components of the drill-string Kriesels et al. (1999). In this context, a clear understanding of drill-string dynamics appears to be crucial to control these vibrations and consequently improve the performance of drilling systems (ROP), prevent any eventual damage and reduce safety risks. This explains why an extensive research effort has been conducted in the last five decades to suppress these undesirable oscillations and in so far reduce the costs of failures and increase the ROP (see the references cited in the survey Leine (1997) and in Saldivar et al. (2016)).

During the drilling process, the operator wants to control the downhole behavior of the drill-string (e.g. reach a given rpm, a given orientation...) and optimize the ROP, while avoiding undesired oscillations. To do so, it is usually possible to impose (using the rotary table) the weight on the drill-string and the torque at the surface. Automated control laws have been designed to solve such control problems Serrarens et al. (1998). Recent control law (which are not simple PID controllers) are usually based on distributed parameter models. More precisely, the drill-string dynamics can be modeled by a set of hyperbolic partial differential equations (namely wave equations) coupled with ODEs through their boundaries Di Meglio and Aarsnes (2015). Such wave equations can then be rewritten as neutral systems. In general, it may not be possible to

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efficiently stabilize such systems using classical control laws. Specific techniques have to be developed as an approximation of delay systems by a finite-dimensional representation (early-lumping) may lead to the same degree of complexity in the control design which can be potentially harmful for the stability of the system. Consequently, different approaches have been developed to provide realistic and effective control laws. A flatness-based approach has for instance been developed in Aarsnes et al. (2018). More recently, a low complexity control law with a reduced number of tuning gains has been proposed in Boussaada et al. (2019a) in the case of a vertical well. The controllers gains are chosen such that the closed-loop system has a particular structure and the corresponding spectrum lies in the complex left-half plane.

In this paper, we modify the control law introduced in Boussaada et al. (2019a), which is based on the MID property, by adding a derivative action, thus introducing an additional degree of freedom that facilitates a targeted placement of the poles of the closed-loop system. The proposed control law only requires the measurement of the states at the boundaries of the system and consequently does not need the design of a state observer. This is of specific interest as the design of an observer for this class of system may be difficult or computationally expensive (see Aarsnes et al. (2019); Auriol et al. (2020) in which state-observers are designed usign the backstepping approach.) Moreover, we analyze the robustness properties of the resulting closed-loop system. More precisely, we show that it is robust with respect to uncertainties on the parameters and delays in the actuation. These results are stated in a general framework of quasi-polynomials functions since, for the proposed control law, the closed-loop system can be expressed using such a structure. It is worth mentioning that the rightmost root for quasipolynomial function corresponding to stable retarded time-delay systems (also in the neutral case under some assumptions) is actually the exponential decay rate of its time-domain solution, (see Mori et al. (1982) for an estimate of the decay rate for stable linear delay systems or Boussaada et al. (2016) for the dominancy properties). This dominancy property presented in Hayes (1950) was used by the authors to design the control law in Boussaada et al. (2019a). The robustness aspects are considered by means of an algebraic analysis in the Laplace domain.

The paper is organized as follows. In Section II, we give the PDE model describing the torsional vibration of an oilwell rotary drilling system. This model is then expressed as a neutral equation. A stabilizing control law that guarantees the damping of the different undesired vibrations is given in Section III. Section IV is devoted to the robustness analysis of the proposed control law. The different results are expressed in a general framework of quasi-polynomials functions. An illustrative example completes the paper in Section V. Some concluding remarks are given in Section VI.

2. PROBLEM FORMULATION

In this section we start by providing an hyperbolic partial differential equations model to describe the torsional and axial motions of the drill-string vibrations. Then, we reduce such a model to a functional differential equation of neutral type.

2.1 A Coupled Axial/Torsional Vibrations Model

Let us consider a vertical well for which the rotary table (located at the top of the drill-string) sets the drill-string into a rotary motion around its main axis. The dynamics of interest can be obtained by assuming elastic deformations and using equations of continuity and state. More precisely, the torsional and axial vibrations can be modeled by a set of hyperbolic PDEs (namely wave equations) coupled through their boundaries. In what follows, we denote L the length of the drill-string, $\Phi(z,t)$ the rotary angle and $U(z,t)$ the longitudinal position. They are are functions of (z,t) evolving in $\{(z,t)|\ z \in [0,L],\ 0 < t < T\}$, the position $z=0$ corresponding to the top of the drill-string and the position $z=L$ to the bottom. We have Boussaada et al. (2012b):

$$\partial_z^2 \Phi(z,t) = \tilde{c}^2 \partial_t^2 \Phi(z,t), \tag{1a}$$
$$\partial_z^2 U(z,t) = c^2 \partial_t^2 U(z,t), \tag{1b}$$

with the boundary conditions

$$GJ\partial_z \Phi(0,t) = \beta \partial_t \Phi(0,t) - u_T(t) \tag{2a}$$
$$GJ\partial_z \Phi(L,t) = -I_B \partial_t^2 \Phi(L,t) - T(\partial_t \Phi(L,t)), \tag{2b}$$

and

$$E\Gamma \partial_z U(0,t) = \alpha \partial_t U(0,t) - u_H(t) \tag{3a}$$
$$E\Gamma \partial_z U(L,t) = -M_B \partial_t^2 U(L,t) - T(\partial_t \Phi(L,t)). \tag{3b}$$

The axial and torsional propagation velocities v_U, v_Φ, are defined as: $v_U = c^{-1}$ and $v_\Phi = \tilde{c}^{-1}$. They depend on the system physical parameters (Young modulus E, shear modulus G and density ρ_a), by means of :$c = \sqrt{\frac{\rho_a}{E}}$ and $\tilde{c} = \sqrt{\frac{\rho_a}{G}}$. The two actuators are denoted u_H and u_T. They respectively correspond to the brake motor control (upward hook force) and to the torque produced by the rotary table motor. The functions $\alpha \partial U_t(0,t)$ corresponds to a friction force of viscous type (where α is the viscous friction coefficient). Finally, the term $\beta \partial_t \Phi(0,t) - u_T(t)$ corresponds to the difference between the motor speed and the rotational speed of the first pipe.

The different geometrical and physical parameters of the drill-string are assumed to be spatially constant. Note that this assumption may be idealistic since the Bottom Hole Assembly (lower part of the drillstring) is usually made of bigger pipes with different physical properties. However, the length of the BHA is usually much lower and its effect can consequently be lumped into the ODE. We denote Γ the shear modulus, J the polar moment of inertia, M_B the mass of the drill-string, I_B the inertia moment [1] of the drill bit. Finally the function T corresponds to the frictional torque resulting from the interaction between the drill bit and the rock. We model it by the following odd function Boussaada et al. (2012a): $T(x) = \frac{2\bar{k}x}{k^2+x^2}$, for which we have the following local approximation:

[1] The inertia moment is such that $I_B = M_B\, r^2$, where r is taken as the averaged radius of drillpipe.

$$T(x) = \frac{2}{\bar{k}}\, x + O\left(x^3\right),$$

For a further discussion on friction models in rotary drilling systems, the reader is invited to see Marquez et al. (2015). This frictional torque is one of the cause of the torsional oscillations that creates the stick-slip phenomenon. However, it is worth mentioning that the stick-slip oscillations may also be caused by a negative difference between static and kinematic along-string Coulomb-type friction Aarsnes and Shor (2018). It emphasizes the action of nonlinear forces along the drill-string (which are combined with the bit rock interaction) in the torsional oscillatory behavior of drilling systems. Although the effect of these side-forces is not really important for vertical wells, they have a significant importance when the well is not vertical. The numerical values of the different parameters are given in Table 1.

Table 1. Numerical values of the system parameters.

Sym.	Param.	Numerical value
L	String length	1000 m
G	Shear modulus	79.3×10^9 N m^{-2}
Γ	Drillstring cross-section	35×10^{-4} m^2
J	Second moment of area	1.19×10^{-5} m^4
I_B	Lumped BHA inertia	89 Kg m^2
ρ_a	Density	8000 Kg m^{-3}
β	Angular momentum	2000 N m s
γ	Damping constant	6,053 10^{-7} $\frac{Nms}{rad}$
$\bar{k}$	Friction top angle	2.6 10^{-3}

2.2 Neutral-type model for the torsional oscillations

In this section we only consider the evolution of the rotary angle $\Phi(z,t)$ that corresponds to the torsional oscillations (1a) with the B.C. (2a)-(2b). Using the method of the characteristics, it is possible to rewrite the state $\Phi(L,t)$ as the solution of a neutral-type equation, thus relating the system variables at both ends of the drilling rod. We can show that there is a one-to-one correspondence between the solutions of the mixed problem for hyperbolic PDE and the initial value problem for the associated system of functional equations Rasvan et al. (1975). This is of interest as it implies that the techniques developed for time delay systems can be used for hyperbolic PDEs.

Namely, as described in Boussaada et al. (2019a), using the linearization of T given above, the system (1a) with the boundary conditions (2a)-(2b) reduces to a to an I/O system which is a *neutral system* of *order 3*:

$$\begin{cases} \left(-\frac{\beta I_B}{\alpha_L 2GJc} - \frac{I_B}{2\alpha_L}\right)\dddot{\Phi}_L(t) + \left(\frac{\beta I_B \alpha_L}{2GJc} - \frac{I_B\alpha_L}{2}\right)\dddot{\Phi}_L(t-2\tau) \\ -\left(\frac{\beta}{2\alpha_L} + \frac{GJc}{2\alpha_L}\right)\ddot{\Phi}_L(t) + \left(\frac{GJc}{2}\alpha_L - \frac{\beta\alpha_L}{2}\right)\ddot{\Phi}_L(t-2\tau) \\ -\left(2\frac{GJc}{2\alpha_L}\zeta + \frac{\beta}{2\alpha_L}\right)\dot{\Phi}_L(t) + \left(2\frac{GJc\zeta\alpha_L}{2} - \frac{\beta\zeta\alpha_L}{2}\right)\dot{\Phi}_L(t-2\tau) \\ + \frac{\zeta^2 GJc}{2\alpha_L}\Phi_L(t) + \frac{\zeta^2 GJc}{2\alpha_L}\Phi_L(t-2\tau) \\ = \frac{1}{\bar{k}}\left(\alpha_L\ddot{\Phi}_L(t-2\tau) + \alpha_L^{-1}\ddot{\Phi}_L(t)\right) - \frac{2}{\bar{k}}\left(\alpha_L\ddot{\Phi}_L(t-2\tau)\right. \\ \left. - \alpha_L^{-1}\ddot{\Phi}_L(t)\right) - \dot{u}_T(t-\tau) - \zeta u_T(t-\tau), \end{cases} \tag{4}$$

The delay $\tau = cL$ represents the total transport time for the torsional wave to travel from one to the other extremity of the drillstring.

3. LOW COMPLEXITY CONTROLLER DESIGN

In this section, we design a stabilizing control law that stabilizes (4). This control law is inspired from Boussaada et al. (2019a) and is based on the *multiplicity induced-dominancy* (MID) property which was underlined in Boussaada et al. (2016, 2018b) for retarded time-delay systems and exploited in several applications Boussaada et al. (2017, 2018a); Boussaada and Niculescu (2018). However, due to the specific structure of the system (neutral), the MID properties for time-delay systems of neutral type had to be adjusted. This has been done in Boussaada et al. (2019a) for instance. Compared to Boussaada et al. (2019a), the control law we propose in this paper includes a action on **the derivative term**. This introduces a supplementary degree of freedom that can be used to enforce the characteristic equation of the closed-loop system to have multiple negative spectral values. Under appropriate conditions, such a multiple root may be the dominant one among the spectrum, guaranteeing the stability of the closed-loop system. Adding this additional degrees of freedom (by means of this derivative term) means that the multiplicity of the dominant root can be increased compared to Boussaada et al. (2019a), thus improving the closed-loop performance. More precisely the control law we consider is defined by $u_T(t) = \kappa_1\,\Phi_L(t-\tau) + \kappa_2\,\dot{\Phi}_L(t-\tau)$, where the delay τ is the same as the intrinsic delay of the nominal system. Consequently, we have $\dot{u}_T(t) = \kappa_1\,\dot{\Phi}_L(t-\tau) + \kappa_2\,\ddot{\Phi}_L(t-\tau)$ and the closed-loop system (4) rewrites

$$\begin{cases} a_3\dddot{\Phi}_L(t) + b_3\dddot{\Phi}_L(t-2\tau) \\ a_2\ddot{\Phi}_L(t) + b_2\ddot{\Phi}_L(t-2\tau) \\ a_1\dot{\Phi}_L(t) + b_1\dot{\Phi}_L(t-2\tau) \\ + a_0\Phi_L(t) + b_0\Phi_L(t-2\tau) = 0, \end{cases} \tag{5}$$

where:

$$\begin{cases} a_3 = -\frac{\beta I_B}{\alpha_L 2GJc} - \frac{I_B}{2\alpha_L},\; b_3 = \frac{\beta I_B\alpha_L}{2GJc} - \frac{I_B\alpha_L}{2}, \\ a_2 = -\frac{\beta}{2\alpha_L} - \frac{GJc}{2\alpha_L} - \frac{3}{\bar{k}\alpha_L},\; b_2 = \kappa_2 + \frac{GJc}{2}\alpha_L - \frac{\beta\alpha_L}{2} + \frac{\alpha_L}{\bar{k}}, \\ a_1 = -\frac{GJc}{\alpha_L}\zeta - \frac{\beta}{2\alpha_L},\; b_1 = \kappa_1 + \kappa_2\zeta + GJc\zeta\alpha_L - \frac{\beta\zeta\alpha_L}{2}, \\ a_0 = \frac{\zeta^2 GJc}{2\alpha_L},\; b_0 = \kappa_1\zeta + \frac{\zeta^2 GJc}{2\alpha_L} \quad \text{where} \\ c = \sqrt{\frac{\rho_a}{G}},\; \zeta = \frac{\gamma}{2c^2},\; \alpha_L = e^{-cL\zeta},\; \tilde{\tau} = 2\tau = 2cL, \end{cases} \tag{6}$$

Such a system is of neutral type (since $a_3 \neq 0$ and $b_3 \neq 0$). Using the Laplace transform, we can rewrite it as a quasipolynomial including one delay.

$$\begin{cases} L(s) = P(s) + Q(s)e^{-s\tilde{\tau}}, \\ P(s) = \sum_{i=0}^{3} a_k s^k, \quad Q(s) = \sum_{i=0}^{3} b_k s^k, \end{cases} \tag{7}$$

The main results can be resumed as follows.

Proposition 1. Boussaada et al. (2019a) The linearized system (5) subject to the controller $u_T(t) = \kappa_1\, \Phi_L(t-\tau) + \kappa_2\, \dot{\Phi}_L(t-\tau)$ has the following properties in closed-loop :

- The maximal multiplicity of any root of the quasipolynomial L given in (7) with arbitrary coefficients $(a_k, b_k)_{0\le k\le 3}$ is bounded by 7.
- If $|\frac{b_k}{a_3}| < 1\ \forall\, 0 \le k \le 3$ and such a maximal multiplicity is reached, then the corresponding spectral value is real and dominant.

The proposed control law u_T introduces two degrees of freedom by means of κ_1 and κ_2. These two tuning parameters can be used to modify the dominant root of L. This dominant root can somehow be seen as an additional tuning parameter (under the constraint that the coefficients (b_0, b_1, b_2) must verify the constraint (5)). Having three degrees of freedom, it may become possible to find a dominant root of order 3. This is a considerable improvement compared to Boussaada et al. (2019a), where the dominant root was simply of order 2. More precisely, we have the following proposition.

Proposition 2. If the system parameters are chosen in Table 1 and if b_0, b_1, b_2 verify (8)-(10), then $s_0 = -3$ is a triple root for (7). In particular, if the parameter values are the ones given in Table 1 then the controller gains satisfy $\kappa_1 = -16929630.76, \kappa_2 = -302236.4149$ and the triple root at $s_0 = -3$ is dominant. In such a case the trivial solution is asymptotically stable.

Sketch of the proof The proof is based on the argument principle applied on the translation of standard Bromwich contour at $s_0 = -3$, which allows to count the roots of the quasipolynomial with $\Re(s) > -3$. by defining a specific integration contour γ. Then an integral over γ is defined as the sum of the integrals over the directed smooth curves that make γ up. See Boussaada et al. (2019b) for details.

It is worth mentioning that the proposed control law only requires the value of the state at the boundary $x = L$. If a sensor located at $x = L$ is available, this implies that we do not need to design a state-observer. This is of specific interest as it means that the proposed approach is not computationally expensive. More precisely, the design of a state observer may require solving a set of PDEs in real time (see Aarsnes et al. (2019); Auriol et al. (2020) for the design of state-observers using the backstepping approach). However, from a physical point of view it may be hard and expensive to have a sensor available at the bottom of the well. Moreover, the frequency of such (noisy) measurements would be extremely low. Thus, for the considered drilling example, the design of an observer may be necessary. We believe that this can be done following a similar approach, which would be much easier to implement compared to Aarsnes et al. (2019) (even if in this reference, the authors consider a more general case of a non-vertical well subject to non-linear side-forces friction terms). However, for other physical applications for which such a sensor would be available, this is an crucial asset of our approach.

4. ROBUSTNESS ASPECTS

In this section, we consider the robustness properties of the control law $u_T(t) = \kappa_1\, \Phi_L(t-\tau) + \kappa_2\, \dot{\Phi}_L(t-\tau)$ we have designed in the previous section. These results will be stated in a general framework. More precisely, let us consider two quasi-polynomial functions $G(s)$ and $K(s)$ respectively defined by

$$G(s) = \sum_{k=0}^{n} a_k s^k + \sum_{k=0}^{n} b_k s^k \mathrm{e}^{-s\tau}, \qquad (11)$$

$$K(s) = \sum_{k=0}^{p} c_k s^k + \sum_{k=0}^{p} d_k s^k \mathrm{e}^{-s\tau}, \qquad (12)$$

where s denotes the Laplace variable, and where the coefficients a_k, b_k, c_k, d_k and τ are real and constant. We assume that $a_n b_n \neq 0$. This means that the equation $G(s) = 0$ corresponds to the characteristic equation associated to a neutral system of order n. Let us consider the system defined in the Laplace domain by

$$G(s)X(s) = u(s), \qquad (13)$$

where X denotes the state of the system and where u denotes the actuation. Let us define the control law u as a feedback with the following structure

$$u(s) = K(s)X(s). \qquad (14)$$

Consequently, the closed-loop system rewrites as a neutral equation

$$(G(s) - K(s))X(s) = 0. \qquad (15)$$

In this section, we will show that if the control operator $K(s)$ guarantees the exponential stability of the closed-loop system (15), then, under some assumptions on the structure of K, this closed-loop system is robust to small uncertainties in the parameters and to delays in the actuation. The robustness with respect to delays in the actuation (*delay-robustness*) requires a nice behavior of the open-loop system at high frequencies. More precisely, we need to guarantee that the principal part of the operator associated to $G(s)$ generates an exponentially stable semigroup. This leads to the following assumption

Assumption 3. The coefficients b_n and a_n verify

$$|b_n| < |a_n|.$$

If this assumption does not hold, then the operator $G(s)$ has an infinite number of zeros in the complex right half plane (RHP) which means that any linear control will lead to a zero delay margin (Logemann et al., 1996, Theorem 1.2) i.e. for any control law u(t) the closed-loop system becomes unstable when there is an (arbitrarily small) delay $\delta > 0$ in the loop. Finally, the approach we propose to prove the existence of robustness margins requires the following assumption

Assumption 4. The degree of the the quasi-polynomial function K is chosen such that $n > p$ (i.e. the order of the open-loop transfer function is larger than the order of the control transfer function).

Let us now assume that the different parameters of the system (i.e. the coefficients that appear in the definition of the function G) are subject to small uncertainties and that there is a delay in the loop. More precisely, let us assume that the real plant operator now rewrites

$$b_0 = -\frac{(9a_0 - 27a_1 + 81a_2 - 243a_3)\tau^2}{2e^{3\tau}} - \frac{(6a_0 - 54a_2 + 324a_3)\tau}{2e^{3\tau}} - \frac{-54e^{3\tau}b_3 + 2a_0 - 54a_3}{2e^{3\tau}}, \tag{8}$$

$$b_1 = -\frac{(3a_0 - 9a_1 + 27a_2 - 81a_3)\tau^2}{e^{3\tau}} - \frac{(a_0 + 3a_1 - 27a_2 + 135a_3)\tau}{e^{3\tau}} - \frac{-27e^{3\tau}b_3 + a_1 - 27a_3}{e^{3\tau}}, \tag{9}$$

$$b_2 = -1/2\frac{(a_0 - 3a_1 + 9a_2 - 27a_3)\tau^2}{e^{3\tau}} - 1/2\frac{(2a_1 - 12a_2 + 54a_3)\tau}{e^{3\tau}} - 1/2\frac{-18e^{3\tau}b_3 + 2a_2 - 18a_3}{e^{3\tau}}. \tag{10}$$

$$G_u(s) = \sum_{k=0}^{n} \bar{a}_k s^k + \sum_{k=0}^{n} \bar{b}_k s^k e^{-s\bar{\tau}}, \tag{16}$$

where for all k we have $\bar{a}_k = a_k + \delta_{a_k}$, $\bar{b}_k = b_k + \delta_{b_k}$, $\bar{c}_k = c_k + \delta_{c_k}$, $\bar{d}_k = d_k + \delta_{d_k}$ and $\bar{\tau} = \tau + \delta_\tau$, the terms δ_{a_k}, δ_{b_k}, δ_{c_k}, δ_{d_k} and δ_τ representing small (unknown) constant uncertainties acting on the different parameters of the systems. We denote κ their common upper-bound. We also consider a delay $\delta > 0$ acting on the actuation. Consequently, the closed-loop system now rewrites

$$(G_u(s) - K(s)e^{-s\delta})X(s) = 0. \tag{17}$$

The next proposition guarantees the existence of robustness margins for the control law $u(t)$ defined in (14).

Proposition 5. Suppose that Assumption 3 and Assumption 4 are satisfied. Suppose that the control law $u(s) = K(s)X(s)$ stabilizes the nominal system (13). Then, there exist $\delta_{\max} > 0$ and $\kappa_0 > 0$ such that for every delay $\delta < \delta_{\max}$ and for every set of uncertainties δ_{a_k}, δ_{b_k}, δ_{c_k}, δ_{d_k} and δ_τ such that $\kappa < \kappa_0$, the state $X(s)$ solution of the uncertain neutral system (17) exponentially converges to zero.

Proof. The characteristic equation associated to the system (17) is given by

$$G_u(s) - K(s)e^{-s\delta} = 0. \tag{18}$$

We need to prove that the solutions of this equation are all located in the complex left half-plane if the delays and uncertainties are small enough. Equation (18) rewrites for $s \neq 0$

$$\bar{a}_n + \bar{b}_n e^{-s\bar{\tau}} = -\frac{1}{s^n}\sum_{k=0}^{n-1}(\bar{a}_k + \bar{b}_k e^{-s\bar{\tau}})s^k + \frac{1}{s^n}e^{-s\delta}\Big(\sum_{k=0}^{p} c_k s^k + \sum_{k=0}^{p} d_k s^k e^{-s\tau}\Big), \tag{19}$$

Since Assumption 3 is satisfied, there exist $\bar{\kappa} > 0$ such that if $\kappa < \bar{\kappa}$, then $|\bar{b}_n| < |\bar{a}_n|$. Thus (see Hale and Lunel (2013)), the left part of (19) is lower-bounded by a constant $\eta > 0$ for $s \in \mathbb{C}^+ = \{s \in \mathbb{C}, \Re(s) \geq 0\}$. Due to Assumption 4 the right part of (19) converges to zero when $|s|$ goes to infinity. Thus, there exists $M_0 > 0$ such that for all $s \in \mathbb{C}^+$ with $|s| > M_0$, the right part of (19) is smaller than $\frac{\eta}{2}$. This implies that (18) does not have a solution on $\mathbb{C}^+$ if $|s| > M_0$. Let us now consider $s \in \mathbb{C}^+$ such that $|s| \leq M_0$. Equation (18) rewrites

$$G(s) - K(s) = (G(s) - G_u(s)) - K(s)(1 - e^{-s\delta}). \tag{20}$$

Since the control law $u(s) = K(s)x(s)$ stabilizes the nominal system, there exists $\eta_1 > 0$ such that for all $s \in \mathbb{C}^+$ we have $|G(s) - K(s)| > \eta_1$. Since $|s| \leq M_0$, the function K is bounded on the considered domain. Consequently, there exists $\delta_{\max} > 0$ such that $|K(s)(1 - e^{-s\delta})| < \frac{\eta_1}{4}$. Similarly, we have the existence of $\tilde{\kappa} > 0$ such that for $\kappa < \tilde{\kappa}$, $|G(s) - G_u(s)| < \frac{\eta_1}{4}$. This implies that (20) does not have any solution on $\mathbb{C}^+$ if $|s| \leq M_0$. Choosing $\kappa_0 = \min \bar{\kappa}, \tilde{\kappa}$ concludes the proof.

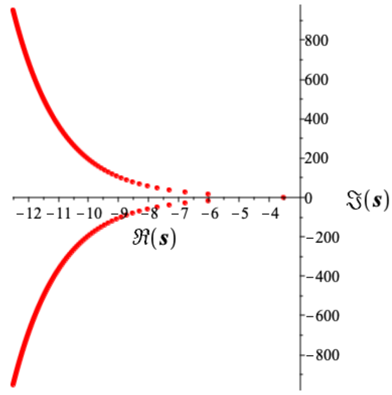

Fig. 1. The spectral values' distribution for equation (5) is illustrated using QPmR toolbox from Vyhlidal and Zítek (2009). We have a triple root at $s = -3$.

This proposition can now be applied for the stabilization of torsional oscillations.

Proposition 6. Consider the neutral system (5) with the control law $u_T(t) = \kappa_1 \Phi_L(t-\tau) + \kappa_2 \dot{\Phi}_L(t-\tau)$. It this control law stabilizes the system (namely if all the roots of the quasipolynomial $L(s)$ defined in (7) have negative real part), then it is robust with respect to uncertainties in the parameters and delays in the loop.

Proof. The control law $u_T(t)$ rewrites (using the Laplace transform) as $u_T(s) = \kappa_1 e^{-s\tau} + \kappa_2 s e^{-s\tau}$. The rest of the proof is a straightforward application of Proposition 5.

5. ILLUSTRATIVE EXAMPLE

In this section, we numerically illustrate our results. The values of the different parameters are defined in Table 1. In Figure 1, we have pictured the spectral values' distribution of the characteristic function (7) for the gains defined in Proposition 2. It is important to notice that the multiple root at $s_0 = -3$ is dominant, which guarantees that the closed loop-system is asymptotically stable. Finally, we have pictured in Figure 2, the distribution of the roots for the characteristic function (7) in presence of an uncertainty acting on the length of the well L. The red branches correspond to a positive uncertainty whose value vary between 0 and 30% of the real value of L. The blue branches correspond to a negative uncertainty. It is important noticing that such an uncertainty modifies all the coefficients a_i (since α_L directly depends on L) and the delay τ. As stated in Proposition 6, the system is robust with respect to this uncertainty, if it is small enough.

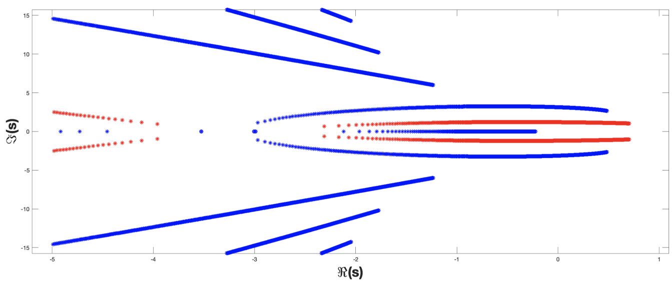

Fig. 2. Sensibility of the spectral values for equation (5) with respect to an uncertainty acting on the length L (positive uncertainty in blue, negative in red)

6. CONCLUDING REMARKS

In this paper, we have used a PDE model of the torsional oscillations of a drilling system. Following the approach given in Boussaada et al. (2019a), this system has been rewritten as a third-order neutral equation with a single delay. We then have designed a reduced complexity delayed stabilizing controller that enables an appropriate pole-placement of the closed-loop system. Compared to Boussaada et al. (2019a), this control law now include a derivative action, then introducing a new degree of freedom. The obtained controller acts for damping oscillation of the wave equation. The proposed control law only requires the measurement of the state at the boundary. If such a measurement is available (which may not be the case for the considered drilling application), then our control law does not require an observer for practical implementation. Finally, this control law has been proved to be robust with respect to uncertainties and delays. This has been done by means of an analysis in the Laplace domain and validated through simulations.

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