Stabilization of Discrete-Time Piecewise Affine Systems in Implicit Representation

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L. Cabral[1], J. M. Gomes da Silva Jr.[1], G. Valmorbida[2]

***Abstract*— This paper addresses the problem of stabilization of discrete-time piecewise affine (PWA) systems. The design of a piecewise affine state feedback control law is studied using an implicit representation based on ramp functions. LMI-based stability conditions, obtained from a piecewise quadratic Lyapunov function and the implicit representation, are stated to assess the global exponential stability of the origin of the closed-loop PWA system. Through appropriate congruence transformations and some structural assumptions, a method to design the control law parameters using semi-definite programming is then proposed.**

I. Introduction

Piecewise Affine (PWA) systems have been used to model nonlinear circuits [12] and other physical systems [14]. They can also represent some classes of switched and hybrid systems [3]. Moreover, some nonlinear functions that arise in dynamic control systems can also be modeled or approximated by PWA functions.

Many different forms to represent PWA systems have been proposed in the literature [6]. The most common and intuitive representation of a discrete-time PWA system is given by the explicit representation [13]

$$x^+ = A_i x + a_i \ , \forall x \in \Gamma_i \tag{1}$$

where Γ_i is the ith set in the partition of the state space, described in general by a finite number of explicit inequalities, x and $x^+ \in \mathbb{R}^n$ are, respectively, the current and the successor state, matrix $A_i \in \mathbb{R}^{n\times n}$ and vector $a_i \in \mathbb{R}^n$ defines the dynamical behavior of the system in region Γ_i. In particular, if the vector field is continuous over the boundary of the partition, we refer to this class as discrete-time Continuous PWA (CPWA) system. Moreover, if $\cup_{i=1}^N \Gamma_i = \mathbb{R}^n$, where N is the number of sets in the partition, the representation in (1) is globally valid.

Considering the above *explicit representation*, many conditions to assess the stability of discrete-time PWA systems have been proposed in the literature. We can cite, for instance, [3] and [2]. In this case, to evaluate the decrease of the Lyapunov function along the trajectories of the system, the possible transitions between regions must be taken into account. In general, all the possible transitions are tested, introducing some conservatism in the conditions. An analysis to determine the possible regions that can be reached from a given set in the partition helps reduce the conservatism in the analysis [1], even though the number of transitions can present combinatorial growth as the number of sets in the partition increases.

To overcome the need for the preliminary reachability analysis, an implicit representation, based on the use of ramp functions, has been recently proposed in [4]. From this representation, it is possible to express piecewise quadratic Lyapunov candidate functions in a compact way and to cast Lyapunov conditions as generalized quadratic inequalities involving ramp functions and their arguments. Furthermore, based on an exact description of the ramp function in terms of linear inequalities and a quadratic identity, it is possible to check these generalized quadratic inequalities using LMIs.

The problem of stabilization has been studied using the explicit representation. For continuous-time PWA systems, [5] casts the design of a piecewise linear state feedback control law as a convex optimization problem, but this is achieved by the restrictive assumption of a common quadratic Lyapunov function to all sets in the partition. In [12] the synthesis of a piecewise affine feedback control law is formulated as an optimization problem subject to a set of quasi-LMIs. However, the method only applies to slab continuous-time PWA systems. For discrete-time systems, [9] considers a piecewise quadratic Lyapunov function to obtain conditions for the synthesis of a piecewise *linear* state feedback control law in terms of a convex optimization problem. However, it should be noticed that the presented stabilization condition applies only when the system is piecewise linear. Due to the affine term, the extension to the PWA case is not as straightforward. Furthermore, as the explicit representation (1) is used, all the possible transitions from one region to another must be considered. This comes from the fact that the closed-loop gains are unknown and thus a reachability analysis cannot be performed to reduce the needed tests and the conservatism. Moreover, diferently from the analysis conditon (see also [7]), the fact that a transition from partition j to i can happen only if $x \in \Gamma_j$ is not taken into account, which is another source of conservatism.

Thanks to the advantages of the representation presented in [4], this paper investigates the global stabilization problem using this implicit representation. With this aim, starting from the Lyapunov inequalities formulated for the stability analysis, we obtain quasi-LMI conditions allowing the

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This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001, by the Brazilian National Council for Research (CNPq) - Grant 307449/2019-0 and by the ANR project HANDY under grant ANR-18-CE40-0010.

synthesis of stabilizing gains through the solution of semi-definite programming problems.

Notation: For a vector $v \in \mathbb{R}^n$, v_i denotes its ith element and $v \succeq 0$ ($v \preceq 0$) denotes elementwise nonnegativity (nonpositivity). For a matrix $M \in \mathbb{R}^{n\times m}$, $M_{(i,j)}$ denotes its (i,j) element while M_{ij} denotes its (i,j) block. Moreover, $M > 0$ ($M \geq 0$) denotes a positive (semi) definite matrix, $M \succeq 0$ denotes an elementwise nonnegative matrix, $\|M\|$ denotes the largest singular value of M and $\text{He}\{M\} \triangleq M + M^T$. The set of diagonal matrices in $\mathbb{R}^{n\times n}$ is represented by $\mathbb{D}^n$.

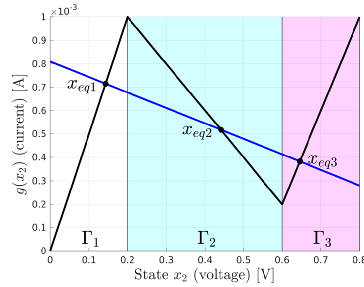

Fig. 2. Piecewise affine characteristic of the nonlinear resistor (black) and load given by the 1.5K resistor (blue) in Figure 1.

II. Problem Statement

Consider a discrete-time PWA system, described with the implicit representation proposed in [4] as

$$x^+ = F_{1o}x + F_{2o}\phi(y(x)) + Bu \tag{2a}$$
$$y(x) = F_3x + F_4\phi(y(x)) + f_5, \tag{2b}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^{n_u}$ is the control input and $y \in \mathbb{R}^{n_y}$ is the argument to the vector-valued ramp function $\phi : \mathbb{R}^{n_y} \to \mathbb{R}^{n_y}$, which is defined elementwise in terms of the ramp function $r : \mathbb{R} \to \mathbb{R}$ as

$$\phi_i(y) = r(y_i) = \begin{cases} 0 \text{ if } y_i < 0 \\ y_i \text{ if } y_i \geq 0 \end{cases} \tag{3}$$

for each $i = 1, ..., n_y$. The system dynamics (2) is defined by the constant matrices $F_{1o} \in \mathbb{R}^{n\times n}$, $F_{2o} \in \mathbb{R}^{n\times n_y}$, $B \in \mathbb{R}^{n\times n_u}$, $F_3 \in \mathbb{R}^{n_y\times n}$, $F_4 \in \mathbb{R}^{n_y\times n_y}$ and vector $f_5 \in \mathbb{R}^{n_y}$.

Equation (2b) defines the activation of affine terms through function ϕ depending on the state value. Hence, there is a direct relation of this equation and the partition regions Γ_i of an explicit representation (see [4] for details). In particular, for each region of the explicit representation, some entries of ϕ will be zero and others will be equal to y_i. For example, consider the nonlinear circuit presented in [12] and shown in Figure 1. The nonlinear resistor characteristic is modeled by a continuous piecewise affine function as depicted in Figure 2, from where we notice the existence of three equilibrium points, one in each set of the partition.

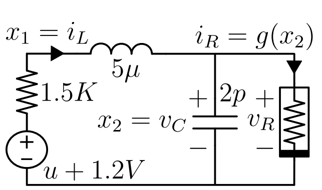

Fig. 1. Circuit with nonlinear resistor [12].

For a sampling period T, a discrete-time explicit PWA representation for the behavior of the system is given by (1) with 3 regions, as follows:

$$\begin{aligned}
&A_1 = \begin{bmatrix} 1-30T & -20T \\ T/20 & 1-T/4 \end{bmatrix},\ a_1 = \begin{bmatrix} 24T \\ 0 \end{bmatrix} \\
&A_2 = \begin{bmatrix} 1-30T & -20T \\ T/20 & 1+T/10 \end{bmatrix},\ a_2 = \begin{bmatrix} 24T \\ -0.07T \end{bmatrix} \\
&A_3 = \begin{bmatrix} 1-30T & -20T \\ T/20 & 1-T/5 \end{bmatrix},\ a_3 = \begin{bmatrix} 24T \\ 0.11T \end{bmatrix},\ B = \begin{bmatrix} 20T \\ 0 \end{bmatrix}, \\
&\Gamma_1 = \{x \in \mathbb{R}^2 \mid \begin{bmatrix} 0 & -1 \end{bmatrix} x \geq -0.2\}, \\
&\Gamma_2 = \left\{x \in \mathbb{R}^2 \mid \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x \succeq \begin{bmatrix} 0.2 \\ -0.6 \end{bmatrix}\right\} \text{ and} \\
&\Gamma_3 = \{x \in \mathbb{R}^2 \mid \begin{bmatrix} 0 & 1 \end{bmatrix} x \geq 0.6\}.
\end{aligned} \tag{4}$$

An implicit representation of the same circuit is given by (2a) and (2b) with

$$F_{1o} = \begin{bmatrix} 1-30T & -20T \\ 0.05T & 1-0.25T \end{bmatrix},\ F_{2o} = \begin{bmatrix} T & 0 & 0 \\ 0 & 50T & -50T \end{bmatrix},$$
$$B = \begin{bmatrix} 20T \\ 0 \end{bmatrix},\ F_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0.007 \\ 0 & 0.006 \end{bmatrix},\ F_4 = 0,\ f_5 = \begin{bmatrix} 24 \\ -0.0014 \\ -0.0036 \end{bmatrix}.$$

Note that the partition is defined by the sets

$$\begin{aligned}
\Gamma_1 &= \{x \in \mathbb{R}^n \mid y_1 \geq 0,\ y_2 < 0,\ y_3 < 0\}, \\
\Gamma_2 &= \{x \in \mathbb{R}^n \mid y_1 \geq 0,\ y_2 \geq 0,\ y_3 < 0\}, \\
\Gamma_3 &= \{x \in \mathbb{R}^n \mid y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0\}.
\end{aligned}$$

For $x \in \Gamma_i$, we have $\phi(y(x)) = \Phi_i y(x)$, where matrices $\Phi_i \in \mathbb{D}^{n_y}$ have the diagonal elements equal to one corresponding to the nonnegative elements in vector y, and zero otherwise. For the previous example, the matrices are

$$\Phi_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \Phi_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \Phi_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence, the relation between the explicit representation (1) and the implicit representation (2) is given by

$$\begin{aligned}
A_i &= F_{1o} + F_{2o}\Phi_i(I - F_4\Phi_i)^{-1}F_3 \\
a_i &= F_{2o}\Phi_i(I - F_4\Phi_i)^{-1}f_5.
\end{aligned} \tag{5}$$

In this work, we are interested in computing a stabilizing control law u such that the global exponential stability of the origin of the closed-loop system is ensured. With this aim, the following nonlinear state feedback control law is considered:

$$u(x) = K_1 x + K_2 \phi(y(x)) \tag{6}$$

with $K_1 \in \mathbb{R}^{n_u \times n}$ and $K_2 \in \mathbb{R}^{n_u \times n_y}$. The closed-loop system composed by (2) and (6) reads:

$$x^+ = F_1 x + F_2 \phi(y(x)) \tag{7a}$$
$$y(x) = F_3 x + F_4 \phi(y(x)) + f_5 \tag{7b}$$

where $F_1 = (F_{1o} + BK_1)$ and $F_2 = (F_{2o} + BK_2)$. Note that the gain K_2 modifies the control action according to the active set Γ_i.

Regarding representation (7) of the closed-loop system, the following assumption is considered.

Assumption 1. *The algebraic loop in* (7b) *is well-posed, i.e. there is an unique solution to the implicit equation* (7b).

Assumption 1 ensures the existence of inverse for $(I - F_4 \Phi_i)$ in (5) and can be verified by applying the following result, whose proof is found in [4].

Proposition 1. *If there exists matrix $X \in \mathbb{D}^{n_y}$ such that*

$$-2X + XF_4 + F_4^T X < 0$$

then the implicit equation (2b) *is well-posed.*

III. Properties of ramp functions

Based on properties of the ramp function, this section states two lemmas for the function ϕ, which will be instrumental to obtain conditions for the stability and stabilization of the closed-loop system.

Since ϕ is defined elementwise in terms of a ramp function, it inherits the following properties from the ramp function, valid for any vector $y \in \mathbb{R}^{n_y}$ [11]:

$$\phi_i(y) \geq 0; \tag{8a}$$
$$(\phi_i(y) - y_i) \geq 0; \tag{8b}$$
$$\phi_i(y)(\phi_i(y) - y_i) = 0. \tag{8c}$$

It should be noticed that these relations apply only to ramp functions, and so its use to obtain stability conditions does not introduce any conservatism. This is a key difference with respect to sector-bounded relations [8], which applies to a broad class of functions.

Let $\xi^T(y) \triangleq \begin{bmatrix} 1 & \phi^T(y) & (\phi(y) - y)^T \end{bmatrix}$. Based on properties given in (8), the following lemmas can be stated.

Lemma 1. *For any symmetric elementwise nonnegative matrix $M = M^T \succeq 0 \in \mathbb{R}^{(1+2n_y)\times(1+2n_y)}$ and $y \in \mathbb{R}^{n_y}$, the function ϕ defined in* (3) *is such that*

$$s_1(M, y) \triangleq \xi^T(y) M \xi(y) \geq 0. \tag{9}$$

Proof. From (8a) and (8b) all elements of the vector $\xi(y)$ are nonnegative. Since each element of M is nonnegative, then (9) holds. ■

Lemma 2. *For any $T \in \mathbb{D}^{n_y}$, $y \in \mathbb{R}^{n_y}$ and the function ϕ defined in* (3)*, it follows that*

$$s_2(T, y) \triangleq \xi(y)^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & T \\ 0 & T & 0 \end{bmatrix} \xi(y) = 0. \tag{10}$$

Proof. Since T is a diagonal matrix, expression $s_2(T, y)$ can be written as $s_2(T, y) = 2\sum_{i=1}^{n_y} T_{(i,i)} \phi_i(y)(\phi_i(y) - y_i)$, which is equal to zero as a consequence of property (8c). ■

IV. Stability conditions

The next two lemmas result from the relation between y and x in (7) and are important to state the stability conditions. Let $\chi^T(x) \triangleq \begin{bmatrix} 1 & x^T & \phi^T(y(x)) & (\phi(y(x)) - y(x))^T \end{bmatrix}$.

Lemma 3. *For any vector $\zeta \in \mathbb{R}^{n_\zeta}$ and matrix $R \in \mathbb{R}^{n_\zeta \times n_y}$ the relation*

$$s_3(R, \zeta, x) \triangleq \zeta^T R Q \chi(x) = 0, \tag{11}$$

with $Q = \begin{bmatrix} f_5 & F_3 & F_4 - I & I \end{bmatrix}$ is verified along the trajectories of the closed-loop system (7).

Proof. Note that from (7b) we have $Q\chi(x) = 0 \; \forall x \in \mathbb{R}^n$. ■

It is possible to state an extended version of Lemma 3 using an augmented form of vector $\chi(x)$ defined as

$$\tilde{\chi}^T(x) = \begin{bmatrix} 1 & x^T & (x^+)^T & \phi^T(\tilde{y}(x)) & (\phi(\tilde{y}(x)) - \tilde{y}(x))^T \end{bmatrix},$$

where $\tilde{y}^T(x) \triangleq \begin{bmatrix} y^T(x) & (y^+(x))^T \end{bmatrix}$. In this case the relation between x and x^+ described by (7a) is taken into account.

Lemma 4. *For any vector $\zeta \in \mathbb{R}^{n_\zeta}$ and matrix $R \in \mathbb{R}^{n_\zeta \times (2n_y + n)}$ the relation*

$$s_4(R, \zeta, x) \triangleq \zeta^T R Q \tilde{\chi}(x) = 0 \tag{12}$$

with

$$Q = \left[\begin{array}{c|cc|cc|cc} f_5 & F_3 & 0 & F_4 - I & 0 & I & 0 \\ f_5 & 0 & F_3 & 0 & F_4 - I & 0 & I \\ 0 & F_1 & -I & F_2 & 0 & 0 & 0 \end{array}\right] \tag{13}$$

is verified along the trajectories of the closed-loop system (7).

Proof. Similarly to the proof of Lemma 3, note that from relations (7a) and (7b), we have $Q\tilde{\chi}(x) = 0 \; \forall x \in \mathbb{R}^n$. ■

The next theorem gives sufficient conditions for the global exponential stability of the origin of system (7). For this, we consider a continuous piecewise quadratic (PWQ) Lyapunov candidate function generically described as follows:

$$V(x) = \begin{bmatrix} x \\ \phi(y) \end{bmatrix}^T \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \begin{bmatrix} x \\ \phi(y) \end{bmatrix} = \begin{bmatrix} x \\ \phi(y) \end{bmatrix}^T P \begin{bmatrix} x \\ \phi(y) \end{bmatrix} \tag{14}$$

with $P_1 \in \mathbb{R}^{n \times n}$, $P_2 \in \mathbb{R}^{n \times n_y}$ and $P_3 \in \mathbb{R}^{n_y \times n_y}$.

Theorem 1. *Consider a PWA system* (7) *with $f_5 \preceq 0$ and $V(x)$ as in* (14)*. If there exist matrices $P = P^T \in \mathbb{R}^{(n+n_y)\times(n+n_y)}$, $T_1 \in \mathbb{D}^{n_y}$, $T_2 \in \mathbb{D}^{2n_y}$, $R_1 \in \mathbb{R}^{(1+n+2n_y)\times n_y}$, $R_2 \in \mathbb{R}^{(1+2n+4n_y)\times(2n_y+n)}$, elementwise nonnegative matrices $M_1 \in \mathbb{R}^{(1+2n_y)\times(1+2n_y)}$ and $M_2 \in$*

$\mathbb{R}^{(1+4n_y)\times(1+4n_y)}$ *and positive scalars* ϵ_1 *and* $\eta \in (0,1)$ *such that*

$$\begin{aligned}(V(x) - \epsilon_1 x^T x) + s_3(R_1, \chi(x), x) + s_2(T_1, y(x)) \\ - s_1(M_1, y(x)) \geq 0\end{aligned} \tag{15}$$

and

$$\begin{aligned}-(V(x^+) - \eta V(x)) + s_4(R_2, \tilde{\chi}(x), x) + s_2(T_2, \tilde{y}(x)) \\ - s_1(M_2, \tilde{y}(x)) \geq 0\end{aligned} \tag{16}$$

with $\chi(x)$*,* $\tilde{y}(x)$ *and* $\tilde{\chi}(x)$ *as previously defined, then the origin of system* (7) *is globally exponentially stable.*

Proof. Since f_5 is assumed to be nonpositive, then $V(x)$ has a finite quadratic upper bound given by $\epsilon_2 \|x\|^2$ [4]. Moreover, applying Lemmas 1, 2 and 3, inequality (15) implies that

$$\epsilon_1 \|x\|^2 \leq V(x) \leq \epsilon_2 \|x\|^2 . \tag{17}$$

On the other hand, from Lemmas 1, 2 and 4, inequality (16) implies that

$$V(x^+) \leq \eta V(x). \tag{18}$$

Since $\eta \in (0,1)$ and $V(x) > 0$, then $\Delta V(x) \triangleq V(x^+) - V(x) < 0$. Moreover, from (18), we conclude that $V(x(k)) \leq \eta^k V(x(0))$, which, from (17), implies that $\|x(k)\| \leq \sqrt{\epsilon_2/\epsilon_1} e^{k \ln(\sqrt{\eta})} \|x(0)\| \ \forall x(0) \in \mathbb{R}^n$, from where the global exponential stability of the origin follows. ■

The conditions (15) and (16) can be written in a matrix form. In this case, given $\eta \in (0,1)$, these conditions become linear matrix inequalities (LMIs) in variables P, T_1, T_2, R_1, R_2, M_1, M_2 and ϵ_1 as stated in the following theorem.

Theorem 2. *Given* $\eta \in (0,1)$*, if there exist matrices* $P \in \mathbb{R}^{(n+n_y)\times(n+n_y)}$*,* $T_1 \in \mathbb{D}^{n_y}$*,* $T_2 \in \mathbb{D}^{2n_y}$*,* $R_1 \in \mathbb{R}^{(1+n+2n_y)\times n_y}$*,* $R_2 \in \mathbb{R}^{(1+2n+4n_y)\times(2n_y+n)}$*,* $M_1 \in \mathbb{R}^{(1+2n_y)\times(1+2n_y)}$*,* $M_2 \in \mathbb{R}^{(1+4n_y)\times(1+4n_y)}$ *and a positive scalar* ϵ_1 *such that the LMIs*

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & P_1 - \epsilon_1 I & P_2 & 0 \\ 0 & P_2^T & P_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - Z_1 + He\{R_1 Q_1\} \geq 0, \tag{19}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta P_1 & 0 & \eta P_2 & 0 & 0 & 0 \\ 0 & 0 & -P_1 & 0 & -P_2 & 0 & 0 \\ 0 & \star & 0 & \eta P_3 & 0 & 0 & 0 \\ 0 & 0 & \star & 0 & -P_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} - Z_2 + He\{R_2 Q_2\} \geq 0 \tag{20}$$

and the elementwise nonnegativity constraints

$$M_1 \succeq 0 \quad and \quad M_2 \succeq 0 \tag{21}$$

with $Q_1 = \begin{bmatrix} f_5 & F_3 & F_4 - I & I \end{bmatrix}$*,* Q_2 *as defined in* (13)*,*

$$Z_1 = \begin{bmatrix} M_{1_{11}} & 0 & M_{1_{12}} & M_{1_{13}} \\ 0 & 0 & 0 & 0 \\ \star & 0 & M_{1_{22}} & M_{1_{23}} - T_1 \\ \star & 0 & \star & M_{1_{33}} \end{bmatrix} \quad and$$

$$Z_2 = \begin{bmatrix} M_{2_{11}} & 0 & 0 & M_{2_{12}} & M_{2_{13}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \star & 0 & 0 & M_{2_{22}} & M_{2_{23}} - T_2 \\ \star & 0 & 0 & \star & M_{2_{33}} \end{bmatrix},$$

are satisfied, then the origin of system (7) *with* $f_5 \preceq 0$ *is globally exponentially stable.*

Proof. By pre and post multiplying (19) by $\chi^T(x)$ and $\chi(x)$, respectively, we obtain (15). Moreover, by pre and post multiplying (20) by $\tilde{\chi}^T(x)$ and $\tilde{\chi}(x)$, respectively, we obtain (16). Constraints (21) ensure the elementwise nonnegativity of matrices M_1 and M_2. ■

V. GLOBAL STABILIZATION

Although Theorem 2 is useful to assess the origin stability of a given CPWA system through a direct LMI feasibility test, the same cannot be done in the stabilization problem. This is due to the product between variables R_2 and Q_2. Note that F_1 and F_2, which are defined from K_1 and K_2, appear in matrix Q_2. To address this problem, we perform a congruence transformation and fix the structure of some matrices. The resulting stabilization condition is proposed in the Theorem below.

Theorem 3. *Given* $\eta \in (0,1)$*, if there exist* $\tilde{P} = \tilde{P}^T \in \mathbb{R}^{(n+n_y)\times(n+n_y)}$*, positive definite* $\tilde{E} = \tilde{E}^T \in \mathbb{R}^{n\times n}$*,* $\tilde{M}_1 \in \mathbb{R}^{(1+2n_y)\times(1+2n_y)}$*,* $\tilde{T}_1 \in \mathbb{D}^{n_y}$*,* $\tilde{M}_2 \in \mathbb{R}^{(1+4n_y)\times(1+4n_y)}$*,* $\tilde{T}_2 \in \mathbb{D}^{2n_y}$*, non-singular symmetric matrices* $W_1 \in \mathbb{R}^{n\times n}$*,* $W_2 \in \mathbb{D}^{n_y}$*,* $W_3 \in \mathbb{D}^{n_y}$*,* $W_4 \in \mathbb{D}^{n_y}$ *and* $W_5 \in \mathbb{D}^{n_y}$*, matrices* $\tilde{K}_1 \in \mathbb{R}^{n_u\times n}$ *and* $\tilde{K}_2 \in \mathbb{R}^{n_u\times n_y}$ *and scalars* α*,* β *and* γ *such that the matrix inequalities*

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \tilde{P}_1 - \tilde{E} & \tilde{P}_2 & 0 \\ 0 & \star & \tilde{P}_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \tilde{Z}_1 + He\{\tilde{R}_1 \tilde{Q}_1\} \geq 0, \tag{22}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta \tilde{P}_1 & 0 & \eta \tilde{P}_2 & 0 & 0 & 0 \\ 0 & 0 & -\tilde{P}_1 & 0 & -\tilde{P}_2 & 0 & 0 \\ 0 & \star & 0 & \eta \tilde{P}_3 & 0 & 0 & 0 \\ 0 & 0 & \star & 0 & -\tilde{P}_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} - \tilde{Z}_2 + He\{\tilde{R}_2 \tilde{Q}_2\} \geq 0 \tag{23}$$

and the elementwise nonnegativity constraints

$$\Pi_1^{-1} \begin{bmatrix} \tilde{M}_{1_{11}} & 0 & \tilde{M}_{1_{12}} & \tilde{M}_{1_{13}} \\ 0 & 0 & 0 & 0 \\ \star & 0 & \tilde{M}_{1_{22}} & \tilde{M}_{1_{23}} \\ \star & 0 & \star & \tilde{M}_{1_{33}} \end{bmatrix} \Pi_1^{-1} \succeq 0, \tag{24}$$

$$\Pi_2^{-1} \begin{bmatrix} \tilde{M}_{2_{11}} & 0 & 0 & \tilde{M}_{2_{12}} & \tilde{M}_{2_{13}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \star & 0 & 0 & \tilde{M}_{2_{22}} & \tilde{M}_{2_{23}} \\ \star & 0 & 0 & \star & \tilde{M}_{2_{33}} \end{bmatrix} \Pi_2^{-1} \succeq 0 \tag{25}$$

are satisfied with $\Pi_1 = \Pi_1^T = diag(1, W_1, W_2, W_5)$*,* $\Pi_2 = \Pi_2^T = diag(1, W_1, W_1, W_2, W_2, W_3, W_4)$*,*

$$\tilde{Z}_1 = \begin{bmatrix} \tilde{M}_{1_{11}} & 0 & \tilde{M}_{1_{12}} & \tilde{M}_{1_{13}} \\ 0 & 0 & 0 & 0 \\ \star & 0 & \tilde{M}_{1_{22}} & \tilde{M}_{1_{23}} - \tilde{T}_1 \\ \star & 0 & \star & \tilde{M}_{1_{33}} \end{bmatrix}, \ \tilde{R}_1 = \begin{bmatrix} 0 \\ 0 \\ \gamma I \\ I \end{bmatrix}$$

$$\tilde{Q}_1 = \begin{bmatrix} f_5 & F_3 W_1 & (F_4 - I)W_2 & W_5 \end{bmatrix},$$

$$\tilde{Z}_2 = \begin{bmatrix} \tilde{M}_{2_{11}} & 0 & 0 & \tilde{M}_{2_{12}} & \tilde{M}_{2_{13}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \star & 0 & 0 & \tilde{M}_{2_{22}} & \tilde{M}_{2_{23}} - \tilde{T}_2 \\ \star & 0 & 0 & \star & \tilde{M}_{2_{33}} \end{bmatrix},$$

$$\tilde{R}_2 = \begin{bmatrix} 0 & 0 & 0 & I & 0 & I & 0 \\ 0 & 0 & 0 & 0 & \beta I & 0 & I \\ 0 & I & \alpha I & 0 & 0 & 0 & 0 \end{bmatrix}^T \quad \textit{and}$$

$$\tilde{Q}_2^T = \begin{bmatrix} f_5^T & f_5^T & 0 \\ (F_3 W_1)^T & 0 & (F_{1o} W_1 + B\tilde{K}_1)^T \\ 0 & (F_3 W_1)^T & -W_1 \\ ((F_4 - I)W_2)^T & 0 & (F_{2o} W_2 + B\tilde{K}_2)^T \\ 0 & ((F_4 - I)W_2)^T & 0 \\ W_3 & 0 & 0 \\ 0 & W_4 & 0 \end{bmatrix},$$

then the gains $K_1 = \tilde{K}_1 W_1^{-1}$ *and* $K_2 = \tilde{K}_2 W_2^{-1}$ *ensure that the origin of the closed-loop system* (7) *is globally exponentially stable.*

Proof. Starting with (20), consider the following particular structure for matrix R_2:

$$R_2 = \begin{bmatrix} 0 & 0 & 0 & W_2^{-1} & 0 & W_3^{-1} & 0 \\ 0 & 0 & 0 & 0 & \beta W_2^{-1} & 0 & W_4^{-1} \\ 0 & W_1^{-1} & \alpha W_1^{-1} & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

After pre and post multiplying (20) by the symmetric matrix Π_2 the term $\Pi_2 R_2 Q_2 \Pi_2$ becomes $\tilde{R}_2 \tilde{Q}_2$ considering the change of variables $\tilde{K}_1 \triangleq K_1 W_1$ and $\tilde{K}_2 \triangleq K_2 W_2$. The first term of (23) is obtained considering $\tilde{P}_1 \triangleq W_1 P_1 W_1$, $\tilde{P}_2 \triangleq W_1 P_2 W_2$ and $\tilde{P}_3 \triangleq W_2 P_3 W_2$. Finally, the term $\tilde{Z}_2$ in (23) is obtained from the following change of variables.

$$\begin{aligned} &\tilde{M}_{2_{11}} \triangleq M_{2_{11}}, \quad \tilde{T}_2 \triangleq \begin{bmatrix} W_2 & 0 \\ 0 & W_2 \end{bmatrix} T_2 \begin{bmatrix} W_3 & 0 \\ 0 & W_4 \end{bmatrix}, \\ &\tilde{M}_{2_{12}} \triangleq M_{2_{12}} \begin{bmatrix} W_2 & 0 \\ 0 & W_2 \end{bmatrix}, \quad \tilde{M}_{2_{13}} \triangleq M_{2_{13}} \begin{bmatrix} W_3 & 0 \\ 0 & W_4 \end{bmatrix}, \\ &\tilde{M}_{2_{22}} \triangleq \begin{bmatrix} W_2 & 0 \\ 0 & W_2 \end{bmatrix} M_{2_{22}} \begin{bmatrix} W_2 & 0 \\ 0 & W_2 \end{bmatrix}, \\ &\tilde{M}_{2_{23}} \triangleq \begin{bmatrix} W_2 & 0 \\ 0 & W_2 \end{bmatrix} M_{2_{23}} \begin{bmatrix} W_3 & 0 \\ 0 & W_4 \end{bmatrix} \text{ and} \\ &\tilde{M}_{2_{33}} \triangleq \begin{bmatrix} W_3 & 0 \\ 0 & W_4 \end{bmatrix} M_{2_{33}} \begin{bmatrix} W_3 & 0 \\ 0 & W_4 \end{bmatrix}. \end{aligned} \tag{26}$$

Consider now (19), but replace the positive scalar ϵ_1 by a positive definite matrix E. This procedure ensures that $V(x)$ is greater than a positive lower bound given by $\lambda_{\min}(E)\|x\|^2$, where $\lambda_{\min}(E)$ is the minimal eigenvalue of E, and allows for a change of variables. Moreover, consider the following structure for matrix $R_1 = \begin{bmatrix} 0 & 0 & \gamma W_2^{-1} & W_5^{-1} \end{bmatrix}^T$. Then, after pre and post multiplying (19) by the symmetric matrix Π_1 the term $\Pi_1 R_1 Q_1 \Pi_1$ becomes $\tilde{R}_1 \tilde{Q}_1$ and the following change of variables is considered to obtain the remaining terms of (22):

$$\begin{aligned} &\tilde{E} \triangleq W_1 E W_1, \ \tilde{M}_{1_{11}} \triangleq M_{1_{11}}, \ \tilde{M}_{1_{12}} \triangleq M_{1_{12}} W_2 \\ &\tilde{M}_{1_{13}} \triangleq M_{1_{13}} W_3, \ \tilde{M}_{1_{22}} \triangleq W_2 M_{1_{22}} W_2, \ \tilde{M}_{1_{23}} \triangleq W_2 M_{1_{23}} W_3 \\ &\tilde{M}_{1_{33}} \triangleq W_3 M_{1_{33}} W_3 \text{ and } \tilde{T}_1 \triangleq W_2 T_1 W_3. \end{aligned} \tag{27}$$

Finally, note that the constraints (24) and (25) ensure that the elementwise constraints in (21) are satisfied, i.e., M_1 and M_2 in Theorem 1 are nonnegative elementwise.

■

Note that conditions in Theorem 3 are nonconvex. There appear products between scalars α, β and γ and some unknown matrices. Also, the elementwise constraints (24) and (25) include products of unknown matrices. In the next section, we propose an algorithm to solve the stabilization problem based on Theorem 3 using convex optimization.

VI. Proposed Algorithm

Since constraints (22) to (25) are nonconvex, it is important to propose an algorithm to solve the feasibility problem defined by such constraints.

First, there is the product between variable matrices $\tilde{R}_1 \tilde{Q}_1$ and $\tilde{R}_2 \tilde{Q}_2$. Since matrices $\tilde{R}_1$ and $\tilde{R}_2$ have only a few scalar variables, then the gridding method proposed in [12] can be used, i.e., define a grid of values for α, β and γ and, for each point in the grid, (22) and (23) are LMIs. The grid is characterized by a minimal value (α_{min}, β_{min}, γ_{min}), a step value (α_s, β_s, γ_s) and a maximum value (α_{max}, β_{max}, γ_{max}) for each variable.

In addition to (22) and (23), we must also satisfy the elementwise constraints (24) and (25). Noting that W_2, W_3, W_4 and W_5 are diagonal matrices, we impose these matrices to be positive or negative definite and then add constraints on the corresponding elements of matrices $\tilde{M}_1$ and $\tilde{M}_2$. We have, therefore, 16 possible cases as described by Table I. For instance, consider case 3 (i.e. $W_5 > 0$, $W_4 > 0$, $W_3 < 0$, $W_2 < 0$). Thus, for this case, from (27) we must impose the following elementwise constraints $\tilde{M}_{1_{11}} \succeq 0$, $\tilde{M}_{1_{12}} \preceq 0$, $\tilde{M}_{1_{13}} \preceq 0$, $\tilde{M}_{1_{22}} \succeq 0$, $\tilde{M}_{1_{23}} \succeq 0$, and $\tilde{M}_{1_{33}} \succeq 0$ to ensure that matrix M_1 is elementwise nonnegative. The same procedure must be applied to $\tilde{M}_2$ following (26) to ensure that matrix M_2 is elementwise nonnegative.

Hence, the idea is to check the feasibility of (22) and (23) in a grid on α, β and γ, considering the elementwise contraints associated to each one of the cases in Table I.

Once a feasible solution is found, the stabilizing gains are given by $K_1 = \tilde{K}_1 W_1^{-1}$ and $K_2 = \tilde{K}_2 W_2^{-1}$, as stated in Theorem 3. Additional performance constraints, such as minimization of η (i.e. maximization of the convergence rate), can be considered to choose the best pair of stabilizing gains among the feasible cases in Table I.

Case	W_5	W_4	W_3	W_2	Case	W_5	W_4	W_3	W_2
0	$>$	$>$	$>$	$>$	8	$<$	$>$	$>$	$>$
1	$>$	$>$	$>$	$<$	9	$<$	$>$	$>$	$<$
2	$>$	$>$	$<$	$>$	10	$<$	$>$	$<$	$>$
3	$>$	$>$	$<$	$<$	11	$<$	$>$	$<$	$<$
4	$>$	$<$	$>$	$>$	12	$<$	$<$	$>$	$>$
5	$>$	$<$	$>$	$<$	13	$<$	$<$	$>$	$<$
6	$>$	$<$	$<$	$>$	14	$<$	$<$	$<$	$>$
7	$>$	$<$	$<$	$<$	15	$<$	$<$	$<$	$<$

TABLE I

TABLE OF CASES TESTED FOR MATRICES W_2 TO W_5

VII. NUMERICAL EXAMPLE

Consider a discrete-time approximation of the system presented in Section 8.1 of [10], obtained with $T = 0.5$ and implicitly represented as in (2) with

$$F_{1o} = \begin{bmatrix} 1 & T \\ 4T & 1-0.25T \end{bmatrix},\ F_{2o} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ T & -T & -T & T \end{bmatrix},\ B = \begin{bmatrix} 0 \\ T \end{bmatrix}$$

$$F_3 = \begin{bmatrix} 0 & 0.4024 \\ 0 & 0.2638 \\ 0 & -0.4024 \\ 0 & -0.2638 \end{bmatrix},\ F_4 = 0,\ \text{and}\ f_5 = \begin{bmatrix} -0.4024 \\ -1.3190 \\ -0.4024 \\ -1.3190 \end{bmatrix}. \quad (28)$$

This system locally approximates the nonlinear function given in [10] by the piecewise function described in Figure 3.

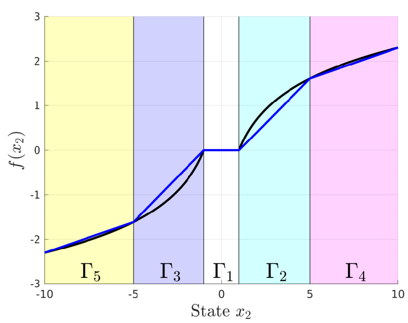

Fig. 3. Nonlinear function (black) and its PWA approximation (blue).

It should be noticed that the origin of the open-loop system (2) is not globally exponentially stable. Thus, applying the method proposed in Section VI with parameters $\eta = 0.99$, $\alpha_{min} = -1.5$, $\alpha_s = 0.5$, $\alpha_{max} = 1.5$, $\beta_{min} = -1.5$, $\beta_s = 0.5$, $\beta_{max} = 1.5$, $\gamma_{min} = -1.5$, $\gamma_s = 0.5$ and $\gamma_{max} = 1.5$ results in the following global stabilizing gains $K_1 = \begin{bmatrix} -5.2350 & -3.9165 \end{bmatrix}$ and $K_2 = \begin{bmatrix} -0.1614 & 0.1243 & 0.1625 & -0.1242 \end{bmatrix}$ for $\alpha = 1.5$, $\beta = 1.0$, $\gamma = 1.5$ and test case 1. Some closed-loop trajectories are shown in Figure 4.

VIII. CONCLUSION

This work addressed the stabilization of discrete-time continuous piecewise affine systems written in a recently proposed implicit representation. This representation was reviewed and sufficient conditions for global stabilization were derived. An algorithm, based on the solution of LMI feasibility problems, was proposed to obtain a

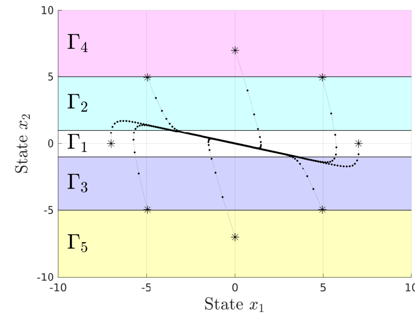

Fig. 4. Example 1: examples of closed-loop trajectories.

stabilizing nonlinear feedback law. Differently from previous approaches in the literature, the enumeration of transitions is not needed and the presence of the affine term is taken into account without further difficulties. Future work shall include the case where the partition is modified by the control law and applications regarding neural networks with ReLU (i.e. ramp) activation functions.

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