Multiplicity-Induced-Dominancy property for second-order neutral differential equations with application in oscillation damping

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Abstract

This paper addresses the exponential stability of linear time-delay systems of neutral type. In general, it is quite a challenge to establish conditions on the parameters of the system in order to guarantee such a stability. Recent works emphasized the link between maximal multiplicity and dominant roots. Indeed, conditions for a given multiple root to be necessarily dominant are investigated, this property is known as Multiplicity-Induced-Dominancy (MID). The aim of this paper is to explore the effect of multiple roots with admissible multiplicities exhibiting, under appropriate conditions, the validity of the MID property for second-order neutral time-delay differential equations with a single delay. The ensuing control methodology is summarized in a five-steps algorithm that can be exploited in the design of higher-order systems. The main ingredient of the proposed method is the dominancy proof for multiple spectral values based on frequency bounds established via integral equations. As an illustration, the stabilization of the classical oscillator benefits from the obtained results.

Keywords. Delay Systems, Neutral Functional Differential Equations, Classical Oscillator, Multiplicity-Induced-Dominancy.

1 Introduction

Delayed systems provide useful models of phenomena arising in various fields such as chemistry, economics, engineering, physics or biology. For more details on time-delay systems and their applications, we refer to [1, 13, 14, 17–19, 26, 27]. Furthermore, it is commonly accepted that second-order linear systems capture the dynamic behavior of many natural phenomena and have found numerous applications in a variety of fields, such as vibration and structural analysis. Stabilization of solutions to such a reduced order model represents a standard test bench to approve of new paradigms and methodologies in control design; see for instance [9].

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The present paper addresses the effect of the delay action on the behavior of solutions corresponding to such second-order dynamical systems. Namely, we investigate the following functional differential equation which extends the study in [9]:

$$\ddot{x}(t)+a_1\dot{x}(t)+a_0x(t)+\alpha_2\ddot{x}(t-\tau)+\alpha_1\dot{x}(t-\tau)+\alpha_0x(t-\tau)=0 \tag{1.1}$$

where the unknown function x is real-valued, $a_0, a_1, \alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}, \alpha_2 \neq 0$, and the delay $\tau > 0$. Since the derivative of highest order appears in both, the delayed term $\ddot{x}(t-\tau)$ and the non-delayed term $\ddot{x}(t)$, equation (1.1) is a delay differential equation of neutral type. Time-delay systems of neutral type, which may have an infinite number of unstable poles, are more difficult to tackle than delay systems of retarded type (i.e. the highest order of derivation is only on the non-delayed function $\ddot{x}(t)$) which exhibit only a finite number of poles in any right half-plane, see for instance [9].

Spectral methods, which investigate the spectrum distribution of the characteristic equations, are a powerful tool for the understanding of the asymptotic behavior of LTI time-delay system solutions. In the Laplace domain, linear systems with delays are described by transfer functions involving quasipolynomials : these quasipolynomials allow the spectral analysis of time-delay systems, they have been widely studied in [8, 12, 34].

The characteristic function of equation (1.1) is the quasipolynomial function $\Delta : \mathbb{C} \to \mathbb{C}$ defined for $s \in \mathbb{C}$ by

$$\Delta(s) = s^2 + a_1 s + a_0 + (\alpha_2 s^2 + \alpha_1 s + \alpha_0)\mathrm{e}^{-\tau s}. \tag{1.2}$$

The exact definition and qualitative properties of a quasipolynomial are recalled in the next section, including the fact that the multiplicity of a root of a quasipolynomial is bounded by the generic Polya and Szegö bound (denoted PS_B), which is equal to the degree of the corresponding quasipolynomial, i.e., the sum of the degrees of the involved polynomials plus the number of delays; see for instance [32, Problem 206.2, page 144 and page 347]. In particular, according to Definition 1, the degree of Δ in (1.2) is equal to 5.

For the exponential behavior of solutions of (1.1), we are interested in the spectral abscissa of the corresponding characteristic function Δ which is the real number $\rho = \sup\{\Re(s) | s \in \mathbb{C}, \Delta(s) = 0\}$; see [[19], Chapter 1, Theorem 6.2] for more details. The number ρ is related to the notion of decay rate of time-delay system solutions. The dominant root of Δ, that is, a root of Δ with the largest real part (see Definition 5), may apply to functions of the form (1.2); see for instance [27]. From a control theory viewpoint, a recent safe control methodology, based on the assignment of the closed-loop dominant solution's decay rate, shows that under appropriate conditions a multiple spectral value is the rightmost. Notice that multiple spectral values for time-delay systems can be characterized using functional Birkhoff/Vandermonde matrices; see for instance [4, 5, 8]. It turns out that, for characteristic quasipolynomials of some time-delay systems, real roots of maximal multiplicity are necessarily dominant, this property is known as Generic Multiplicity-Induced-Dominancy (GMID); see [6]. The link between multiple (not necessarily of maximal multiplicity) spectral values and their dominance, baptised Multiplicity-Induced-Dominancy (MID) in [9], has been first hinted at in [31] for some low-order cases without any attempt to address the general question; see also [20] for the specific scalar first-order equations. Recent developments pursue the investigation of the MID property mainly in the single-delay case, see for instance [3, 9–11, 23–25, 33]. In [22], the stabilization via delayed proportional derivative-acceleration feedback and predictor feedback of the inverted pendulum is considered, where the critical length of the pendulum that limits stabilization is obtained owing to the MID property; see also [29]. It is also shown in [28] that the MID-based approach provides the critical delay, and the associated control gains are easily deduced from the characteristic equation and its derivatives.

Even though the GMID is completely characterized in [6], in general, the limits of the MID property remain an open question and further developments are required to improve the understanding of its mechanisms and benefits for a control purpose.

Three main leads remain to be addressed for the MID property : the multi-delay case, spectral values with non maximal admissible multiplicities and the neutral case.

To the best of our knowledge, the multi-delay case was first investigated in [16], where the MID property is proved to hold for the first-order retarded scalar equation with two delays. Next, in the context of spectral values with strictly intermediate admissible multiplicities, one may cite [9] where a discriminant-based parametric MID was investigated in the second-order retarded case with spectral values of codimensions three and four, and [2] where sufficient and necessary conditions are provided for the MID to hold in nth-order retarded systems with a finite dimensional part corresponding to realrooted plants. Further, the neutral case was addressed in some particular cases ; see [3,23,25]. As a matter of fact, the MID has been fully characterized, in the case where maximal multiplicity is reached (GMID), for the first-order neutral equation in [25], and for the second-order in [3] and for nth-order systems in [6]. However, for spectral values with strictly intermediate admissible multiplicities, the only contributions are provided in [23]. Indeed, the MID property is extended to codimension four in second-order time-delay neutral systems, and a systematic method for a PID stabilizing tuning for low-order delayed plants is proposed.

In the present paper, we aim at improving the understanding and the characterization of the MID property for second-order neutral delay equations in the presence of real spectral values with any admissible multiplicity, which is a question of interest from an application viewpoint; see [28,29].

The sequel of the paper is organized as follows. Section 2 presents some prerequisite pertaining to quasipolynomials, and recent results for time-delay equations. Section 3 states a design methodology exploiting the MID property, the classical steps leading to the proof are recalled through a comprehensive example, the first-order neutral equation with a single delay. The main result is presented in Section 4, where a classification of admissible multiplicities for second-order neutral time-delay differential equation with a single delay is provided. Section 5 is dedicated to the proof of the main result. Finally, Section 6 is dedicated to the illustration of the obtained results on the stabilization of the classical oscillator.

2 Prerequisites

In the study of linear systems with delay, we deal with transfer functions involving quasipolynomials, which are defined hereafter.

Definition 1. *A quasipolynomial is a particular entire function $\Delta : \mathbb{C} \to \mathbb{C}$ which may be written as follows*

$$\Delta(s) = \sum_{i=0}^{k} P_i(s)\, \mathrm{e}^{-\tau_i s}, \tag{2.1}$$

where k is a positive integer, τ_i $(i = 0..k)$ are pairwise distinct non-negative real numbers and P_i $(i = 0..k)$ are polynomials of degree $d_i \geq 0$. The degree D of the quasipolynomial Δ is equal the sum of the degrees of the involved polynomials P_i plus the number of delays, i.e.,

$$D = k + \sum_{i=0}^{k} d_i.$$

An important result in the literature, known as Polya-Szegö bound, shows that there exists a link between the degree of a quasipolynomial and the number of its roots in horizontal strips of the complex plane.

Proposition 2. *[32, Problem 206.2, page 144 and page 347]. Let Δ be a quasipolynomial of degree D as in* (2.1)*, and α, $\beta \in \mathbb{R}$ be such that $\alpha \leq \beta$. If M is the number of roots of Δ contained in the set $\{s \in \mathbb{C} \,|\, \alpha \leq \Im(s) \leq \beta\}$ counting multiplicities, then*

$$\frac{(\tau_k - \tau_0)(\beta - \alpha)}{2\pi} - D \leq M \leq \frac{(\tau_k - \tau_0)(\beta - \alpha)}{2\pi} + D.$$

Furthermore, for a given root $s_0 \in \mathbb{C}$ of a quasipolynomial Δ, one obtains the following link between the multiplicity of s_0 and the degree of Δ.

Corollary 3. *Let Δ be a quasipolynomial of degree D. Then, any root $s_0 \in \mathbb{C}$ of Δ exhibits a multiplicity at most equal to D.*

Remark 4. *Corollary 3 is obtained immediately by letting $\alpha = \beta = \Im(s_0)$ in Proposition 2. Notice also that Polya-Szegö bound has been recovered in [4] using a constructive approach based on functional Birkhoff matrices. Furthermore, if some coefficients of the polynomials P_i defined in* (2.1) *vanish, then a sharper bound for the multiplicity is provided in [4].*

In what follows, we give a precise definition of the dominant root.

Definition 5. *A spectral eigenvalue s_0 is said to be a dominant (respectively, strictly dominant) root of Δ, if one has $\Re(\tilde{s}) \leq \Re(s_0)$ (respectively, $\Re(\tilde{s}) < \Re(s_0)$) for any $\tilde{s} \in \mathbb{C} \backslash \{s_0\}$, a distinct eigenvalue of Δ.*

In this paper, $\Re(s)$ and $\Im(s)$ designate respectively the real and imaginary part of the complex root s. The next proposition presents important recent result providing insights on spectrum distribution for neutral delay systems.

Proposition 6. *[30]. The generic form of the transfer function of a neutral delay system is*

$$G(s) = \frac{r(s)}{P_0(s) + P_\tau(s)\mathrm{e}^{-s\tau}}$$

where P_0, P_τ and r are real polynomials such that $\deg P_0 = \deg P_\tau$ and $\tau > 0$. Let $\alpha = \lim_{|s| \to \infty} \frac{P_0(s)}{P_\tau(s)}$

- *If $0 < |\alpha| < 1$, then G has an infinite number of unstable poles in the right half-plane;*
- *If $|\alpha| > 1$, then G has a finite number of unstable poles.*

Notice that sufficient conditions for the dominancy of simple spectral values has been proposed in [15] in the case of first-order scalar neutral equation.

Lemma 7. *[15]. Consider a characteristic equation of the form*

$$Q(s) = s\left(1 + \sum_{l=1}^{m} c_l\,\mathrm{e}^{-\sigma_l s}\right) - a - \sum_{j=1}^{k} b_j\,\mathrm{e}^{-h_j s} \tag{2.2}$$

where $a, b_j (j = 1, ..., k), c_l (l = 1, ..., m)$ are real numbers and $h_j (j = 1, ..., k)$, $\sigma_l (l = 1, ..., k)$ are positive real numbers.
Given equation (2.2)*, we introduce a function $V : \mathbb{R} \to \mathbb{R}$, defined by,*

$$V(s) = \sum_{l=1}^{m} |c_l|\,(1 + |s|\sigma_l)\,\mathrm{e}^{-\sigma_l s} + \sum_{j=1}^{k} |b_j|\mathrm{e}^{-h_j s}, \quad s \in \mathbb{R}.$$

Suppose that there exists a real zero s_0 of equation (2.2)*. If $V(s_0) < 1$, then s_0 is a real simple dominant zero of* (2.2).

Note that the extension of the above result to second-order delay equations remains a challenging endeavor.

3 MID methodology on a toy model

The MID property consists of the conditions under which a given multiple complex zero of a quasipolynomial is dominant. For instance, in the generic quasipolynomial case, the real root of maximal multiplicity is necessarily the dominant (GMID). However, multiple roots with intermediate admissible multiplicities may be dominant or not. Thanks to this property, an ensued control strategy is proposed in [2, 9], which consists in assigning a root with an admissible multiplicity once appropriate conditions guaranteeing its dominancy are established. Furthermore, the MID property may be used to tune standard controllers. For instance, in [23] it is applied to the systematic tuning of the stabilizing PID controller of a first order plant. Here, we aim at assigning dominant multiple real roots with admissible codimensions.

The proof of the MID property consists of five steps. First, we establish conditions on the parameters of the system guaranteeing the existence of a multiple root. Second, an affine change of variable of the characteristic equation is performed in order to reduce the said quasipolynomial to a normalized form; the desired multiple root becomes 0 and the delay 1. Next, under the latter normalization, the characteristic equation may be easily factorized in terms of an integral expression. Hence, we derive a bound on the imaginary part of roots of the normalized quasipolynomial in the complex right half-plane. Lastly, a certification of the dominance of the multiple root is demonstrated.

In the following, Algorithm 1 is a pseudo-code listing the instructions to be followed to target an suitable frequency bound.

To illustrate the proof of the methodology of the MID property described above, we consider a model of phenomena in the bio-sciences describing the dynamics of a vector-borne disease. It is based on a simple scalar delay differential equation with a positive single delay τ. In its linearized version, the infected host population $x(t)$ is governed by

$$\dot{x}(t) + a_0 x(t) + a_1 x(t-\tau) = u(t) \tag{3.1}$$

where u is the delayed output-feedback: $u(t) = (a_1 - \alpha_0)x(t-\tau) - \alpha_1 \dot{x}(t-\tau)$, α_0, α_1 are real coefficients, and $a_1 > 0$ is called the contact rate; it represents the contact number between infected and uninfected populations. Assume that the infection of the host recovery proceeds exponentially at a rate of $-a_0 > 0$. The characteristic equation of (3.1) is the quasipolynomial function of degree 3, defined by

$$\Delta(s) = s + a_0 + (\alpha_1 s + \alpha_0)\,\mathrm{e}^{-\tau s}. \tag{3.2}$$

The first-order neutral equation is treated in the context of delay differential-algebraic systems in [25]. In the following, we shall illustrate the dominancy proof following the methodology previously described.

Step 1. (Forcing multiplicity) The real s_0 is a root of multiplicity 3 of Δ if, and only if, the coefficients a_0, α_0, α_1, the root s_0 and the delay τ satisfy the relations below

$$a_0 = -s_0 - \frac{2}{\tau}, \quad \alpha_0 = \left(-s_0 + \frac{2}{\tau}\right)\mathrm{e}^{s_0 \tau}, \quad \alpha_1 = \mathrm{e}^{\tau s_0}. \tag{3.3}$$

Algorithm 1: Estimation of the MID frequency bound in second-order neutral time-delay differential equations

Input: $\tilde{\Delta}(z) = P_0(z) + P_1(z)\mathrm{e}^{-z}$; `// Normalized quasipolynomial`

`// Initialization`

$\mathrm{ord} = 0$; `// ord: order of truncation of the Taylor expansion of` $\mathrm{e}^{2x} = \underbrace{\underbrace{1}_{\mathrm{ord}=0} + 2x}_{\mathrm{ord}=1} + 2x^2 + \frac{4x^3}{3} + \ldots$;

$dominance = false$;

$\exists z_0 = x + i\omega \in \mathbb{R}_+^* + i\mathbb{R}_+^*$ s.t. $\tilde{\Delta}(z_0) = 0$;

$|P_0(x+i\omega)|^2 \mathrm{e}^{2x} = |P_1(x+i\omega)|^2$;

while $\sim$ dominance **do**

  ord =ord+1;

  $F(\omega) = |P_1(x+i\omega)|^2 - |P_0(x+i\omega)|^2 \mathrm{T}_{ord}(\mathrm{e}^{2x}) > 0$; `//` $\mathrm{T}_{ord}(\mathrm{e}^{2x})$`: Taylor expansion of` e^{2x} `of order` $= ord$

  $\omega^2 = \Omega$;

  $G(\Omega) = a(x)\Omega^2 + b(x)\Omega + c(x)$; `//` $a(x) \neq 0$, $G(\Omega) = F(\omega)$

$\Omega^{\pm}(x) = \frac{-b(x) \pm \sqrt{b^2(x) - 4a(x)c(x)}}{2a(x)}$; `//` $\Omega^{\pm}(x)$ `depends on free parameters denoted by` **`param`** `hereafter`

if $\max_x(\max_{\mathbf{param}}(\Omega^{\pm}(x))) < \pi^2$ **then**

  $dominance = true$;

Output: Frequency bound ;

Step 2. (Normalization) Performing the translation and scaling of the spectrum by the following change of variables $\tilde{\Delta}(z) = \tau\Delta(\frac{z}{\tau} + s_0)$ for $z \in \mathbb{C}$, we get the following normalized characteristic equation $\tilde{\Delta}(z) = z + b_0 + (\beta_1 z + \beta_0)\,\mathrm{e}^{-z}$ with relations (3.3) normalized as follows

$$b_0 = \tau\,(a_0 + s_0)\,, \quad \beta_0 = \tau\,(\alpha_1 s_0 + \alpha_0)\,\mathrm{e}^{-\tau s_0}, \quad \beta_1 = \alpha_1\,\mathrm{e}^{-\tau s_0}. \tag{3.4}$$

Step 3. (Integral representation) The real root s_0 is a root of multiplicity 3 of Δ if, and only if, 0 is a root of multiplicity 3 of $\tilde{\Delta}$. As a matter of fact, since $\tilde{\Delta}$ is a quasipolynomial of degree 3, zero is a root of multiplicity 3 of $\tilde{\Delta}$ if, and only if, $\tilde{\Delta}(0) = \tilde{\Delta}'(0) = \tilde{\Delta}''(0) = 0$. The latter identities yield a linear system whose unique solution is $(b_0, \beta_0, \beta_1) = (-2, 2, 1)$. From relations (3.4), one concludes that s_0 is a root of multiplicity 3 of Δ if, and only if, relations (3.3) hold. Moreover, under the latter conditions, the quasipolynomial (3.2) reduces to $\tilde{\Delta}(z) = P_0(z) + P_1(z)\,\mathrm{e}^{-z}$ where $P_0(z) = z - 2$ and $P_1(z) = z + 2$. Hence, the quasipolynomial $\tilde{\Delta}$ admits the following Fredholm integral representation

$$\tilde{\Delta}(z) = \int_0^1 q(t)\,\mathscr{K}(z,t)\,\mathrm{d}t$$

where

$$q(t) = t\,(1-t)\,, \quad \mathscr{K}(z,t) = z^3 \mathrm{e}^{-tz}$$

which is easily verified via an integration by parts.

Step 4. (Frequency bound) Assume that $z_0 = x_0 + i\,\omega_0 \in \mathbb{R}_+ + i\mathbb{R}_+$ is a root of $\tilde{\Delta}$, so that $\tilde{\Delta}(z_0) = 0$ if, and only if, $|P_0(x_0 + i\omega_0)|^2\,\mathrm{e}^{2x_0} = |P_1(x_0 + i\omega_0)|^2$. Considering a truncation of order 1 of the exponential term e^{2x}, the latter is lower bounded by $1 + 2x$. Next, define $F(x,\omega) = |P_1(x+i\omega)|^2 - (1+2x)\,|P_0(x+i\omega)|^2$ where $F > 0$ for any $x > 0$. The zeros of F are characterized by the first order polynomial $G(\Omega = \omega^2) = -2x\Omega - 2x^3 + 8x^2$. The polynomial function G admits a single real root $\Omega_0(x) = -x\,(x-4)$, which reaches a maximum value at $x^* = 2$. As a result, Ω_0 is bounded by $\Omega^* = 4 < \pi^2$. Thus, one obtains the desired frequency bound, $0 < \omega \leq 2 < \pi$.

Step 5. (Dominancy) The purpose of the frequency bound is to prove the dominancy by a contradiction argument. For this purpose, assume that there exists $z_0 \in \mathbb{R}_+ + i\mathbb{R}_+$ root of $\tilde{\Delta}$. Then, the integral representation yields $\int_0^1 t\,(1-t)\,\mathrm{e}^{-t\,z_0}\,\mathrm{d}t = 0$, the imaginary part of which is

$$\int_0^1 t\,(1-t)\,\mathrm{e}^{-t\,x}\sin(\omega t)\,\mathrm{d}t = 0.$$

Now, the frequency bound $0 < \omega \leq \pi$ of the previous step entails that the function $t \mapsto t\,(1-t)\,\mathrm{e}^{-xt}\sin(\omega t)$ is strictly positive in $(0,1)$, thereby contradicting the last equality.

4 Statement of the main result

The main result in this paper presents a classification of admissible multiplicities for a given root of the quasipolynomial (1.2).

Theorem 8. *Consider the quasipolynomial function* Δ *defined in* (1.2).

(a) *GMID : spectral value of maximal admissible multiplicity*

(a) The real s_0 *is a root of multiplicity* 5 *of* Δ *if, and only if, the coefficients* $a_0, a_1, \alpha_0, \alpha_1, \alpha_2$*, the root* s_0 *and the delay* τ *satisfy the following relations*

$$\begin{cases} a_1 = -2s_0 - \frac{6}{\tau}, \quad a_0 = s_0^2 + \frac{6}{\tau}s_0 + \frac{12}{\tau^2}, \\ \alpha_2 = -\mathrm{e}^{\tau s_0}, \quad \alpha_1 = \left(2s_0 - \frac{6}{\tau}\right)\mathrm{e}^{\tau s_0}, \quad \alpha_0 = -\left(s_0^2 - \frac{6}{\tau}s_0 + \frac{12}{\tau^2}\right)\mathrm{e}^{\tau s_0}. \end{cases} \tag{4.1}$$

(b) If relations (4.1) *are satisfied then* s_0 *is necessarily a dominant root of* Δ.

(b) *MID : codimension 4*

(a) Consider $d = {a_1}^2 - 4\,a_0$ *the discriminant of the finite dimensional part of the dynamical system defined by* Δ*. The quasipolynomial function* (1.2) *admits a real root at*

$$s_\pm = \frac{1}{\tau}\left(-\frac{a_1\tau}{2} - 3 \pm \frac{1}{2}\sqrt{\tau^2 d + 12}\right) \tag{4.2}$$

of multiplicity 4 *if, and only if, the coefficients* α_0, α_1 *and* α_2 *satisfy the following relations*

$$\begin{cases} \alpha_0 = \left(\left(\frac{{a_1}^2\tau}{2} - \tau\,a_0 + 6a_1 + \frac{42}{\tau}\right)s_\pm + \frac{\tau a_0 a_1}{2} + \frac{3{a_1}^2}{2} + 8\,a_0 + \frac{30a_1}{\tau} + \frac{54}{\tau^2}\right)\mathrm{e}^{\tau s_\pm}, \\ \alpha_1 = \left((a_1\tau + 12)\,s_\pm + 2\,\tau\,a_0 + 8\,a_1 + \frac{18}{\tau}\right)\mathrm{e}^{\tau s_\pm}, \quad \alpha_2 = \left(2 + \tau\,\left(s_\pm + \frac{a_1}{2}\right)\right)\mathrm{e}^{\tau s_\pm}. \end{cases} \tag{4.3}$$

(b) If the relations above are satisfied, and a_1, a_0 *satisfy the lower bounds* $a_0 \geq -\frac{6}{\tau^2}$ *and* $a_1 \geq -\frac{6}{\tau}$*, then* s_+ *is a dominant root of* Δ.

(c) *MID : codimension 3*

(a) The real number s_0 is a root of multiplicity 3 *of* Δ *if, and only if, the following relations hold*

$$\begin{cases} \alpha_0 = -\frac{1}{2}\left(\tau^2 a_1 {s_0}^3 + \tau^2 {s_0}^4 + \tau^2 a_0 {s_0}^2 + 2\tau {s_0}^3 - 2\tau a_0 s_0 + 2a_0\right) e^{\tau s_0}, \\ \alpha_1 = \left(\tau^2 a_1 {s_0}^2 + \tau^2 {s_0}^3 + \tau^2 a_0 s_0 + \tau a_1 s_0 + 3\tau {s_0}^2 - \tau a_0 - a_1\right) e^{\tau s_0}, \\ \alpha_2 = -\frac{1}{2}\left(\tau^2 a_1 s_0 + \tau^2 {s_0}^2 + a_0 \tau^2 + 2a_1\tau + 4\tau s_0 + 2\right) e^{\tau s_0}. \end{cases} \tag{4.4}$$

(b) If the relations above hold and a_1, a_0 satisfy the lower bounds $a_0 \geq \frac{\varepsilon}{4\tau^2}$ and $a_1 \geq 0$, where $\varepsilon = (-10\sqrt{2}-16)\sqrt{16\sqrt{2}-22} + 16\sqrt{2} + 20$, then the real root s_0 chosen as follows

$$\begin{cases} s_0 \in \left(-\frac{a_1}{2} - \frac{\sqrt{d+\frac{\varepsilon}{\tau^2}}}{2}, -\frac{a_1}{2} + \frac{\sqrt{d+\frac{\varepsilon}{\tau^2}}}{2}\right) & \textit{if } d < 0, \\ s_0 \in \left(-\frac{a_1}{2} - \frac{\sqrt{d+\frac{\varepsilon}{\tau^2}}}{2}, -\frac{a_1}{2} - \frac{\sqrt{d}}{2}\right) \cup \left(-\frac{a_1}{2} + \frac{\sqrt{d}}{2}, -\frac{a_1}{2} + \frac{\sqrt{d+\frac{\varepsilon}{\tau^2}}}{2}\right) & \textit{otherwise}, \end{cases}$$

is a dominant root of Δ.

From a control theory viewpoint, if instantaneous access to the state variables is not available, one option is to consider delayed controllers. In our case, the aim is to stabilize solutions of the control system $\ddot{x}(t) + a_1\dot{x}(t) + a_0 x(t) = u(t)$ by using a delayed feedback controller $u(t) = -\alpha_2\ddot{x}(t-\tau) - \alpha_1\dot{x}(t-\tau) - \alpha_0 x(t-\tau)$. Notice that, such an idea has already been proposed in [9] with controller $u(t) = -\alpha_1\dot{x}(t-\tau) - \alpha_0 x(t-\tau)$, by exploiting the MID property for retarded differential equation, the above result extends such an idea to neutral equations. Furthermore, Theorem 8 offers a certified tuning of the controller's parameters allowing to assign the closed-loop dominant spectral value based on the MID strategy with appropriate admissible multiplicity. This can be done by taking into account the discriminant of the open-loop characteristic function as discussed in [9], see also [2]. Such a control strategy is part of a more general framework called partial pole placement, see for instance [7].

5 Proof of the main result

The proof of item (a) (GMID) in Theorem 8 is detailed in [3]. The normalization of the characteristic function Δ gives $\tilde{\Delta}(z) = z^2 - 6z + 12 - (z^2 + 6z + 12)e^{-z}$. Next, the integral factorization of $\tilde{\Delta}$ is computed to be $\tilde{\Delta}(z) = \frac{z^5}{2}\int_0^1 t^2(t-1)^2 e^{-zt}dt$. The dominance proof is established by providing an adequate frequency bound ($\omega_0 < \pi$), where the considered truncation is of order 3, to show that a non-zero root of $\tilde{\Delta}$ with non-negative real part cannot exist.

Item (b) is well presented in [23]. In a similar way, the normalization of the characteristic function Δ provides

$$\tilde{\Delta}(z) = z^2 + (\rho - 6)z - 3\rho + 12 + \left[(\rho/2 - 1)z^2 + (2\rho - 6)z + 3\rho - 12\right]e^{-z}, \tag{5.1}$$

where $\rho = \sqrt{12 + ({a_1}^2 - 4a_0)\tau^2}$. The integral representation of the characteristic function is $\tilde{\Delta}(z) = \frac{z^4}{2}\int_0^1 t(1-t)(t(\rho-4)+2)e^{-tz}dt$. The dominance of s_+ as a root of Δ is equivalent to the dominance of $z = 0$ as a root of $\tilde{\Delta}$. Consider $z_0 = x_0 + i\omega_0 \in \mathbb{R}_+ + i\mathbb{R}_+$ as a root of $\tilde{\Delta}(z) = P_0(z) + P_1(z)e^{-z}$ as defined in (5.1), with $P_0(z) = z^2 + (\rho - 6)z - 3\rho + 12$ and $P_1(z) = \left(\frac{\rho}{2} - 1\right)z^2 + (2\rho - 6)z + 3\rho - 12$, so that $|P_0(x_0 + i\omega_0)|^2 e^{2x_0} = |P_1(x_0 + i\omega_0)|^2$. Now, define the function $F_\rho(x, \omega) = |P_1(x + i\omega)|^2 - (1 + 2x)|P_0(x + i\omega)|^2$, where $F_\rho > 0$ since $e^{2x} > 1 + 2x$ for any $x > 0$;

the order of the considered truncation order in this case is equal to 1. The zeros of F_ρ can be characterized by the quadratic polynomial

$$G_\rho(x,\Omega) = a_\rho(x)\,\Omega^2 + b_\rho(x)\,\Omega + c_\rho(x) \tag{5.2}$$

where $\Omega = \omega^2$, $a_\rho(x) = \left(\rho^2 - 4\rho - 8x\right)/4$, $b_\rho(x) = x^2\left(\rho^2 - 12\rho - 8x + 48\right)/2$, and $c_\rho(x) = -2x^5 + x^4(\rho-8)(\rho-12)/4 + 24\,(\rho-4)x^3 + 18\,(\rho-4)^2 x^2$. The discriminant of G_ρ is positive under the condition $\rho \in (2\sqrt{3},4)$ for $x > 0$. The polynomial function G_ρ admits two real roots denoted by $\Omega_\rho^\pm$, where Ω_ρ^+ is the greater solution (positive signal). Using the fact that $\rho \in (2\sqrt{3},4)$, the solution Ω_ρ^+ is upper-bounded by

$$\Omega^+(x) = -x^2 - 3\sqrt{3}x + \frac{15}{2}x + \sqrt{(-228x+468)\sqrt{3} + 4x^2 + 369x - 810}$$

which depends only on x and reaches its maximum at $x^* \approx 2.139$. Thus, $\omega^2 = \Omega_\rho^+(x) < \Omega^+(x^*) \approx 4.961 < \pi^2$, i.e., $\omega < \pi$.

The completion of the proof of Theorem 8 (item (c)) is presented in the sequel; it follows the methodology already described in detail in section 3 and applied to the toy model (3.1); see also Algorithm 1.

5.1 Forcing multiplicity and normalization of the characteristic function

This section covers Step 1 and 2 of the methodology. The following lemma gives a normalization of the quasipolynomial function Δ admitting a triple real root, which corresponds to conditions (4.4).

Lemma 9. *Let $s_0 \in \mathbb{R}$ and consider the quasipolynomial $\tilde{\Delta} : \mathbb{C} \to \mathbb{C}$ obtained from Δ in* (1.2) *by the following change of variables $\tilde{\Delta}(z) = \tau^2 \Delta\left(\frac{z}{\tau} + s_0\right)$, $z \in \mathbb{C}$, then*

$$\begin{cases} \tilde{\Delta}(z) = \left((-\delta/2 - 1 - \upsilon)\,z^2 + (-\delta - \upsilon)\,z - \delta\right)\mathrm{e}^{-z} + z^2 + \upsilon\,z + \delta, \\ \delta = \tau^2\left({s_0}^2 + a_1 s_0 + a_0\right), \\ \upsilon = \tau\,(2\,s_0 + a_1)\,. \end{cases} \tag{5.3}$$

Proof. It follows immediately from the normalization that s_0 is a root of multiplicity 3 of Δ if, and only if, 0 is a root of multiplicity 3 of $\tilde{\Delta}$. As a matter of fact, zero is a root of multiplicity 3 of $\tilde{\Delta}$ if, and only if, $\tilde{\Delta}(0) = \tilde{\Delta}'(0) = \tilde{\Delta}^{(2)}(0) = 0$. Hence, we obtain the linear system $b_0 + \beta_0 = -\beta_0 + \beta_1 + b_1 = 2 + \beta_0 - 2\beta_1 + 2\beta_2 = 0$ whose solution is $(\beta_0, \beta_1, \beta_2) = (-b_0, -b_0 - b_1, -1 - \frac{b_0}{2} - b_1)$, where

$$\begin{cases} b_0 = \left({s_0}^2 + a_1\,s_0 + a_0\right)\tau^2, \quad b_1 = 2\,\tau\,\left(s_0 + \frac{1}{2}\,a_1\right) \\ \beta_0 = \tau^2\,\left(\alpha_2\,{s_0}^2 + \alpha_1\,s_0 + \alpha_0\right)\,\mathrm{e}^{-s_0\,\tau}, \quad \beta_1 = 2\,\tau\,\left(\alpha_2\,s_0 + \frac{1}{2}\,\alpha_1\right)\,\mathrm{e}^{-s_0\,\tau}, \quad \beta_2 = \alpha_2\,\mathrm{e}^{-s_0\,\tau}, \end{cases} \tag{5.4}$$

which completes the proof. ■ □

5.2 Factorization of the normalized characteristic function

This section covers Step 3 of the methodology. The quasipolynomial $\tilde{\Delta}$ defined in (5.3) can be factorized as

$$\tilde{\Delta}(z) = z^3 \int_0^1 q_{\delta,\upsilon}(t)\,\mathrm{e}^{-tz}\,\mathrm{d}t \quad \text{where} \quad q_{\delta,\upsilon}(t) = \frac{\delta}{2}t^2 + \upsilon\,t + 1. \tag{5.5}$$

In our approach, the sign constancy of the polynomial $q_{\delta,\upsilon}$ defined previously for $t \in (0,1)$ is necessary. Therefore, the following lemma gives regions in the parameter space guaranteeing the sign constancy of $q_{\delta,\upsilon}$ for $t \in (0,1)$; see Figure 5.1.

Lemma 10. *Let $q_{\delta,\upsilon}$ be the quadratic polynomial with respect to t defined by* (5.5). *Then, $q_{\delta,\upsilon}$ has a constant sign for $t \in (0,1)$ if, and only if, $(\delta,\upsilon) \in R_q = R_q^1 \cup R_q^2 \cup R_q^3$ where*

$$R_q^1 = \left\{(\delta,\upsilon) \in \mathbb{R}^2 : \delta > 0, -\frac{\delta}{2} - 1 \le \upsilon \le -\delta\right\} \cup \left\{(\delta,\upsilon) \in \mathbb{R}^2 : \delta > 0, -\sqrt{2\delta} < \upsilon\right\}, \quad (5.6)$$

$$R_q^2 = \left\{(\delta,\upsilon) \in \mathbb{R}^2 : \delta < 0, \upsilon \ge -1 - \frac{\delta}{2}\right\} \quad \text{and} \quad R_q^3 = \left\{(\delta,\upsilon) \in \mathbb{R}^2 : \delta = 0, \upsilon \ge -1\right\}. \quad (5.7)$$

Proof. The polynomial $q_{\delta,\upsilon}$ admits two roots given by $t^{\pm} = \left(-\upsilon \pm \sqrt{\upsilon^2 - 2\delta}\right)/\delta$. Since, for $\upsilon^2 - 2\delta < 0$, the polynomial $q_{\delta,\upsilon}$ does not admit real roots, then $q_{\delta,\upsilon}$ has a constant sign in $(0,1)$. If $\upsilon^2 - 2\delta \ge 0$, then $q_{\delta,\upsilon}$ admits two real roots $t^{\pm}$; sub-cases are to be considered with respect to the sign of δ.

(a) If $\delta > 0$, then $t^- \le t^+$ and the assumption $\upsilon^2 - 2\delta \ge 0$ is equivalent to $\upsilon \le -\sqrt{2\delta}$ or $\upsilon \ge \sqrt{2\delta}$. Since $\delta > 0$, one has $\upsilon^2 - 2\delta < \upsilon^2$, so that

$$\left(-|\upsilon| + \sqrt{\upsilon^2 - 2\delta}\right)/\delta < 0.$$

The latter inequality is split in two cases.

(a) If $\upsilon \ge \sqrt{2\delta}$, then $t^+ < 0$. As a result, $q_{\delta,\upsilon}$ has no roots in $(0,1)$ which guarantees its sign constancy.

(b) If $\upsilon \le -\sqrt{2\delta}$, then $t^- > 0$. In this case, we need to look for conditions guaranteeing that $t^- \ge 1$. Since $\delta > 0$, we have $-\upsilon - \sqrt{\upsilon^2 - 2\delta} \ge \delta$, so that$\upsilon^2 - 2\delta \le (\delta + \upsilon)^2$. We conclude that $-1 - \frac{\delta}{2} \le \upsilon$. Consequently, $t^- \ge 1$ if, and only if, $-\frac{\delta}{2} - 1 \le \upsilon \le -\delta$ for all $\delta > 0$.

As a conclusion, if $\delta > 0$, then the quadratic polynomial $q_{\delta,\upsilon}$ has constant sign for $t \in (0,1)$ if, and only if, $(\delta,\upsilon) \in \mathbb{R}_q^1$.

(b) If $\delta < 0$, then the assumption $\upsilon^2 - 2\delta \ge 0$ is obviously satisfied and we can notice that $t^- > 0$, $t^+ < 0$ and $t^- > t^+$. In this case, we need to look for conditions under which $t^- \ge 1$ which is equivalent to $\sqrt{\upsilon^2 - 2\delta} \ge -\delta - \upsilon$.

Now, consider two cases.

(a) If $-\delta - \upsilon \ge 0$, then $\sqrt{\upsilon^2 - 2\delta} \ge -\delta - \upsilon$ is equivalent to $-\frac{\delta}{2} - 1 \le \upsilon \le -\delta$, for all $\delta < 0$.

(b) If $-\delta - \upsilon < 0$, then $\sqrt{\upsilon^2 - 2\delta} \ge -\delta - \upsilon$ is immediately satisfied, so that $t^- \ge 1$ if $\upsilon > -\delta$, for all $\delta < 0$.

As a conclusion, if $\delta < 0$, then the quadratic polynomial $q_{\delta,\upsilon}$ has constant sign for $t \in (0,1)$ if, and only if, $(\delta,\upsilon) \in \mathbb{R}_q^2$.

(c) If $\delta = 0$, then the quadratic polynomial reduces to $q_{\delta,\upsilon}(t) = \upsilon t + 1$ which reduces to 1 for $\upsilon = 0$.

Next, if we assume that $\upsilon \ne 0$, then $q_{\delta,\upsilon}$ admits one real root given by $t_0 = -\frac{1}{\upsilon}$. As a matter of fact, one has $t_0 < 0$ when $\upsilon \le 0$ and $t_0 \ge 1$ when $-1 \le \upsilon < 0$. Hence, $q_{\delta,\upsilon}$ has constant sign for $t \in (0,1)$ if, and only if, $(\delta,\upsilon) \in R_q^3$.

The announced result is proved. ■ □

5.3 Parametric characterization of candidate regions for MID

Now, we follow Algorithm 1 in order to tackle Step 4 of the methodology. Let $z_0 = x_0 + i\omega_0 \in \mathbb{R}_+ + i\mathbb{R}_+$ be a root of $\tilde{\Delta}(z) = P_0(z) + P_1(z)\,\mathrm{e}^{-z}$ as defined in (5.3), where $P_0(z) = z^2 + \upsilon z + \delta$, $P_1(z) = (-\delta/2 - 1 - \upsilon)z^2 + (-\delta - \upsilon)z - \delta$ and z_0 satisfy the following equality $|P_0(x_0 + i\omega_0)|^2\mathrm{e}^{2x_0} = |P_1(x_0 + i\omega_0)|^2$. Since $\mathrm{e}^{2x} > \mathrm{T}_{ord}(\mathrm{e}^{2x})$ for any $x \in \mathbb{R}_+$ for truncation orders $ord \in \{0, 1\}$, function $F_{\delta,\upsilon}(x,\omega) = |P_1(x+i\omega)|^2 - |P_0(x+i\omega)|^2\,\mathrm{T}_{ord}(\mathrm{e}^{2x})$ satisfies $F_{\delta,\upsilon}(x_0,\omega_0) > 0$. Moreover, the zeros of $F_{\delta,\upsilon}$ can be characterized by the following quadratic polynomial of degree 2 in $\Omega = \omega^2$

$$G_{\delta,\upsilon}(x,\Omega) = a_{\delta,\upsilon}(x)\,\Omega^2 + b_{\delta,\upsilon}(x)\,\Omega + c_{\delta,\upsilon}(x), \tag{5.8}$$

where coefficients $a_{\delta,\upsilon}, b_{\delta,\upsilon}, c_{\delta,\upsilon}$ depend on the lower bound T_{ord} provided by the truncation order ord.

5.3.1 Order zero truncation

In this case, $\mathrm{T}_0 = 1$, hence the $G_{\delta,\upsilon}$ coefficients are given by

$$\begin{cases} a_{\delta,\upsilon} = \frac{(2\upsilon+\delta+4)(2\upsilon+\delta)}{4}, \quad b_{\delta,\upsilon}(x) = 2\,a_{\delta,\upsilon}\,x^2 + b_{1,\delta,\upsilon}\,x, \\ c_{\delta,\upsilon}(x) = a_{\delta,\upsilon}\,x^4 + b_{1,\delta,\upsilon}\,x^3 + c_{2,\delta,\upsilon}\,x^2 + c_{1,\delta,\upsilon}\,x, \end{cases} \tag{5.9}$$

with, $b_{1,\delta,\upsilon} = \delta^2 + 3\,\upsilon\,\delta + 2\,\upsilon^2 + 2\,\delta, \quad c_{1,\delta,\upsilon} = 2\,\delta^2, \quad \text{and} \quad c_{2,\delta,\upsilon} = 2\,(2\,\upsilon + \delta)\,\delta.$

Recall that in our approach, the condition of constancy of the sign of $q_{\delta,\upsilon}$ is necessary. In addition, we need to guarantee the positivity of $G_{\delta,\upsilon}$, i.e., one has to investigate conditions on the signs of $a_{\delta,\upsilon}$ as well as the discriminant of $G_{\delta,\upsilon}$ which is defined by the following second degree polynomial in x

$$D_{\delta,\upsilon}(x) = \left(-4\,a_{\delta,\upsilon}\,c_{2,\delta,\upsilon} + {b^2}_{1,\delta,\upsilon}\right)x^2 - \left(4\,a_{\delta,\upsilon}\,c_{1,\delta,\upsilon}\right)x. \tag{5.10}$$

Let define

$$\Upsilon_{\delta,\upsilon} = -4\,a_{\delta,\upsilon}\,c_{2,\delta,\upsilon} + b^2_{1,\delta,\upsilon}. \tag{5.11}$$

The following lemma provides an analysis of the sign of $\Upsilon_{\delta,\upsilon}$.

Lemma 11. *Consider $\Upsilon_{\delta,\upsilon}$ given by* (5.11)*, and let*

$$\upsilon_1 = \delta_+ - \frac{1}{4}\sqrt{(\alpha_+ + \beta_+\delta)\,\delta}, \quad \upsilon_2 = \delta_- - \frac{1}{4}\sqrt{(\alpha_- - \beta_-\delta)\,\delta}, \tag{5.12}$$

$$\upsilon_3 = \delta_- + \frac{1}{4}\sqrt{(\alpha_- - \beta_-\delta)\,\delta}, \tag{5.13}$$

$$\upsilon_4 = \delta_+ + \frac{1}{4}\sqrt{(\alpha_+ + \beta_+\delta)\,\delta}, \quad \delta_1 = -\frac{\alpha_+}{\beta_+}, \quad \delta_2 = \frac{\alpha_-}{\beta_-}. \tag{5.14}$$

where

$$\alpha_\pm = 16(3 \pm 2\sqrt{2}), \quad \beta_\pm = 12\sqrt{2} \pm 17, \quad \delta_\pm = \frac{\delta}{4} \pm \frac{\delta}{\sqrt{2}}. \tag{5.15}$$

Then,

- $\Upsilon_{\delta,\upsilon} > 0 \iff (\delta,\upsilon) \in R_{\Upsilon^+}$, *with*

$$R_{\Upsilon^+} = R_1^{++} \cup R_2^{++} \cup R_3^{++} \cup R_1^{-+} \cup R_2^{-+} \cup R_3^{-+} \cup R_4^{-+} \cup R_5^{-+} \cup R_6^{-+},$$

where

$$
\begin{aligned}
R_1^{++} &= \left\{(\delta,\upsilon)\in\mathbb{R}^2 : \delta>0, \upsilon<\upsilon_1\right\},\\
R_2^{++} &= \left\{(\delta,\upsilon)\in\mathbb{R}^2 : \delta>0, \upsilon_2<\upsilon<\upsilon_3\right\},\\
R_3^{++} &= \left\{(\delta,\upsilon)\in\mathbb{R}^2 : \delta>0, \upsilon>\upsilon_4\right\},\\
R_1^{-+} &= \left\{(\delta,\upsilon)\in\mathbb{R}^2 : \delta_1<\delta<0\right\},\\
R_2^{-+} &= \left\{(\delta,\upsilon)\in\mathbb{R}^2 : \delta_2<\delta<\delta_1, \upsilon>\upsilon_4\right\},\\
R_3^{-+} &= \left\{(\delta,\upsilon)\in\mathbb{R}^2 : \delta_2<\delta<\delta_1, \upsilon<\upsilon_1\right\},\\
R_4^{-+} &= \left\{(\delta,\upsilon)\in\mathbb{R}^2 : \delta<\delta_2, \upsilon>\upsilon_4\right\},\\
R_5^{-+} &= \left\{(\delta,\upsilon)\in\mathbb{R}^2 : \delta<\delta_2, \upsilon<\upsilon_1\right\},\\
R_6^{-+} &= \left\{(\delta,\upsilon)\in\mathbb{R}^2 : \delta<\delta_2, \upsilon_2<\upsilon<\upsilon_3\right\}.
\end{aligned}
$$

- $\Upsilon_{\delta,\upsilon}<0 \iff (\delta,\upsilon)\in R_{\Upsilon^-}$, *with*

$$
R_{\Upsilon^-} = R_1^{+-}\cup R_2^{+-}\cup R_1^{--}\cup R_2^{--}\cup R_3^{--}
$$

where

$$
\begin{aligned}
R_1^{+-} &= \left\{(\delta,\upsilon)\in\mathbb{R}^2 : \delta>0, \upsilon_1<\upsilon<\upsilon_2\right\},\\
R_2^{+-} &= \left\{(\delta,\upsilon)\in\mathbb{R}^2 : \delta>0, \upsilon_3<\upsilon<\upsilon_4\right\},\\
R_1^{--} &= \left\{(\delta,\upsilon)\in\mathbb{R}^2 : \delta_2<\delta<\delta_1, \upsilon_1<\upsilon<\upsilon_4\right\},\\
R_2^{--} &= \left\{(\delta,\upsilon)\in\mathbb{R}^2 : \delta<\delta_2, \upsilon_3<\upsilon<\upsilon_4\right\}\\
R_3^{--} &= \left\{(\delta,\upsilon)\in\mathbb{R}^2 : \delta<\delta_2, \upsilon_1<\upsilon<\upsilon_2\right\}.
\end{aligned}
$$

Proof. We compute the expression of $\Upsilon_{\delta,\upsilon}$ in terms of δ and υ,

$$
\Upsilon_{\delta,\upsilon} = -2\,(2\,\upsilon+\delta+4)\,(2\,\upsilon+\delta)^2\,\delta + \left(\delta^2+3\,\upsilon\,\delta+2\,\upsilon^2+2\,\delta\right)^2.
$$

As a fourth-degree polynomial with respect to υ, $\Upsilon_{\delta,\upsilon}$ admits 4 roots.

(a) **Case** $\delta>0$: The roots are real such that $\upsilon_1<\upsilon_2<\upsilon_3<\upsilon_4$, see (5.12-5.14), and $\Upsilon_{\delta,\upsilon} = 4\,(\upsilon-\upsilon_1)\,(\upsilon-\upsilon_2)\,(\upsilon-\upsilon_3)\,(\upsilon-\upsilon_4)$. As a result,

- $\Upsilon_{\delta,\upsilon}>0 \iff (\delta,\upsilon)\in R_1^{++}\cup R_2^{++}\cup R_3^{++}$;
- $\Upsilon_{\delta,\upsilon}<0 \iff (\delta,\upsilon)\in R_1^{+-}\cup R_2^{+-}$

(b) **Case** $\delta<0$: Consider δ_1 and δ_2 given in (5.14). In this case, υ_1 and υ_4 are well defined for $\delta\in(-\infty,\delta_1)$, as for υ_2 and υ_3 are well defined for $\delta\in(-\infty,\delta_2)$. Notice that υ_1 and υ_4 form a parabola of vertex (δ_1,δ_+), and that υ_2 and υ_3 form a parabola of vertex (δ_2,δ_-), this leads to

- $\Upsilon_{\delta,\upsilon}>0 \iff (\delta,\upsilon)\in R_1^{-+}\cup R_2^{-+}\cup R_3^{-+}\cup R_4^{-+}\cup R_5^{-+}\cup R_6^{-+}$;
- $\Upsilon_{\delta,\upsilon}<0 \iff (\delta,\upsilon)\in R_1^{--}\cup R_2^{--}\cup R_3^{--}$. ■

□

We are now able to characterize the regions in the parameter space guaranteeing the positivity of the discriminant $D_{\delta,\upsilon}$.

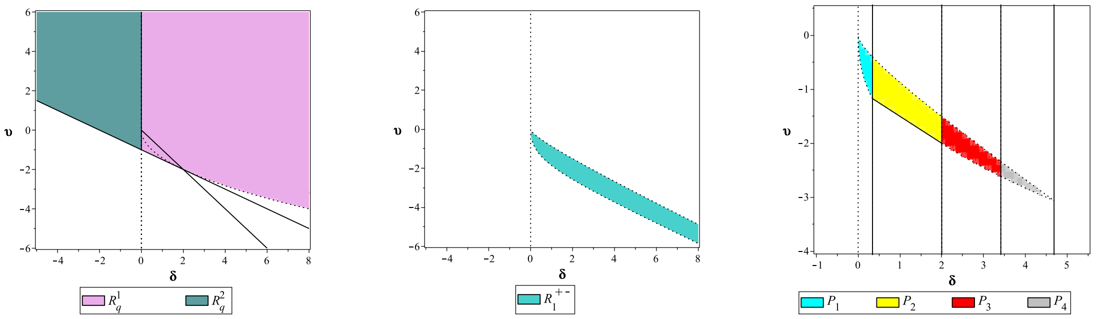

Figure 5.1: The left plot represents region R_q in terms of the parameters (δ, υ) as defined in (5.6) and (5.7). The middle plot represents region $R_{D^+} = R_1^{+-}$. The last plot shows a zoom on the regions P_i, $i = 1..4$.

Lemma 12. *If the expression of* $\Upsilon_{\delta,\upsilon}$ *defined in* (5.11) *is negative, then the discriminant* $D_{\delta,\upsilon}$ *defined in* (5.10) *is positive for* $x \in \left(0, 4\,a_{\delta,\upsilon}\,c_{1,\delta,\upsilon}/(-4\,a_{\delta,\upsilon}\,c_{2,\delta,\upsilon} + b_{1,\delta,\upsilon}^2)\right)$ *if, and only if,* $(\delta, \upsilon) \in R_1^{+-}$.

Proof. Considering $D_{\delta,\upsilon}$ as a polynomial of degree 2 with respect to x, it admits two roots $x_1 = 0$ and $x_2 = 4\,a_{\delta,\upsilon}\,c_{1,\delta,\upsilon}/(-4\,a_{\delta,\upsilon}\,c_{2,\delta,\upsilon} + b_{1,\delta,\upsilon}^2)$. The term $a_{\delta,\upsilon}$ as a polynomial of degree 2 with respect to υ, admits 2 real roots given by $\upsilon \in \{-\delta/2 - 2, -\delta/2\}$, so that $a_{\delta,\upsilon}$ is negative if, and only if,

$$(\delta, \upsilon) \in R_{a^-} = \left\{(\delta, \upsilon) \in \mathbb{R}^2 : -\frac{\delta}{2} - 2 < \upsilon < -\frac{\delta}{2}\right\}$$

and positive elsewhere. Note that the region R_{a^-} is of interest in our analysis. Now, we investigate the sign of x_2 taking into account the signs of $a_{\delta,\upsilon}$ and $\Upsilon_{\delta,\upsilon}$. Namely, since coefficient $c_{2,\delta,\upsilon} > 0$, then x_2 is positive if, and only if, $\left(a_{\delta,\upsilon} > 0 \quad \text{and} \quad \Upsilon_{\delta,\upsilon} > 0\right)$ or $\left(a_{\delta,\upsilon} < 0 \quad \text{and} \quad \Upsilon_{\delta,\upsilon} < 0\right)$, so that x_2 is positive if, and only if, $(\delta, \upsilon) \in R_{a^-} \cap \left\{R_1^{+-} \cup R_2^{+-} \cup R_1^{--} \cup R_2^{--} \cup R_3^{--}\right\} = R_1^{+-}$. If $\Upsilon_{\delta,\upsilon} > 0$, then $D_{\delta,\upsilon} > 0$ for $x \in (-\infty, 0) \cup (x_2, +\infty)$. As a result, if $\Upsilon_{\delta,\upsilon} < 0$, then $D_{\delta,\upsilon} > 0$ for $x \in (0, x_2)$ if, and only if, $(\delta, \upsilon) \in R_1^{+-}$, as expected. ■ □

In the sequel, we are interested in the parameter region guaranteeing the sign constancy of $q_{\delta,\upsilon}$ and the positivity of $D_{\delta,\upsilon}$, which corresponds to $R_q \cap R_{D^+}$. More precisely, $R_q \cap R_{D^+} = \bigcup_{i=1}^4 P_i$, where

$$\begin{cases} P_1 = \left\{(\delta, \upsilon) \in \mathbb{R}^2 : \delta \in \left(0, 2/(3+2\sqrt{2})\right], \upsilon \in \left(\upsilon_1, \upsilon_2\right)\right\}, \\ P_2 = \left\{(\delta, \upsilon) \in \mathbb{R}^2 : \delta \in \left(2/(3+2\sqrt{2}), 2\right], \upsilon \in \left[-1 - \delta/2, \upsilon_2\right)\right\}, \\ P_3 = \left\{(\delta, \upsilon) \in \mathbb{R}^2 : \delta \in \left(2, 2+\sqrt{2}\right], \upsilon \in \left(-\sqrt{2\delta}, \upsilon_2\right)\right\}, \\ P_4 = \left\{(\delta, \upsilon) \in \mathbb{R}^2 : \delta \in \left(2+\sqrt{2}, \left(\sqrt{2} - 4 + \sqrt{16\sqrt{2} - 22}\right)^2 (3+2\sqrt{2})/4\right), \right. \\ \left. \upsilon \in \left(-\sqrt{2\delta}, \upsilon_2\right)\right\} \end{cases} \tag{5.16}$$

where υ_1 and υ_2 are defined in (5.12).

5.3.2 Order one truncation

In this case, $\mathrm{T}_1 = 1+2x$ (see Algorithm 1), which in turn allows us to define the quadratic polynomial (5.8) of degree 2 in $\Omega = \omega^2$, where

$$
\begin{aligned}
a_{\delta,\upsilon}(x) &= -2x + \frac{(\delta+2\upsilon)(\delta+2\upsilon+4)}{4},\\
b_{\delta,\upsilon}(x) &= -4x^3 + \frac{\left(\delta^2+4\delta\upsilon+4\upsilon^2+4\delta\right)x^2}{2} + \delta\,(\delta+3\upsilon+6)x,\\
c_{\delta,\upsilon}(x) &= -2x^5 + \frac{\left(\delta^2+4\delta\upsilon+4\upsilon^2+4\delta-8\upsilon\right)x^4}{4} + \delta\,(\delta+3\upsilon-2)x^3 + 2\delta^2x^2.
\end{aligned}
$$

In order to guarantee the positivity of $G_{\delta,\upsilon}$, one has to investigate conditions on the signs of $a_{\delta,\upsilon}$ as well as the discriminant of $G_{\delta,\upsilon}$ denoted in the sequel $\tilde{D}_{\delta,\upsilon}(x) = x^2 D_{\delta,\upsilon}(x)$ where

$$
\begin{aligned}
D_{\delta,\upsilon}(x) = &-16\left(-\upsilon^2+4\delta\right)x^2 + 8\delta\left(\delta^2+3\delta\upsilon+\upsilon^2+6\delta+2\upsilon\right)x - \delta^4 - 2\delta^3\upsilon\\
&+\delta^2\upsilon^2+4\delta^3+20\delta^2\upsilon+36\delta^2.
\end{aligned}
\tag{5.17}
$$

As a matter of fact, let $\Omega_{1,2}$ be the two real solutions of $G_{\delta,\upsilon}(x,\Omega)=0$, then $G_{\delta,\upsilon}$ is positive if, and only if,

$$
\begin{aligned}
&\left(D_{\delta,\upsilon} < 0 \text{ and } a_{\delta,\upsilon} > 0\right) \text{ or}\\
&\left(D_{\delta,\upsilon} > 0 \text{ and } a_{\delta,\upsilon} > 0 \text{ and } \Omega \in \mathbb{R} - (\Omega_1,\Omega_2)\right) \text{ or}\\
&\left(D_{\delta,\upsilon} > 0 \text{ and } a_{\delta,\upsilon} < 0 \text{ and } \Omega \in (\Omega_1,\Omega_2)\right).
\end{aligned}
$$

Note that in the first and second cases, $G_{\delta,\upsilon}$ is unbounded which is not of interest in our method. Hence, we only keep the third set of conditions. Since the coefficient in front of x in the expression of $a_{\delta,\upsilon}$ is negative and independent of δ and υ, then $a_{\delta,\upsilon}$ is negative for $x \in (x^*, +\infty)$, where $x^* = (\delta+2\upsilon)(\delta+2\upsilon+4)/8$.

The next lemma provides a characterization of regions in the parameter space guaranteeing the positivity of $D_{\delta,\upsilon}$.

Lemma 13. *Let $D_{\delta,\upsilon}$ be the parametric polynomial defined in* (5.17)*. Then, $D_{\delta,\upsilon}$ is positive*

- *for $x \in (-\infty, \min\limits_{\delta,\upsilon}(x^-,x^+)) \cup (\max\limits_{\delta,\upsilon}(x^-,x^+), +\infty)$, if $(\delta,\upsilon) \in R_{d^+} \cap R_{A^+}$,*
- *for $x \in (\min\limits_{\delta,\upsilon}(x^-,x^+), \max\limits_{\delta,\upsilon}(x^-,x^+))$, if $(\delta,\upsilon) \in R_{d^+} \cap R_{A^-}$,*

where $x^{\pm} = \left(-8\delta\left(\delta^2+3\delta\upsilon+\upsilon^2+6\delta+2\upsilon\right) \pm \sqrt{\mathrm{d}(\delta,\upsilon)}\right)/(32\upsilon^2-128\delta)$.

Proof. To investigate the sign of $D_{\delta,\upsilon}$, we first study the sign of its leading coefficient that we denote by $A(\delta,\upsilon) = 16\upsilon^2 - 64\delta$, a polynomial in δ of degree 1, which admits one positive root at $\delta = \frac{\upsilon^2}{4}$. As a result, $A(\delta,\upsilon)$ is positive if, and only if, $(\delta,\upsilon) \in R_{A^+} = \left\{(\delta,\upsilon) \in \mathbb{R}^2 : \delta \le \upsilon^2/4\right\}$ and negative if $(\delta,\upsilon) \in R_{A^-} = \mathbb{R}^2 - R_{A^+}$. To study the sign of $D_{\delta,\upsilon}$, as a polynomial in x of degree 2, we analyse its discriminant which is given by

$$
d(\delta,\upsilon) = 64\,(\delta+2\upsilon+4)\left(\delta^3+4\delta^2\upsilon+4\delta\upsilon^2+4\delta^2+8\delta\upsilon-8\upsilon^2+36\delta\right)\delta^2.
$$

Now, consider $d_1(\delta,\upsilon) = \delta+2\upsilon+4$, as a polynomial in δ of degree 1, it admits one real root at $\delta = -2\upsilon-4$. Next, consider

$$
d_2(\delta,\upsilon) = \left(\delta^3+4\delta^2\upsilon+4\delta\upsilon^2+4\delta^2+8\delta\upsilon-8\upsilon^2+36\delta\right),
$$

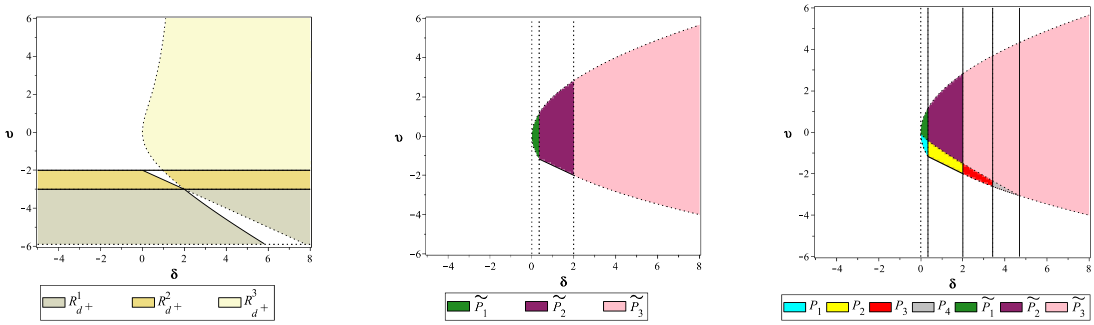

Figure 5.2: The left plot represents region R_{d^+}. The middle plot represents regions $\tilde{P}_1$, $\tilde{P}_2$ and $\tilde{P}_3$. The last plot shows that regions P_i, $i = 1..4$ obtained for an order zero truncation have been recovered and enlarged when we increased the truncation order.

as polynomial in δ of degree 3, it admits two conjugate roots and, for $\upsilon \in (-1-2\sqrt{6}, +\infty)$ one real root $\delta^* = \frac{1}{3}\Gamma - \frac{4}{3}(-\upsilon^2 - 2\upsilon + 23)/\Gamma$, where

$$\Gamma = \big(8\,\upsilon^3 + 132\,\upsilon^2 + 600\,\upsilon + 584 \\ +12\sqrt{12\,\upsilon^5 + 213\,\upsilon^4 + 1284\,\upsilon^3 + 2988\,\upsilon^2 + 3456\,\upsilon + 7776}\Big)^{\frac{1}{3}}$$

In the sequel, we shall assume that $\upsilon \in (-1-2\sqrt{6}, +\infty)$ which allows to conclude that $d(\delta, \upsilon)$ is positive if,

$$(\delta, \upsilon) \in R_{d^+} = \left\{\{(-\infty, \delta^*] \cup (-2\,\upsilon - 4, \infty)\} \times \left(-1-2\sqrt{6}, -3\right]\right\} \\ \cup \{(-\infty, -2\,\upsilon - 4] \times (-3, -2]\} \cup \{(\delta^*, \infty) \times (-3, \infty)\}.$$

■ □

The set of interest here is $R_{d^+} \cap R_{A^-}$ to which we shall add conditions guaranteeing the sign constancy of $q_{\delta,\upsilon}$. Hence, we characterize the intersection $R_q \cap R_{d^+} \cap R_{A^-}$ as

$$\bigcup_{i=1}^{3} \tilde{P}_i = \left\{\left(0, 2/(3+2\sqrt{2})\right] \times \left(-2\sqrt{\delta}, 2\sqrt{\delta}\right)\right\} \\ \cup \left\{\left(2/(3+2\sqrt{2}), 2\right] \times \left[-1-\delta/2, 2\sqrt{\delta}\right)\right\} \cup \left\{(2, +\infty] \times \left(-\sqrt{2\,\delta}, 2\sqrt{\delta}\right)\right\}. \quad (5.18)$$

5.4 Frequency bound

After having characterized the candidate regions, we present the main technical ingredient for the analysis of the frequency bound, which achieves Step 4 of the methodology.

Lemma 14. *Let $\tilde{\Delta} = \tilde{\Delta}_{\delta,\upsilon}$ be the quasipolynomial given in* (5.3)*, with $(\delta, \upsilon) \in \tilde{P}_1 \cup \tilde{P}_2 \cup P_3 \cup P_4$, where the regions $\tilde{P}_1$, $\tilde{P}_2$, P_3 and P_4 are given by* (5.18) *and* (5.16) *respectively. If $\tilde{\Delta}$ has a root $z_0 \in \mathbb{R}_+ + i\mathbb{R}_+$, then $0 < \Im(z_0) < \pi$. In addition, the root z_0 may be properly assigned.*

Proof. Consider $(\delta,\upsilon)\in\tilde{P}_1\cup\tilde{P}_2\cup P_3\cup P_4$. Since the discriminant of the polynomial function $G_{\delta,\upsilon}$ defined in (5.8-5.9) is positive, then $G_{\delta,\upsilon}$ admits the following two real roots

$$\begin{aligned}\Omega^{\pm}_{\delta,\upsilon}(x) = &-\frac{\left(\delta^2 x+4\,\upsilon x\delta+4\,\upsilon^2 x+2\,\delta^2+6\,\delta\,\upsilon+4\,\delta\,x-8x^2+12\,\delta\right)x}{\delta^2+4\,\delta\,\upsilon+4\,\upsilon^2+4\,\delta+8\,\upsilon-8\,x}\\ &\pm\frac{\sqrt{\tilde{D}_{\delta,\upsilon}(x)}}{\frac{\delta^2}{2}+2\,\delta\,\upsilon+2\,\upsilon^2+2\,\delta+4\,\upsilon-4\,x,}\end{aligned} \tag{5.19}$$

where $\Omega^{+}_{\delta,\upsilon}$ denotes the greater solution (positive signal). We deal with each of the considered regions separately hereafter.

(a) **Region $\tilde{P}_1$.** Since $\upsilon\in\left(-2\sqrt{\delta},2\sqrt{\delta}\right)$ and $x>0$, the solution $\Omega^{+}_{\delta,\upsilon}$ is upper bounded with respect to υ by the parameter expression

$$\Omega^{+}_{\delta}(x)=\left(h_{1,\delta}(x)-2x\sqrt{h_{2,\delta}(x)}\right)/h_{3,\delta}(x)$$

where

$$\begin{aligned}h_{1,\delta}(x) &= 8x^3+\left(-\delta^2+8\,\delta^{\frac{3}{2}}-20\,\delta\right)x^2+\left(-2\,\delta^2+12\,\delta^{\frac{3}{2}}-12\,\delta\right)x,\\ h_{2,\delta}(x) &= x\left(8\,\delta^3+48\,\delta^{\frac{5}{2}}+80\,\delta^2+32\,\delta^{\frac{3}{2}}\right)-\delta^4-4\,\delta^{\frac{7}{2}}+8\,\delta^3+40\,\delta^{\frac{5}{2}}+36\,\delta^2,\\ h_{3,\delta}(x) &= -8x+\delta^2-8\,\delta^{\frac{3}{2}}+20\,\delta-16\sqrt{\delta}.\end{aligned}$$

Next, we obtain the following parameter-free upper bound for Ω^{+}_{δ}

$$\begin{aligned}\Omega^{+}(x)= &\frac{8x^3}{-8x-4}\\ &+\frac{2x}{8x+4}\sqrt{\frac{x\left(111872\sqrt{2}+158208\right)}{\left(1+\sqrt{2}\right)^{11}}+\frac{1989312+1406656\sqrt{2}}{\left(1+\sqrt{2}\right)^{15}}}\end{aligned}$$

which reaches a maximum value at $x^*\approx 0.5791$.
Thus, $\omega^2=\Omega^{+}_{\delta,\upsilon}(x)\le\Omega^{+}(x^*)\approx 0.3970<\pi^2$, i.e., $\omega_0<\pi$.
Now, we shall detail the assignment of s_0. From (5.3), we infer that $\upsilon=-\sqrt{\left(a_1^2-4\,a_0\right)\,\tau^2+4\,\delta}$ which, combined with $\upsilon\in(-2\sqrt{\delta},2\sqrt{\delta})$, leads to

$$-4\,\delta<\left(a_1^2-4\,a_0\right)\,\tau^2<0. \tag{5.20}$$

The above inequality represents the condition of compatibility in terms of a_1 and a_0 for the assignment of s_0. We shall assume henceforth that it holds. Now, since $0<\delta\le 2/(3+2\sqrt{2})$, from (5.3) we get $0<\tau^2\left(s_0{}^2+a_1 s_0+a_0\right)\le 2(3+2\sqrt{2})$. We consider each inequality separately below.

(a) **Condition $\tau^2\left(s_0{}^2+a_1s_0+a_0\right)>0$.** Solving $\tau^2\left(s_0{}^2+a_1s_0+a_0\right)=0$ with respect to s_0, two roots are obtained $s^{\pm}_{0,A}=-\frac{a_1}{2}\pm\frac{1}{2}\sqrt{a_1{}^2-4\,a_0}$. So that, if $a_1{}^2-4\,a_0<0$, there is no sign change and the set of solutions is $\mathbb{R}$. Otherwise, $a_1{}^2-4\,a_0\ge 0$ and, consequently, $s_0\in(-\infty,s^{-}_{0,A})\cup(s^{+}_{0,A},+\infty)$.

(b) **Condition** $\tau^2\left(s_0{}^2+a_1s_0+a_0\right)\le\frac{2}{3+2\sqrt{2}}$**.** Solving $\tau^2\left(s_0{}^2+a_1s_0+a_0\right)=\frac{2}{3+2\sqrt{2}}$ with respect to s_0, we obtain two roots

$$s_{0,B}^{\pm}=-\frac{a_1}{2}\pm\frac{1}{2}\sqrt{a_1{}^2-4a_0+(-16\sqrt{2}+24)/\tau^2}.$$

From (5.20) and the fact that δ is upper bounded by $2/(3+2\sqrt{2})$, we show that $a_1{}^2-4a_0+(-16\sqrt{2}+24)/\tau^2>0$. Hence, the set of solutions is $s_0\in\left[s_{0,B}^-,s_{0,B}^+\right]$.

Now, bearing in mind that $s_{0,B}^-<s_{0,A}^-<s_{0,A}^+<s_{0,B}^+$, the intersection between the set of solutions for both cases is $s_0\in\left[s_{0,B}^-,s_{0,B}^+\right]$ if $a_1{}^2-4a_0<0$, and $s_0\in\left[s_{0,B}^-,s_{0,A}^-\right)\cup\left(s_{0,A}^+,s_{0,B}^+\right]$ otherwise. Finally, for the exponential decay s_0 has to be negative, so we impose that $s_{0,B}^+<0$, i.e., $a_0\ge(-4\sqrt{2}+6)/\tau^2$ and $a_1\ge 0$.

(b) **Region $\tilde{P}_2$.** Since $\upsilon\in\left[-1-\delta/2,2\sqrt{\delta}\right)$ and $x>0$, the solution $\Omega_{\delta,\upsilon}^+$ is upper bounded with respect to υ by the parameter expression $\Omega_\delta^+(x)$ which is upper bounded with respect to δ by the parameter-free expression

$$\Omega^+(x)=\frac{1}{-8x-4}\left(8x^3+\frac{\left(-20-8\sqrt{2}\right)x^2}{3+2\sqrt{2}}+\frac{\left(-32-24\sqrt{2}\right)x}{\left(3+2\sqrt{2}\right)^2}\right.$$
$$\left.-2x\sqrt{192+x\left(384+256\sqrt{2}\right)+128\sqrt{2}}\right)$$

which reaches a maximum value at $x^*\approx 1.9018$. Thus, $\omega^2=\Omega_{\delta,\upsilon}^+(x)<\Omega^+(x^*)\approx 6.7190<\pi^2$, so that $\omega_0<\pi$.

To assign the root s_0 in this case, we proceed as with the previous region $\tilde{P}_1$ and conclude that if $a_0\ge\frac{2}{\tau^2}$ and $a_1\ge 0$, then $s_0\in\left[s_{0,C}^-,s_{0,B}^-\right)\cup\left(s_{0,B}^+,s_{0,C}^+\right]$ where $s_{0,C}^{\pm}=-\frac{a_1}{2}\pm\frac{1}{2}\sqrt{a_1{}^2-4a_0+8/\tau^2}$.

(c) **Region $\tilde{P}_3$.** Consider $\delta\in(2,\vartheta)$, with $\vartheta>0$ and follow the same procedure as with previous regions. The table below

ϑ	2.001	2.2	2.3	2.5
Ω^+	8.7083	9.7402	10.2747	11.3748

emphasizes the fact that an interesting frequency bound may be found only for a positive δ close to 2, which is not interesting for continuing the next step. Unfortunately for $\delta\in(2,\infty)$, the dominancy of s_0 cannot be concluded unless the order of truncation of the exponential term is increased as in Algorithm 1 in order to obtain an adequate frequency bound.

(d) **Region $P_3\cup P_4$.** Since $\upsilon\in\left(-\sqrt{2\delta},\upsilon_2\right)$ and $x>0$, the solution $\Omega_{\delta,\upsilon}^+$ is upper bounded with respect to υ by the parameter expression Ω_δ^+, a function the expression of which we have avoided writing because of its length, this function Ω_δ^+ itself can be upper bounded with respect to δ by the parameter-free expression $\Omega^+(x)=\left(h_1(x)-2x\sqrt{h_2(x)}\right)/(-8x-4)$

where

$$h_1(x) = 8x^3 + \left(\left(-4\sqrt{2}+6\right)\sqrt{41-28\sqrt{2}} + 40\sqrt{2} - 66\right)x^2 - 8x,$$

$$h_2(x) = -64x^2 + x\left(\left(-3184\sqrt{2}-4512\right)\sqrt{16\sqrt{2}-22} + 2720\sqrt{2} + 3680\right)$$
$$+ \left(4016\sqrt{2}+5664\right)\sqrt{16\sqrt{2}-22} - 3104\sqrt{2} - 4384.$$

The latter expression of Ω^+ reaches a maximum value at $x^* \approx 1.5514$. Thus, $\omega^2 = \Omega^+_{\delta,\upsilon}(x) < \Omega^+(x^*) \approx 5.1031 < \pi^2$, i.e., $\omega_0 < \pi$. To assign the root s_0 in this case, we analyse in a similar way as in the previous cases and we conclude that if $a_0 \geq \left[(-10\sqrt{2}-16)\sqrt{16\sqrt{2}-22}+16\sqrt{\right.$ and $a_1 \geq 0$ are satisfied, then we are able to assign the root s_0 such that $s_0 \in (s^-_{0,D}, s^-_{0,C}) \cup (s^+_{0,C}, s^+_{0,D})$, where

$$s^{\pm}_{0,D} = -\frac{a_1}{2} \pm \frac{1}{2}\sqrt{a_1{}^2 - 4a_0 + \left[(-10\sqrt{2}-16)\sqrt{16\sqrt{2}-22} + 16\sqrt{2}+20\right]/\tau^2}.$$

The proof is complete. ■

□

Remark 15.

(a) *Our approach gives sufficient conditions for the dominance which are valid in regions P_i, $i = 3,4$ and in regions $\tilde{P}_i$, $i = 1,2$ which contain respectively P_i, $i = 1,2$. For each of the aforementioned regions, a frequency bound of interest ($\omega < \pi$) was obtained. For region $\tilde{P}_3$, the truncation order needs to be increased.*

(b) *Note that the set of conditions guaranteeing the MID obtained with a truncation of order $k+1$ contains the set of conditions guaranteeing the MID with a truncation of order k. As a result, higher orders of truncation shall lead to wider ranges on the conditions.*

5.5 Conclusion of the proof of Theorem 8 (item (c))

After characterizing regions for which a frequency bound of interest was found, we can complete the proof of Theorem 8 ; this corresponds to Step 5 of the methodology.

Proof. From the subsections 5.1 and 5.2, the normalization of Δ is given by $\tilde{\Delta}$ in (5.3), while the factorization of $\tilde{\Delta}$ is defined in (5.5). Using relations (5.4), one concludes that s_0 is a root of multiplicity 3 of Δ if, and only if, relations (4.4) hold, thereby ending the proof of the item ((c)a). To show ((c)b), we use the technical results previously proved. Consider $(\delta,\upsilon) \in \tilde{P}_1 \cup \tilde{P}_2 \cup P_3 \cup P_4$, the proof of the dominance is based on a contradiction. To do so, assume that there exists $z_0 \in \mathbb{C}$ root of $\tilde{\Delta}$ satisfying $\Re(z_0) > 0$. Write $z_0 = x_0 + i\omega_0$ and using the fact that z_0 is a non-zero root of $\tilde{\Delta}$, one may infer from (5.5) by taking the imaginary part, that $\int_0^1 \left(\frac{\delta}{2}t^2 + t\upsilon + 1\right)\sin(t\,\omega_0)\,\mathrm{e}^{-t x_0}\,\mathrm{d}t = 0$. Since $\omega_0 < \pi$ from Lemma 14, the function $t \mapsto \left(\frac{\delta}{2}t^2 + t\upsilon + 1\right)\sin(t\,\omega_0)$ is strictly positive in $(0,1)$, which contradicts the above equality as required to end the proof.■ □

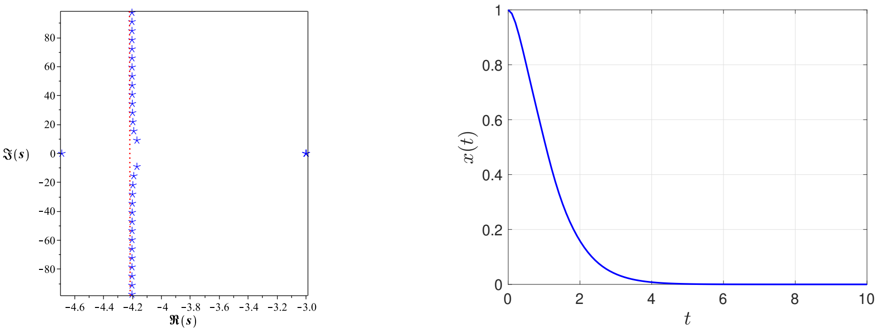

Figure 6.1: For $\omega = 2$, $\eta = \frac{1}{9}$ and $\tau = 1$, the left plot exhibits the spectrum distribution of the quasipolynomial Δ where the assigned rightmost triple root at $s_0 = -3$ and the roots with large modulus are asymptotic to a vertical line $\Re(s) \approx -\log|\alpha|/\tau \approx -4.2$ ([30]), and the second plot illustrates the oscillator response with initial condition taken to be $\varphi(t) = 1$ for $t \in [-\tau, 0)$.

6 Illustrative example: Classical oscillator

Consider the classical oscillator control problem:

$$\ddot{x}(t) + 2\,\eta\,\omega\,\dot{x}(t) + \omega^2\,x(t) = u(t), \tag{6.1}$$

with u as the delayed output-feedback as proposed in [21]: $u(t) = -\alpha_2\,\ddot{x}(t-\tau) - \alpha_1\,\dot{x}(t-\tau) - \alpha_0\,x(t-\tau)$, η is the damping factor such that $0 < \eta < 1$, ω describes the natural frequency. The characteristic equation corresponds to (6.1) is defined by

$$\Delta(s) = s^2 + 2\,\eta\,\omega\,s + \omega^2 + (\alpha_2\,s^2 + \alpha_1\,s + \alpha_0)\,\mathrm{e}^{-\tau s}. \tag{6.2}$$

Following item (c) in Theorem 8, it shows that the real number s_0 is a root of multiplicity 3 of the quasipolynomial function (6.2) if, and only if, the following relations hold

$$\begin{cases} \alpha_0 = -\frac{1}{2}\left(2\,\omega^2 + \left(2\,\eta\,\omega\,{s_0}^3 + \omega^2{s_0}^2 + {s_0}^4\right)\tau^2 - \left(2\,\omega^2 s_0 - 2\,{s_0}^3\right)\tau\right)\mathrm{e}^{\tau s_0}, \\ \alpha_1 = \left(-2\,\eta\,\omega + \left(2\,\eta\,\omega\,{s_0}^2 + \omega^2 s_0 + {s_0}^3\right)\tau^2 + \left(2\,\eta\,\omega\,s_0 - \omega^2 + 3\,{s_0}^2\right)\tau\right)\mathrm{e}^{\tau s_0}, \\ \alpha_2 = -\frac{1}{2}\left(2 + \left(2\,\eta\,\omega\,s_0 + \omega^2 + {s_0}^2\right)\tau^2 + \left(4\,\eta\,\omega + 4\,s_0\right)\tau\right)\mathrm{e}^{\tau s_0}. \end{cases} \tag{6.3}$$

The normalization and the integral representation of the characteristic function (6.2) are defined in (5.3) and (5.5) respectively, where in this case $\delta = \tau^2\,\left({s_0}^2 + 2\,\eta\,\omega\,s_0 + \omega^2\right)$ and $\upsilon = 2\,\tau\,(s_0 + \eta\,\omega)$. Notice that $\delta > 0$ for all s_0 since $\delta = \tau^2\,\left((s_0 + \eta\,\omega)^2 + \omega^2(1-\eta^2)\right) > 0$. Finally, we conclude that if $\omega \geq \max\left\{\frac{\delta}{\tau^2} - \frac{\upsilon_2^2}{4\,\tau^2}, \frac{1}{\tau}\sqrt{\frac{2}{3+2\sqrt{2}}}\right\}$ is satisfied, then we are able to assign the root s_0 in the interval $-\eta\,\omega - \frac{1}{\tau}\sqrt{B} \leq s_0 \leq -\eta\,\omega + \frac{1}{\tau}\sqrt{B}$, with $B = \frac{2}{3+2\sqrt{2}} - \tau^2\,\omega^2(1-\eta^2) > 0$. The left plot in Figure 6.1 illustrates the roots of $\tilde{\Delta}$ computed numerically using `Maple`, while the right Figure in 6.1 presents a temporal simulation with the same choice of ω and η.

7 Further remarks on the MID: case of multiplicity 2

Consider the quasipolynomial function Δ defined in (1.2). The real number s_0 is a root of multiplicity 2 of the quasipolynomial function Δ if, and only if, for $\alpha_2 = \gamma_2\,\mathrm{e}^{\tau s_0}$ the following relations

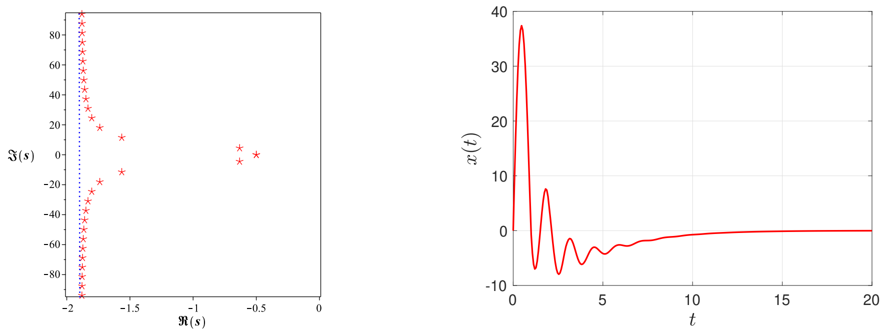

Figure 7.1: The left plot illustrates for $\omega = 2$, $\eta = 0.02$ the rightmost double root of Δ at $s_0 = -0.5$ and the roots with large modulus asymptotic to the vertical line $\Re(s) \approx -\log|\alpha|/\tau \approx -1.88$. The right plot is the time-response of a particular solution of (6.1) with the same choice of coefficients ; the initial condition is taken to be $\varphi(t) = 0$ for $t \in [-\tau, 0)$.

hold

$$\begin{cases} \alpha_0 = \left(\tau {s_0}^3 + (\tau a_1 + \gamma_2 + 1)\, {s_0}^2 + \tau a_0 s_0 - a_0\right) \mathrm{e}^{\tau s_0}, \\ \alpha_1 = \left(-\tau {s_0}^2 + (-\tau a_1 - 2\gamma_2 - 2)\, s_0 - \tau a_0 - a_1\right) \mathrm{e}^{\tau s_0}. \end{cases} \tag{7.1}$$

The normalized quasipolynomial is given by $\tilde{\Delta}(z) = \left(\gamma_2 z^2 - (\delta + \upsilon)\, z - \delta\right) \mathrm{e}^{-z} + z^2 + \upsilon z + \delta$ with $\delta = \tau^2 \left({s_0}^2 + a_1 s_0 + a_0\right)$ and $\upsilon = \tau\,(2 s_0 + a_1)$. The integral representation is given by $\tilde{\Delta}(z) = z^2 \left(1 + \upsilon + \gamma_2 + \frac{\delta}{2} - z \int_0^1 \left(-\frac{\delta t^2}{2} + \upsilon\,(1-t) + \gamma_2 + \frac{\delta}{2}\right) \mathrm{e}^{-tz}\, \mathrm{d}t\right)$ which is not the standard factorization. In fact, it is a more general form as the one described for instance in [24]. In the case of multiplicity 2, the normalized polynomial admits 3 free parameters which makes the analytic proof of the MID property quite delicate. However, we claim that even in in such a case, one is able to numerically exploit such a property for rightmost spectral value assignment as is exhibited by the next example.

Consider the classical oscillator control problem (6.1). Let $\omega = 2$ and $\eta = 0.02$, we choose $s_0 = -0.5$, $\tau = 1$ and $\gamma_2 = -0.25\,\mathrm{e}^{0.5}$. Then, $\alpha_2 \approx -0.25$ and, owing to relations (7.1), we compute $\alpha_1 \approx -2.1471$ and $\alpha_0 \approx -3.5891$.

8 Conclusion

In this paper, we have treated the multiplicity-induced-dominancy (MID) property for second order time-delay differential equations of neutral type with single-delay, i.e., the corresponding characteristic function is a quasipolynomial of degree 5. We present an algorithm as well as an overview of classification of admissible multiplicities for this class of equations. First, necessary and sufficient conditions are established, in which a real root of the characteristic function of maximal multiplicity 5 is necessarily dominant. Next, necessary and sufficient conditions have been provided in order to ensure that a given root of multiplicity 4 is the rightmost root of the characteristic function. For the case of multiplicity 3, we only provide sufficient conditions for the dominance where the number of free parameters is 2. In the latter case, we used first a truncation of the exponential function of order 0, which led to some regions where the MID property holds. To illustrate the use of the proposed algorithm, we further extended the validity area of the MID property by increasing the truncation up to order one, this allowed to enlarge the region of validity of the MID obtained with truncation of order 0. Finally, for the multiplicity 2, as the number of free parameters increases (3 free parameters), the computations become quite cumbersome from a symbolic point of view, but for the time being we used numerical approaches which can give sufficient conditions for the dominance. The obtained results have been illustrated through the delayed stabilization of the classical oscillator.

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