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# Insights on Pole-Placement of Dynamical Systems by PID Control with Guaranteed Delay Robustness

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Abstract The PID control is favored in controlling industrial processes for its ease of implementation. In this paper, the *multiplicity-induced-dominancy* property is used in the design of stabilizing PID controllers for some delayed reduced-order plants. More precisely, the controllers gains are tuned using the multiplicity's algebraic constraints allowing to assign analytically the closed-loop solutions' decay rate. Furthermore, the robustness of the control against uncertain delays is also addressed. An illustrative example completes the presentation.

*Keywords:* feedback system, time delay, PID control, robust stabilization.

## 1. INTRODUCTION

Linear systems with commensurate delays (all delays are multiple of a nominal delay) are described in the Laplace domain by transfer functions involving quasi-polynomials and then possibly admit an infinite number of poles. Studying the stability properties of retarded systems (they admit a finite number of poles in any right half-plane) is much easier than studying those of neutral systems which may have an infinite number of poles, in chains asymptotic to vertical axes possibly located in the open right half-plane or clustering the imaginary axis from left or right. Both situations prevent to get exponential stability for these systems. A subclass of neutral systems of interest is the one with all asymptotic axes in the open left half-plane guaranteing that there is a finite number of poles in an extended right half-plane. For such systems, the concept of $\alpha-$stability will play an important role.

In the Laplace domain, a number of effective methods have been proposed, see for instance Bellman and Cooke (1963); Cooke and van den Driessche (1986); Walton and Marshall (1987); Stépán (1989); Hale and Lunel (1993); Michiels and Niculescu (2007); Olgac and Sipahi (2002); Sipahi et al. (2011).

Even with the significant advances that have been reported on the topic of *Delay systems*, the question of determining conditions on the equation parameters that guarantee asymptotic stability of solutions of linear time-invariant time-delay systems remains an open question.

Once stability conditions are established; further questions related to performance occur. What is about the estimation of the corresponding rightmost roots of the system characteristic equation? Such a rightmost root corresponds to the so-called $\alpha-$stability problem, itself is related to the solution's decay rate. Also a fundamental measure of robust stabilization against uncertain time delays is the so-called *delay margin*, which addresses a central issue in the study of feedback stabilization of time-delay systems: What is the largest range of delay so that there exists a single controller that can stabilize the delay plant within that entire range? This question is also longstanding and remains open except in particular cases, see for instance Ma and Chen (2019).

PID controllers have been extensively used to control and regulate industrial processes which are typically modeled by reduced-order dynamics. In Ma and Chen (2019), the delay margin achievable using PID controllers for reduced order linear time-invariant (LTI) systems subject to variable, unknown time delays is investigated. An explicit expressions of the exact delay margin is carried out and its upper bounds achievable by a PID controller for low-order delay systems with unknown constant and possibly time-varying delays. The effect of non minimum phase zeros is also investigated and the fundamental limits of delay within which a PID controller may robustly stabilize a delay process is emphasized.

In recent works, the characterization of multiple spectral values for time-delay systems of retarded type were established using a Birkhoff/Vandermonde-based approach; see for instance Boussaada and Niculescu (2016b,a, 2014); Boussaada et al. (2016). In particular, in Boussaada and Niculescu (2016a), it is shown that the admissible multiplicity of the zero spectral value is bounded by the generic *Polya and Szegö bound* denoted $PS_B$, which is merely the *degree* of the corresponding quasipolynomial [1], see for instance Pólya and Szegő (1972). In Boussaada and Niculescu (2016b), it is shown that a given crossing imaginary root with a non vanishing frequency never reaches $PS_B$ and a sharper bound for its admissible multiplicities is established. Moreover, in Boussaada et al. (2016), the manifold corresponding to a multiple root for scalar time-delay equations defines a stable manifold for the steady state. An example of a scalar retarded equation with two delays is studied in Boussaada and Niculescu (2016b) where it is shown that the multiplicity of real spectral values may reach the $PS_B$. In addition, the corresponding system has some further interesting properties: (i) it is asymptotically stable, (ii) its spectral abscissa (rightmost root) corresponds to this maximal allowable multiple root located on the imaginary axis. Such observations enhance the outlook of further exhibiting the existing links between the maximal allowable multiplicity of some negative spectral value reaching the quasipolynomial degree and the stability of the trivial solution of the corresponding dynamical system. This interesting property induced by multiplicity appears also in optimization problems since such a multiple spectral value is indeed the rightmost root, see also Vanbiervliet et al. (2008).

It is worth noting that the rightmost root for quasipolynomial function corresponding to stable retarded time-delay systems (also in the neutral case under some assumptions) is actually the exponential decay rate of its time-domain solution, see for instance Mori et al. (1982) for an estimate of the decay rate for stable linear delay systems. To the best of our knowledge, the first time an analytical proof of the dominancy of a spectral value for the scalar equation with a single delay was presented in Hayes (1950). The dominancy property is further explored and analytically shown in scalar delay equations in Boussaada et al. (2016), then in second-order systems controlled by a delayed proportional is proposed in Boussaada et al. (2017) where its applicability in damping active vibrations for a piezo-actuated beam is proved. An extension to the delayed proportional-derivative controller case is studied in Boussaada et al. (2018) where the dominancy property is parametrically characterized. We emphasize that the idea of using roots assignment for controller-design for time-delay system is not new. As a matter of fact, an analytical/numerical stabilization method for retarded time-delay systems related to the classical pole-placement method for ordinary differential equations is proposed in Michiels et al. (2002), see also Zitek et al. (2013) for further insights on pole-placement methods for retarded time-delays systems with proportional-integral-derivative controller-design.

The present work is a natural continuation of Boussaada and Niculescu (2014, 2016a); Ma et al. (2020), it aims at proposing a systematic PID controller tuning to achieve the asymptotic stability of a first order linear time-invariant dead-time plant by extending and exploiting the *Multiplicity-Induced-Dominancy* property for quasipolynomial functions corresponding to neutral systems with single delay. Furthermore, it extends Ma et al. (2020) by considering the robustness of such a design against uncertain delays.

The remaining paper is organized as follows. Some prerequisites in complex analysis and the problem formulation are presented in section 2 . The main results are presented in Sections 3 and 4. In the first, a PID tuning based on the MID property is provided. Section 4 is dedicated to the robustness of such a design against uncertain delay. An illustrative example is provided in Section 5. A conclusion ends the paper.

## 2. PREREQUISITES AND PROBLEM FORMULATION

The principle argument is a consequence of Cauchy theorem and connects the winding number of a curve with the number of zeros and poles of a given complex variable function inside that curve. More precisely, it asserts that the integral of the ratio of the single valued complex variable functions $f'(z)/f(z)$ on a single passage along a closed path in the positive sense (in counter clockwise direction) is equal to $2i\pi(N-P)$ where $N$ (respectively $P$) is the sum of multiplicities of zeros (respectively of poles) of the function $f$ enclosed in such a contour. A natural and direct application of the principle of the argument is the stability analysis of dynamical systems. As a matter of fact, in the frequency domain, showing the asymptotic stability of the trivial solution of a given dynamical system amounts to prove that the zeros of the corresponding characteristic equation are located in the open left-half complex plane. The principle of argument is applied to determine the stability of closed loop system by choosing a closed path which encircles the entire right half complex plane in counter clockwise direction.

The next theorem follows directly from Krall (1964) and (Partington and Bonnet, 2004, Prop. 2.1) which gives an explicit localisation of the spectrum chain's asymptote for quasipolynomial functions corresponding to the following neutral systems.

$$G(s) = \frac{r(s)}{Q_0(s) + Q_\tau(s)\mathrm{e}^{-s\tau}} \tag{1}$$

such that $\deg(r) \leq \deg(Q_0) = \deg(Q_\tau)$.

*Theorem 1.* Let $\alpha = \lim_{|s|\to\infty} Q_\tau(s)/Q_0(s)$

1. If $|\alpha| < 1$ then the poles of $G$ of large modulus are asymptotic to a vertical line $\Re(s) \approx \log(|\alpha|)/\tau$ in the left-half plane. The number of poles of $G$ in the right of $\Re(s) = \log(|\alpha|)/\tau + \epsilon$ is finite for any fixed $\epsilon > 0$.
2. If $|\alpha| > 1$ then $G$ has infinitely many unstable poles, asymptotic to a vertical line $\Re(s) \approx \log(|\alpha|)/\tau$ which is the right-half plane.

So, under the condition $|\alpha| < 1$ one is able to prove that at most a finite number of poles of $G$ are located in the right-half plane. By scaling, this property extends to any parallel to the imaginary axis located at the right of the spectrum chain's asymptote.

[1] The quasipolynomial degree is exactly the number of the involved polynomials plus their degree minus one

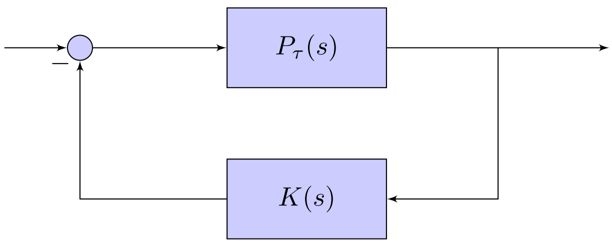


Figure 1. A feedback control system

*Remark 2.* The above result naturally apply to any transfer function

$$G(s)=\frac{r(s)\mathrm{e}^{-s\,T}}{Q_0(s)+Q_\tau(s)\mathrm{e}^{-s\tau}} \quad \text{for any} \quad T>0. \tag{2}$$

### *2.1 Problem formulation*

Consider the feedback control system depicted in Figure.1 where $P(s)$ is a reduced order plant which is subject to an unknown delay. Thus, the corresponding transfer function is given by

$$P_\tau(s)=P_0(s)\,e^{-\tau\,s} \tag{3}$$

where $P_0(s)$ is a delay-free plant. Our aim in this paper is to provide an easy way in tuning the standard PID gains $(k_i,\,k_p,\,k_d)$ achieving the stabilization of the closed-loop system. More precisely, the approach consists in intentionally impelling an appropriate multiplicity for a given root of the quasipolynomial function such that the induced algebraic constraints on the quasipolynomial's coefficients leads to robustly stabilize the considered unstable delayed plant. This approach showed its efficiency in studying stability and stabilization of retarded time-delay systems. However, in the present paper context, the obtained characteristic equation in closed-loop system corresponds to a delay system of neutral type, a class for which the stability analysis becomes more involved. As a matter of fact, the structure of the controller we consider is given by:

$$K_{PID}(s)=k_p+k_d\,s+\frac{k_i}{s}, \tag{4}$$

Consider the open-loop transfer function:

$L_0(s)=P_0(s)K_{PID}(s)$

In order for the system to achieve disturbance attenuation and for the open loop gain to roll off at high frequencies, we make the following assumption.

*Assumption:* (i) $|L_0(0)|>1$ (ii) $|L_0(\infty)|<1$

## 3. PID STABILIZING DESIGN FOR FIRST-ORDER DELAYED PLANTS

First-order plants are typical benchmarks usually considered for controlling industrial processes. In this aim, let consider the unstable plant $P_0(s)=\frac{1}{s-p}$ where $p$ is a positive pole and define the delay margin $\tau_{PID}=\sup\{\mu\geq 0:$ There exists some $K_{PID}(s)$ that stabilizes $P_\tau(s)\ \forall\tau\in[0,\mu)\}$.

It is shown in Silva et al. (2002) that $\tau_{PID}=\frac{2}{p}$ , see also Ma and Chen (2019).

The resulting closed-loop plant is given by:

$$M(s)=\frac{\left(k_p\,s+k_i+s^2k_d\right)\mathrm{e}^{-s\tau}}{s^2-sp+\mathrm{e}^{-s\tau}k_p\,s+\mathrm{e}^{-s\tau}k_i+\mathrm{e}^{-s\tau}s^2k_d}. \tag{5}$$

Since we are dealing with stability aim, let us focus on the corresponding characteristic equation:

$$\begin{cases}\Delta(s)=Q_0(s)+Q_\tau(s)\mathrm{e}^{-s\tau} \quad \text{where}\\ Q_0(s)=s^2-sp \quad \text{and} \quad Q_\tau(s)=k_ds^2+k_p\,s+k_i.\end{cases} \tag{6}$$

Notice that the degree of the quasipolynomial function defined in (6) is equal to 5. So, using a result from Pólya and Szegő (1972) one asserts that 5 is the generic bound of the multiplicity of any root of (6). The following theorem provides a sharper bound on the multiplicity of the corresponding zeros as well as a systematic manner to tune the parameters $k_p$, $k_i$, $k_d$ such that the closed-loop system (5) becomes stable.

*Theorem 3.* i) For arbitrary real parameters $k_p$, $k_i$, $k_d$ and arbitrary positive delay $\tau$, the multiplicity of a given root of the quasipolynomial function (6) is bounded by 4.

ii) The quasipolynomial (6) admits a multiple real spectral value at

$$s_\pm=\frac{\tau\,p-6\pm\sqrt{\tau^2p^2+12}}{2\tau} \tag{7}$$

with algebraic multiplicity 4 if, and only if,

$$\begin{cases}k_d=\dfrac{(4+2\,\tau\,s_\pm-\tau\,p)\,\mathrm{e}^{\tau\,s_\pm}}{2},\\ k_p=-\dfrac{\left((8\,\tau+\tau^2s_\pm)\,p-18-12\,\tau\,s_\pm\right)\mathrm{e}^{\tau\,s_\pm}}{\tau},\\ k_i=\dfrac{\left((\tau s_\pm+3)\,\tau^2p^2+(-12\tau s_\pm-60)\,\tau p+108+84\tau s_\pm\right)\mathrm{e}^{\tau s_\pm}}{2\tau^2}\end{cases} \tag{8}$$

iii) If $s=s_+$ is a quadruple root of (6) then it is also the corresponding rightmost root and the corresponding tuning (8) is stabilizing.

**Proof.** First, the vanishing of the quasipolynomial $\Delta$ given in (6) yields the elimination of the exponential term as a rational function in $s$:

$$\mathrm{e}^{-\tau\,s}=\frac{-s^2+ps}{k_d\,s^2+k_p\,s+k_i} \tag{9}$$

Next, to investigate potential roots with algebraic multiplicity 4, one substitutes of the obtained equality (9) in the ideal generated by the first three derivatives of $\Delta$. This allows to investigate a variety of three algebraic equations in 6 unknowns $k_i$, $k_p$, $k_d$, $\tau$, $p$, $s$. Using standard elimination techniques, one obtains the following set of admissible solutions:

1. $k_i=k_p=k_d=0.$
2. $k_i=s=0.$
3. $k_p=-2\,\frac{\left((\tau s_\pm+8)\tau p-12\,\tau\,s_\pm-18\right)k_d}{\tau\left(\tau\,(2s_\pm-p)+4\right)}$,

   $k_i=\frac{\left((\tau s_\pm+3)\tau^2p^2+\left(-12\,\tau s_\pm-60\right)\tau p+84\,\tau\,s_\pm+108\right)k_d}{\left(2\,\tau\,s_\pm+4-\tau\,p\right)\tau^2}$ where the explicit expression of $s_\pm$ is given by (7).

Observe that the first solution corresponds to the open-loop system while the second solution is inconsistent with respect to the transcendental-term elimination (9). So that, these two solutions are discarded. Furthermore, substituting conditions of the third solution in (9) yields the explicit values of the gain $k_d$ allowing to tune the parameters as provided in (8). Note that, when forcing multiplicity 5 complex gains and delay are obtained.

Next to show that $s_+$ is the dominant root of the quasipolynomial $\Delta$ (with coefficients satisfying (8)), one has to apply *the principle of the argument*. However, it is necessary first to prove that only a finite number of roots of $\Delta$ may occur in the right-half complex plane $\mathbb{C}^+_{s_+} = \{s \in \mathbb{C}, \text{ s.t. } \Re(s) > s_+\}$. According to Theorem 1, one has to check that $\tau\, s_+ > \log(|\alpha|)$, see also (Partington and Bonnet, 2004, Prop. 2.1). In our case, $\alpha$ is nothing but $k_d$. So that,

$$\begin{aligned}\ln(k_d) &= \ln\left( \left(\frac{\sqrt{\tau^2p^2+12}}{2} - 1\right) e^{-1/2\,\tau\,p - 3 + 1/2\,\sqrt{\tau^2p^2+12}}\right)\\ &= \tau\, s_+ - \ln(2) + \ln(\sqrt{\tau^2p^2+12} - 2)\end{aligned}$$

Since, $\tau < \tau_{PID} = \frac{2}{p}$, then $\sqrt{\tau^2p^2+12} < 4$. So that, $\log(k_d) < \tau\, s_+$ which proves that only a finite number of roots of $\Delta$ may occur in the right-half complex plane $\mathbb{C}^+_{s_+} = \{s \in \mathbb{C}, \text{ s.t. } \Re(s) > s_+\}$.

Equivalently, by using the following scaling $s \to z + s_+$ and the new parametrization $\delta = \sqrt{\tau^2p^2+12}$, one has to show the dominancy of zero spectral value for the following quasipolynomial function:

$$\begin{aligned}\tilde{\Delta}(z) =& z^2 + \frac{\delta-6}{\tau} z + \frac{12-3\,\delta}{\tau^2}\\ &+ \left(\frac{\delta-2}{2} z^2 + \frac{2\,\delta-6}{\tau} z + \frac{3\,\delta-12}{\tau^2}\right) e^{-z\tau}\end{aligned} \tag{10}$$

Obviously, $z = 0$ is a root of (10) with multiplicity 4. To apply the principle argument on the standard Bromwich contour, which allows to counting the roots of the quasipolynomial (10) on the right half-plane, one needs first to introduce a deflation eliminating the roots on the imaginary axis. To do so, let us first investigate nonzero imaginary roots for (10). Assume that there exists $\omega > 0$ such that $z = i\omega$ is a root of (10). Let define $R(\omega) = \Re(\tilde{\Delta}(i\omega))$ and $S(\omega) = \Im(\tilde{\Delta}(i\omega))$, which gives :

$$\begin{cases} R(\omega) = \left(\frac{2-\delta}{2}\omega^2 + \frac{3\,\delta-12}{\tau^2}\right)\cos(\omega\,\tau) \\ \qquad + \frac{(2\,\delta-6)\,\omega\,\sin(\omega\,\tau)}{\tau} - \omega^2 + \frac{12-3\delta}{\tau^2}, \\ S(\omega) = \left(\frac{\delta-2}{2}\omega^2 + \frac{12-3\,\delta}{\tau^2}\right)\sin(\omega\,\tau) \\ \qquad + \frac{\omega\,(2\,\delta-6)\cos(\omega\,\tau)}{\tau} + \frac{\delta-6}{\tau}\omega. \end{cases} \tag{11}$$

This means that for any $z = i\omega$ a root of (10) one has:

$$R(\omega) = 0, \quad S(\omega) = 0.$$

Some algebraic manipulations allow to eliminate the trigonometric functions. Next, using the standard trigonometric identity $\cos^2(\omega\,\tau) + \sin^2(\omega\,\tau) = 1$ we get

$$\begin{aligned}0 =& \Omega^2\delta\,(\delta-4)\times\\ &\Big((\delta^2 - 4\,\delta + 4)\Omega^2 + (4\,\delta^2 - 24\,\delta + 48)\Omega\\ &+36\,\delta^2 + 576 - 288\,\delta\Big)\end{aligned} \tag{12}$$

where $\Omega = \tau^2\omega^2$. Since $\tau < \tau_{PID} = 2/p$, then , one shows that no positive solution exists for both cases $(\star_\pm)$.

Now, one is able to apply the principle of the argument to investigate the dominancy of $s_+$ as a root of $\Delta$, given by (6) such that its coefficients satisfy (8), which is equivalent to investigate the dominancy of zero as a root of $\tilde{\Delta}$ given by (10). Clearly, apart from zero, $\tilde{\Delta}$ and $\hat{\Delta}(s) = \frac{\tilde{\Delta}(s)}{s^4}$ have the same roots.

*Remark 4.* Notice that if (8) is satisfied and $s = s_+$ is a root of (6) then $\Delta$ can be normalized first using the change of variable $z = s - s_+$ then by scaling $\lambda = z\,\tau$ which allows to:

$$\begin{aligned}\bar{\Delta}(\lambda) =& \lambda^2 + (\delta-6)\,\lambda - 3\,\delta + 12\\ &+ \left(\frac{\delta-2}{2}\lambda^2 + (2\,\delta-6)\lambda - 12 + 3\,\delta\right) e^{-\lambda},\end{aligned} \tag{13}$$

with $\delta = \sqrt{p^2\tau^2+12}$. So, $s_+$ is the dominant root of $\Delta$ is equivalent to say that the roots of $\bar{\Delta}$ are located in the left-half complex plane. Interestingly, $\bar{\Delta}$ can be written in the following compact integral form,

$$\bar{\Delta}(\lambda) = \lambda^4 \int_0^1 \left(\frac{4-\delta}{2}t^3 + \frac{\delta-6}{2}t^2 + t\right) e^{-t\lambda}\, dt,$$

Further, $\bar{\Delta}$ satisfies:

$$\frac{\bar{\Delta}(\lambda)}{\lambda^4} = \frac{(\delta-4)\,\Gamma(2)^2\,M(2,4,-\lambda)}{2\Gamma(4)} + \frac{\Gamma(2)\Gamma(3)\,M(3,5,-\lambda)}{\Gamma(5)},$$

where $M(a,b,z)$ is nothing but the hypergeometric functions solution of the Kummer differential equation, see for instance Abramowitz and Stegun (1964). Other quasipolynomials written under confluent hypergeometric functions appeared in Boussaada et al. (2016, 2017); Mazanti et al. (2021) in investigating the dominant root of quasipolynomial functions corresponding to retarded delay systems.

## 4. ROBUSTNESS OF THE DESIGN AGAINST UNCERTAIN DELAY

The following result addresses the delay robustness with respect to uncertain delay of a stabilizing PID controller as defined in Theorem 3:

*Theorem 5.* i) For any $\tau < \tau_{PID} = \frac{2}{p}$ with $(k_d,\, k_p,\, k_i)$ given in (8) one has:

$$0 < k_d < 1, \quad k_p > p, \quad k_i > 0. \tag{14}$$

The PID controller $K_{PID}(s)$ stabilizes $P_\tau(s)$ for all $\tau \in [0, \bar{\tau})$ where

$$\bar{\tau} = \frac{\tan^{-1}(\frac{\omega_0}{p})}{\omega_0} + \frac{\tan^{-1}(\frac{k_d\omega_0 - \frac{k_i}{\omega_0}}{k_p})}{\omega_0} \tag{15}$$

with $\omega_0$ is given by:

$$2\omega_0 = \frac{k_p^2 - 2k_dk_i - p^2}{1-k_d^2} + \sqrt{\left(\frac{{k_p}^2 - 2k_dk_i - p^2}{1-k_d^2}\right)^2 + 4\frac{k_i^2}{1-k_d^2}} \tag{16}$$

ii) For $\tau \to \tau_{PID}$ with $(k_d,\, k_p,\, k_i)$ given in (8) one has:

$$k_d \to 1,\ k_p \to p,\ k_i \to 0. \tag{17}$$

Under this circumstance, $s_+ \to 0$. The PID controller $K_{PID}(s)$ stabilizes $P_\tau(s)$ for all $\tau \in [0, \tau_{PID})$.

**Proof.** To prove $i$), we first note that

$$\tau\, s_+ = \frac{\tau\, p - 6 + \sqrt{(\tau\, p)^2 + 12}}{2}. \tag{18}$$

To examine the controllers gains, let denote $\tau\, p = \chi$. Then for $\tau < \frac{2}{p}$, one has $\chi \in [0, 2)$. Accordingly, one has

$$\tau\, s_+ = \frac{\chi - 6 + \sqrt{\chi^2 + 12}}{2}.$$

which allows to rewrites (8) as:

$$k_d = f(\chi)\mathrm{e}^{\tau\, s_+},\; k_p = \frac{g(\chi)}{\tau}\mathrm{e}^{\tau\, s_+},\; k_i = \frac{h(\chi)}{\tau^2}\mathrm{e}^{\tau\, s_+}.$$

where

$$\begin{cases} f(\chi) = \dfrac{\sqrt{\chi^2+12}}{2} - 1, \\ g(\chi) = \left(6 - \dfrac{\chi}{2}\right)\sqrt{\chi^2+12} - \dfrac{\chi^2}{2} + \chi - 18, \\ h(\chi) = \left(\dfrac{\chi^2}{2} - 6\,\chi + 42\right)\sqrt{\chi^2+12} \\ \qquad + \dfrac{\chi^3}{2} - 6\,\chi^2 + 18\,\chi - 144 \end{cases}$$

First, for any $\chi < 2$, we have $f(\chi) < 1$. Since $\mathrm{e}^{\tau\, s_+} < 1$ then $k_d = f(\chi)\mathrm{e}^{\tau\, s_+} < 1$. Next, $k_d > 0$ since $f(\chi) > 0$ for any $\chi \in [0, 2)$.

Consider the function

$$\hat{g}(\chi) = g(\chi)\mathrm{e}^{\frac{\chi - 6 + \sqrt{\chi^2+12}}{2}} - \chi = g(\chi)\mathrm{e}^{g_1(\chi))} - \chi$$

We prove below that $\hat{g}$ is monotonically increasing on $[0, 2)$. For this purpose we consider its two first derivatives

$$\hat{g}'(\chi) = \frac{(\chi - 5)\sqrt{\chi^2+12} - \chi^2 + 5\,\chi - 8}{2}\mathrm{e}^{g_1(\chi)} - 1$$

and

$$\hat{g}''(\chi) = \left(\frac{-\chi^2 + 3\,\chi - 3}{2} + \frac{\chi^3 - 3\,\chi^2 + 9\,\chi - 18}{2\sqrt{\chi^2+12}}\right)\mathrm{e}^{g_1(\chi)}$$

One easily shows that $\hat{g}''$ is negative since both of the second-order polynomials $g_2(\chi) = -\chi^2 + 3\,\chi - 3$ and $g_3(\chi) = \chi^3 - 3\,\chi^2 + 9\,\chi - 18$ are negative on the interval $[0, 2)$. In other words, $\hat{g}'$ is a decreasing function, which in turn means that $\inf_{\chi\in[0,2)} \hat{g}'(\chi) = g'(2) = 0$. As such, on $[0, 2)$, $\hat{g}'(\chi) > 0$ which implies that $\hat{g}$ is monotonically increasing, and

$$\hat{g}(\chi) = g(\chi)\mathrm{e}^{\frac{\chi - 6 + \sqrt{\chi^2+12}}{2}} - \chi = g(\chi)\mathrm{e}^{g_1(\chi))} - \chi > \hat{g}(0) > 0.$$

or equivalently

$$g(\chi)\mathrm{e}^{g_1(\chi))} > \chi$$

Substituting now $\chi = \tau\, p$, we have $k_p = \frac{g(\chi)}{\tau}\, e^{\tau\, s_+} > p$.

Finally, consider the parameter $k_i$. For this aim, we show that $h(\chi) > 0$. The first derivative of $h$ reads:

$$h'(\chi) = 3\,\frac{(\chi - 2)\left((\chi - 6)\sqrt{\chi^2+12} + \chi^2 - 6\,\chi + 24\right)}{2\sqrt{\chi^2+12}}.$$

It is obvious that the denominator $h'$ is positive and the first factor of its numerator is negative. Let investigate the properties of the second factor of $h'$. Assume first that there exists $\chi_0 \in [0, 2)$ such that $(\chi_0 - 6)\sqrt{{\chi_0}^2 + 12} + {\chi_0}^2 - 6\,\chi_0 + 24 = 0$. and consider the second-order polynomial

$$h_1(\chi) = \chi^2 + (\xi - 6)\,\chi - 6\,\xi + 24$$

with $\xi = \sqrt{{\chi_0}^2 + 12}$. Thus, necessarily $\xi$ is in the interval $\sqrt{12} < \xi < 4$. To investigate the sign of $h_1$ in the prescribed intervals $(\xi, \chi) \in (\sqrt{12}, 4) \times (0, 2)$ one computes the discriminant of $h_1$ which is given by $\delta = \xi^2 + 12\,\xi - 60$ which vanishes only at $\xi_\pm = -6 \pm 4\sqrt{6}$ so that only $\xi_+ \approx 3.464101616$ is of interest. However, solving $\sqrt{\chi^2+12} = \xi_+$ gives the pair of solutions $\chi_\pm = -6\sqrt{2} \pm 4\sqrt{3} \notin (0, 2)$ which are discarded. In conclusion, the discriminant $\delta$ of $h_1$ does not vanish, so that $h_1|_{\xi=\sqrt{{\chi_0}^2+12}}$ keeps a constant sign in the interval $[0, 2)$ which guarantees the negativity of $h'$. This shows that $h$ is a decreasing function. Since $h(2) = 0$ then $h$ is strictly positive in the interval $[0, 2)$.

We have thus proved $0 < k_d < 1$, $k_p < p$ and $k_i > 0$. The rest of the proof follows directly from Theorem 3.1 Ma and Chen (2019).

The second item $ii$) follows directly by substituting $\tau = \frac{2}{p}$.

*Remark 6.* This result provides a very useful justification of the selection of $k_d$, $k_p$, $k_i$ in (8). First, it shows that this selection will not introduce any nonminimum phase zero. Secondly, it actually guarantees some level of system's robustness against uncertain delays.

## 5. ILLUSTRATIVE EXAMPLE

Consider the plant depicted in Figure 1 where $P_\tau(s)$ is a delayed first order instable plant and $K(s)$ is a PID controller. So that the performance properties of the closed-loop plant are characterized via the transfert function (5) and its stability is characterized by (6). Assume the $K(s)$ admits an instable pole at $p = 1$ and a delay $\tau = 1$. One tunes the controller gains as prescribed in (8) and such that $s_+$ is a root of (6), which gives:

$$\begin{aligned} k_d &= 1/2\,\left(-2 + \sqrt{13}\right)\mathrm{e}^{-5/2+1/2\,\sqrt{13}}, \\ k_i &= 1/2\,\left(-\frac{263}{2} + \frac{73}{2}\sqrt{13}\right)\mathrm{e}^{-5/2+1/2\,\sqrt{13}}, \\ k_p &= -\left(\frac{35}{2} - 11/2\,\sqrt{13}\right)\mathrm{e}^{-5/2+1/2\,\sqrt{13}}, \end{aligned} \tag{19}$$

This guarantees that $s_+ = -5/2 + 1/2\sqrt{13}$ is a root of the quasipolynomial function (6) with multiplicity 4. Obviously, $s_+$ is negative and its dominancy is asserted by Theorem 3. Furthermore, since $\hat{\Delta}$ is analytic inside and on a closed contour $\gamma$, and is not zero on $\gamma$, then the number of its within $\gamma$ is equal to the number of times the image curve of $\gamma$ under the mapping $\omega = \hat{\Delta}(z)$ encircles the origin in the $\omega$-plane. The point $\omega = \hat{\Delta}(z)$ moves around as depicted in Fig 2, where $\theta$ increases from $-\pi/2$ to $\pi/2$, and $y$ increases from $-R$ to $R$. Since the curve does not encircles the origin, the argument of $\hat{\Delta}$ does not increase, see for instance Bellman and Cooke (1963).

## 6. CONCLUSION

This paper presented a systematic method for PID controller design able to stabilize first order unstable plants.

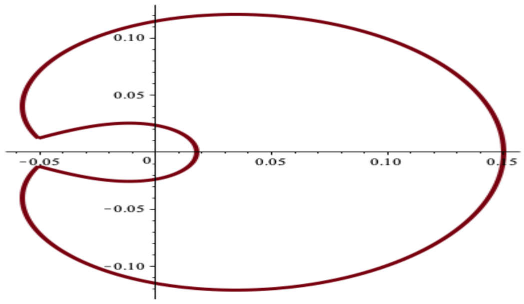


Figure 2. The argument variation: the behavior the point $\omega$ defined by the mapping $\omega = \hat{\Delta}(z)$.

It is shown that the proposed design is robust against delay uncertainties and satisfies the requirement for rolling off at high frequencies. The method is based on the multiplicity-induced-dominancy property extended to linear functional differential equations of neutral type. The proposed method can be extended to higher order systems with potential application in systems with propagation. An extended version of this work with complete proofs and further discussions will be available soon.

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