Stabilizing output-feedback control law for hyperbolic systems using a Fredholm transformation

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***Abstract*—For most networked systems found in the literature, the actuated boundary is usually located at one end. In this paper, we first consider the stabilization of a chain of two interconnected subsystems, actuated at the *in-between* boundary. Each subsystem corresponds to coupled hyperbolic partial differential equations. Such in-domain actuation leads to higher complexity, and represents a significant difference with existing results.**

Then, starting from a classical controllability condition, we design a state feedback control law for the considered class of systems. The proposed approach is based on the backstepping methodology. However, to deal with the complex structure of the system, we use Fredholm integral transforms instead of classical Volterra transforms. We prove the invertibility of such transforms using an original operator framework. The well-posedness of the backstepping kernel equations defining the transformations is also shown with the same arguments. By using a similar procedure, we are then able to design a Luenberger-type observer. Finally, we use the state estimation in the stabilizing controller to obtain an output-feedback law, and some test cases complete the paper.

***Index Terms*—Distributed parameter systems, hyperbolic systems, observer design, backstepping methodology, underactuated networks.**

I. INTRODUCTION

THE control of networks of Partial Differential Equations (PDEs) is an active research topic. This class of systems is naturally encountered in multiple applications as traffic flows [1], electrical networks [2], [3], density-flow systems [4]–[6], or unsteady flows on open canals [7]. Constructive control designs for PDE systems often require specific structural assumptions: not fully interconnected systems but cascades for instance, or several independent actuators [8]. To envision the most general real applications, the questions of the controllability for such networks of PDEs have to be considered.

Recently, the backstepping approach has enabled breakthroughs for the stabilization of interconnected systems of PDEs. Delay robust stabilization of a chain of two interconnected subsystems of hyperbolic PDEs has, for instance, been obtained by rewriting the network as a simple neutral system with distributed delays [9]. This approach has been extended to a chain with an arbitrary number of subsystems in [10]. Other types of interconnected systems have also been considered [11]–[13], including, more recently, a chain of many interconnected PDE systems coupled at one end with an ODE [14].

In most (if not all) of the above contributions, the actuator is located at one end of the chain. Although such a configuration covers a wide range of applications, as drilling pipes or UAV-cable-payload structures, there are several situations for which the actuator is located at an arbitrary node of the chain. For example, when developing traffic control strategies on vast road networks, the actuator (ramp metering) can be located at a crossroad. This situation has been considered in [1] in a simple configuration, where in particular some boundary coupling terms were equal to zero. We can also cite the case of micro-endoscopes actuated by Electro-Active Polymers which can be modeled by Timoshenko beam equations with actuation on a small portion inside the system [15]. In a first approximation, the actuator can be considered as quasi-punctual.

Having an actuator located at one of the intersection nodes of the chain raises challenging controllability questions. In most cases, such interconnected systems may not be controllable, and appropriate controllability conditions need to be derived. Solving such a problem is a necessary step towards the stabilization of complex networks and underactuated systems. Similar challenges hold regarding observability problems.

In this paper, we focus on the stabilization and estimation of a system of two interconnected 2×2 hyperbolic subsystems. It is actuated at the in-between boundary. The application of the classical backstepping methodology (using Volterra transforms) leads to a reformulation of the system as two coupled transport equations with integral couplings at the unactuated boundary. Such system can be expressed as a time-delay system with distributed actuation dynamics or distributed measurement. Interestingly, this class of problem appeared in [16], where the authors applied a dynamic inversion procedure using predictor-based techniques. However, this approach is allowed only if the resulting actuator transfer function has no pole on the complex right-half plane. This stability requirement seems to be a significant limitation: one could conceivably still stabilize the system as long as there are no unstable modes in the system corresponding to transmission zeros of the control operator. Here, using the preliminary analysis that has been done on the corresponding time-delay systems class discussed in [17], [18], we overcome this limitation and present a new approach to stabilize the considered class of systems. Indeed, recent works emphasized the interest of taking advantages of conversions between different representations [19].

The proposed methodology requires a natural controllability assumption and is based on the design of an appropriate Fredholm integral transform. The Volterra transforms traditionally used in the backstepping methodology do not offer enough degrees of freedom to deal with this kind of interconnections. Using an operator framework adjusted from [20], we show that the existence and the invertibility of this new transform is a consequence of the controllability condition. To the best of the authors' knowledge, this methodology is a novelty in the literature and is a milestone towards the backstepping-based design of stabilizing control laws for underactuated systems. Unlike in [17], an output-feedback law is proposed here.

The strategy can be resumed as follows. First, we use two Volterra transforms and several changes of variables to rewrite the system under consideration as two heterodirectional linear hyperbolic PDEs, with integral coupling terms at

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Manuscript received November, 2021; revised April 2022.

the unactuated boundary. Next, inspired by the backstepping methodology, we map the resulting actuated system to a stable target system to design the control law. Unlike traditional approaches [21], we need to use a Fredholm integral transform to get rid of the integral coupling terms.Contrary to Volterra transforms, its existence and invertibility are not guaranteed. Several results in the literature deal with the invertibility of Fredholm transforms when kernels have a specific structure, for instance, lower diagonal matrices [22], or specific boundary conditions [23]. As these conditions are not fulfilled here, we use an operator framework inspired by [20]. We show that the invertibility of the transform is a consequence of a natural controllability condition. To prove its existence, we show that the kernels are solutions of a Fredholm equation that can be solved using the previously introduced operator framework. The observer design is constructed by analogy.

The layout of this paper follows the aforementioned steps of the strategy. In Section II, we present the system under consideration, the overall strategy as well as several assumptions. Then, in Section III, we design a full-state feedback controller using integral transforms. Next, we follow a similar methodology to design an observer state for the system (Section IV). The resulting output-feedback control law is presented in Section V. Some illustrative simulation results are given in Section VI. Finally, the paper ends with some perspectives in Section VIII. More computational details are given in Appendix A.

Notations. For all $a, b, \nu \in [0,1]$, define the *characteristic function* $\mathbb{1}_{[a,b]}(\nu)$, as the function equal to 1 if $\nu \in [a,b]$, and equal to 0 elsewhere. Denote $\mathcal{S} \in [0,1]^2$ the unit square, $\mathcal{T}^- = \{(x,y) \in [0,1]^2 x \geq y\}$ its lower triangular part and $\mathcal{T}^+ = \{(x,y) \in [0,1]^2, x \leq y\}$ its upper triangular part. The Hilbert space of square integrable functions is denoted $L^2([0,1];\mathbb{R}) \doteq L^2(0,1)$, and the space of piece-wise continuous functions defined on $[0,1]$ (resp. on the unit square) is denoted $C_{pc}(0,1)$ (resp. $C_{pc}(\mathcal{S})$). For any $(u,v) \in C^0([0,T];L^2(0,1))^2$, the L^2-norm is defined by $\|(u,v)\|_{L^2} = (\sum_{i=1}^2 \|u_i\|_{L^2}^2 + \|v_i\|_{L^2}^2)^{\frac{1}{2}}$. When not necessary, the time dependency may be omitted. The Sobolev space of L^2-functions whose derivative is in L^2 is denoted $H^1([0,1];\mathbb{R})$ [24].

II. System presentation

A. System under consideration

In this paper, we consider a system composed of two scalar hyperbolic PDE subsystems interconnected through their boundaries. This class of system may appear in the case of oil production systems made of networks of pipes or traffic network systems for instance. However, contrary to previous results in the area, see, e.g. [10], as illustrated in Figure 1, the considered system is here actuated at the junction. Each subsystem $i \in \{1,2\}$ is modeled by

$$\partial_t u_i(t,x) + \lambda_i \partial_x u_i(t,x) = \sigma_i^+(x) v_i(t,x), \quad (1)$$
$$\partial_t v_i(t,x) - \mu_i \partial_x v_i(t,x) = \sigma_i^-(x) u_i(t,x), \quad (2)$$

with σ_i^+, σ_i^- two continuous in-domain coupling functions. With no loss of generality, we assume normalized state variables such that $t > 0$, $x \in [0,1]$. In addition, for the sake of simplicity, transport velocities $\lambda_i > 0$, $\mu_i > 0$ are assumed to be constant, but the proposed approach still holds for space-varying terms. The two subsystems are interconnected through their boundaries

$$u_1(t,0) = q_{11} v_1(t,0), \quad v_2(t,1) = \rho_{22} u_2(t,1), \quad (3)$$
$$v_1(t,1) = V(t) + \rho_{11} u_1(t,1) + \rho_{12} v_2(t,0), \quad (4)$$
$$u_2(t,0) = q_{22} v_2(t,0) + q_{21} u_1(t,1). \quad (5)$$

The different couplings terms q_{ij} and ρ_{ij} are assumed to be constant. The real-valued actuation $V(t)$ is located at the right boundary of the first subsystem. We assume that we measure the opposite boundary of the unactuated subsystem $y(t) = v_2(t,0)$.

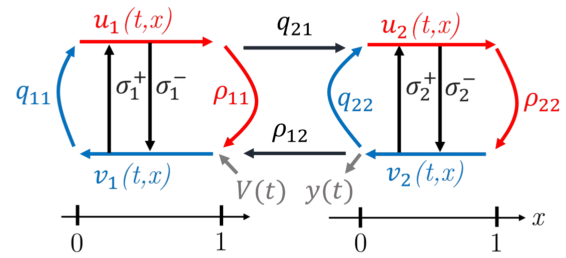


Fig. 1. Schematic representation of the system (1)-(5)

We denote $u_i^0(\cdot) = u_i(0,\cdot), v_i^0(\cdot) = v_i(0,\cdot) \in H^1([0,1],\mathbb{R})$ the initial conditions associated to (1)-(2). They satisfy the compatibility equations (3)-(5). The existence of solutions in H^1 for the open-loop system in the sense of the L^2-norm is guaranteed by [25, Appendix A]. The proposed framework is more general than the one introduced in [1] where several coupling terms were equal to zero. We see the proposed system as an excellent test case before generalizing results to more intricate networks. Indeed, before dealing with complex networks of hyperbolic systems (that could be not scalar), it is crucial to fully understand the difficulties that can arise when the actuator is not located at one end of the chain. For an interconnection of two scalar hyperbolic systems with boundary input (or output), there are only two possible locations for the actuator: at the end of the chain (a case that has already been solved in the literature) or between the two subsystems. The fact that the actuator is here located at the junction makes difficult the application of classical methods as the recursive methodology developed in [14]. More precisely, the re-circulation induced by the couplings between the two subsystems may create some unstable loops that prevent the stabilization. Consequently, having an "in-between actuation" leads to a completely different control design. Therefore, to design a stabilizing feedback law, some controllability assumptions are needed. Finally, as mentioned in the introduction, the proposed class of systems can model traffic networks [1]. In the case of non-scalar states (which will be the purpose of future work), it can also model micro-endoscopes actuated by Electro-Active Polymers.

B. Structural assumptions

We first make some structural assumptions on the boundary couplings of the interconnected system. The conditions that follow can be directly verified.

Assumption 1: The boundary coupling coefficient q_{21} does not equal 0.

This first assumption is crucial for stabilizing the whole system. In the case where $q_{21} = 0$, it is not possible to act on subsystem 2 using the control input on subsystem 1; thus, without this assumption, it would be impossible to stabilize the potentially unstable subsystem 2. Also, in this case, subsystem 1 is undetectable if only the available measurement $v_2(t,0)$ is used. This assumption is therefore necessary to design any output feedback law.

Assumption 2: The boundary coupling coefficients q_{11} and ρ_{22} do not equal 0.
If $q_{11}=0$, the control input acts on subsystem 2 through distributed terms only. Note that the backstepping methodology proposed here cannot be adjusted to deal with this case. Similar considerations arise in the observer design when $\rho_{22}=0$. It is so far a limitation of this approach[1].

Finally, we make the following (delay-) robustness assumption

Assumption 3: The coupling coefficients $|\rho_{11}q_{11}|$ and $|\rho_{22}q_{22}|$ are strictly less than 1.
The fact that the coefficient $q'_{22}=\rho_{22}q_{22}$ belongs to $(-1,1)$ implies that subsystem 2 has a finite number of unstable roots [26]. This restriction on coefficient $|q'_{22}|$ is slightly stronger than the necessary condition for delay-robust stabilisation given in [27]. Similarly, the condition $|\rho_{11}q_{11}|<1$ is used to guarantee that subsystem 1 has a finite number of unstable roots on the right-half plane. These two conditions could be related to the general robustness assumptions stated in [9].

In addition to these three general assumptions, some specific spectral controllability and spectral observability assumptions are added in Sections III-B3 and IV-B3 respectively.

C. Structure simplification

Under Assumption 2, we first make a change of variables to simplify the design of a stabilizing control law. We consider the bijective transformation

$$u'_1(t,x)=u_1(t,x),\ v'_1(t,x)=q_{11}v_1(t,x),$$
$$u'_2(t,x)=\rho_{22}u_2(t,x),\ v'_2(t,x)=v_2(t,x),$$

such that (1)-(2) hold for the new state (u'_i,v'_i) with the new coupling terms $\sigma_i'^{\pm}$ defined by

$$\sigma_1'^{+}(x)\doteq\frac{1}{q_{11}}\sigma_1^{+}(x),\ \sigma_1'^{-}(x)\doteq q_{11}\sigma_1^{-},$$
$$\sigma_2'^{+}(x)\doteq\rho_{22}\sigma_2^{+}(x),\ \sigma_2'^{-}(x)\doteq\frac{1}{\rho_{22}}\sigma_2^{-}.$$

The boundary conditions are now written as follows:

$$u'_1(t,0)=v'_1(t,0),\quad v'_2(t,1)=u'_2(t,1), \tag{6}$$
$$v'_1(t,1)=q_{11}(V(t)+\rho_{11}u'_1(t,1)+\rho_{12}v'_2(t,0)), \tag{7}$$
$$\begin{aligned}u'_2(t,0)&=\rho_{22}(q_{22}v'_2(t,0)+q_{21}u'_1(t,1))\\&=q'_{22}v'_2(t,0)+q'_{21}u'_1(t,1).\end{aligned} \tag{8}$$

We can now define $V_S(t)=q_{11}(V(t)+\rho_{12}v_2(t,0)+\rho_{11}u_1(t,1))$, such that (7) rewrites as $v'_1(t,1)=V_S(t)$. With the changes above, we have two unitary boundary couplings and have included some boundary couplings in the control input. This will simplify the analysis. Although there is now a cascade structure from the first subsystem to the second one, the stabilization problem is fundamentally different from the one studied in [14]. This structure difference would clearly appear when considering the natural extension of three interconnected subsystems for which the actuator would be located on the second subsystem. In the proposed design, the controller V_S will take the form of an integral law with piecewise continuous kernels. Thus, the control input will be continuous. It ensures the well-posedness of the closed-loop system (1)-(2) with the boundary conditions (6)-(8) in the H^1 state-space as long as the initial compatibility conditions are verified. The proof follows from [25, Appendix A] with minor adjustments (using Lumer-Philipps theroem). Alternatively, since the original closed-loop system (1)-(2) will be mapped to a simple target system (namely (42)-(44)) using invertible bounded transformations, we can show that the well-posedness of this final target system (directly using [25, Appendix A]) and deduce the well-posedness of the original closed-loop system. However, the fact that the control law contains non strictly proper terms $\rho_{12}v_2(t,0)$ and $\rho_{11}u_1(t,1)$ may lead to some robustness issues [28]. To avoid this problem, we can combine it with a well-tuned low-pass filter, as proposed in [16]. However, the robustness aspects are out of the scope of this paper and will not be considered here.

[1]More precisely, the case $q_{11}=0$ implies a "distributed only" effect of the actuation. When solving the kernel equations, the resulting Fredholm equations become degenerate and the proposed techniques do not apply.

D. Overall strategy

As mentioned in the introduction, the objective of this paper is to design an output-feedback control law $V(t)$ that exponentially stabilizes the system in the sense of the L^2-norm, i.e., we want to find $V(t)$ such that there exist $\nu>0,\ C_0\geq 1$, for all $(u_i^0,v_i^0)\in H^1([0,1];\mathbb{R}^2)$ verifying the compatibility conditions, we have $\|(u,v)\|_{L^2}\leq C_0\mathrm{e}^{-\nu t}\|(u^0,v^0)\|_{L^2}$.

The control strategy we propose is based on the backstepping methodology. It is schematically presented in Figure 2, and can be resumed as follows:

- First, an invertible Volterra transform $\mathcal{L}_i$ is applied on each subsystem (states (u'_i,v'_i)) mapping them on simpler intermediate subsystems (states (α_i,β_i)), without in-domain couplings. Due to this first integral change of variables, integral coupling terms appear at the boundary. A change of variables allows rewriting the system as two heterodirectional hyperbolic PDEs (state (w,z)). However, there still remains integral coupling terms at the boundary $x=0$ for the uncontrolled system (more precisely in $\alpha_2(t,0)$) ;
- Second, following the approach given in [17], this new system is mapped to an exponentially stable target system (state (υ,ψ)), by removing the aforementioned integral terms. This is done by using an appropriate invertible Fredholm integral transform $\mathcal{N}$;
- Third, the existence and invertibility of the backstepping transformation $\mathcal{N}$ is shown;
- Finally, a stabilizing full-state feedback control law is designed for the original system, whose well-posedness in closed-loop can easily be assessed.

The observer design strategy follows a similar strategy described in Section IV. The observer state is then used in Section V to provide an output-feedback controller.

III. Full-state feedback control law design

In this section, a full-state feedback control law that exponentially stabilizes the system (1)-(5) is proposed, based on the strategy detailed above.

A. First target system without in-domain couplings

1) Volterra transform: Consider now two invertible integral transforms $\mathcal{L}_i,\ i\in\{1,2\}$ acting on $H^1([0,1];\mathbb{R}^2)$ such that

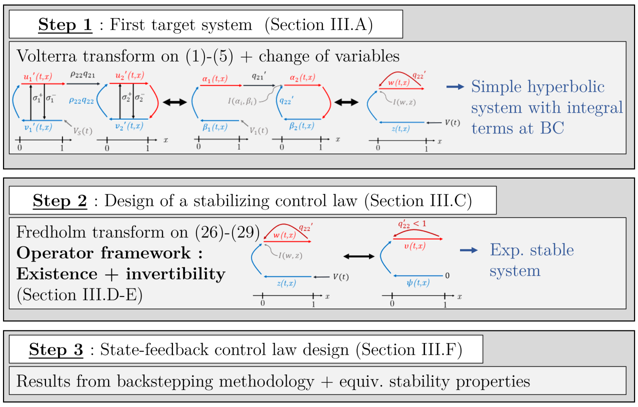


Fig. 2. Schematic representation of the control strategy

$\begin{pmatrix} u_i' \\ v_i' \end{pmatrix} = \mathcal{L}_i \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}$. More precisely, we have

$$\begin{cases} u_1' = \alpha_1 - \int_0^x L_1^{11}(x,y)\alpha_1(y) + L_1^{12}(x,y)\beta_1(y)\mathrm{d}y, \\ v_1' = \beta_1 - \int_0^x L_1^{21}(x,y)\alpha_1(y) + L_1^{22}(x,y)\beta_1(y)\mathrm{d}y, \end{cases} \tag{9}$$

$$\begin{cases} u_2' = \alpha_2 - \int_x^1 L_2^{11}(x,y)\alpha_2(y) + L_2^{12}(x,y)\beta_2(y)\mathrm{d}y, \\ v_2' = \beta_2 - \int_x^1 L_2^{21}(x,y)\alpha_2(y) + L_2^{22}(x,y)\beta_2(y)\mathrm{d}y, \end{cases} \tag{10}$$

where L_1^{ij} (resp. L_2^{ij}) are bounded piecewise continuous functions defined on the lower part of the unit square $\mathcal{T}^-$ (resp. on $\mathcal{T}^+$). The kernels satisfy the following set of equations

$$\lambda_i\partial_x L_i^{11}(x,y) + \lambda_i\partial_y L_i^{11}(x,y) = \sigma_i'^{+}(x)L_i^{21}(x,y), \tag{11}$$
$$\lambda_i\partial_x L_i^{12}(x,y) - \mu_i\partial_y L_i^{12}(x,y) = \sigma_i'^{+}(x)L_i^{22}(x,y), \tag{12}$$
$$\mu_i\partial_x L_i^{21}(x,y) - \lambda_i\partial_y L_i^{21}(x,y) = -\sigma_i'^{-}(x)L_i^{11}(x,y), \tag{13}$$
$$\mu_i\partial_x L_i^{22}(x,y) + \mu_i\partial_y L_i^{22}(x,y) = -\sigma_i'^{-}(x)L_i^{12}(x,y), \tag{14}$$

with boundary conditions

$$L_1^{12}(x,x) = -\frac{\sigma_1'^{+}(x)}{\lambda_1+\mu_1}, \quad L_1^{21}(x,x) = \frac{\sigma_1'^{-}(x)}{\lambda_1+\mu_1}, \tag{15}$$
$$L_1^{11}(x,0) = \frac{\mu_1}{\lambda_1}L_1^{12}(x,0), \quad L_1^{22}(x,0) = \frac{\lambda_1}{\mu_1}L_1^{21}(x,0), \tag{16}$$
$$L_2^{12}(x,x) = \frac{\sigma_2'^{+}(x)}{\lambda_2+\mu_2}, \quad L_2^{21}(x,x) = -\frac{\sigma_2'^{-}(x)}{\mu_2+\lambda_2}, \tag{17}$$
$$L_2^{11}(x,1) = \frac{\mu_2}{\lambda_2}L_2^{12}(x,1), L_2^{22}(x,1) = \frac{\lambda_2}{\mu_2}L_2^{21}(x,1). \tag{18}$$

These two sets of equations admit a unique continuous solution [29]. The integral transform $\mathcal{L}_i$, $i \in \{1,2\}$ is a bounded (and therefore continuous) operator from $H^1([0,1];\mathbb{R}^2)$ to $H^1([0,1];\mathbb{R}^2)$. The transformation is invertible as it is a Volterra transform [30]. The inverse transforms $\mathcal{L}_i^{-1}$ have the same structure.
The two Volterra transforms map the original system to

$$\partial_t\alpha_i(t,x) + \lambda_i\partial_x\alpha_i(t,x) = 0, \tag{19}$$
$$\partial_t\beta_i(t,x) - \mu_i\partial_x\beta_i(t,x) = 0, \tag{20}$$

with the boundary conditions

$$\alpha_1(t,0) = \beta_1(t,0), \quad \beta_1(t,1) = V_1(t), \tag{21}$$
$$\alpha_2(t,0) = q_{22}'\beta_2(t,0) + q_{21}'\alpha_1(t,1) + \mathcal{I}(\alpha_i,\beta_i), \tag{22}$$
$$\beta_2(t,1) = \alpha_2(t,1). \tag{23}$$

The resulting integral boundary couplings and control law are defined by

$$\begin{aligned} \mathcal{I}(\alpha_i,\beta_i) &= -q_{21}'\int_0^1 L_1^{11}(1,y)\alpha_1(y) + L_1^{12}(1,y)\beta_1(y)\mathrm{d}y \\ &+ \int_0^1 (L_2^{11}(0,y) - q_{22}'L_2^{21}(0,y))\alpha_2(y) \\ &\quad + (L_2^{12}(0,y) - q_{22}'L_2^{22}(0,y))\beta_2(y)\mathrm{d}y, \\ V_1(t) &= V_S(t) + \int_0^1 L_1^{21}(1,y)\alpha_1(y) + L_1^{22}(1,y)\beta_1(y)\mathrm{d}y. \end{aligned} \tag{24}$$

Denote by $(\alpha_i^0(\cdot),\beta_i^0(\cdot))^T = \mathcal{L}_i^{-1}((u_i'(0,\cdot),v_i'(0,\cdot))^T) \in H^1([0,1],\mathbb{R}^2)$ the initial conditions associated to (19)-(20). They satisfy the compatibility equations (21)-(23) (with $V_1 \equiv 0$). This first target system (19)-(23) is therefore composed of two transport equations, but presents integral terms (24) which may be sources of instabilities at the boundary $x = 0$.

2) Change of variables: To simplify the problem, (19)-(23) can be rewritten as a single 2×2 system whose state is denoted $(z(t,x),w(t,x))$. Indeed, each subsystem can be independently considered as a transport equation with a propagation time $\tau_i = \frac{1}{\lambda_i} + \frac{1}{\mu_i}$, and a velocity $\Lambda_i = \frac{\mu_i\lambda_i}{\lambda_i+\mu_i}$. Let us define the new set of coordinates $(w(t,x),z(t,x))$ by

$$\begin{cases} w(t,x) = \mathbb{1}_{[0,x_2)}(x)\alpha_2(t,\frac{x}{x_2}) + \mathbb{1}_{[x_2,1]}(x)\beta_2(t,\frac{x-1}{x_2-1}), \\ z(t,x) = q_{21}'\Big(\mathbb{1}_{[0,x_1)}(x)\alpha_1(t,1-\frac{x}{x_1}) \\ \qquad\qquad + \mathbb{1}_{[x_1,1]}(x)\beta_1(t,\frac{x-x_1}{1-x_1})\Big), \end{cases} \tag{25}$$

with $x_i = \frac{\mu_i}{\lambda_i+\mu_i}$. It can be shown that if α_2 and β_2 are in $H^1([0,1];\mathbb{R})$, then $w(t) \in H^1([0,1];\mathbb{R})$ due to the condition $\beta_2(t,1) = \alpha_2(t,1)$ (which gives the continuity for $x = x_2$) [24, Section 8.2]. The converse obviously holds. The same holds for the functions (α_1,β_1) and z. The new states $(w(t,x),z(t,x))$ satisfy the following set of equations

$$\partial_t w(t,x) + \Lambda_2\partial_x w(t,x) = 0, \tag{26}$$
$$\partial_t z(t,x) - \Lambda_1\partial_x z(t,x) = 0, \tag{27}$$

with

$$w(t,0) = z(t,0) + q_{22}'w(t,1) + \int_0^1 N_w(y)w(t,y) + N_z(y)z(t,y)\mathrm{d}y, \tag{28}$$
$$z(t,1) = q_{21}'V_1(t) = V_1'(t). \tag{29}$$

The integral coupling terms are defined by

$$\begin{aligned} N_w(x) &= \mathbb{1}_{[0,x_2)}(x)\frac{1}{x_2}(L_2^{11}(0,\frac{x}{x_2}) - q_{22}'L_2^{21}(0,\frac{x}{x_2})) \\ &+ \mathbb{1}_{[x_2,1]}(x)\frac{1}{1-x_2}(L_2^{12}(0,\frac{1-x}{1-x_2}) - q_{22}'L_2^{22}(0,\frac{1-x}{1-x_2})), \\ N_z(x) &= -\Big(\mathbb{1}_{[0,x_1)}(x)\frac{1}{x_1}L_1^{11}(1,1-\frac{x}{x_1}) \\ &\qquad + \mathbb{1}_{[x_1,1]}(x)\frac{1}{1-x_1}L_1^{12}(1,\frac{x-x_1}{1-x_1})\Big). \end{aligned}$$

Note that N_w (resp. N_z) is continuous by definition of x_i and due to the boundary conditions (16) and (18). In the following, we use system (26)-(29) to design the control law.

B. Operator formulation

We follow the backstepping methodology to design a control law V_1' that stabilizes the system (26)-(29). Due to the presence of integral coupling terms in (28), we cannot use a Volterra transform to map the system to an exponentially stable target system. We need more degrees of freedom, and therefore use a Fredholm integral transform, whose kernels are defined on $\mathcal{S}$. However, it is well-known that such integral transforms are not always invertible [30]. We then show that the invertibility of the transform is related to a controllability assumption for our system. The proof follows the approach proposed by [25] and relies on an operator framework.

1) Reformulation of system (26)-(29)*:* in the abstract form

$$\frac{d}{dt}\begin{pmatrix} w \\ z \end{pmatrix} = A\begin{pmatrix} w \\ z \end{pmatrix} + BV_1', \tag{30}$$

where we can identify the operators A and B through their adjoints by taking formally the canonical scalar product of (30) with smooth test functions and comparing with the weak formulation [17]. The operator A is thus defined by

$$\begin{aligned} A : &D(A) \subset L^2([0,1],\mathbb{R}^2) \to L^2([0,1],\mathbb{R}^2) \\ &\begin{pmatrix} w \\ z \end{pmatrix} \longmapsto \begin{pmatrix} -\Lambda_2 w_x(x) \\ \Lambda_1 z_x(x) \end{pmatrix}, \end{aligned} \tag{31}$$

with

$$D(A) = \{(w,z) \in H^1(0,1)^2 |\ z(1) = 0,$$
$$w(0) = z(0) + q_{22}' w(1) + \int_0^1 N_w(y)w(y) + N_z(y)z(y)\mathrm{d}y\}.$$

Its adjoint A^* is defined by

$$\begin{aligned} A^* : &D(A^*) \subset L^2([0,1],\mathbb{R}^2) \to L^2([0,1],\mathbb{R}^2) \\ &\begin{pmatrix} w \\ z \end{pmatrix} \longmapsto \begin{pmatrix} \Lambda_2 w_x(x) + \Lambda_2 N_w(x) w(0) \\ -\Lambda_1 z_x(x) + \Lambda_2 N_z(x) w(0) \end{pmatrix}, \end{aligned} \tag{32}$$

with $D(A^*) = \{(w,z) \in H^1([0,1],\mathbb{R}^2) |\ w(1) = q_{22}' w(0),\ z(0) = \frac{\Lambda_2}{\Lambda_1} w(0)\}$. The operator $B \in \mathcal{L}^2(\mathbb{R}, D(A^*)')$ is defined by $< BV_1', \begin{pmatrix} w \\ z \end{pmatrix} >= \Lambda_1 z(1) V_1'$, and its adjoint $B^* \in \mathcal{L}(D(A^*),\mathbb{R})$ by

$$B^*\begin{pmatrix} w \\ z \end{pmatrix} = \Lambda_1 z(1). \tag{33}$$

The operator A is well-posed and densely defined in $L^2([0,1],\mathbb{R}^2)$. Adjusting the approach of [25, Appendix A] to handle the integral terms, based on the Lumer-Philips theorem, it is possible to show that A generates a C^0-semigroup. Since A^* is closed, its domain $D(A^*)$ is a Hilbert space, equipped where the norm $\|(w,z)\|_{D(A^*)} = (\|(w,z)\|_{L^2}^2 + \|A^*(w,z)\|_{L^2}^2)^{1/2}, (w,z) \in D(A^*)$,
with $\|\cdot\|_{D(A^*)}$ and $\|\cdot\|_{L^2}$ are equivalent norms on $D(A^*)$. Since the control law V_1' resulting from our approach will be an integral operator, the closed-loop is well-posed and there exists a unique solution to (26)-(29) in $H^1([0,1];\mathbb{R}^2)$. Following the approach of [20], we could have shown that B is admissible. However, the well-posedness of the closed-loop system will be shown by proving the well-posedness of a target system (namely (42)-(44)) and using invertible bounded transformations.

2) Generalities on Fredholm integral operators: The stabilization of the PDE system (26)-(29) is done using a integral transform of the Fredholm type. More precisely, consider an operator $\mathcal{T} : L^2([0,1],\mathbb{R}^2) \to L^2([0,1],\mathbb{R}^2)$ defined by

$$\mathcal{T}\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} - \int_0^1 K(x,y)\begin{pmatrix} u(y) \\ v(y) \end{pmatrix} dy, \tag{34}$$

where $K \in C_{pc}(\mathcal{S})$ is bounded piecewise continuous on $\mathcal{S}$. Note that the integral part has a regularizing effect, such that $\forall (u,v)^T \in L^2([0,1],\mathbb{R}^2)$, $\int_0^1 K(x,y)(u(y),v(y))^T \in H^1([0,1],\mathbb{R}^2)$. Unlike Volterra integral transformations, Fredholm transformations are not always invertible [30]. The following lemma (adjusted from [20, Lemma 2.2, Proposition 2.6]) guarantees (under several conditions) the invertibility of such an integral operator.

Lemma 1: Consider two operators $\mathcal{A}, \mathcal{B}$, such that $D(\mathcal{A}) \subset L^2([0,1],\mathbb{R}^2)$ and a Fredholm integral operator $\mathcal{T} : L^2([0,1],\mathbb{R}^2) \to L^2([0,1],\mathbb{R}^2)$ as defined by (34). Assume

(a) $\ker(\mathcal{T}) \subset D(\mathcal{A})$,
(b) $\ker(\mathcal{T}) \subset \ker(\mathcal{B})$,
(c) $\forall z \in \ker(\mathcal{T}),\ \mathcal{T}\mathcal{A}z = 0$,
(d) $\forall s \in \mathbb{C},\ \ker(s - \mathcal{A}) \cap \ker(\mathcal{B}) = \{0\}$.

Then, the operator $\mathcal{T}$ is invertible. Moreover, its inverse is a Fredholm integral operator whose kernels inherit the same regularity properties.

Proof 1: The proof follows the steps of [20, Lemma 2.2, Proposition 2.6]. Since the integral part of $\mathcal{T}$ is a compact operator, the Fredholm alternative [24] implies that $\dim\ker(\mathcal{T}) < \infty$. Suppose that $\ker(\mathcal{T}) \neq \{0\}$. Due to condition (a), for all $z \in \ker(\mathcal{T})$ $\mathcal{A}z$ is well-defined, and condition (c) implies that $\ker(\mathcal{T})$ is stable by $\mathcal{A}$, that is to say, for all $z \in \ker(\mathcal{T})$, $\mathcal{A}z \in \ker(\mathcal{T})$. Since $\ker(\mathcal{T})$ is finite-dimensional and not reduced to $\{0\}$, the restriction $\mathcal{A}_{|\ker(\mathcal{T})}$ of $\mathcal{A}$ to $\ker(\mathcal{T})$ has at least one eigenvalue $\nu \in \mathbb{C}$. Let ζ be the corresponding eigenfunction. Thus, $\zeta \in \ker(\nu - \mathcal{A})$ and $\zeta \in \ker(\mathcal{B})$ by condition (b). This is in contradiction with condition (d). Thus, $\ker(\mathcal{T}) = \{0\}$ and $\mathcal{T}$ is injective. Using the Fredholm alternative [24], we obtain that $\mathcal{T}$ is invertible. The fact that the inverse operator is a Fredholm integral operator whose kernels inherit the same regularity properties comes from [20, Section 2.4]. □

Remark 1: Note that the operator $\mathcal{T}$ is invertible if and only if its adjoint operator $\mathcal{T}^*$ is invertible. In some cases, the invertibility of the adjoint operator is easier to prove.

3) Spectral controllability assumption: Considering the four assumptions of Lemma 1, we note that the conditions $(a) - (c)$ only depend on the choice of the integral operator $\mathcal{T}$. However, the condition (d) corresponds to a fundamental property of the system that does not depend on the operator. Therefore, we make the following assumption.

Assumption 4: The operators A^* defined in (32) and B^* defined in (33) satisfy

$$\forall s \in \mathbb{C}, \ker(s - A^*) \cap \ker(B^*) = \{0\}.$$

Assumption 4 is a controllability condition that is similar to the one given in [20]. It is related to the *approximate controllability* of the system and has been introduced by [31] in a much larger setting. We believe that as proposed in [20], it could possibly be verified through a spectral analysis. Interestingly, using a time-delay systems formalism, we propose below a complex analysis version of this assumption.

Denote $\phi(t) = w(t,0)$. Applying the method of characteristics to the transport equations (26)-(27), we obtain

$$\phi(t) = q'_{22}\phi(t-\tau_2) + \int_0^{\tau_2} \Lambda_2 N_w(\Lambda_2\nu)\phi(t-\nu)\mathrm{d}\nu \quad (35)$$
$$+ V_1'(t-\tau_1) + \int_0^{\tau_1} \Lambda_1 N_z(1-\Lambda_1\nu)V_1'(t-\nu)\mathrm{d}\nu.$$

This corresponds to the general class of *integral delay equation* considered in [17]. Let us formally take the Laplace transform of (35) (with zero initial condition). We have $F_2(s)\phi(s) = F_1(s)V_1(s)$, where the holomorphic function F_2 and F_1 are defined by

$$F_2(s) = 1 - q'_{22}\mathrm{e}^{-\tau_2 s} - \int_0^{\tau_2} \Lambda_2 N_w(\Lambda_2\nu)\mathrm{e}^{-\nu s}d\nu, \quad (36)$$
$$F_1(s) = \mathrm{e}^{-\tau_1 s} + \int_0^{\tau_1} \Lambda_1 N_z(1-\Lambda_1\nu)\mathrm{e}^{-\nu s}d\nu. \quad (37)$$

To ensure that $F_2(s), F_1(s)$ cannot simultaneously be equal to zero, we are lead to the following[2] *spectral-like controllability* assumption [33], [34]:

Assumption 5: For all $s \in \mathbb{C}$, $\mathrm{rank}[F_2(s), F_1(s)] = 1$.
We can show that Assumption 5 implies Assumption 4.

Lemma 2: Under Assumption 5, Assumption 4 is satisfied.

Proof 2: Consider $s \in \mathbb{C}$ and $(w,z) \in \ker(s-A^*) \cap \ker(B^*)$. Since $(w,z) \in \ker(B^*)$, we have $z(1) = 0$. Since $(w,z) \in \ker(s-A^*)$, we have

$$sw(x) = \Lambda_2 w'(x) + \Lambda_2 N_w(x)w(0),$$
$$sz(x) = -\Lambda_1 z'(x) + \Lambda_2 N_z(x)w(0),$$

with the boundary conditions $w(1) = q'_{22}w(0), \Lambda_1 z(0) = \Lambda_2 w(0)$. Solving these two equations, we obtain

$$w(x) = \mathrm{e}^{\frac{s}{\Lambda_2}x}w(0) - w(0)\int_0^x N_w(\nu)\mathrm{e}^{\frac{s}{\Lambda_2}(x-\nu)}d\nu, \quad (38)$$
$$z(x) = \mathrm{e}^{-\frac{s}{\Lambda_1}x}z(0) + z(0)\int_0^x N_z(\nu)\mathrm{e}^{-\frac{s}{\Lambda_1}(x-\nu)}d\nu. \quad (39)$$

Using $z(1) = 0,\ w(1) = q'_{22}w(0)$, and evaluating (38)-(39) in $x = 1$, one gets $w(0)F_2(s) = 0,\ z(0)F_1(s) = 0$.
Using Assumption 5, we cannot simultaneously have $F_2(s) = 0$ and $F_1(s) = 0$. It prevents pole-zero cancellation from V_1 to ϕ. Thus, we either have $w(0) = 0$ or $z(0) = 0$. Since $(w,z) \in D(A^*)$, $w(0) = z(0) = 0$ and $(w,z) = (0,0)$. □
Assumption 4 is therefore a direct consequence of the spectral-like controllability assumption. Note numerical methods for locating the zeros of analytical functions have been developed [35], and there now exists algorithms to this problem. Assumption 5 can then be numerically verified, using, for instance, the software package ZEAL [36].

To give more mathematical details, function F_1 cannot vanish if the imaginary part of s is large enough (due to Riemann-Lebesgue's lemma) or if $\Re(s) \to -\infty$. Similarly, F_2 cannot vanish if $\Re(s) \to +\infty$. Thus, common zeros can be detected on a compact set whose bounds are numerically computed from the system parameters. So, on any arbitrary subset of this compact set, the number of zeros can be predetermined using curve integrals and Cauchy's argument principle [37].

Note that, in practice, numerical convergence in the computation of the Fredholm transform kernels (34) will not be achieved if Assumption 5 is not satisfied.

[2]Intuitively, as pointed out by [32], the spectral controllability simply says that all its finite-dimensional modal subsystems are controllable in the usual sense.

C. Constructive design of a stabilizing control law

In this section, we adjust the strategy developed in [17] to design a full-state feedback controller for system (26)-(29).

1) Presentation of the target system: Following the backstepping method, we want to map the PDE system (26)-(29) to a stable target system. Denote the target state as (υ, ψ). Next, define the integral operator $\mathcal{N}$ of the form (34) with kernels $N^{ij} \in C_{pc}(\mathcal{S})$, verifying $\begin{pmatrix} w(x) \\ z(x) \end{pmatrix} = \mathcal{N}\begin{pmatrix} \upsilon(x) \\ \psi(x) \end{pmatrix}$, such that

$$w(x) = \upsilon(x) - \textstyle\int_0^1 N^{11}(x,y)\upsilon(y) + N^{12}(x,y)\psi(y)dy, \quad (40)$$
$$z(x) = \psi(x) - \textstyle\int_0^1 N^{21}(x,y)\upsilon(y) + N^{22}(x,y)\psi(y)dy. \quad (41)$$

The target state satisfies the following set of equations

$$\partial_t \upsilon(t,x) + \Lambda_2 \partial_x \upsilon(t,x) = 0, \quad (42)$$
$$\partial_t \psi(t,x) - \Lambda_1 \partial_x \psi(t,x) = 0, \quad (43)$$

with the boundary conditions

$$\upsilon(t,0) = \psi(t,0) + q'_{22}\upsilon(t,1), \quad \psi(t,1) = 0. \quad (44)$$

Note that this system corresponds to the system (26)-(29) without the integral term in (28). Denote now $(\upsilon^0(\cdot), \psi^0(\cdot))^T = \mathcal{N}^{-1}((w^0(\cdot), z^0(\cdot))^T) \in H^1([0,1], \mathbb{R}^2)$ the initial conditions associated to (42)-(43). They satisfy the compatibility equations (44). The target system (42)-(44) is well-posed [25, Appendix A]. It is exponentially stable in the sense of the L^2-norm. Indeed, due to the propagation of the boundary condition, ψ converges to 0 in finite time. For $t > \frac{1}{\Lambda_1}$, the first boundary condition becomes $\upsilon(t,0) = q'_{22}\upsilon(t,1) = q'_{22}\upsilon(t-\frac{1}{\Lambda_2}, 0)$. According to [25], [26], the system converges to 0 and is exponentially stable since $|q'_{22}| < 1$ (Assumption 1).
We now need to show that it is possible to map the system (26)-(29) to this target system using a bounded invertible transform. To this end, the first step is to rewrite it by using an operator formulation. We have $\frac{d}{dt}\begin{pmatrix} \upsilon \\ \psi \end{pmatrix} = A_0 \begin{pmatrix} \upsilon \\ \psi \end{pmatrix}$, where A_0 satisfies (31), and is defined on $D(A_0) = \{(\upsilon,\psi) \in H^1([0,1],\mathbb{R}^2)|\ \upsilon(0) = \psi(0) + q'_{22}\upsilon(1),\ \psi(1) = 0\}$. Its adjoint A_0^* is defined on $D(A^*)$ by

$$A_0^* : D(A^*) \subset L^2([0,1],\mathbb{R}^2) \to L^2([0,1],\mathbb{R}^2)$$
$$\begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} \Lambda_2 u_x(x) \\ -\Lambda_1 v_x(x) \end{pmatrix}. \quad (45)$$

2) Kernel equations: To map the original system (26)-(29) to the target system (42)-(44), the kernels of the Fredholm integral transform $\mathcal{N}$ must satisfy a set of equations. Following the backstepping methodology, we derive the expression of (υ,ψ) with respect to x and t and integrate by parts. Plugging the resulting expressions into the target system, we obtain:

$$\partial_x N^{11}(x,y) + \partial_y N^{11}(x,y) = 0, \quad (46)$$
$$\Lambda_2 \partial_x N^{12}(x,y) - \Lambda_1 \partial_y N^{12}(x,y) = 0, \quad (47)$$
$$\Lambda_1 \partial_x N^{21}(x,y) - \Lambda_2 \partial_y N^{21}(x,y) = 0, \quad (48)$$
$$\partial_x N^{22}(x,y) + \partial_y N^{22}(x,y) = 0, \quad (49)$$

with the boundary conditions

$$N^{11}(x,0) = \tfrac{\Lambda_1}{\Lambda_2}N^{12}(x,0),\ N^{22}(x,0) = \tfrac{\Lambda_2}{\Lambda_1}N^{21}(x,0), \quad (50)$$
$$N^{11}(x,1) = q'_{22}N^{11}(x,0),\ N^{21}(x,1) = q'_{22}N^{21}(x,0). \quad (51)$$

This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2022.3184405

Evaluating (40) in $x=0$, one gets

$$N_w(y) - \int_0^1 N_w(\nu)N^{11}(\nu,y) + N_z(\nu)N^{21}(\nu,y)\mathrm{d}\nu = -N^{11}(0,y) + N^{21}(0,y) + q_{22}'N^{11}(1,y), \quad (52)$$

$$N_z(y) - \int_0^1 N_w(\nu)N^{12}(\nu,y) + N_z(\nu)N^{22}(\nu,y)\mathrm{d}\nu = -N^{12}(0,y) + N^{22}(0,y) + q_{22}'N^{12}(1,y). \quad (53)$$

To ensure the well-posedness of the problem, we add the two following boundary conditions

$$N^{12}(x,1) = 0, \quad N^{22}(x,1) = 0. \quad (54)$$

These boundary conditions do not correspond to a degree of freedom. Setting them to zero is necessary to satisfy condition (b) of Lemma 1, and further guarantee the invertibility of $\mathcal{N}$ and its boundedness. We have the following theorem

Theorem 1: The set of equations (46)-(54) admits a unique solution in $C_{pc}(\mathcal{S};\mathbb{R}^{2\times 2})$.

We cannot apply classical methods [38] to prove the existence of a unique solution to (46)-(54), partly due to the integral terms in the boundary conditions (52)-(53). The proof of Theorem 1 requires technical computations, and is proposed below.

D. Well-posedness of the kernel equations

In this section, the existence of a unique solution to the kernels equations (46)-(54) is proved. First, we express all the kernels as functions of the boundary functions $N^{12}(0,y)$ and $N^{21}(0,y)$. We show that the existence of $N^{12}(0,\cdot), N^{21}(0,\cdot)$ implies the existence of all kernels on $\mathcal{S}$. Moreover, they share the same regularity properties. Then, we show that $N^{12}(0,\cdot), N^{21}(0,\cdot)$ are defined by an integral equation of the form (34). We can use Lemma 1 to conclude on the existence, boundedness and uniqueness of the kernels. For brevity and clarity, all the proofs are given in Appendix A. Unlike [17] (where a similar proof was proposed), we do not restrain here to the case $\Lambda_1 < \Lambda_2$.

1) Kernels reduction:

Lemma 3: For all $(x,y) \in \mathcal{S}$, $(i,j) \in \{1,2\}^2$, $N^{ij}(x,y)$ can be expressed as functions of $N^{12}(0,\cdot)$ and $N^{21}(0,\cdot)$.

Proof 3: The explicit expression is given by (A.1)-(A.3) in Appendix A. We apply the method of characteristics to the transport equations (46)-(49) to express $N^{ij}(x,y)$ on $\mathcal{S}$ as functions of the corresponding boundary value. We then use the boundary conditions (50)-(54) to express all the kernels as functions of $N^{12}(0,\cdot)$ and $N^{21}(0,\cdot)$. In the case $\Lambda_2 > \Lambda_1$, it is necessary to divide the definition domain into different subparts and to iterate this procedure to cover $\mathcal{S}$ entirely. □

2) Integral formulation: Define the new variable $\bar{N}^{12}(y) \doteq N^{12}(0,y) - \mathbb{1}_{[0,1-\frac{\Lambda_1}{\Lambda_2}]}(y)q_{22}'N^{12}(0,y+\frac{\Lambda_1}{\Lambda_2})$. As shown in Appendix A, this change of variable is a bijection. We then rewrite $N^{21}(0,y)$ and $\bar{N}^{12}(y)$ as the solutions of two integral equations. After some technical computations given in Appendix A, we show that they satisfy

$$\begin{pmatrix} N_w(y) \\ N_z(y)\end{pmatrix} = \begin{pmatrix} N^{21}(0,y) \\ -\bar{N}^{12}(y)\end{pmatrix} - \int_0^1 \begin{pmatrix} -\mathcal{I}_{12}(\nu,y) & \mathcal{I}_{11}(\nu,y) \\ -\mathcal{I}_{22}(\nu,y) & \mathcal{I}_{21}(\nu,y)\end{pmatrix} \begin{pmatrix} N^{21}(0,\nu) \\ -\bar{N}^{12}(\nu)\end{pmatrix} d\nu, \quad (55)$$

with $\mathcal{I}_{ij}$ four bounded piecewise continuous coupling terms depending on N_w, N_z, defined by (A.6)-(A.9) and (A.14).

3) Operator formulation: From equation (55), we state

Theorem 2: The Fredholm integral operator $\mathcal{Q}$ of form (34) defined by

$$\mathcal{Q}: L^2([0,1],\mathbb{R}^2) \to L^2([0,1],\mathbb{R}^2) \quad (56)$$
$$\begin{pmatrix} u(y) \\ v(y)\end{pmatrix} \mapsto \begin{pmatrix} u(y) \\ v(y)\end{pmatrix} - \int_0^1 \begin{pmatrix} -\mathcal{I}_{12}(\nu,y) & \mathcal{I}_{11}(\nu,y) \\ -\mathcal{I}_{22}(\nu,y) & \mathcal{I}_{21}(\nu,y)\end{pmatrix} \begin{pmatrix} u(\nu) \\ v(\nu)\end{pmatrix} \mathrm{d}\nu$$

is boundedly invertible.

Proof 4: The proof is given in Appendix A. □

4) Well-posedness of the kernel equations: We then have all arguments to prove Theorem 1.

Proof 5: By (55), we have $Q\begin{pmatrix} N^{21}(0,\cdot) \\ -\bar{N}^{12}(\cdot)\end{pmatrix} = \begin{pmatrix} N_w \\ N_z\end{pmatrix}$. The invertibility of the operator $\mathcal{Q}$ given by Theorem 2 implies the existence and uniqueness of $N^{12}(0,y), \bar{N}^{21}(y)$ in $L^2([0,1],\mathbb{R})$, and therefore the existence of $N^{12}(0,y), N^{21}(0,y)$. Since the kernels $\mathcal{I}_{ij}$ are piecewise continuous, the integral operator $\mathcal{Q}^{-1}$ has a regularizing effect. Since N_w and N_z are piecewise continuous, $\begin{pmatrix} N^{12}(0,y) \\ -\bar{N}^{21}(y)\end{pmatrix} = \mathcal{Q}^{-1}\begin{pmatrix} N_w(y) \\ N_z(y)\end{pmatrix}$ are in fact defined in $C_{pc}([0,1];\mathbb{R}^2)$. According to Lemma 3, the four kernels N^{ij} are then uniquely defined in $C_{pc}(\mathcal{S})$. □

E. Invertibility of the Fredholm transform

We now show that the Fredholm integral transform $\mathcal{N}$ is boundedly invertible.

Theorem 3: Consider the Fredholm integral operator $\mathcal{N}$ of the form (34) defined on $L^2([0,1];\mathbb{R}^2)$, with kernels defined on $C_{pc}(\mathcal{S})$ as the unique solution of (46)-(54). Then the operator $\mathcal{N}$ is boundedly invertible.

Proof 6: The adjoint operator $\mathcal{N}^*$ associated to $\mathcal{N}$, is also of the form (34). We have

$$\mathcal{N}^*(\begin{pmatrix} \upsilon(x) \\ \psi(x)\end{pmatrix}) = \begin{pmatrix} \upsilon(x) \\ \psi(x)\end{pmatrix} - \int_0^1 \bar{N}(y,x)^T \begin{pmatrix} \upsilon(y) \\ \psi(y)\end{pmatrix} dy.$$

Due to the regularity of the integral and of the kernels N^{ij}, we have $\ker(\mathcal{N}^*) \subset H^1([0,1],\mathbb{R}^2)$. Taking any $z \in \ker(\mathcal{N}^*)$, and evaluating it in $x=0$, $x=1$, we directly obtain conditions $(a),(b)$ of Lemma 1. Since $\mathcal{N}$ maps the original system (26)-(29) to the target system (42)-(44), we have for all $z \in \ker(B^*)$, $\mathcal{N}^*A^*z = A_0^*\mathcal{N}^*z$ (see [20] for instance). From (b), we therefore obtain condition (c). Condition (d) does not depend on the operator $\mathcal{N}$. We can then conclude that $\mathcal{N}^*$ is invertible, and so is $\mathcal{N}$.

The inverse operator $\mathcal{N}^{-1}$ associated to $\mathcal{N}$ is of form (34), with kernels $\check{N}$ defined on $C_{pc}(\mathcal{S};\mathbb{R}^{2\times 2})$ as the unique solution of $\check{N}(x,y) = -N(x,y) + \int_0^1 N(x,\nu)\check{N}(\nu,y)\mathrm{d}\nu$. □

The function $\check{N}$ can be numerically computed using a fixed-point method. It is necessary to compute the control law $V_1'(t)$.

F. Stabilizing control law

Using the inverse transform, we define the full-state feedback controller $V_1'(t)$ by

$$V_1'(t) = -\int_0^1 \check{N}^{21}(1,\nu)w(\nu,t) + \check{N}^{22}(1,\nu)z(\nu,t)\mathrm{d}\nu. \quad (57)$$

We can then compute the control law $V(t)$ stabilizing the initial system

$$V(t) = \frac{1}{q_{11}}V_S(t) - \rho_{12}v_2(t,0) - \rho_{11}u_1(t,1), \quad (58)$$

with

$$V_S(t) = -\frac{1}{q'_{21}}\Big(\int_0^1 x_2\check{N}^{21}(1,x_2\nu)\alpha_2(t,\nu) + (1-x_2)\check{N}^{21}(1,1-(1-x_2)\nu)\beta_2(t,\nu)\mathrm{d}\nu\Big) - \int_0^1 [L_1^{21}(1,\nu)+x_1\check{N}^{22}(1,x_1(1-\nu))]\alpha_1(t,\nu) + [L_1^{22}(1,\nu)+(1-x_1)\check{N}^{22}(1,x_1+(1-x_1)\nu)]\beta_1(t,\nu)\mathrm{d}\nu. \quad (59)$$

Since the two Volterra backstepping transforms $\mathcal{L}_1, \mathcal{L}_2$ are invertible, we can express the control law (59) as a function of the original states (u_i, v_i). We can conclude this section with the following theorem

Theorem 4: The state-feedback control law $V(t)$ defined by (58) exponentially stabilizes the hyperbolic system (1)-(5) in the sense of the L^2-norm.

Proof 7: First, let us show that state-feedback control law $V_1'(t)$ defined by (57) exponentially stabilizes the hyperbolic system (26)-(29) in the sense of the L^2-norm. Any initial condition of (26)-(29) in $H^1([0,1];\mathbb{R}^2)$ is mapped to an initial condition for (42)-(44) in $H^1([0,1];\mathbb{R}^2)$. The target system (42)-(44) admits a unique solution with adequate regularity. As justified earlier, it is exponentially stable in the sense of the L^2-norm. Due to the bounded invertibility of the Fredholm integral transform $\mathcal{N}$ (Theorem 1) in $H^1([0,1];\mathbb{R}^2)$, the intermediate system (26)-(29) admits a unique solution with desired regularity. With the control law $V_1'(t)$, the hyperbolic system (26)-(29) and the target system (42)-(44) share the same stability properties. It is straightforward to express α_i and β_i as functions of w and z. Therefore, the convergence of (w,z) to zero at an exponential rate immediately implies the exponential stability of (α_i,β_i). Due to the bounded invertibility of the Volterra integral transforms $\mathcal{L}_i$, the original states (u_i,v_i) share the same stability properties. □
This proof can be easily adjusted to show that the well-posedness of the target system (42)-(44) implies the well-posedness of the closed-loop system (1)-(5).

IV. Observer design

In this section, we design a state observer for the system (1)-(5), using the measurement $y(t) = v_2(0,t)$. We use a strategy similar to the one used in Section III. The design of the observer will be done on a simpler target system.

A. Target system

1) Volterra transform and kernel equations: Define two integral transforms $\mathcal{M}_i$, $i\in\{1,2\}$ on $L^2([0,1];\mathbb{R}^2)$ such that $\begin{pmatrix}u_i\\ v_i\end{pmatrix} = \mathcal{M}_i\begin{pmatrix}a_i\\ b_i\end{pmatrix}$. More precisely, we have

$$\begin{cases} u_1 = a_1 + \int_x^1 M_1^{11}(x,y)a_1(y) + M_1^{12}(x,y)b_1(y)\mathrm{d}y,\\ v_1 = b_1 + \int_x^1 M_1^{21}(x,y)a_1(y) + M_1^{22}(x,y)b_1(y)\mathrm{d}y,\end{cases} \quad (60)$$

$$\begin{cases} u_2 = a_2 + \int_0^x M_2^{11}(x,y)a_2(y) + M_2^{12}(x,y)b_2(y)\mathrm{d}y,\\ v_2 = b_2 + \int_0^x M_2^{21}(x,y)a_2(y) + M_2^{22}(x,y)b_2(y)\mathrm{d}y,\end{cases} \quad (61)$$

where the kernels M_1^{ij} (resp. M_2^{ij}) are piecewise continuous bounded functions defined on $\mathcal{T}^+$ (resp. $\mathcal{T}^-$). They satisfy the same set of equations (11)-(14) as kernels L_i (except that the coupling terms are now $\sigma_i^{\pm}$), with the boundary conditions

$$M_1^{12}(x,x) = -\frac{\sigma_1^+(x)}{\lambda_1+\mu_1},\ M_1^{21}(x,x) = \frac{\sigma_1^-(x)}{\lambda_1+\mu_1},$$
$$M_1^{11}(0,y) = q_{11}M_1^{21}(0,y),\ M_1^{22}(0,y) = \frac{1}{q_{11}}M_1^{12}(0,y),$$
$$M_2^{12}(x,x) = \frac{\sigma_2^+(x)}{\lambda_2+\mu_2},\ M_2^{21}(x,x) = -\frac{\sigma_2^-(x)}{\mu_2+\lambda_2},$$
$$M_2^{11}(1,y) = \frac{1}{\rho_{22}}M_2^{21}(1,y),\ M_2^{22}(1,y) = \rho_{22}M_2^{12}(1,y).$$

These two sets of equations admit a unique piecewise continuous solution [39]. Applying the transformation (60) to the first subsystem (respectively (61) to the second subsystem), we obtain the target system

$$\partial_t a_i(t,x) + \lambda_i\partial_x a_i(t,x) = H_i^a(x)a_1(t,1) + F_i^a(x)b_2(t,0) + K_i^a(x)V(t), \quad (62)$$
$$\partial_t b_i(t,x) - \mu_i\partial_x b_i(t,x) = H_i^b(x)a_1(t,1) + F_i^b(x)b_2(t,0) + K_i^b(x)V(t), \quad (63)$$

with the boundary conditions

$$a_1(t,0) = q_{11}b_1(t,0), \quad (64)$$
$$b_1(t,1) = \rho_{11}a_1(t,1) + \rho_{12}b_2(t,0) + V(t), \quad (65)$$
$$a_2(t,0) = q_{22}b_2(t,0) + q_{21}a_1(t,1), \quad (66)$$
$$b_2(t,1) = \rho_{22}a_2(t,1). \quad (67)$$

Denote $(a_i^0(\cdot),b_i^0(\cdot))^T = \mathcal{M}_i^{-1}((u_i^0(\cdot),v_i^0(\cdot))^T) \in H^1([0,1];\mathbb{R}^2)$ the initial conditions associated to (62)-(63). They satisfy the compatibility equations (64)-(67). The in-domain coupling terms $F_i^a, F_i^b, H_i^a, H_i^b$ are defined by the set of equations

$$H_1^*(x) + \int_x^1 M_1^{i1}(x,\nu)H_1^a(\nu) + M_1^{i2}(x,\nu)H_1^b(\nu)\mathrm{d}\nu = \lambda_1 M_1^{i1}(x,1) - \mu_1\rho_{11}M_1^{i2}(x,1), \quad (68)$$
$$F_1^*(x) + \int_x^1 M_1^{i1}(x,\nu)F_1^a(\nu) + M_1^{i2}(x,\nu)F_1^b(\nu)\mathrm{d}\nu = -\mu_1\rho_{12}M_1^{i2}(x,1), \quad (69)$$
$$H_2^*(x) + \int_0^x M_2^{i1}(x,\nu)H_2^a(\nu) + M_2^{i2}(x,\nu)H_2^b(\nu)\mathrm{d}\nu = -\lambda_2 q_{21}M_2^{i1}(x,0), \quad (70)$$
$$F_2^*(x) + \int_0^x M_2^{i1}(x,\nu)F_2^a(\nu) + M_2^{i2}(x,\nu)F_2^b(\nu)\mathrm{d}\nu = \mu_2 M_2^{i2}(x,0) - \lambda_2 q_{22}M_2^{i1}(x,0), \quad (71)$$

with $i=1$ if $*=a$, and $i=2$ if $*=b$. The coupling terms K are defined by

$$\begin{pmatrix}K_1^a(x)\\ K_1^b(x)\end{pmatrix} = \mathcal{M}_1^{-1}\Big(\begin{pmatrix}-\mu_1 M_1^{12}(x,1)\\ -\mu_1 M_1^{22}(x,1)\end{pmatrix}\Big),\ \begin{pmatrix}K_2^a(x)\\ K_2^b(x)\end{pmatrix} = 0. \quad (72)$$

The Volterra integral equations (68)-(72) admit a unique solution in $L^2(0,1)$. Due to the piecewise continuity of the kernels M^{ij} and the regularizing property of the integral operator, H_i^*, F_i^*, K_1^* are actually piecewise continuous functions.

2) Change of variables: Consider a new set of coordinates $(\omega(t,x),\gamma(t,x))$ given by

$$\begin{cases}\omega(t,x)=q_{21}(q_{11}\mathbb{1}_{[0,\xi_1)}(x)b_1(t,1-\frac{x}{\xi_1}) \\ \qquad +\mathbb{1}_{[\xi_1,1]}(x)a_1(t,\frac{x-\xi_1}{1-\xi_1})), \\ \gamma(t,x)=\mathbb{1}_{[0,\xi_2)}(x)b_2(t,\frac{x}{\xi_2}) \\ \qquad +\mathbb{1}_{[\xi_2,1]}(x)\rho_{22}a_2(t,\frac{x-1}{\xi_2-1}),\end{cases} \tag{73}$$

with $\xi_i=\frac{\lambda_i}{\lambda_i+\mu_i}=1-x_i$. Note that the boundary value in $\gamma(0,t)$ corresponds to $y(t)$. These new variables satisfy the following set of equations

$$\partial_t\omega(t,x)+\Lambda_1\partial_x\omega(t,x)=H_1(x)\omega(t,1)+F_1(x)\gamma(t,0) + K_1(x)V(t), \tag{74}$$

$$\partial_t\gamma(t,x)-\Lambda_2\partial_x\gamma(t,x)=H_2(x)\omega(t,1)+F_2(x)\gamma(t,0), \tag{75}$$

where $\Lambda_i=\frac{\mu_i\lambda_i}{\mu_i+\lambda_i}$ is defined in Section III-A2, and where the functions verify

$$H_1(x)=q_{11}\mathbb{1}_{[0,\xi_1)}(x)H_1^b(1-\frac{x}{\xi_1})+\mathbb{1}_{[\xi_1,1]}(x)H_1^a(\frac{x-\xi_1}{1-\xi_1}),$$

$$F_1(x)=q_{21}(q_{11}\mathbb{1}_{[0,\xi_1)}(x)F_1^b(1-\frac{x}{\xi_1})+\mathbb{1}_{[\xi_1,1]}(x)F_1^a(\frac{x-\xi_1}{1-\xi_1})),$$

$$H_2(x)=\frac{1}{q_{21}}(\mathbb{1}_{[0,\xi_2)}(x)H_2^b(\frac{x}{\xi_2})+\mathbb{1}_{[\xi_2,1]}(x)\rho_{22}H_2^a(\frac{x-1}{\xi_2-1})),$$

$$F_2(x)=\mathbb{1}_{[0,\xi_2)}(x)F_2^b(\frac{x}{\xi_2})+\mathbb{1}_{[\xi_2,1]}(x)\rho_{22}F_2^a(\frac{x-1}{\xi_2-1}),$$

$$K_1(x)=q_{21}(\mathbb{1}_{[0,\xi_1)}(x)q_{11}K_1^b(1-\frac{x}{\xi_1})+\mathbb{1}_{[\xi_1,1]}(x)K_1^a(\frac{x-\xi_1}{1-\xi_1})).$$

They satisfy the boundary conditions

$$\omega(t,0)=\rho_{11}q_{11}\omega(t,1)+q_{21}\rho_{12}q_{11}\gamma(t,0)+q_{21}q_{11}V(t), \tag{76}$$

$$\gamma(t,1)=q_{22}\rho_{22}\gamma(t,0)+\omega(t,1). \tag{77}$$

B. Observer and error state

1) Definition: In this subsection, we define an observer for the system (74)-(77). Classically, it is a copy of the original system with input injection terms (Luenberger-type observer). Interestingly, we see that the measurement $y(t)$ corresponds to $\gamma(t,0)=b_2(t,0)=v_2(t,0)$. The different changes of variables turned the measurement at the in-between boundary into a classical measurement at one end of the resulting system. The observer state $(\hat{\omega},\hat{\gamma})$ satisfies the following set of equations

$$\partial_t\hat{\omega}(t,x)+\Lambda_1\partial_x\hat{\omega}(t,x)=H_1(x)\hat{\omega}(t,1)+F_1(x)\hat{\gamma}(t,0) + K_1(x)V(t)+G_1(x)(\hat{\gamma}(t,0)-y(t)), \tag{78}$$

$$\partial_t\hat{\gamma}(t,x)-\Lambda_2\partial_x\hat{\gamma}(t,x)=H_2(x)\hat{\omega}(t,1)+F_2(x)\hat{\gamma}(t,0) + G_2(\hat{\gamma}(t,0)-y(t)), \tag{79}$$

with the boundary conditions

$$\hat{\omega}(t,0)=\rho_{11}q_{11}\hat{\omega}(t,1)+q_{21}\rho_{12}q_{11}\gamma(t,0)+q_{21}q_{11}V(t), \tag{80}$$

$$\hat{\gamma}(t,1)=q_{22}\rho_{22}\gamma(t,0)+\hat{\omega}(t,1). \tag{81}$$

Since the boundary conditions of the observer system contain non strictly proper terms $\gamma(t,0)$ corresponding to the measurement, it may lead to some robustness issues [40]. To avoid this problem, we could low-pass filter the output $y(t)$.

Finally, we define the error state $(\tilde{\omega},\tilde{\gamma})=(\omega,\gamma)-(\hat{\omega},\hat{\gamma})$. It satisfies the set of PDEs

$$\partial_t\tilde{\omega}(t,x)+\Lambda_1\partial_x\tilde{\omega}(t,x)=H_1(x)\tilde{\omega}(t,1)+G_1'(x)\tilde{\gamma}(t,0), \tag{82}$$

$$\partial_t\tilde{\gamma}(t,x)-\Lambda_2\partial_x\tilde{\gamma}(t,x)=H_2(x)\tilde{\omega}(t,1)+G_2'(x)\tilde{\gamma}(t,0), \tag{83}$$

where $G_i'\doteq F_i+G_i$ are two bounded piecewise continuous functions in $C_{pc}(0,1)$, and the boundary conditions

$$\tilde{\omega}(t,0)=\rho_{11}q_{11}\tilde{\omega}(t,1),\quad \tilde{\gamma}(t,1)=\tilde{\omega}(t,1). \tag{84}$$

Our objective is to determine the gains G_i' such that the error system (82)-(84) is exponentially stable.

2) Operator framework: We rewrite system (82)-(84) in the abstract form

$$\frac{d}{dt}\begin{pmatrix}\tilde{\omega}\\ \tilde{\gamma}\end{pmatrix}=\tilde{A}\begin{pmatrix}\tilde{\omega}\\ \tilde{\gamma}\end{pmatrix}+\mathcal{G}\tilde{C}\begin{pmatrix}\tilde{\omega}\\ \tilde{\gamma}\end{pmatrix}, \tag{85}$$

where the operator $\tilde{A}$ is defined by

$$\begin{aligned}\tilde{A}:&D(\tilde{A})\subset L^2([0,1],\mathbb{R}^2)\to L^2([0,1],\mathbb{R}^2)\\ &\begin{pmatrix}\tilde{\omega}\\ \tilde{\gamma}\end{pmatrix}\longmapsto\begin{pmatrix}-\Lambda_1\tilde{\omega}_x(x)+H_1(x)\tilde{\omega}(1)\\ \Lambda_2\tilde{\gamma}_x(x)+H_2(x)\tilde{\omega}(1)\end{pmatrix},\end{aligned} \tag{86}$$

with $D(\tilde{A})=\{(\tilde{\omega},\tilde{\gamma})\in H^1([0,1],\mathbb{R}^2)|\ \tilde{\omega}(0)=\rho_{11}q_{11}\tilde{\omega}(1),\ \tilde{\gamma}(1)=\tilde{\omega}(1)\}$.The operator $\tilde{A}$ is well posed and densely defined [25]. We can already draw a parallel with the definition of operator $\tilde{A}$ in (86) and the adjoint operator A^* defined in Section III by (32). The trace operator $\tilde{C}$ is defined by

$$\begin{aligned}\tilde{C}:&D(\tilde{A})\subset L^2([0,1],\mathbb{R}^2)\to\mathbb{R}\\ &(\tilde{\omega}\quad\tilde{\gamma})^T\longmapsto\tilde{\gamma}(0),\end{aligned} \tag{87}$$

and the operator $\mathcal{G}$ is defined by

$$\begin{aligned}&\mathcal{G}:\mathbb{R}\to C_{pc}(0,1)^2\\ &x\mapsto(G_1'.x\quad G_2'.x)^T.\end{aligned}$$

3) Spectral observability condition: Similarly to what has been done in Section III-B3, we need to formulate an observability assumption to guarantee the possibility to estimate the PDE states.

Assumption 6: The operators $\tilde{A}$ and $\tilde{C}$ respectively defined by (86) and (87) satisfy for any $s\in\mathbb{C}$

$$\ker(s-\tilde{A})\cap\ker(\tilde{C})=\{0\}. \tag{88}$$

This is analogous to the *controllability* Assumption 4. Rewriting equations (74)-(75) in the time-delay framework, we can reformulate this assumption using holomorphic functions. Define the functions

$$\tilde{F}_1(s)=1-\rho_{11}q_{11}\mathrm{e}^{-\tau_1 s}-\int_0^{\tau_1}H_1(\Lambda_1\nu)\mathrm{e}^{(\nu-\tau_1)s}d\nu, \tag{89}$$

$$\tilde{F}_2(s)=e^{-\tau_2 s}+\mathrm{e}^{\tau_2 s}\int_0^{\tau_2}H_2(\Lambda_2\nu)\mathrm{e}^{-\nu s}d\nu. \tag{90}$$

Using the variation of constant formula, and taking the Laplace transform in (86), we obtain $\tilde{\omega}(1)\tilde{F}_1(s)=\tilde{\omega}(1)\tilde{F}_2(s)=0$. We have the following *spectral observability* assumption

Assumption 7: For all $s\in\mathbb{C}$, $\mathrm{rank}[\tilde{F}_1(s),\tilde{F}_2(s)]=1$. Expressing the solutions of (82)-(84) using the variation of constants formula, we show that the Assumptions 6 and 7 are equivalent. The proof is analogous to the one of Lemma 2 ans is omitted here. Assumption 7 can be checked in the same way as Assumption 5.

C. Constructive design of the observer gains

In this section, we design the observer gains G_1', G_2' to stabilize the error system (82)-(84). We use a Fredholm integral transform to map this error system to a stable target system. The proof of existence and invertibility of such a transform follows the same strategy as the one presented in III.

1) Presentation of the target system: Following the backstepping methodology given in [41], we map the error system (82)-(84) to a stable target system with equivalent stability properties. Consider the candidate target system

$$\partial_t \tilde{\zeta}(t,x) + \Lambda_1 \partial_x \tilde{\zeta}(t,x) = 0, \quad (91)$$

$$\partial_t \tilde{\eta}(t,x) - \Lambda_2 \partial_x \tilde{\eta}(t,x) = 0, \quad (92)$$

with the boundary conditions

$$\tilde{\zeta}(t,0) = \rho_{11} q_{11} \tilde{\zeta}(t,1), \quad \tilde{\eta}(t,1) = \tilde{\zeta}(t,1). \quad (93)$$

Denote $(\tilde{\zeta}^0, \tilde{\eta}^0)$ in $H^1([0,1];\mathbb{R}^2)$ the initial conditions associated to (91)-(92) satisfying the compatibility conditions (93). The well-posedness of the error system (91)-(93) implies the one of the error system (78)-(81) and consequently of the observer system (82)-(84). This target system (91)-(93) is exponentially stable in the sense of the L^2-norm, since $|\rho_{11} q_{11}| < 1$ by Assumption 3 [25]. Define now the Fredholm integral transform $\mathcal{K}$ of the form (34), such that $\begin{pmatrix} \tilde{\zeta} \\ \tilde{\eta} \end{pmatrix} = \mathcal{K} \begin{pmatrix} \tilde{\omega} \\ \tilde{\gamma} \end{pmatrix}$. More precisely, we have

$$\tilde{\zeta}(t,x) = \tilde{\omega}(t,x) - \int_0^1 K^{11}(x,\nu)\tilde{\omega}(t,\nu) + K^{12}(x,\nu)\tilde{\gamma}(t,\nu)\mathrm{d}\nu, \quad (94)$$

$$\tilde{\eta}(t,x) = \tilde{\gamma}(t,x) - \int_0^1 K^{21}(x,\nu)\tilde{\omega}(t,\nu) + K^{22}(x,\nu)\tilde{\gamma}(t,\nu)\mathrm{d}\nu, \quad (95)$$

where $K^{ij},\ i,j \in \{1,2\}$ are four bounded piecewise continuous functions defined on $\mathcal{S}$.

2) Kernel equations: Following the backstepping methodology, we show that the kernels K^{ij} must satisfy the following set of equations

$$\partial_x K^{11}(x,y) + \partial_y K^{11}(x,y) = 0, \quad (96)$$

$$\partial_x K^{12}(x,y) - \frac{\Lambda_2}{\Lambda_1} \partial_y K^{12}(x,y) = 0, \quad (97)$$

$$\partial_x K^{21}(x,y) - \frac{\Lambda_1}{\Lambda_2} \partial_y K^{21}(x,y) = 0, \quad (98)$$

$$\partial_x K^{22}(x,y) + \partial_y K^{22}(x,y) = 0, \quad (99)$$

where we have

$$H_1(x) + \Lambda_1(K^{11}(x,1) - \rho_{11}q_{11}K^{11}(x,0)) - \Lambda_2 K^{12}(x,1) = \int_0^1 K^{11}(x,\nu)H_1(\nu) + K^{12}(x,\nu)H_2(\nu)\mathrm{d}\nu, \quad (100)$$

$$H_2(x) + \Lambda_1(K^{21}(x,1) - \rho_{11}q_{11}K^{21}(x,0)) - \Lambda_2 K^{22}(x,1) = \int_0^1 K^{21}(x,\nu)H_1(\nu) + K^{22}(x,\nu)H_2(\nu)\mathrm{d}\nu, \quad (101)$$

and the boundary conditions

$$K^{11}(0,y) = \rho_{11}q_{11}K^{11}(1,y),\ K^{12}(0,y) = \rho_{11}q_{11}K^{12}(1,y),$$
$$K^{11}(1,y) = K^{21}(1,y), \quad K^{22}(1,y) = K^{12}(1,y). \quad (102)$$

To these conditions, we add the two following boundary conditions,

$$K^{21}(0,y) = 0, \quad K^{22}(0,y) = 0. \quad (103)$$

The boundary conditions (103) are necessary to ensure that condition (b) of Lemma 1 is satisfied for the operator $\mathcal{K}$. If we manage to show that (96)-(103) admit a solution, we will be able to prove that (82)-(84) can be mapped to (91)-(93). Indeed, differentiating (94)-(95) with respect to time and space, integrating by parts, and using the fact that the state $(\tilde{\omega}, \tilde{\gamma})$ verifies (82)-(84), we directly obtain the target system (91)-(93).

3) Well-posedness of kernel equations: The proof of the existence of a solution to (96)-(103) derives from the proof of Theorem 1 given in Section III-D (see Appendix A). Indeed, let us define the kernels $\tilde{N}^{ij}$ on $\mathcal{S}$ by

$$\tilde{N}^{11}(x,y) = K^{11}(1-y,1-x), \quad (104)$$

$$\tilde{N}^{12}(x,y) = \frac{\Lambda_1}{\Lambda_2} K^{21}(1-y,1-x), \quad (105)$$

$$\tilde{N}^{21}(x,y) = \frac{\Lambda_2}{\Lambda_1} K^{12}(1-y,1-x), \quad (106)$$

$$\tilde{N}^{22}(x,y) = K^{22}(1-y,1-x). \quad (107)$$

The kernels $\tilde{N}^{ij}$ satisfy the same set of PDEs (46)-(49) than kernels N^{ij} (defining the invertible Fredholm integral transform $\mathcal{N}$). Moreover, they satisfy the same boundary conditions (50), (51) and (54) (the only difference being the name of the coupling coefficient (q'_{22} or $\rho_{11}q_{11}$), that are both strictly less than 1 by Assumption 3). Finally, the kernels $\tilde{N}^{ij}$ satisfy similar integral equations

$$\tilde{N}_w(y) - \int_0^1 \tilde{N}_w(\nu)\tilde{N}^{11}(\nu,y) + \tilde{N}_z(\nu)\tilde{N}^{21}(\nu,y)\mathrm{d}\nu = -\tilde{N}^{11}(0,y) + \tilde{N}^{21}(0,y) + \rho_{11}q_{11}\tilde{N}^{11}(1,y),$$

$$\tilde{N}_z(y) - \int_0^1 \tilde{N}_w(\nu)\tilde{N}^{12}(\nu,y) + \tilde{N}_z(\nu)\tilde{N}^{22}(\nu,y)\mathrm{d}\nu = -\tilde{N}^{12}(0,y) + \tilde{N}^{22}(0,y) + \rho_{11}q_{11}\tilde{N}^{12}(1,y),$$

with $\tilde{N}_w(y) \doteq \frac{1}{\Lambda_1}H_1(1-y)$ and $\tilde{N}_z(y) \doteq \frac{1}{\Lambda_2}H_2(1-y)$. Under the spectral observability Assumption 6, we prove the well-posedness and the existence of kernels $\tilde{N}^{ij}$ on $\mathcal{S}$, following the approach given in Appendix A. Since the change of variables (104)-(107) is invertible, we immediately state the well-posedness of (96)-(103). Since we have $(H_1, H_2) \in C_{pc}(0,1)^2$, and due to the regularizing properties of the integral operator, the kernel equations (96)-(103) admit a unique piecewise continuous solution on $\mathcal{S}$.

4) Invertibility of the Fredholm transform: Similarly to what was done in Section III-E, we have the following:

Theorem 5: The Fredholm integral transform $\mathcal{K}$ whose kernels are defined by (96)-(103) is invertible.

Proof 8: We show that this operator satisfies the conditions of Lemma 1. □

5) Definition of the observer gains: Following the backstepping procedure, we obtain the expressions of the observer gains G_i'. Indeed, in order to map the original system (82)-(84) to the target system (91)-(93), the observer gains must satisfy the integral equations

$$\begin{aligned} G_1'(x) - \int_0^1 K^{11}(x,\nu)G_1'(\nu) + K^{12}(x,\nu)G_2'(\nu)\mathrm{d}\nu \\ = -\Lambda_2 K^{12}(x,0), \\ G_2'(x) - \int_0^1 K^{21}(x,\nu)G_1'(\nu) + K^{22}(x,\nu)G_2'(\nu)\mathrm{d}\nu \\ = -\Lambda_2 K^{22}(x,0), \\ \iff \quad \mathcal{K}(\begin{pmatrix} G_1'(x) \\ G_2'(x) \end{pmatrix}) = \begin{pmatrix} -\Lambda_2 K^{12}(x,0) \\ -\Lambda_2 K^{22}(x,0) \end{pmatrix}. \end{aligned} \quad (108)$$

Since $\mathcal{K}$ is invertible, the observer gains G_1' and G_2' defined by (108) exist and are uniquely defined as piecewise continuous functions on $[0, 1]$. They satisfy

$$\begin{pmatrix} G_1'(x) \\ G_2'(x) \end{pmatrix} = \mathcal{K}^{-1} \begin{pmatrix} -\Lambda_2 K^{12}(x,0) \\ -\Lambda_2 K^{22}(x,0) \end{pmatrix}. \quad (109)$$

D. Convergence of the observer state

We can now show the convergence of the observer state $(\hat{\omega}, \hat{\gamma})$ to the real state (ω, γ). First, we have the following:

Lemma 4: Any solution $(\tilde{\omega}, \tilde{\gamma})$ of (82)-(84) converges to zero in the sense of the L^2-norm.

Proof 9: System (91)-(93) is exponentially stable in the sense of the L^2-norm. Since the backstepping transform $\mathcal{K}$ is bounded and invertible by Theorem 5, system (82)-(84) shares equivalent stability properties. □

Thus, the error system (82)-(84) is exponentially stable. The observer state (78)-(81) defined with gains $G_i = G_i' - F_i$ converges towards the initial state (ω, γ). We then define observer states for (a_i, b_i) by

$$\hat{a}_1(t,x) = \frac{1}{q_{21}}\hat{\omega}(t, \xi_1 + (1-\xi_1)x), \quad (110)$$

$$\hat{a}_2(t,x) = \frac{1}{\rho_{22}}\hat{\gamma}(t, 1-(1-\xi_2)x), \quad (111)$$

$$\hat{b}_1(t,x) = \frac{1}{q_{21}q_{11}}\hat{\omega}(t, \xi_1(1-x)), \ \hat{b}_2(t,x) = \hat{\gamma}(t, \xi_2 x). \quad (112)$$

Using the Volterra integral transforms $\mathcal{M}_i$ (60)-(61), we then define observer states for the initial states (u_i, v_i) by

$$\begin{pmatrix} \hat{u}_i \\ \hat{v}_i \end{pmatrix} = \mathcal{M}_i \begin{pmatrix} \hat{a}_i \\ \hat{b}_i \end{pmatrix}. \quad (113)$$

We have the following theorem:

Theorem 6: The state estimates $(\hat{u}_i, \hat{v}_i)$ defined by (113) converge towards the original states (u_i, v_i) in the sense of the L^2-norm.

Due to space restrictions, the proof is omitted. It is worth mentioning that it is a direct consequence of the properties of the Volterra integral transforms $\mathcal{M}_i$.

V. Output-Feedback control law

We can now combine the state observer designed in Section IV with the full state feedback control law $V(t)$ designed in Section III-F, to obtain an output feedback controller. We can state the main theorem of this article:

Theorem 7: The output-feedback control law $V(t, \hat{u}_1(t), \hat{u}_2(t), \hat{v}_1(t), \hat{v}_2(t))$ defined by

$$\hat{V}(t) = \frac{1}{q_{11}}\hat{V}_S(t) - \rho_{12}y(t) - \rho_{11}\hat{u}_1(t,1), \quad (114)$$

with

$$\hat{V}_S(t) = -\tfrac{1}{q_{12}} \int_0^1 \begin{pmatrix} x_2 \check{N}^{21}(1, x_2\nu) \\ (1-x_2)\check{N}^{22}(1, 1-(1-x_2)\nu) \end{pmatrix}^T \mathcal{L}_2^{-1} \begin{pmatrix} \hat{u}_2(t,\nu) \\ \hat{v}_2(t,\nu) \end{pmatrix} \mathrm{d}\nu$$
$$+ \int_0^1 \begin{pmatrix} L_1^{21}(1,\nu) - x_1\check{N}^{21}(1, x_1(1-\nu)) \\ L_1^{22}(1,\nu) - (1-x_1)\check{N}^{21}(1, x_1 + (1-x_1)\nu) \end{pmatrix}^T \mathcal{L}_1^{-1} \begin{pmatrix} \hat{u}_1(t,\nu) \\ \hat{v}_1(t,\nu) \end{pmatrix} \mathrm{d}\nu$$

exponentially stabilizes system (1)-(5) in the sense of the L^2-norm.

Proof 10: Similarly to what has been done in [42], we define $(\hat{u}_i, \hat{v}_i) = (\hat{u}_i - u_i + u_i, \hat{v}_i - v_i + v_i) = (-\tilde{u}_i + u_i, -\tilde{v}_i + v_i)$. By linearity of the integral operators, we obtain

$$\hat{V}(t) = V(t) + \tilde{V}(t), \quad (115)$$

where $\tilde{V}(t) = -\frac{1}{q_{11}}\tilde{V}_S(t) + \rho_{11}\tilde{u}_1(t,1)$, is the difference between the output feedback law and the previously designed state feedback law. By Lemma 4, and since the control integral operator is bounded, we have $|\tilde{V}_S(t)| \underset{t\to\infty}{\longrightarrow} 0$ and $\|\tilde{u}_1(t)\|_{L^2} \underset{t\to\infty}{\longrightarrow} 0$ as the error states converge to zero. Thus, the term $\tilde{V}(t)$ can be seen as a disturbance that converges to zero. Using Theorem 4, and the input-to-state stability of the system (as it is done in [42], [43] for two equations), we can conclude to the exponential stability of the system. Indeed, the closed-loop system would rewrite as a neutral system subject to a disturbance that goes to zero [44]. Applying the variations of constants formula yields the expected result. □

VI. Applications and extensions

Although applied to a specific case of a chain of two interconnected hyperbolic PDE subsystems, the approach proposed in this paper can be extended to other classes of systems.

A. General classes of integral delay equations

Besides time-delay systems, the approach given in this article can be used to stabilize systems represented by a general class of Integral Delay Equations (IDE), as done in [17]. This is the consequence of the strong links between hyperbolic systems and time-delay systems of neutral type.

B. Underactuated 1+2 linear hyperbolic system

Another class of systems that can be stabilized with this approach are underactuated 1+2 hyperbolic systems, where only one of the two leftward-convecting equations is actuated. Such system was stabilized in [16], under a more restrictive assumption, since the authors assumed exponentially stable actuation dynamics, which is not the case here. This application case corresponds to a state $w(t,x) = (u(t,x), v_1(t,x), v_2(t,x))^T$ satisfying

$$\partial_t w(t,x) + \Lambda \partial_x w(t,x) = \Sigma(x) w(t,x), \quad (116)$$

where the different arguments evolve in $\{(t,x) \text{ s.t. } t > 0, x \in [0,1]\}$, and with the following boundary conditions

$$u(t,0) = q_1 v_1(t,0) + q_2 v_2(t,0), \quad (117)$$

$$v_1(t,1) = \rho_1 u(t,1) + V(t), \ v_2(t,1) = \rho_2 u(t,1). \quad (118)$$

The diagonal matrix Λ is given by $\Lambda = \mathrm{diag}(\lambda, -\mu_1, -\mu_2)$, where the different velocities λ, μ_1, μ_2 are assumed to be constant and positive, as the boundary couplings q_1, q_2, ρ_1 and ρ_2. The components of the matrix Σ are continuous functions. Thus the proposed methodology is of high interest for the stabilization of underactuated hyperbolic systems.

VII. Simulation results

In this section, we give some simulation results to illustrate the relevance of the stabilizing output-feedback control law proposed in this paper. The control strategy was implemented using Matlab. We simulated our system on a time scale of 30s, with 101 space-discretization points in $[0,1]$. The numerical values of the parameters are $\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = \begin{pmatrix}2\\ 1\end{pmatrix}$, $\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix} = \begin{pmatrix}1.3\\ 1.8\end{pmatrix}$, $\begin{pmatrix}\sigma_1^+\\ \sigma_2^+\end{pmatrix} = \begin{pmatrix}-0.2\\ -0.3\end{pmatrix}$, $\begin{pmatrix}\sigma_1^-\\ \sigma_2^-\end{pmatrix} = \begin{pmatrix}0.2\\ 0.3\end{pmatrix}$, $\begin{pmatrix}q_{11} & *\\ q_{21} & q_{22}\end{pmatrix} = \begin{pmatrix}0.9 & *\\ 1 & 0.4\end{pmatrix}$, $\begin{pmatrix}\rho_{11} & \rho_{12}\\ * & \rho_{22}\end{pmatrix} = \begin{pmatrix}0.3 & 0.8\\ * & 0.9\end{pmatrix}$. The initial conditions of the states are constant functions $u_i(0,.) = 0.1,\ v_i(0,.) = 0.2$. The observer values are initialized to 0. Assumptions 1-3 are obviously satisfied. Assumptions 5 and 7 have been verified numerically.

Beforehand, the kernel of the invertible Volterra transforms $\mathcal{L}_i, \mathcal{M}_i$ and Fredholm transforms $\mathcal{N}, \mathcal{K}$ (and their inverse) are computed using the successive approximation technique [45]. Their values are stored in matrices whose dimension is directly defined by the number of discretization points (here 101). As illustrated on Figure 3, the computation time becomes very important when the space step gets smaller.

ϵ	Iter. \ p	20	50	100
10^{-5}	11	2.01	27.9	247.3
10^{-8}	17	3.39	44.1	362.1
10^{-10}	21	3.91	61.9	462.3

Fig. 3. Evolution of computation time (in sec) for kernels N^{ij}, for different precision (ϵ) and space step (1/p) of the domain.

Then, the functions H_i^*, F_i^*, K_1^*, G_i are computed using the same method. The integral terms are approximated using a trapezoidal method. All the values are computed off-line and do not need to be updated while running the closed-loop simulations. If needed, functions are interpolated using the linear method `interp1`.

Next, we can simulate the evolution of the system using the classical finite volume method based on a Godunov scheme [46]. As illustrated on Figure 4 (blue curve), the parameters are chosen such that the whole interconnected system remains unstable in open-loop. In presence of the control law (114) represented on Figure 5, the system (u,v) becomes exponentially stable. Indeed, as illustrated in Figure 4 (red), its L^2-norm converges to zero.

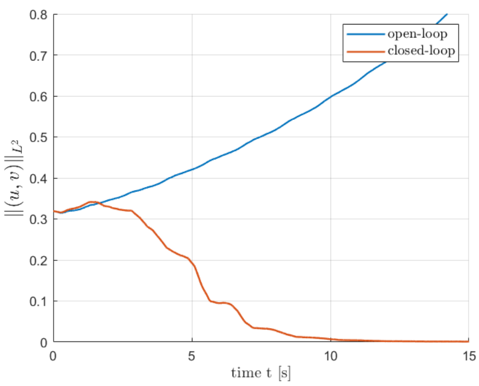


Fig. 4. Evolution of the L^2-norm of the state in closed-loop and open-loop.

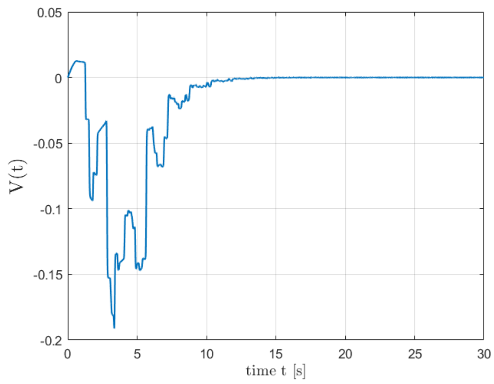


Fig. 5. Evolution of the control effort V(t).

VIII. Conclusion

In this paper, we proposed a new approach to stabilize a chain of two interconnected hyperbolic PDE subsystems for which the actuator and the measurement are located at the in-between boundary. In the proposed methodology, we first designed a full-state feedback controller, by using classical Volterra transforms and a change of variables to rewrite the chain as a scalar hyperbolic system. We then used the backstepping approach to map this PDE system to a simple (exponentially stable) target system. However, the configuration considered in the paper required a Fredholm transform, which is not always invertible. We proved the invertibility of the Fredholm transform using an operator framework inspired by [17], [20]. The well-posedness of the kernels defining the Fredholm transform was proved using the same ideas. Second, we used a similar approach to design a state observer. This led to an output-feedback controller, whose performances have been illustrated by some numerical simulations. The proposed approach paves the way for future contributions on networks with actuation inside the graph structure. We believe that this approach could be combined with [14] to tackle a wider diversity of physical systems with an arbitrary number of PDEs or ODEs. It is also a milestone towards the stabilization of under-actuated systems. In the coming period, we wish to extend our results to non-scalar systems or to more complex networks. We will also consider more complex cases where some of the boundary couplings are equal to zero resulting in degenerate Fredholm equations.

Acknowledgment

The authors would like to thank the Associate Editor and Reviewers for their relevant remarks and comments. They contributed greatly to the overall quality of this paper.

Appendix A
Well-posedness of kernel equations of a Fredholm Transform

A. *Proof of Lemma 3*

Let us show that $N^{ij}(x,y)$, $(i,j) \in \{1,2\}^2$ can be expressed on $\mathcal{S}$ as functions of $N^{12}(0,\cdot)$ and $N^{21}(0,\cdot)$. Applying the method of characteristics on the transport equations (46)-(49), we can express N^{ij} on $\mathcal{S}$ as functions of their boundary values.

First, for kernels N^{11}, N^{22}, the slope of the characteristics

This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and
content may change prior to final publication. Citation information: DOI 10.1109/TAC.2022.3184405

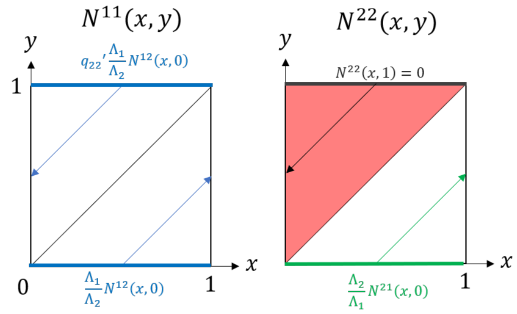


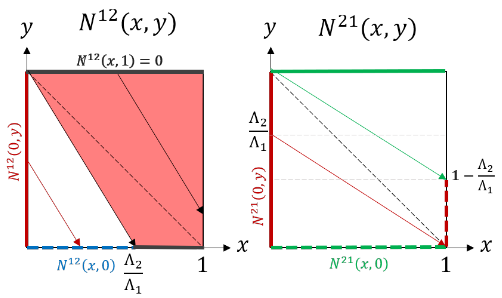
Fig. 6. Representation of the kernels N^{11}, N^{22}

does not depend on Λ_i, as illustrated on Figure 6. The kernels are entirely defined by their boundary values in $y = 1$ and $y = 0$. Using the boundary conditions (50)-(54), direct computations give

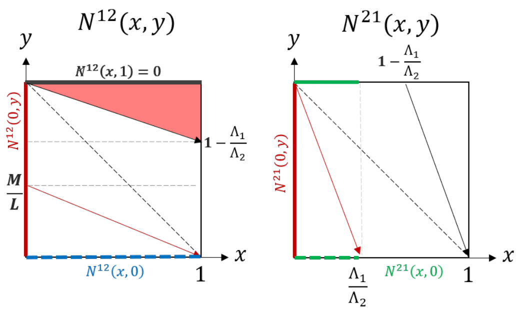
$$N^{11}(x,y) = \frac{\Lambda_1}{\Lambda_2}(\mathbb{1}_{[0,y]}(x)q'_{22}N^{12}(x-y+1,0) + \mathbb{1}_{[y,1]}(x)N^{12}(x-y,0)), \quad (A.1)$$

$$N^{22}(x,y) = \frac{\Lambda_2}{\Lambda_1}\mathbb{1}_{[y,1]}(x)N^{21}(x-y,0). \quad (A.2)$$

To express the two other kernels as functions of their boundary terms, we need to have a closer look to their characteristics, whose slope depends on the ratio $\frac{\Lambda_1}{\Lambda_2}$. The case $\Lambda_2 = \Lambda_1$ is the easiest to handle, since the characteristic lines are parallel to the antidiagonal of $\mathcal{S}$.
In the other cases, the characteristic lines for kernels N^{21}, N^{12} do not divide $\mathcal{S}$ into two equal triangular domains, as illustrated on Figure 7.



(a) $\Lambda_2 < \Lambda_1$



(b) $\Lambda_2 > \Lambda_1$

Fig. 7. Representation of kernels N^{21}, N^{12}

In particular, in the case $\Lambda_2 > \Lambda_1$ the boundary condition $N^{12}(x,1) = 0$ defines the values of $N^{12}(1,y)$ for the triangular domain $x \in [0,1],\ y \in [1-\frac{\Lambda_2}{\Lambda_1}x, 1]$ only, and the boundary condition $N^{21}(0,y)$ directly defines the kernels' values for the triangular domain $x \in [0, \frac{\Lambda_1}{\Lambda_2}], y \in [0, 1-\frac{\Lambda_2}{\Lambda_1}x]$ only. One can note that the boundary condition $N^{12}(x,1) = 0, \quad \forall x \in [0,1]$ propagates along the characteristic lines, such that N^{12} is equal to 0 on the right upper part of $\mathcal{S}$, as illustrated by the red domains on Figure 7.
To determine the values on $\mathcal{S}$ in that case, we use an iterative procedure. Let us define p as the unique integer verifying $p\frac{\Lambda_1}{\Lambda_2} \leq 1 < (p+1)\frac{\Lambda_1}{\Lambda_2}$. We can divide the square $\mathcal{S}$ into different sub-domains $D_k, k \in [\![0, p+1]\!]$, as illustrated on Figure 8.

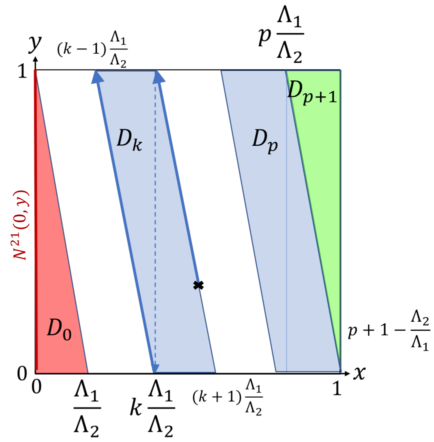


Fig. 8. Different domains in the expression of N^{21}

More precisely, we have:

- $D_0 = \{0 \leq y \leq 1, 0 \leq x \leq \frac{\Lambda_1}{\Lambda_2}(1-y)\}$,
- $\forall k \in [\![1, p-1]\!], D_k = \{0 \leq y \leq 1, \frac{\Lambda_1}{\Lambda_2}(k-y) \leq x < \frac{\Lambda_1}{\Lambda_2}(k+1-y)\}$,
- $D_p = \{0 \leq y \leq 1, \frac{\Lambda_1}{\Lambda_2}(p-y) \leq x < \min(1, 1-\frac{\Lambda_1}{\Lambda_2}y)\}$,
- $D_{p+1} = \{p+1-\frac{\Lambda_2}{\Lambda_1}x \leq y \leq 1, p\frac{\Lambda_1}{\Lambda_2} \leq x \leq 1\}$.

Note that when $\Lambda_2 < \Lambda_1$ we have $p = 0$. Integrating along the characteristic lines, and using (51), we obtain by iteration $\forall k \in [\![0, p+1]\!]$,

$$\forall (x,y) \in D_{k,k\leq p+1},\ N^{21}(x,y) = q'^{k}_{22}N^{21}(0, y-k+\frac{\Lambda_2}{\Lambda_1}x).$$

In the same way, we can express kernel N^{12} as a function of $N^{12}(0,y)$. We have, for all $(x,y) \in \mathcal{S}$

$$N^{12}(x,y) = \mathbb{1}_{[0,\frac{\Lambda_2}{\Lambda_1}(1-y)]}(x)N^{12}(0, y+\frac{\Lambda_1}{\Lambda_2}x). \quad (A.3)$$

This concludes the proof.

B. Integral formulation

In this subsection, we rewrite $N^{21}(0,y)$ and $N^{12}(0,y)$ as the solutions of two integral equations of the form (55) and give the explicit expression of the terms $\mathcal{I}_{ij}$. Notice first that using the transport equation (46) in (50), we obtain $N^{11}(0,y) - q'_{22}N^{11}(1,y) = 0$ which simplifies (52). Then, we have $N^{22}(0,y)=0, \quad \forall y \in [0,1]$ and $N^{12}(1,y)=0, \quad \forall y \in [\max(0,1-\frac{\Lambda_1}{\Lambda_2}),1]$. We therefore have

$$N_w(y) - \int_0^1 N_w(\nu)N^{11}(\nu,y) + N_z(\nu)N^{21}(\nu,y)\mathrm{d}\nu = N^{21}(0,y),$$
$$N_z(y) - \int_0^1 N_w(\nu)N^{12}(\nu,y) + N_z(\nu)N^{22}(\nu,y)\mathrm{d}\nu = -N^{12}(0,y) + \mathbb{1}_{[0,1-\frac{\Lambda_1}{\Lambda_2}]}(y)q'_{22}N^{12}(0,y+\frac{\Lambda_1}{\Lambda_2}).$$

We decompose the integral terms into subdomains (depending on p) to express the kernels N^{ij} as functions of the boundary values $N^{12}(0,\cdot)$ and $N^{21}(0,\cdot)$. We obtain

$$N_w(y) = N^{21}(0,y) - \int_0^1 I_{11}(\nu,y)(-N^{12}(0,\nu)) - I_{12}(\nu,y)N^{21}(0,\nu)\mathrm{d}\nu, \tag{A.4}$$
$$N_z(y) = -(N^{12}(0,y) - \mathbb{1}_{[0,1-\frac{\Lambda_1}{\Lambda_2}]}(y)q'_{22}N^{12}(0,y+\frac{\Lambda_1}{\Lambda_2})) - \int_0^1 I_{21}(\nu,y)(-N^{12}(0,\nu)) - I_{22}(\nu,y)N^{21}(0,\nu)\mathrm{d}\nu, \tag{A.5}$$

where

$$I_{11}(\nu,y) = \mathbb{1}_{[0,1]}(\nu)(\mathbb{1}_{[0,\frac{\Lambda_1}{\Lambda_2}(1-y)]}(\nu)N_w(y+\frac{\Lambda_2}{\Lambda_1}\nu) + \mathbb{1}_{[\frac{\Lambda_1}{\Lambda_2}(1-y),\frac{\Lambda_1}{\Lambda_2}]}(\nu)q'_{22}N_w(y-1+\frac{\Lambda_2}{\Lambda_1}\nu)) \tag{A.6}$$
$$I_{12}(\nu,y) = \frac{\Lambda_1}{\Lambda_2}[\mathbb{1}_{[y,1]}(\nu)N_z(\frac{\Lambda_1}{\Lambda_2}(\nu-y)) + \sum_{k=1}^{p}\mathbb{1}_{[0,\frac{\Lambda_2}{\Lambda_1}-k+y]}(\nu)q'^k_{22}N_z(\frac{\Lambda_1}{\Lambda_2}(\nu-y+k)) + \mathbb{1}_{[p+1+\frac{\Lambda_1}{\Lambda_2},1]}(y)\mathbb{1}_{[0,\frac{\Lambda_2}{\Lambda_1}-(p+1)+y]}(\nu)q'^{p+1}_{22}N_z(\tfrac{\Lambda_1}{\Lambda_2}(\nu-y+p+1))], \tag{A.7}$$
$$I_{21}(\nu,y) = \mathbb{1}_{[y,y+\frac{\Lambda_1}{\Lambda_2}]}(\nu)\mathbb{1}_{[0,1]}(\nu)\frac{\Lambda_2}{\Lambda_1}N_w(\frac{\Lambda_2}{\Lambda_1}(\nu-y)), \tag{A.8}$$
$$I_{22}(\nu,y) = \sum_{k=0}^{p}\mathbb{1}_{[0,\frac{\Lambda_2}{\Lambda_1}(1-y)-k]}(\nu)q'^k_{22}N_z(y+\frac{\Lambda_1}{\Lambda_2}(\nu+k)). \tag{A.9}$$

The computations to obtain the terms I_{ij} rely on Fubini's theorem. To rewrite the integral equations (A.4)-(A.5) using an integral operator of the form (34), we need to get rid of the term $\mathbb{1}_{[0,1-\frac{\Lambda_1}{\Lambda_2}]}(y)q'_{22}N^{12}(0,y+\frac{\Lambda_1}{\Lambda_2})$ in (55). Let f be a bounded function, and define the function $\bar{f}$, such that for all $y \in [0,1]$ we have

$$\bar{f}(y) = f(y) - \mathbb{1}_{[0,1-\frac{\Lambda_1}{\Lambda_2}]}(y)q'_{22}f(y+\frac{\Lambda_1}{\Lambda_2}). \tag{A.10}$$

This yields the following lemma:

Lemma 5: The operator $\bar{\cdot}$ defined by (A.10) is invertible. More precisely, we have

$$f(y) = \sum_{k=0}^{p}q'^k_{22}\mathbb{1}_{[0,1-k\frac{\Lambda_1}{\Lambda_2}]}(y)\bar{f}(y+k\frac{\Lambda_1}{\Lambda_2}). \tag{A.11}$$

Proof 11: Formula (A.11) is obtained by an iterative approach. Let us take $y \in [0,1]$, and assume that $\Lambda_2 > \Lambda_1$ (else, the change of variables is equal to the identity and the proof is straightforward). We have

$$\begin{cases} \bar{f}(y) = f(y), \quad \text{if } 1-\frac{\Lambda_1}{\Lambda_2} < y \leq 1, \\ \bar{f}(y) = f(y) - q'_{22}f(\underbrace{y+\frac{\Lambda_1}{\Lambda_2}}_{\geq \frac{\Lambda_1}{\Lambda_2}}), \quad \text{if } 0 \leq y \leq 1-\frac{\Lambda_1}{\Lambda_2}. \end{cases}$$

Then, if $1-\frac{\Lambda_1}{\Lambda_2} \leq \frac{\Lambda_1}{\Lambda_2} \iff \frac{\Lambda_2}{\Lambda_1} < 2 \iff p=1$, we directly have $f(y) = \bar{f}(y) + q'_{22}\bar{f}(y+\frac{\Lambda_1}{\Lambda_2})$. Else, we need to iterate $p-1$ more times the operation, which successively add the terms $q'^k_{22}\mathbb{1}_{[0,1-k\frac{\Lambda_1}{\Lambda_2}]}(y)\bar{f}(y+k\frac{\Lambda_1}{\Lambda_2})$. We finally obtain (A.11). □

Defining, $\bar{N}^{12}(y) = N^{12}(0,y) - \mathbb{1}_{[0,1-\frac{\Lambda_1}{\Lambda_2}]}(y)q'_{22}N^{12}(0,y+\frac{\Lambda_1}{\Lambda_2})$, we can rewrite (A.4)-(A.5) as

$$N_w(y) = N^{21}(0,y) - \int_0^1 \bar{I}_{11}(\nu,y)(-\bar{N}^{12}(\nu)) - I_{12}(\nu,y)N^{21}(0,\nu)\mathrm{d}\nu, \tag{A.12}$$
$$N_z(y) = -\bar{N}^{12}(y) - \int_0^1 \bar{I}_{21}(\nu,y)(-\bar{N}^{12}(\nu)) - I_{22}(\nu,y)N^{21}(0,\nu)\mathrm{d}\nu. \tag{A.13}$$

Using the expression (A.11) in the integral terms, we can define the new coupling terms $\bar{I}_{j1}, \ j \in \{1,2\}$ by

$$\bar{I}_{j1}(\nu,y) = \sum_{k=0}^{p}q'^k_{22}\mathbb{1}_{[\frac{\Lambda_1}{\Lambda_2}k,1]}(\nu)I_{j1}(\nu-\frac{\Lambda_1}{\Lambda_2}k,y). \tag{A.14}$$

Remark 2: Note that in the case $\Lambda_2 \leq \Lambda_1$, the change of variables (A.11) is the identity.

We can finally define on $\mathcal{S}$ four bounded functions $\mathcal{I}_{11},\mathcal{I}_{21},\mathcal{I}_{12},\mathcal{I}_{22}$ introduced in (56) by $\mathcal{I}_{j2} = I_{j2}$ and $\mathcal{I}_{j1} = \bar{I}_{j1}$ $j \in \{1,2\}$ (A.7)-(A.9),(A.14).

C. Proof of Theorem 2

In this section, we prove the invertibility of the Fredholm integral operator $\mathcal{Q}$ of the form (34) defined in (56). Similarly to the proof 8, it relies on Lemma 1. Indeed, the four functions $\mathcal{I}_{ij}$ are bounded, such that the integral part of $\mathcal{Q}$ is a compact operator. By [24, Theorem 6.6] (Fredholm alternative), we have $\dim\ker(\mathcal{Q}) < \infty$. Let us show that conditions $(a)-(d)$ are verified.

First, conditions $(a),(b)$ are proved by evaluating the components of the kernel $R(x,y)$ in $y=0$ and $y=1$. We obtain

$$\mathcal{I}_{11}(0,\nu) = \sum_{k=0}^{p}q'^k_{22}\mathbb{1}_{[k\frac{\Lambda_1}{\Lambda_2},1]}(\nu)N_w(\frac{\Lambda_2}{\Lambda_1}\nu-k), \tag{A.15}$$
$$\mathcal{I}_{12}(0,\nu) = \frac{\Lambda_1}{\Lambda_2}\sum_{k=0}^{p}q'^k_{22}\mathbb{1}_{[0,\frac{\Lambda_2}{\Lambda_1}-k]}(\nu)N_z(\frac{\Lambda_1}{\Lambda_2}(\nu+k)), \tag{A.16}$$
$$\mathcal{I}_{11}(1,\nu) = q'_{22}\mathcal{I}_{11}(0,\nu), \quad \mathcal{I}_{12}(1,\nu) = q'_{22}\mathcal{I}_{12}(0,\nu), \tag{A.17}$$
$$\mathcal{I}_{21}(0,\nu) = \frac{\Lambda_2}{\Lambda_1}\mathcal{I}_{11}(0,\nu), \quad \mathcal{I}_{22}(0,\nu) = \frac{\Lambda_2}{\Lambda_1}I_{12}(0,\nu). \tag{A.18}$$

Let us take $z = \begin{pmatrix} f \\ g \end{pmatrix} \in \ker(\mathcal{Q})$, s.t for all $x \in [0,1]$, we have

$$\begin{pmatrix} f(x) \\ g(x) \end{pmatrix} = \begin{pmatrix} \int_0^1 -\mathcal{I}_{12}(\nu,x)f(\nu) + \mathcal{I}_{11}(\nu,x)g(\nu)\mathrm{d}\nu \\ \int_0^1 -\mathcal{I}_{22}(\nu,x)f(\nu) + \mathcal{I}_{21}(\nu,x)g(\nu)\mathrm{d}\nu \end{pmatrix}.$$

Due to the regularizing property of the integral, we have $\ker(\mathcal{Q}) \subset H^1([0,1],\mathbb{R}^2)$. The boundary condition (A.17) gives $f(1) = q'_{22}f(0)$, and (A.18) give $f(0) = \frac{\Lambda_1}{\Lambda_2}g(0)$, such that $z \in D(A^*)$.

Next, we evaluate the coupling terms $\mathcal{I}_{21},\mathcal{I}_{22}$ in $y=1$. We obtain $\mathcal{I}_{21}(1,\nu) = \mathcal{I}_{22}(1,\nu) = 0$. We then have $\Lambda_1 g(1) = 0$, such that $z \in \ker(B^*)$.

We now need to prove that $\ker(\mathcal{Q})$ is stable by A^* (condition (c)), *i.e* $\forall z \in \ker(\mathcal{Q}), \mathcal{Q}A^*z = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. We have

$$A^*z = \begin{pmatrix} \Lambda_2 f'(y) + \Lambda_2 f(0) N_w(y) \\ -\Lambda_1 g'(y) + \Lambda_2 f(0) N_z(y) \end{pmatrix}.$$

We compute the derivative of functions $(f, g) \in \ker(\mathcal{Q})$ on one side, and we integrate by parts in the integral terms on the other side. Some computations are given below. On the first component of $\mathcal{Q}A^*z$, we need to show that

$$\Lambda_2 f'(y) + \Lambda_2 f(0) N_w(y) + \int_0^1 \mathcal{I}_{12}(\nu, y)(\Lambda_2 f'(\nu) + \Lambda_2 f(0) N_w(\nu))\mathrm{d}\nu \\ - \int_0^1 \mathcal{I}_{11}(\nu, y)(-\Lambda_1 g'(\nu) + \Lambda_2 f(0) N_z(\nu))\mathrm{d}\nu = 0. \quad \text{(A.19)}$$

Let us check that the terms in $f(0)$ are compensated, that is to say,

$$f(0) \int_0^1 \mathcal{I}_{12}(y, \nu)\Lambda_2 N_u(\nu) - \mathcal{I}_{11}(y, \nu)\Lambda_2 N_v(\nu))\mathrm{d}\nu = 0.$$

Due to the presence of characteristic functions, we obtain two sums of integral terms in $N_w(\cdot) \times N_z(\cdot)$. By a change of variables in the second term, we get the equality.
Next, we compute separately $\Lambda_1 \int_0^1 \mathcal{I}_{11}(y, \nu) g'(\nu)\mathrm{d}\nu$ and $\Lambda_2 \int_0^1 \mathcal{I}_{12}(y, \nu) f'(\nu)\mathrm{d}\nu$. Once again, we decompose the integral on different subdomains to get rid of the characteristic function. We integrate by parts and use the fact that $f(1) = q'_{22} f(0)$, $g(1) = 0$, $f(0) = \frac{\Lambda_2}{\Lambda_1} g(0)$ to simplify some terms. Finally, we compute the derivative of f. We have $f(y) = \int_0^1 \mathcal{I}_{11}(y, \nu) g(\nu) - \mathcal{I}_{12}(y, \nu) g(\nu)\mathrm{d}\nu$, by definition of $z \in \ker(\mathcal{Q})$. We then verify that all the terms are compensated using several changes of variables in the integral terms and Fubini's theorem.
In a second time, we follow the same steps to show that the second component of $\mathcal{Q}A^*z$ vanishes, that is

$$-\Lambda_1 g'(y) + \Lambda_2 f(0) N_z(y) + \int_0^1 \mathcal{I}_{22}(\nu, y)(\Lambda_2 f'(\nu) + \Lambda_2 f(0) N_w(\nu))\mathrm{d}\nu \\ - \int_0^1 \mathcal{I}_{21}(\nu, y)(-\Lambda_1 g'(\nu) + \Lambda_2 f(0) N_z(\nu))\mathrm{d}\nu = 0.$$

Once again, we show that

$$f(0) \int_0^1 \mathcal{I}_{22}(y, \nu)\Lambda_2 N_u(\nu) - \mathcal{I}_{21}(y, \nu)\Lambda_2 N_v(\nu))\mathrm{d}\nu = 0$$

using a change of variables $(\eta = \frac{\Lambda_2}{\Lambda_1}(\nu - y) - k)$.
Next, we compute separately the other integral terms and use integration by parts. The integral term $\int_0^1 \mathcal{I}_{22}(y, \nu)\Lambda_2 f'(\nu)\mathrm{d}\nu$ rewrites

$$\sum_{k=0}^{p} \int_0^{\frac{\Lambda_2}{\Lambda_1}(1-y)-k} \mathbb{1}_{[0,1]}(\nu) q_{22}'^{k} N_z\left(\frac{\Lambda_1}{\Lambda_2}(\nu + k) + y\right)\Lambda_2 f'(\nu)\mathrm{d}\nu.$$

We get rid of the characteristic function by decomposing into different integration domains, as illustrated on Figure 9 a). Let us define the decreasing sequence $y_k = 1 - \frac{\Lambda_1}{\Lambda_2}(k+1)$, $k \in [\![0, p+1]\!]$. We decompose the integral term according to the value of y relative to y_k. We factorize all terms in $f(0)$ resulting from the integration by parts to obtain $-\Lambda_2 f(0) N_z(y)$.
Integral term $\int_0^1 \mathcal{I}_{21}(y, \nu)\Lambda_1 g'(\nu)\mathrm{d}\nu$ rewrites

$$-\Lambda_2 \sum_{k=0}^{p} q_{22}^{k} \int_{y+k\frac{\Lambda_1}{\Lambda_2}}^{y+(k+1)\frac{\Lambda_1}{\Lambda_2}} \mathbb{1}_{[0,1]}(\nu) N_w\left(\frac{\Lambda_2}{\Lambda_1}(\nu - y) - k\right) g'(\nu)\mathrm{d}\nu.$$

Following the same procedure, we decompose the integration domain as illustrated on Figure 9 b). Finally, we compute

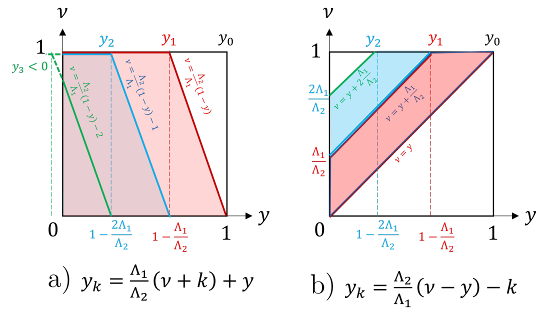


Fig. 9. Representation of the integration domain for $2 < \frac{\Lambda_2}{\Lambda_1} < 3, p = 2$

the derivative of g using condition (b) and the expression of $\ker(\mathcal{Q})$. It proves that $\forall z \in \ker(\mathcal{Q}),\ \mathcal{Q}A^*z = 0$. The condition (d) is given by Lemma 2 and derives from spectral controllability of the system (Assumption 5). Using the arguments given in the proof of Lemma 1, we obtain that $\mathcal{Q}$ is invertible. This concludes the proof.

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