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Lyapunov Function computation for Periodic Linear Hybrid Systems via Handelman, Polya and SoS approaches: A comparative study

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Abstract: We propose a method for the stability analysis of linear hybrid systems with periodic jumps. The method relies on the solution to polynomial inequalities based on the Handelman decomposition. Compared to existing approaches, such as sum-of-squares (SoS) and Polya's theorem, the proposed method reduces the computation time to obtain stability certificates.

Keywords: Periodic linear hybrid systems; Lyapunov stability analysis; polynomial methods; convex optimization; semidefinite programming.

1. INTRODUCTION

Hybrid Dynamical Systems allow to model systems composed of continuous-time processes interconnected to discrete-time digital devices. The additional complexity introduced by combining continuous and discrete-time dynamics requires the development of mathematical tools for the stability analysis to go beyond the analysis of purely continuous- or discrete-time systems Goebel et al. (2009, 2012); Cox et al. (2012). The particular case studied in this paper is the class of linear hybrid systems with periodic jumps.

A Lyapunov-based approach for stability of linear hybrid systems with periodic jumps is to impose the decrease of the Lyapunov function after the continuous-time dynamics (flows) followed by the the discrete-time evolution (jumps). This approach allows us to formulate stability analysis and the design of stabilizing periodic control laws based on the solution to differential matrix inequalities, as illustrated in Galeani et al. (2015). When using a polynomial parametrization of the functions in the differential matrix inequalities these inequalities become polynomial matrix inequalities which are required to be non-negative in a compact interval.

In a different context, the stability of nonlinear polynomial systems can be carried out by searching polynomial Lyapunov functions. The existence of these polynomial functions is a necessary and sufficient condition for local stability (over bounded sets) of exponentially stable systems, Peet (2009). Handelman's theorem introduced by Handelman et al. (1988) has been used as an alternative to assess the non-negativity of polynomial expressions. Handelman decomposition exploits the fact that positive polynomials can be recast in linear combinations of elements from its corresponding Handelman basis. Some works exploiting this decomposition in Kamyar et al. (2014); Briat (2013); Sassi et al. (2015) illustrate the possibility of using Handelman theorem for the stability analysis of linear and nonlinear systems as well as to compute inner estimates to regions of attraction. Moreover, Polya's theorem can be exploited to analyze the non-negativity of polynomials Kamyar et al. (2013); Kamyar and Peet (2013). However, as Kamyar et al. (2014) points out, the growth of the associated optimization problems on the parameters of the polynomial can prevent the application of these methods. Indeed, to construct a polynomial Lyapunov function given by a polynomial of degree six for a system composed of ten variables, the Polya's theorem approach requires to set up a semidefinite programming with $\approx 10^8$ variables and $\approx 10^5$ constrains.

In this paper, we search for polynomial parametrization on Handelman's bases, Polya's theorem and Sum-of-Squares decompositions, to conclude about the non-negativity of a polynomial parametrization of differential matrix inequalities.

The paper is structured as follows. After some preliminaries on definitions and notations related to the Handelman decomposition that are established in Section 2, the stability analysis of linear hybrid systems with periodic jumps is addressed in Section 3 through Lyapunov-based stability conditions recast in terms of differential matrix inequalities. Numerical examples are provided in Section 4 to illustrate the potential of the proposed approach. Finally, some conclusions and potential future work are drawn in Section 5.

Notation: The identity matrix of dimension n is I_n and the the matrix of zeros of dimension $n \times m$ is $0_{n\times m}$. The set symmetric and positive definite matrices is denoted by $\mathbb{S}^n_{\succ 0}$. Periodic linear hybrid systems solutions are piecewise absolutely continuous functions $\mathcal{X}(.,.)$ that are almost everywhere differentiable and satisfy a differential equation (flow dynamics) $\dot{\mathcal{X}} = f(\mathcal{X})$ when $(t,k) \in \mathcal{T}$ is such that $t \in (k\tau,(k+1)\tau)$, and satisfy a difference equation (jump dynamics) $\mathcal{X}^+ = g(\mathcal{X})$ when $(t,k) \in \mathcal{T}$ is such that $t = (k+1)\tau$. For brevity, the notation $x_{[k]}$, $x_{<k>}$ may be adopted in place of $x(k\tau,k)$ and $x((k+1)\tau,k)$ to refer to the value at the beginning and at the end, respectively, of the k^{th} flow time interval.

2. PRELIMINARIES

In this section, we give conditions to check the non-negativity of a polynomial. These conditions will be instrumental in the stability conditions of next section.

Definition 1. (Scalar Polynomial): A scalar and mono-variable polynomial $p : \mathbb{R} \to \mathbb{R}$ of degree n is defined by

$$p(\theta) = p_n\theta^n + p_{n-1}\theta^{n-1} + \ldots + p_1\theta + p_0,$$

where p_i for all $i = 1,\ldots,n$ correspond to the polynomial coefficients. A polynomial $p(\theta)$ is said to be strictly positive within an interval $\mathcal{I} = \{\theta \in \mathbb{R} : \theta \in [0,1]\}$, if $p(\theta) > 0$ for all $\theta \in \mathcal{I}$.

Definition 2. (Convex Polytope, Grünbaum et al. (1997)) An n-dimensional convex polytope $\mathcal{P} \in \mathbb{R}^n$ composed of a set of K vertices $\mathcal{V} := \{v_j \in \mathbb{R}, j = 1,\ldots,K\}$ is defined by

$$\mathcal{P} := \{\theta \in \mathbb{R} : \theta = \sum_{j=1}^{K} \beta_j v_j,\ \ \sum_{j=1}^{K} \beta_j = 1,\ \ \forall j = 1,\ldots,K\}.$$

The interval $\mathcal{I}$ is an convex polytope and it can be expressed as

$$\mathcal{P} := \{\theta \in \mathbb{R} : e_i^\top\theta + f_i \geq 0,\ \ \forall\ \ i = 1,2\}, \tag{1}$$

with $e_1 = 1$, $f_1 = 0$, $e_2 = -1$ and $f_2 = 1$ being affine coefficients of the convex polytope $\mathcal{P}$ defined within $\mathcal{I}$.

Definition 3. (Handelman basis) The Handelman basis of degree d_m associated to the convex polytope in (1) is defined by

$$\begin{aligned}\mathcal{B}(\mathcal{P},d_m) :=&\{\rho(\theta) : \rho(\theta) = \prod_{j=1}^{2}(e_j^\top\theta + f_j)^{\alpha_j},\\ &\alpha = (\alpha_1,\alpha_2) \in \mathbb{N}^2, |\alpha|_1 \leq d_m\},\\ =&\{1,\theta,(1-\theta),\theta^2,\theta(1-\theta),(1-\theta)^2,\ldots,(1-\theta)^{d_m},\theta^{d_m}\}.\end{aligned}$$

From the above definitions, we formulate conditions for the non-negativity in a polytope $\mathcal{P}$ of a polynomial expressed in terms of the Handelman basis associated to $\mathcal{P}$.

2.1 Handelman Decomposition

The following theorem exploits the Handelman basis to provide a condition for the non-negativity of a polynomial in a set $\mathcal{P}$.

Theorem 1. Assume that a polynomial $p(\theta) : \mathbb{R} \to \mathbb{R}$ is strictly positive on the convex set (1), that is, $p(\theta) > 0, \forall\theta \in \mathcal{P}$. There exists a degree d_m, and a set of coefficients $c_l \in \mathbb{R}_{\geq 0}$, such that p is expressed as

$$p(\theta) := \sum_{l=1}^{M} c_l \prod_{j=1}^{K}(e_j^\top\theta + f_j)^{\alpha_j},$$

where M corresponds to the number of elements in $\mathcal{B}(\mathcal{P},d_m)$.

From the above theorem, we can conclude that if a given polynomial $p(\theta)$ is strictly positive on the set $\mathcal{P}$, it can be decomposed as a sum of non-negative polynomials on $\mathcal{P}$ multiplied by non-negative coefficients. Given a polynomial expressed, or decomposed, in the Handelman basis of a polytope $\mathcal{P}$ with non-negative coefficients, it is clear that the polynomial is non-negative in $\mathcal{P}$. See Handelman et al. (1988) for more details on the proofs of Theorem 1.

Remark 1. The existence of Handelman decomposition is only guaranteed if the polynomial is strictly positive on $\mathcal{P}$. ⋆

Example 1 (Scalar Polynomial) Consider the following strictly positive polynomial in a single varaible $p_1(\theta) : \mathbb{R} \to \mathbb{R}$,

$$p_1(\theta) = 2\theta^2 - \frac{1}{4}\theta + 1,$$

where $\theta \in \mathcal{P} = [0,1]$. The polynomial $p_1(\theta)$ can be decomposed into the following polynomial expression in terms of the Handelman basis

$$p_1(\theta) = c_1 + c_2\theta + c_3(1-\theta) + \ldots + c_M(1-\theta)^{d_m}.$$

We then look for conditions on the coefficients c_l for the non-negativity of the polynomial following the conditions of Theorem 1, obtained from the identity

$$c_1 + c_2\theta + c_3(1-\theta) + \ldots + c_M(1-\theta)^{d_m} = 2\theta^2 - \frac{1}{4}\theta + 1,$$

The coefficients of the decomposition can then be computed by solving the corresponding system of linear equations with non-negativity constraints on the coefficients c_l. For the case $d_m = 3$, we obtain

$$\begin{aligned} c_2 - c_3 + 2c_5 + c_8 - 3c_9 + c_{10} &= -\frac{1}{4},\\ c_4 + c_5 + c_7 - 2c_8 + 3c_9 - c_{10} &= 2,\\ c_1 + c_3 + c_6 + c_9 &= 1,\\ c_6 - c_7 - c_8 - c_9 &= 0.\end{aligned}$$

The above set of equations and the non-negativity of $c_l, \forall l = 1,\ldots,10$ are taken as constraints of a linear programming (LP). The LP is then solved by standard LP solvers (e.g. Gurobi Optimization, LLC (2021), MOSEK ApS (2019), Cplex (2009)). We illustrate solutions of the corresponding LPs of the above example in Table 1 for different values of the degree d_m obtained with MOSEK ApS (2019).

Example 2 (Polynomial Matrix) Consider a polynomial symmetric matrix $F_1 : \mathbb{R} \to \mathbb{S}^{2\times 2}$

$$F_1(\theta) = \begin{bmatrix} 3+3\theta & \theta-1 \\ \theta-1 & 2 \end{bmatrix}.$$

Table 1. *Handelman coefficients c_l for different values of maximum polynomial degree d_m.*

d_m	c_1	c_2	c_3	c_5	c_6	c_7	c_{10}	c_{14}	c_{19}
2	0	0.75	1.00	2.00	0	0	0	0	0
3	0.88	0	0.13	0	0.13	1.88	0	0	0
4	0.67	1.08	0	0	0.33	0	1.00	0.33	0
5	0.67	1.08	0	0	0	0.33	0	1.00	0.33

To test whether $F_1(\theta) > 0, \forall\theta \in \mathcal{I}$ by using the Theorem 1, we express the positive definiteness of the matrix $F_1(\theta)$ by the equivalent condition $f_p(\theta, x_1, x_2) > 0 \ \forall x \in \mathbb{R}^2 \setminus \{0\}$, $\forall\theta \in \mathcal{I}$ with

$$\begin{aligned} f_p(\theta, x_1, x_2) &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^\top \begin{bmatrix} 3+3\theta & \theta-1 \\ \theta-1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= 3x_1^2(1+\theta) + 2x_2x_1(\theta-1) + 2x_2^2. \end{aligned} \quad (3)$$

Since the above expression is homogeneous on x, it suffices to verify $f_p(\theta, x_1, x_2) > 0 \ \forall x \in \{x \in \mathbb{R}^2 : |x_j| \le 1 \quad \forall i = 1, 2\}$, which is a set containing the unit ball. Therefore, by using Theorem 1, the positive-definiteness of the $F_1(\theta)$ on the polytope (1), can be concluded through the verification of the positivity of $f_p(\theta, x_1, x_2)$ on the following convex polytope

$$\begin{aligned} \mathcal{P}_m = \{(\theta, x_1, x_2) \in \mathbb{R}^3 : \theta \ge 0, \ \ \theta - 1 \ge 0, \\ -1 < x_1 < 1, \ \ -1 < x_2 < 1\}. \end{aligned}$$

By constructing a Handelman basis for the polytope $\mathcal{P}_m$, we can proceed as in the previous example to obtain a Handelman decomposition to $f_p(\theta, x_1, x_2)$ via the solution of an LP. However, we can notice that to recast a polynomial matrix of dimension m in terms of a scalar expression, is necessary to add m auxiliary variables, increasing the complexity and solving time of the corresponding LP and making the approach impracticable for a high-dimensional polynomial matrices.

To avoid the increase of the number of variables describing the polynomial, we can formulate the Handelman decomposition by solving a semidefinite programming (SDP), as detailed in the following theorem.

Theorem 2. Assume that a polynomial symmetric matrix $F(\theta) \in \mathbb{S}^{n\times n}$ is positive definite on the convex set (1), that is $F(\theta) \succ 0, \forall\theta \in \mathcal{P}$. There exists a degree d_m, and a set of positive definite matrices $C_l \in \mathbb{S}^n_{\ge 0}$, such that $F(\theta)$ can be expressed as

$$F(\theta) = \sum_{l=1}^{M} C_l \prod_{j=1}^{K} (e_j^\top \theta + f_j)^{\alpha_j}.$$

where M corresponds to the number of elements in $\mathcal{B}(\mathcal{P}, d_m)$.

See Handelman et al. (1988) for more details on Handelman decomposition.

Example 2 cont. (Polynomial Matrix) By exploiting the Theorem 2 it is possible to find positive definite matrices $C_l \in \mathbb{S}^2_{\succ 0}$ for $l = 1, \dots, M$, such that $F_1(\theta)$ is decomposed into the following expression,

$$\begin{aligned} F_1(\theta) = &\underbrace{\begin{bmatrix} 1.4417 & -0.3769 \\ -0.3769 & 0.6085 \end{bmatrix}}_{C_1} + \underbrace{\begin{bmatrix} 3.1257 & 0.2391 \\ 0.2391 & 0.7100 \end{bmatrix}}_{C_2} \theta \\ &+ \underbrace{\begin{bmatrix} 0.8688 & -0.3788 \\ -0.3788 & 0.7426 \end{bmatrix}}_{C_3} (1-\theta) + \underbrace{\begin{bmatrix} 0.6896 & -0.2443 \\ -0.2443 & 0.6489 \end{bmatrix}}_{C_4} (\theta-1)^2 \\ &+ \underbrace{\begin{bmatrix} 1.4326 & 0.1377 \\ 0.1377 & 0.6815 \end{bmatrix}}_{C_5} \theta^2 + \underbrace{\begin{bmatrix} 2.1222 & -0.1066 \\ -0.1066 & 1.3303 \end{bmatrix}}_{C_6} \theta(1-\theta), \end{aligned}$$

for $d_m = 2$. Where the matrices C_l were obtained by solving the associated SDP with MOSEK ApS (2019).

2.2 Sum-of-Squares (SoS) Decomposition

The sum-of-squares approach replaces conditions for a polynomial to be non-negative by the sufficient condition that the polynomial is decomposed into a sum of squared polynomials.

Theorem 3. (Sum-of-Squares decomposition) A polynomial $p(\theta) : \mathbb{R} \to \mathbb{R}$ is a sum-of-squares of polynomials if and only if there exist a positive semidefinite matrix G, called *Gram matrix*, such that,

$$p(\theta) = z(\theta)^\top G z(\theta),$$

where $z(\theta) = [1, \theta, \dots, \theta^{d_m}]$ is the vector of monomials of degree up to d_m.

Therefore, if we can find a vector of monomials $z(\theta)$ and a positive semidefinite matrix G, the non-negativity of $p(\theta)$ is guaranteed.

For the examples in the previous section, the scalar polynomial $p_1(\theta)$ can be decomposed as a sum-of-squares as follows

$$\begin{aligned} p_1(\theta) &= \begin{bmatrix} 1 \\ \theta \end{bmatrix}^\top \begin{bmatrix} 1 & -0.125 \\ -0.125 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ \theta \end{bmatrix} \\ &= p_{11}^2(\theta) + p_{12}^2(\theta), \end{aligned}$$

with $p_{11}(\theta) = 0.9987 - 0.05183\theta$ and $p_{12}(\theta) = 1.4133\theta - 0.05183$. Also, the expression (3) related to the polynomial matrix $F_1(\theta)$ can be lower bounded as $f_p(\theta, x_1, x_2) \ge x^\top F_{lb}(\theta) x$. The eigenvalues λ_1 and λ_2 of $F_{lb}(\theta)$ are illustrated in Figure 1.

It thus appears evident that the polynomial $p_1(\theta)$ is not negative and the polynomial matrix $F_1(\theta)$ is positive semidefinite within the interval $\theta \in [0, 1]$. See (Parrilo, 2000, Section 4.2) for more details on sum-of-squares programming.

In the rest of the paper, we look for a local certificate of the non-negativity of a polynomial $p(\theta)$ by searching for an SoS decomposition in the interval $\mathcal{I}$ by verifying whether

$$p(\theta) - (1-\theta)\theta g(\theta), \quad (4)$$

is an SoS polynomial with $g(\theta)$ an SoS polynomial. If there exists an SoS decomposition of $p(\theta) - (1-\theta)\theta g(\theta)$, it is immediate that $p(\theta)$ is not negative for all $\theta \in [0, 1]$, namely, the set where $(1-\theta)\theta \ge 0$.

2.3 Polya's Theorem

Polya's theorem, allows to check whether a polynomial $p(\theta)$ is non-negative as follows.

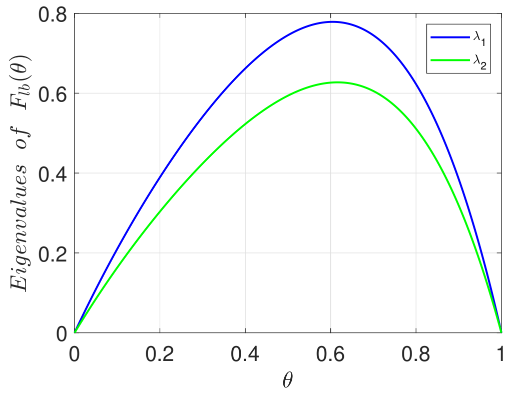

Fig. 1. *Eigenvalues of the lower bound matrix $F_{lb}(\theta)$ within the interval $\theta \in [0, 1]$.*

Theorem 4. If a polynomial $p : \mathbb{R}^N \to \mathbb{R}$ is positive for all $\theta \in \{\theta \in \mathbb{R}^N \mid \theta_i \geq 0, \sum_{i=1}^N \theta_i = 1\}$, then there exists an integer d_m, such that all coefficients of the polynomial

$$(\theta_1 + \theta_2 + \ldots + \theta_N)^{d_m} p(\theta)$$

are strictly positive.

We refer the reader to Pólya et al. (1934) for the proofs of Theorem 4.

Since, in this paper, we study scalar polynomials in the interval $\theta \in [0, 1]$ on variable θ, we shall introduce two variables θ_1 and θ_2, thus $N = 2$ to describe the set $[0, 1]$ as above. Namely we use $\{(\theta_1, \theta_2) \in \mathbb{R}^2 \mid \theta_1 \geq 0, \theta_2 \geq 0, \theta_1 = 1 - \theta_2\}$ and set $\theta = \theta_1$ to limit its value to the interval $[0, 1]$. This way, the conditions for the positivity of the scalar example given by $p_1(\theta)$ above are that all the coefficients of the polynomial

$$\bar{p}_1(\theta) = (\theta_1 + \theta_2)^{d_m}(2\theta_1^2 - \frac{1}{4}\theta_1 + 1), \tag{5}$$

are positive for some integer d_m. For the matrix example above we generalize Polya's theorem to the verification of the positive semi-definiteness of all matrices multiplying the monomials in the polynomial matrix expression

$$\begin{aligned}\bar{F}_1(\theta) &= (\theta_1 + \theta_2)^{d_m} F_1(\theta_1) \\ &= (\theta_1 + \theta_2)^{d_m} \begin{bmatrix} 3 + 3\theta_1 & \theta_1 - 1 \\ \theta_1 - 1 & 2 \end{bmatrix}\end{aligned}$$

for some degree d_m.

3. PROBLEM STATEMENT

In this section, a class of linear hybrid systems with periodic time domains is introduced, and its stability analysis is formulated through differential matrix inequalities based upon Lyapunov-based conditions detailed in Goebel et al. (2012); Cox et al. (2012); Galeani et al. (2015).

3.1 Linear Hybrid Systems with periodic jumps

Consider a particular class of linear hybrid systems with all solutions defined on the same *periodic hybrid time domain* given by

$$\mathcal{T} := \{(t, k) : t \in [k\tau, (k + 1)\tau], k \in \mathbb{N}\},$$

with $\tau > 0$ given and t denoting the current value of continuous-time and k the number of jumps already produced.

Consider the linear hybrid system, defined by

$$\dot{x} = Ax(t, k) + Bu_F, \tag{6a}$$
$$x^+ = Ex_{<k>} + Fu_J, \tag{6b}$$

where $x \in \mathbb{R}^n$ corresponds to the state of the linear hybrid system (6). Moreover, $A \in \mathbb{R}^{n\times n}$, $E \in \mathbb{R}^{n\times n}$, with $u_F \in \mathbb{R}^{m_c}$ denoting the continuous-time control input and $u_J \in \mathbb{R}^{m_d}$ representing the an *impulsive* control input.

From the theory of linear hybrid systems formulated in Goebel et al. (2012) (see Chapter 3), the stability conditions for the origin of the system can be established by a Lyapunov function $V(x) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying positivity constraints and the decrease along the trajectories of the system. In addition, in the hybrid context imposing decrease of $V(x)$ with respect to the flow dynamics (6a) as well as the jump dynamics (6b) with the same time-invariant quadratic function $V(x) = x^\top Px$, leads in general to restrictive constraints, as detailed in (Zaccarian et al., 2011, Section 4). In the following, the stability properties of system (6) are studied as time-invariant quadratic Lyapunov functions defined for the equivalent discrete-time system.

3.2 Stability Analysis

As detailed in Galeani et al. (2015) the stability analysis of (6) can be characterised by the variation of $V(x) = x^\top Px$ over one period, that is

$$\Delta_\tau(V(x)) = \underbrace{\int_{k\tau}^{(k+1)\tau} \dot{V}(x)d\xi}_{\Delta_c(V(x))} + \underbrace{x_{[k+1]}^\top - x_{\langle k\rangle}^\top Px_{\langle k\rangle}}_{\Delta_d(V(x))}, \tag{7}$$

where $\Delta_c(V(x))$ and $\Delta_d(V(x))$ correspond to the variation of $V(x)$ over the continuous-time dynamics and after a jump, respectively. Moreover, setting $\theta = \xi\tau$, the expression (7) can be re-written as follows

$$\Delta_\tau(V(x)) = \int_0^1 2x^\top \tau P\dot{x} + x_{[k+1]}^\top Px_{[k+1]} - x_{\langle k\rangle}^\top Px_{\langle k\rangle} d\theta.$$

thus transforming the integral into the interval $[0, 1]$ and assuming $u_F = u_J = 0$, we have the following theorem.

Theorem 5. (Stability Analysis, Galeani et al. (2015)) Consider the linear hybrid system (6) its origin is asymptotically stable if there exists a quadratic Lyapunov function $V(x) = x^\top Px$ with $P \in \mathbb{S}_{>0}^{n\times n}$, matrix-valued functions $R(\theta), H(\theta) : [0, 1] \to \mathbb{S}^{n\times n}$ and a matrix $W \in \mathbb{R}^{n\times n}$, such that

$$\frac{d}{d\theta}R(\theta) + R^\top(\theta)A + A^\top R(\theta) \leq -\epsilon I_{n\times n}, \tag{8a}$$

$$\begin{bmatrix} R(0) & R(0)E \\ E^\top R(0) & -R(1) \end{bmatrix} \leq 0_{n\times n}, \tag{8b}$$

hold for all $\theta \in [0, 1]$, with $R(\theta) = P + H(\theta)$, $W = P$ and $H(0) = 0$.

See Galeani et al. (2015)([Propositions 2.1 and 3.1] and the references therein) for more details on the proof of Theorem 5. With the above theorem, the conditions for stability of linear hybrid systems with periodic jumps are expressed as the set of inequalities.

A solution to (8), namely matrices P and H can be obtained by imposing these matrices to be polynomials and by further imposing the inequalities to be strict. The resulting polynomial inequality can then be solved by computing the coefficients of a Handelman decomposition, following Theorem 2. It is important to notice that to exploit Theorem 2 we impose strict inequality to (8).

Remark 2. Note that the computation of $V(x) = x^\top Px$ detailed in this section can also be addressed by solving the equation $E^\top e^{A^\top \tau} P e^{A\tau} E - P + Q = 0$, where Q is any positive definite matrix. However, the formulation using inequalities makes it possible to cope with uncertainties, as for instance, modeled by polytopes.

⋆

4. NUMERICAL EXAMPLES

Numerical examples illustrate the proposed strategy to solve the matrix inequalities (8). The computation time to obtain the Handelman decomposition that gives the quadratic Lyapunov function is compared against the sum-of-squares programming and Polya's theorem results. For this numerical example, the toolbox CVX, Grant and Boyd (2014), and SOSTOOLS, Papachristodoulou et al. (2021), along with the solver SeDuMi Sturm (1999) were used to generate the results presented below with tolerances $\epsilon = 10^{-6}$.

For this purpose, consider a linear hybrid system with periodic jumps given by

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x(t,k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_F, \tag{9a}$$

$$x^+ = \begin{bmatrix} 2 & 0 \\ 1 & \frac{1}{2} \end{bmatrix} x_{\langle k \rangle}, \tag{9b}$$

$$u_F = [-44.57 \;\; -30.05]\, x(t,k), \tag{9c}$$

where $x \in \mathbb{R}^2$ and a stabilizing control u_F is given by the above expression.

It appears evident that, in open-loop, x_1 is continuous-time exponentially stable and discrete-time unstable, whereas x_2 is flow unstable and jump exponentially stable. Using the stabilizing continuous-time control law u_F, we obtain the converging time histories of the system states, as illustrated in Figure 2.

Figure 3 shows the comparison between the time required to solve polynomial relaxations of the differential matrix inequalities in (8) for each approach: the green line concerns the results obtained by using the Handelman decomposition, whereas the blue and red line correspond to the solving time needed if we exploit the sum-of-squares and Polya's approaches, respectively. This figure illustrates the impact of increasing the maximum polynomial degree d_m on the solving time of the corresponding semidefinite programming for each method. We can note that by using the sum-of-squares and Polya's approaches, the computation time required to solve the inequalities (8) is considerably larger than the time required by the Handelman decomposition for the same maximum polynomial degree. The improvement in solving time is even more relevant when a higher maximum polynomial degree d_m is considered. Moreover, Figure 4 compares the Lyapunov

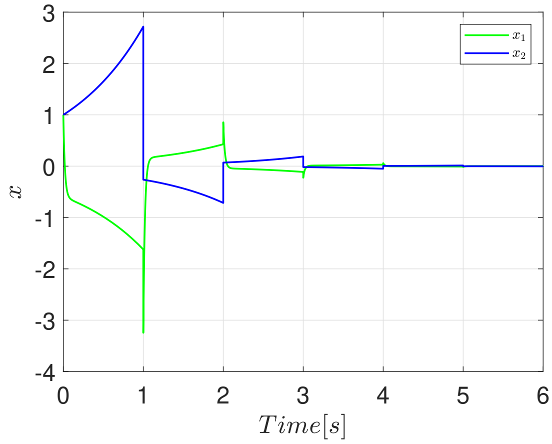

Fig. 2. *Time histories of the state of system (9) in closed-loop with the proposed stabilizing controller u_F.*

function $V(x) = x^\top Px$ as a function of time obtained for each one of those approaches pointing out that the solutions do not coincide, and that the monotonic decay of such solutions happens after a pair of a flow and a jump. All computations were performed on a DELL L8TQN4S laptop with Intel 2.7 GHz i7-7500U CPU and 16 GB RAM. Overall, the average improvement in solving time using Handelman decomposition rather than sum-of-squares and Polya's theorem was ≈ 79.79% and ≈ 52.58%, respectively.

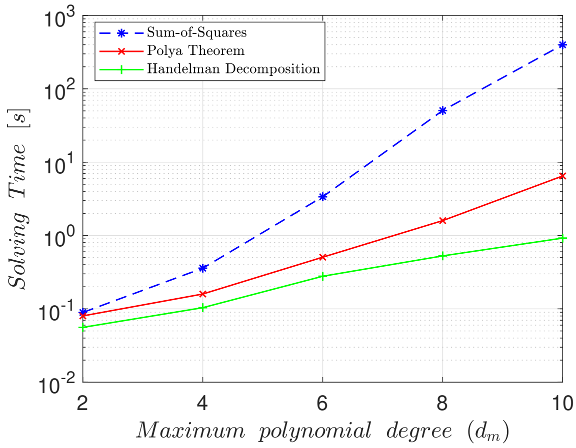

Fig. 3. *Comparison between the solving time needed to solve (8) by using Handelman decomposition, sum-of-squares and Polya's theorem considering different values of d_m.*

5. CONCLUSIONS AND FUTURE WORK

The stability analysis of linear hybrid systems with periodic jumps was considered. By exploiting the fact that non-negativity analysis of polynomials can be recast in terms of Handelman decomposition, a method was proposed to solve the differential matrix inequalities related to the stability analysis of such linear hybrid systems. Numerical examples illustrate the potential of the proposed method, showing a considerable reduction in computation time compared to other approaches allowing to verify polynomial non-negativity. The potential of the proposed

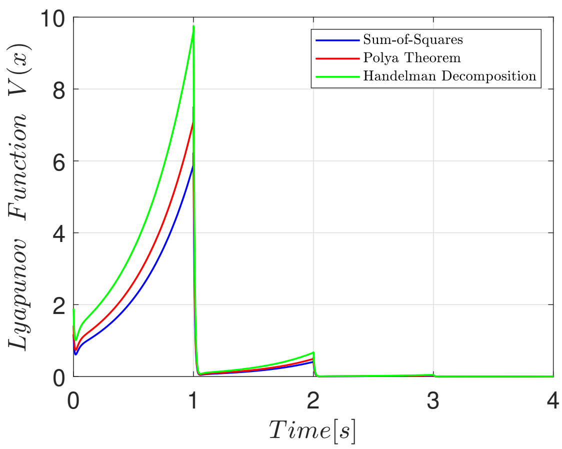

Fig. 4. *Comparison between the Lyapunov functions obtained to solve* (8) *by using Handelman decomposition, sum-of-squares and Polya's theorem considering* $d_m = 2$.

method should be fully exploited for systems of larger dimension for which existing methods, such as Polya's theorem in Kamyar et al. (2013) and sum-of-squares programming in Powers (2011), are still of limited application.

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