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Abstract: In this paper, we propose a full state-feedback boundary control strategy for a one-dimensional wave-like equation with spatially varying parameters and indefinite damping coefficient. We consider Dirichlet boundary conditions at one end of the spatial domain and actuation at the other end. The control design relies on the backstepping methodology and aims at assigning the distributed damping (which determines the decay rate of the solutions) of the closed-loop system. The problem is formulated using the port-Hamiltonian system framework that allows the introduction of tuning parameters with clear physical interpretations for both backstepping transformations and achievable closed-loop behavior. The overall design is carried out on the vibrating string system example. Simulations illustrate the performance of the controller.

Keywords: infinite dimensional systems; Port-Hamiltonian Systems; backstepping methodology.

1. INTRODUCTION

The interaction of physical systems with their environment can be seen as a power flow through interaction ports. From this observation, (Maschke and van der Schaft, 1992) introduced three decades ago the notion of *Port-Hamiltonian* systems (PHS), extending the Hamiltonian formulation to open physical systems. This formalism has then been extended to distributed parameter systems in (van der Schaft and Maschke, 2002), and has been since widely used to model and control systems described by Partial Differential Equations (PDEs) (Le Gorrec et al., 2004, 2005; Villegas, 2007; Jacob, 2012).

Since this framework takes into account physical properties (dissipation, interconnections), it can be used to design boundary controllers efficiently exploiting the physical properties of the system. For instance, Passivity-based control (PBC) design, initially proposed for finite dimensional non linear systems (Ortega et al., 2008), consists in using structural invariants and damping injection to modify the closed-loop properties of the system. This control design technique has been extended to boundary controlled PHS in (Macchelli et al., 2017). In this case it has been shown that the closed-loop properties can only be partially modified using the available information at the boundary.

In this paper, we consider a one-dimensional wave-like equation with varying parameters and possible indefinite (negative, positive or null) damping characterizing the motion of a string. This system is clamped at one end and actuated at the other end. Depending on the damping values, this system can be either exponentially stable or unstable in open-loop (Cox and Zuazua, 1994; Freitas and Zuazua, 1996). The controller is generally used to damp out the oscillations or to improve the closed-loop performance. Our objective is to design a dynamic boundary feedback such that the closed loop system is equivalent to a target system known to have satisfying stability properties.

The backstepping methodology has been proved to be an efficient method for stabilizing hyperbolic systems (Krstic et al., 2008, 2006). The main idea is to map the original system into an exponentially stable target system using a boundedly invertible integral transform. However, the adequate target system is obtained on a case by case basis, sometimes facing difficulties. When possible, the simplest exponentially stable system is chosen, without paying attention to the intrinsic properties of the original system.

Since the PHS framework naturally emphasizes the physical properties of the system we aim at showing in this paper that it can be used against this main drawback of the backstepping approach. Specifically for linear hyperbolic PDEs, the Port-Hamiltonian formalism has been efficiently used to prove the existence of solutions or determine adequate boundary conditions guaranteeing the well-posedness of the associated boundary control system (Zwart et al., 2010). It also helps inferring observability or controllability properties (Jacob, 2012). Therefore, it could be of great interest to investigate further on developing state feedback boundary controllers using the backstepping approach and taking advantages of this formalism. In this respect in (Ramirez et al., 2017), the authors considered a general class of linear PHS, and used a coordinate transformation to derive an invertible mapping with a target system containing in-domain dissipative terms and homogeneous boundary conditions. However, in the case of an initial system which already contains a dissipative term, and with spatially varying parameters, a similar multiplicative operator cannot be used.

In this paper, we propose an innovative approach taking advantage of the backstepping methodology and the Port-Hamiltonian framework for a simple *toy-system*. We choose an exponentially stable target system based on energy considerations. We propose a new form of integral transform to map the initial system to this target system. We prove that such a transformation exists and is invertible and bounded to guarantee that the resulting control law imposes the desired dynamics to the initial plant.

The organization of the paper is as follows. First, in Section 2, we present the system under consideration and the control objective. Then, in Section 3, we present the step-by-step approach and the backstepping transform used. Finally, the performance of the controller is illustrated in Section 4. Some concluding remarks and perspectives end this paper (Section 5).

Notations

We denote $C^1([0,1])$ the space of real differentiable functions defined on $[0,1]$ with a continuous derivative. We denote $C^1([0,1])^+ \subset C^1([0,1])$, the subset of $C^1([0,1])$ that contains positive functions (i.e. functions that are in $C^1([0,1])$ and that are positive for all $x \in [0,1]$). Let $\mathcal{H} \in C^1([0,1];\mathbb{R}^{2\times 2})^+$, a positive diagonal matrix. We denote $\chi \doteq L^2([0,1];\mathbb{R}^2)$ the Hilbert-space equipped with the inner-product $< u,v >_\chi = \frac{1}{2}\int_0^1 u(x)^T \mathcal{H}(x) v(x) \mathrm{d}x$, and we denote $\|u\|_\chi$ the associated norm (which is equivalent to the standard L^2-norm). We denote the lower triangular part of the unit square as $\mathcal{T}^- = \{(x,y) \in [0,1]^2 | \ 0 \leq y \leq x\}$. For any space B, Id_B corresponds to the identity operator for the space. When there is no ambiguity, the subscript may be omitted. Similarly, the time and/or space dependency may be omitted.

2. SYSTEM UNDER CONSIDERATION

2.1 Vibrating string model

Let us consider a vibrating string clamped at one end and actuated at the other. Such a system is schematically pictured in Figure 2.1. We denote $w(x,t)$ the vertical position of the string at point x and time $t > 0$. The spatial domain is normalized so that $x \in [0,1]$, the left end (that is clamped) corresponds to $x = 0$, while the actuator is located at $x = 1$.

For all $x \in [0,1]$ and all $t > 0$, the state $w(x,t)$ satisfies

$$\rho(x)\frac{\partial^2 w}{\partial t^2}(x,t) = \frac{\partial}{\partial x}\left(E(x)\frac{\partial w}{\partial x}(x,t)\right) - \kappa(x)\frac{\partial w}{\partial t}(x,t), \tag{1}$$

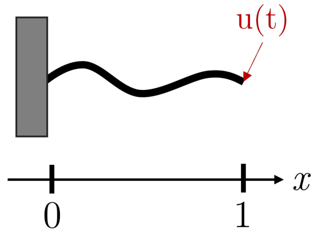

Fig. 1. Schematic representation of the system

with $\rho(x), E(x) \in C^1([0,1])^+$ being the mass density and the Young's modulus. They may vary along the string. Even if in the considered case $\kappa(x) \in C^0([0,1])$ is positive, the results of this paper apply to the more general case where $\kappa(x)$ can be positive, negative (anti-damping) or equal to zero. At the fixed end of the string, no movement is allowed such that $\frac{\partial w}{\partial t}|_{x=0}(t) = 0$. We exert a force $u(t)$ (control input) on the opposite end, *i.e.* $E(1)\frac{\partial w}{\partial x}|_{x=1}(t) = u(t)$. Initially, the position of the string is given by $w(x,0) = w_0(x) \in C^1([0,1])$. The energy of this system is defined by

$$\mathcal{E}(t) = \frac{1}{2}\int_0^1 \left(\rho(x)\left(\frac{\partial w}{\partial t}(x,t)\right)^2 + E(x)\left(\frac{\partial w}{\partial x}(x,t)\right)^2\right)\mathrm{d}x. \tag{2}$$

Using the dynamics of the string (1) and the specific boundary conditions, we can integrate by parts, and obtain the derivative of the energy

$$\frac{d\mathcal{E}}{dt}(t) = \frac{\partial w}{\partial t}\Big|_{x=1} u(t) - \int_0^1 \left(\kappa(x)\left(\frac{\partial w}{\partial t}\right)^2\right)\mathrm{d}x. \tag{3}$$

Thus, the boundary control can be used to guarantee that the energy of the system is strictly decreasing.

2.2 Port-Hamiltonian formulation

To rewrite the model as a Port-Hamiltonian system, we introduce the *Hamiltonian density* matrix $\mathcal{H}(x) = \mathrm{diag}(E(x), \frac{1}{\rho(x)})$ and the following state variables $X = (X_1, X_2)$ with

$$\begin{cases} X_1(x,t) = \dfrac{\partial w}{\partial x}(x,t) & : \text{strain.} \\ X_2(x,t) = \rho(x)\dfrac{\partial w}{\partial t}(x,t) & : \text{momentum density.} \end{cases} \tag{4}$$

In this formalism, the dynamics of the damped string (1) rewrite

$$\frac{\partial}{\partial t}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{\partial}{\partial x}\left(\dfrac{1}{\rho(x)}\cdot\right) \\ \dfrac{\partial}{\partial x}(E(x)\cdot) & -c(x) \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \tag{5}$$

where $c(x) = \frac{\kappa(x)}{\rho(x)}$. Following the approach proposed in (Jacob, 2012), Equation (5) rewrites

$$\frac{\partial X}{\partial t}(x,t) = P_1\frac{\partial}{\partial x}(\mathcal{H}(x)X(x,t)) + G_0(\mathcal{H}(x)X(x,t)),$$

with $P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $G_0 = \begin{pmatrix} 0 & 0 \\ 0 & -c(x)\rho(x) \end{pmatrix}$. The *boundary effort* e_∂ and *boundary flow* f_∂ are given by

$$\begin{cases} e_\partial = \frac{1}{\sqrt{2}}((\mathcal{H}X)(1) + (\mathcal{H}X)(0)) \\ f_\partial = \frac{1}{\sqrt{2}}(P_1(\mathcal{H}X)(1) - P_1(\mathcal{H}X)(0)) \end{cases} \quad (6)$$

The boundary conditions rewrite

$$W_B \begin{pmatrix} f_\delta \\ e_\delta \end{pmatrix} = \begin{pmatrix} u(t) \\ 0 \end{pmatrix}, \text{ with } W_B = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}. \quad (7)$$

We define the operator $A \doteq P_1 \frac{\partial}{\partial x}\mathcal{H} + G_0\mathcal{H}$, with domain

$$D(A) = \{X \in \chi | \; \mathcal{H}X \in H^1([0,1];\mathbb{R}^2), W_B \begin{pmatrix} f_\delta \\ e_\delta \end{pmatrix} = 0\}.$$

Then, A is the infinitesimal generator of a C_0-semigroup on χ. For any control input $u \in C^2([0,T],\mathbb{R})$, for any initial conditions given by $X_1(x,0) = \frac{dw_0}{dx}$, $X_2(x,0) = \rho(x)\frac{\partial w}{\partial t}|_{t=0}(x)$ satisfying the corresponding compatibility conditions, there exists a unique classical solution of (5)-(7) (Jacob, 2012, Lemma 13.2.1). Note that in this formalism, the energy of the system (or *Hamiltonian*) rewrites $\mathcal{E}(t) = \|X\|_\chi^2$.

2.3 Control objective

In the *power balance* (2), the actuation at the boundary of the spatial domain impacts the change of internal power. In the case $\kappa > 0$, the system is already stable as its energy is strictly decreasing in open-loop. However, we can use the control input $u(t)$ to fasten its stabilization. In case of wave equation stemming from the linearization of an unstable system ($\kappa < 0$), the control input can be used for stabilization purposes.

As mentioned in the introduction, we already have examples of boundary feedback controllers stabilizing system (5)-(7). In this paper, we do not simply want to stabilize the system but instead impose a specific decay rate to the energy of the system $\mathcal{E}$, using *a distributed damping assignment*. More precisely, we want to design a control law $u(t)$ making the dynamics of X equivalent to the dynamics of $\bar{X} = (\bar{X}_1, \bar{X}_2)$ satisfying

$$\frac{\partial}{\partial t}\begin{pmatrix} \bar{X}_1 \\ \bar{X}_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial}{\partial x}\left(\frac{1}{\rho(x)}\cdot\right) \\ \frac{\partial}{\partial x}(E(x)\cdot) & -K \end{pmatrix} \begin{pmatrix} \bar{X}_1 \\ \bar{X}_2 \end{pmatrix}, \quad (8)$$

with boundary conditions

$$\bar{X}_1(1,t) = 0, \; \bar{X}_2(0,t) = 0. \quad (9)$$

This system is well-posed for any smooth initial conditions $\bar{X}_1(x,0) = \bar{X}_{10}(x)$, $\bar{X}_2(x,0) = \bar{X}_{20}(x)$ and exponentially stable. We define the new damping coefficient $K > 0$. For sake of simplicity, we considered here that the coefficient $K > 0$ is constant. However, the proposed approach could be extended for spatially-varying damping term $K(x)$. In closed-loop the energy decreases proportionally to it

$$\frac{d\bar{\mathcal{E}}}{dt} = -K \int_0^1 \left(\frac{\bar{X}_2(x,t)}{\rho(x)}\right)^2 dx. \quad (10)$$

3. CONTROLLER DESIGN

To find the full-state feedback controller $u(t)$ satisfying the control objective, we propose a specific integral transform of the form $\mathcal{T} : \mathcal{C} + \int_0^x \mathcal{N} dx$ mapping system (5)-(7) to (8)-(9). We prove that this transform is boundedly invertible. It is an extension of the classical backstepping transform to Port Hamiltonian systems. Indeed, we aim to map the original system (5)-(7) into the target system (8)-(9). Although this transformation can be obtained directly, it can be decomposed into several classical elementary transforms (exponential changes of variables and classical backstepping coordinates changes). To facilitate the generalization of the proposed approach to other systems, we first present this more intuitive step-by-step approach.

3.1 Insights on the transform

In this section, we briefly present the methodology we used to find the form of the adequate transform $\mathcal{T}$. Due to space restrictions, we only give here the main ideas. The global integral transform will be presented in the next section.

To simplify the notations, we introduce the following functions $\lambda(x) = \sqrt{E(x)/\rho(x)}$, $\gamma(x) = \sqrt{E(x)\rho(x)}$ and $\delta(x) = \left(\frac{\rho'}{\rho} + \frac{E'}{E}\right)(x)$, $\delta_1^3(x) = \left(3\frac{\rho'}{\rho} - \frac{E'}{E}\right)(x)$, $\delta_3^1(x) = \left(\frac{\rho'}{\rho} - 3\frac{E'}{E}\right)(x)$.

First change of variables: Riemann coordinates. First, we rewrite system (5)-(7) as two coupled transport equations using an invertible change of variables (Riemann coordinates). Indeed, since the matrix $A(x) = \begin{pmatrix} 0 & 1/\rho(x) \\ E(x) & 0 \end{pmatrix}$ admits two eigenvalues $\pm\lambda(x)$, there exists $\mathcal{P}(x) \in C^1([0,1],\mathbb{R}^{2\times 2})$ invertible, such that $A(x) = \mathcal{P}(x)\Lambda(x)\mathcal{P}(x)^{-1}$, with $\Lambda(x) = \text{diag}(-\lambda(x),\lambda(x))$. The new state variables $\begin{pmatrix} \xi^+ \\ \xi^- \end{pmatrix} = \mathcal{P}^{-1}(x)\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ satisfy two heterodirectional hyperbolic PDEs with in-domain spatially varying coupling terms

$$\frac{\partial}{\partial t}\begin{pmatrix} \xi^+ \\ \xi^- \end{pmatrix} = \Lambda(x)\frac{\partial}{\partial x}\begin{pmatrix} \xi^+ \\ \xi^- \end{pmatrix} + \begin{pmatrix} \sigma^{++} & \sigma^{+-} \\ \sigma^{-+} & \sigma^{--} \end{pmatrix}\begin{pmatrix} \xi^+ \\ \xi^- \end{pmatrix}, \quad (11)$$

with boundary conditions

$$\xi^+(0,t) = q_0\xi^-(0,t), \quad (12)$$

$$\xi^-(1,t) = r_1\xi^+(1,t) + u_1(t), \quad (13)$$

where $q_0 = \frac{1}{\gamma(0)}$, $r_1 = -\gamma(1)$ $u_1(t) = \frac{\sqrt{2}}{\lambda(1)}u(t)$. The in-domain couplings $\sigma^{\cdot\cdot} \in C^0([0,1])$ are continuous functions depending on the system parameters. They are defined by

$$\begin{cases} \sigma^{++}(x) = \frac{1}{2}\left(-c(x) + \frac{\lambda}{2}\delta_3^1(x)\right), \\ \sigma^{+-}(x) = \frac{1}{2\rho(x)}\left(\frac{c(x)}{\lambda(x)} - \frac{1}{2}\delta(x)\right), \\ \sigma^{-+}(x) = \frac{E(x)}{2}\left(\frac{c(x)}{\lambda(x)} + \frac{1}{2}\delta(x)\right), \\ \sigma^{--}(x) = \frac{1}{2}\left(-c(x) - \frac{\lambda}{2}\delta_1^3(x)\right). \end{cases}$$

Inspired by the constructive backstepping approach (Krstic, 2008), we use a Volterra integral transform to move the in-domain couplings to the actuated boundary.

Exponential change of variables Before doing so, we apply an exponential change of variables to suppress the diagonal coupling terms. Define the new set of variables $\begin{pmatrix} \bar{\xi}^+(x,t) \\ \bar{\xi}^-(x,t) \end{pmatrix} = \text{diag}(f(x)e^{I_c(x)}, g(x)e^{-I_c(x)})\begin{pmatrix} \xi^+(x,t) \\ \xi^-(x,t) \end{pmatrix}$, with

f, g two continuous functions defined on $[0,1]$ by $f(x) = \sqrt{\frac{E(x)\lambda(x)}{E(0)\lambda(0)}}$ and $g(x) = \sqrt{\frac{\lambda(x)\rho(0)}{\rho(x)\lambda(0)}}$ and where $I_c(x) = \int_0^x \frac{c}{2\lambda(s)} ds$.

The new variables satisfy the following equations

$$\frac{\partial}{\partial t}\begin{pmatrix}\bar{\xi}^+\\ \bar{\xi}^-\end{pmatrix} = \Lambda(x)\frac{\partial}{\partial x}\begin{pmatrix}\bar{\xi}^+\\ \bar{\xi}^-\end{pmatrix} + \begin{pmatrix}0 & \bar{\sigma}^+\\ \bar{\sigma}^- & 0\end{pmatrix}(x)\begin{pmatrix}\bar{\xi}^+\\ \bar{\xi}^-\end{pmatrix}, \quad (14)$$

with $\bar{\sigma}^\pm \in C^0([0,1])$ defined by $\bar{\sigma}^+(x) = e^{2I_c(x)}\frac{f(x)}{g(x)}\sigma^{+-}(x)$ and $\bar{\sigma}^-(x) = e^{-2I_c(x)}\frac{g(x)}{f(x)}\sigma^{-+}(x)$. They rewrite as

$$\bar{\sigma}^+(x) = \frac{1}{2\gamma(0)}e^{2I_c(x)}(c - \frac{\lambda}{2}\delta(x)),$$
$$\bar{\sigma}^-(x) = \frac{\gamma(0)}{2}e^{-2I_c(x)}(c + \frac{\lambda}{2}\delta(x)),$$

The boundary conditions are given by

$$\bar{\xi}^+(0,t) = q_0\bar{\xi}^-(0,t),$$
$$\bar{\xi}^-(1,t) = \bar{r}_1\bar{\xi}^+(1,t) + \bar{U}(t), \quad (15)$$

where $\bar{r}_1 = -\gamma(0)e^{-2I_c(1)}$, $\bar{U}(t) = g(1)e^{-I_c(1)}u_1(t)$.

Volterra integral transform Next, we use a classical invertible Volterra integral transform of the second kind $\mathrm{Id} + \int_0^x K$ to replace the in-domain coupling terms by adequate terms that correspond to the ones we would have obtained performing the change of variables $\mathcal{P}^{-1}$ and the exponential change of coordinates on the system (8). More precisely, we define the new state variables as

$$\begin{pmatrix}\alpha^+\\ \alpha^-\end{pmatrix} = \begin{pmatrix}\bar{\xi}^+\\ \bar{\xi}^-\end{pmatrix} - \int_0^x \begin{pmatrix}K^{++} & K^{+-}\\ K^{-+} & K^{--}\end{pmatrix}(x,y)\begin{pmatrix}\bar{\xi}^+\\ \bar{\xi}^-\end{pmatrix}(y)dy,$$

with kernels $K^{\pm\mp}$ defined on $\mathcal{T}^-$. Following the backstepping methodology (Vazquez et al., 2011), one can easily show that there exists a unique set of kernels such that the new variables verify

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha^+\\ \alpha^-\end{pmatrix} = \Lambda(x)\frac{\partial}{\partial x}\begin{pmatrix}\alpha^+\\ \alpha^-\end{pmatrix} + \begin{pmatrix}0 & b^+\\ b^- & 0\end{pmatrix}(x)\begin{pmatrix}\alpha^+\\ \alpha^-\end{pmatrix}, \quad (16)$$

with the boundary conditions

$$\alpha^+(0,t) = q_0\alpha^-(0,t), \alpha^-(1,t) = a_1\alpha^+(1,t). \quad (17)$$

The two continuous functions $b^\pm$ are defined on $[0,1]$ by

$$b^+(x) = \frac{1}{2\gamma(0)}\left(K - \frac{\lambda(x)}{2}\delta(x)\right)e^{2I_K(x)},$$
$$b^-(x) = \frac{\gamma(0)}{2}\left(K + \frac{\lambda(x)}{2}\delta(x)\right)e^{-2I_K(x)},$$

with $I_K(x) = \frac{1}{2}\int_0^x \frac{K}{\lambda(s)}ds$, and $a_1 = -\gamma(0)e^{-2I_K(1)}$.

The control input is given by

$$\begin{aligned}\bar{U}(t) =& (a_1 - \bar{r}_1)\bar{\xi}^+(1,t)\\ &+ \int_0^1 (K^{-+}(1,y) - a_1K^{++}(1,y))\bar{\xi}^+(y)\\ &\quad + (K^{--}(1,y) - a_1K^{+-}(1,y))\bar{\xi}^-(y)\mathrm{d}y.\end{aligned}$$

Note that system (16)-(17) corresponds to system (14)-(15) where the parameter c has been replaced by K. Using the invertibility of the different transformations, it is straightforward to express $\bar{U}(t)$ as a function of the original states X_i.

Second exponential transform and inverse change of variables. Finally, we define another exponential transform by

$$\begin{pmatrix}\bar{\alpha}^+\\ \bar{\alpha}^-\end{pmatrix} = \begin{pmatrix}f(x)^{-1}e^{-I_K(x)} & 0\\ 0 & g(x)^{-1}e^{I_K(x)}\end{pmatrix}\begin{pmatrix}\alpha^+\\ \alpha^-\end{pmatrix}. \quad (18)$$

We can show that $\mathcal{P}^{-1}(x)\begin{pmatrix}\bar{\alpha}^+\\ \bar{\alpha}^-\end{pmatrix}$ corresponds to the expected target system (8)-(9) in the Port Hamiltonian formalism. This step-by-step strategy is illustrated in Fig. 2.

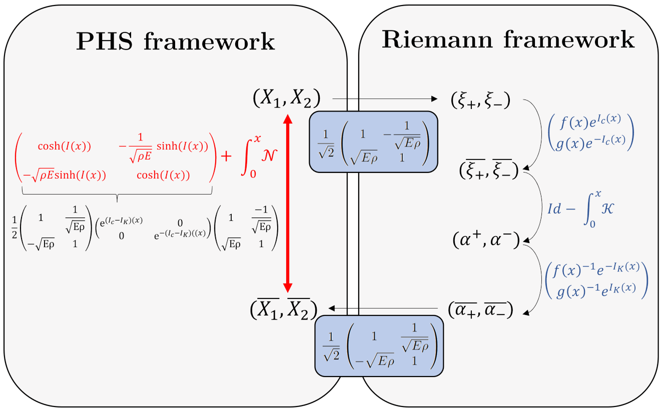

Fig. 2. Presentation of the two control strategies

3.2 *Overall transform*

We now present a specific integral transform that directly maps the original system (5) into the target system (8). It is obtained as a composition of all the transformations presented in the last section.

Definition Let us introduce the function C defined by

$$\begin{aligned}C:\ & [0,1] \to C^1([0,1];\mathbb{R}^{2\times 2})\\ & x \mapsto \begin{pmatrix}\cosh(I(x)) & -1/\gamma(x)\sinh(I(x))\\ -\gamma(x)\sinh(I(x)) & \cosh(I(x))\end{pmatrix}\end{aligned}$$

where the function I is defined by $I: x \mapsto I_c(x) - I_K(x) \in C^1([0,1])$. Note that for all $x \in [0,1]$, $\det(C(x)) = 1$ such that $C(x)$ admits a unique inverse $C^{-1}(x)$. We then define the invertible operator

$$\mathcal{C}: \begin{array}{c}\chi \to \chi\\ X \mapsto C \times X.\end{array}$$

We finally introduce the integral operator $\mathcal{T}: \chi \to \chi$,

$$\mathcal{T}(X)(x) = \mathcal{C}(X)(x) + \int_0^x N(x,y)X(y)dy, \quad (19)$$

where N is a bounded function defined on $\mathcal{T}^-$.

Invertibility of $\mathcal{T}$ Our objective is to find the appropriate full-state feedback controller $u(t)$ mapping (5)-(7) to (8)-(9) using $\mathcal{T}$ defined by (19), *i.e* $\begin{pmatrix}\bar{X}_1\\ \bar{X}_2\end{pmatrix} = \mathcal{T}\begin{pmatrix}X_1\\ X_2\end{pmatrix}$. To guarantee that both systems share the same stability properties, the proposed transformation must be boundedly invertible. This is straightforward as $\mathcal{T}$ is a composition of the invertible transform $\mathcal{C}$ with an invertible Volterra integral transform of the second kind (Yoshida, 1960). The inverse transform verifies

$$\mathcal{T}^{-1}\begin{pmatrix}u(x)\\ v(x)\end{pmatrix} = \mathcal{C}^{-1}\begin{pmatrix}u(x)\\ v(x)\end{pmatrix} - \int_0^x \bar{N}(x,y)\begin{pmatrix}u(y)\\ v(y)\end{pmatrix}dy, \quad (20)$$

where $\bar{N}$ is a bounded function defined on $\mathcal{T}^-$ by

$$\bar{N}(x,y) = C^{-1}(x)N(x,y)C^{-1}(y) - \int_y^x N(x,s)\bar{N}(s,y)ds.$$

3.3 Kernel equations

Our objective is to map (5)-(7) to (8)-(9). Differentiating Equation (19) with respect to time and space and integrating by parts, we obtain the following set of kernel equations:

$$\frac{1}{\rho(x)}N_x^{21} + E(y)N_y^{12} = \frac{\rho'(x)}{\rho(x)^2}N^{21}(x,y), \quad (21)$$

$$\frac{1}{\rho(y)}N_y^{11} + \frac{1}{\rho(x)}N_x^{22} = \frac{\rho'(x)}{\rho(x)^2}N^{22}(x,y) - c(y)N^{12}(x,y),$$

$$E(x)N_x^{11} + E(y)N_y^{22} = KN^{21}(x,y) - E'(x)N^{11}(x,y),$$

$$E(x)N_x^{12} + \frac{1}{\rho(y)}N_y^{21} = -E'(x)N^{12}(x,y) + (K - c(y))N^{22}(x,y), \quad (22)$$

with the boundary conditions

$$N^{12}(x,0) = 0,\ N^{22}(x,0) = 0, \quad (23)$$

$$N^{11}(x,x) = N^{22}(x,x) - \frac{1}{2\lambda(x)}(c(x)+K)\sinh\alpha(x),$$

$$N^{21}(x,x) = \rho E\left(N^{12}(x,x) + \frac{1}{2E}(c(x)-K)\cosh\alpha(x) - \frac{1}{2\lambda\rho}\left(\frac{E'}{E} + \frac{\rho'}{\rho}\right)\sinh\alpha(x)\right). \quad (24)$$

We have the following theorem

Theorem 1. The system (21)-(22) with boundary conditions (23)-(24) admits a unique continuous solution N defined on $\mathcal{T}^-$.

Proof. To prove the well-posedness of the kernel equations, we rewrite (21)-(24) using the formalism of (Di Meglio et al., 2018, Theorem 3.2). Define a new set of kernels $K^i,\ i \in [\![1,4]\!]$ on $\mathcal{T}^-$ by

$$\begin{pmatrix} K^1 \\ K^2 \\ K^3 \\ K^4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \frac{\gamma(y)}{\gamma(x)} \\ -\frac{\gamma(x)}{\gamma(y)} & 0 & 0 & 1 \\ 0 & -\gamma(y)\gamma(x) & 1 & 0 \\ 0 & 1 & \frac{1}{\gamma(y)\gamma(x)} & 0 \end{pmatrix} \begin{pmatrix} N^{11} \\ N^{12} \\ N^{21} \\ N^{22} \end{pmatrix}. \quad (25)$$

The kernels K^i satisfy

$$\epsilon_i(x)\partial_x K^i + \nu_i(y)\partial_y K^i = \sum_{j=1}^4 \Sigma_{ij}(x,y)K^j(x,y), \quad (26)$$

with $\epsilon(x) = (1\ 1\ 1\ 1)^T \lambda(x),\ \nu(y) = (1\ {-1}\ {-1}\ 1)^T \lambda(y)$, and the continuous in-domain coupling terms are defined by (27). Each kernel K^i has boundary conditions on $\Omega_i \subset \partial\mathcal{T}^-$ of the form

$$K_i|_{\Omega_i} = f_i + \sum_{j=1}^4 \Gamma_{ij}K_j|_{\Omega_i} \quad (28)$$

$$\text{with } \begin{cases} \Omega_1 = \Omega_4 = \{(x,y) \in \mathcal{T}^- | y = 0\}, \\ \Omega_2 = \Omega_3 = \{(x,y) \in \mathcal{T}^- | y = x\}, \end{cases}$$

with $\Gamma_{12}(x) = -\frac{\gamma(0)}{\gamma(x)}$, $\Gamma_{43}(x) = \frac{1}{\gamma(0)\gamma(x)}$, $f_2(x) = \frac{c(x)+K}{2\lambda(x)}\sinh(\alpha(x))$, $f_3(x) = \frac{\rho(x)}{2}((c-K)\cosh(\alpha(x)) - \lambda(x)\delta(x)\sinh(\alpha(x))$, the other coupling terms being equal to zero. We show that the assumptions of (Di Meglio et al., 2018, Theorem 3.2) are verified. Thus, the system (26)-(28) admits a unique bounded solution K. The regularity of the solution derives from the one of the couplings.

3.4 Controller design

We can now design the stabilizing control law. We have the following theorem.

Theorem 2. The initial system (5)-(7) with the control law

$$u(t) = \gamma(1)\tanh(I(1))X_2(1,t) - \frac{E(1)}{\cosh(I(1))}\int_0^1 N^{11}(1,y)X_1(y,t) + N^{12}(1,y)X_2(y,t)dy \quad (29)$$

has the same dynamics as (8)-(9).

Proof. The proof is a direct application of the backstepping methodology. ■

We therefore obtain a full-state feedback controller for the initial system, that stabilizes it with the desired damping cœfficient.

Comment on robustness It is worth noticing that the control law (29) involves the pointwise term $\gamma(1)\tanh(I(1))$ $X_2(1,t)$. This corresponds to the cancellation of the reflection term $(a_1 - \bar{r}_1)\bar{\xi}^+(1,t)$ to obtain the target system (16) in the step by step approach. As shown in (Auriol and Di Meglio, 2019), this may have major consequences regarding the robustness margins of the closed-loop system since the corresponding feedback law is not strictly proper. To avoid any robustness issue, we must have $|q_0(a_1 - \bar{r}_1)| + |q_0\bar{r}_1| < 1$ which implies the condition

$$e^{-2I_c(1)} + |e^{-2I_K(1)} - e^{-2I_c(1)}| < 1. \quad (30)$$

Note that (30) is never satisfied in the case $\kappa \le 0$. Indeed, the terms neglected in the linearization can effect positively the robustness of the real system.

4. SIMULATION RESULTS

In this section, we illustrate the performance of our controller in simulations. Consider the wave equation (1) with constant cœfficients $\rho = 936\text{kg.m}^{-3}, E = 4.14\text{GPa}$. The initial string position is $w_0(x) = 0.1x$. We simulated system (14)-(15) on a time horizon of 20s using a Godounov Scheme (LeVeque, 2002) ($CFL = 1$). The space domain $[0,1]$ is discretized with a mesh of 200 points. Beforehand, the kernels $K^{\pm\mp}$ are computed offline using a fixed-point algorithm. The control input is computed at each time step using (29). The integral term is approximated using `trapz`.

We first consider the undamped case, *i.e.* $c = 0$. Note that in this case the robustness property is not satisfied, and the purpose is simply to illustrate the performance of our control strategy. As illustrated in Figure 3, the energy of the open-loop system is constant, while the energy of the closed-loop system decays to zero with the same rate as the one of the target system (dotted black line). From

$$
\begin{cases}
\Sigma_{1j}(x,y) = \left(\frac{1}{4}((\lambda\delta_3^1)(x) + (\lambda\delta)(y)) \;\; \frac{1}{4}(\frac{\gamma(y)\delta(x)}{\rho(x)} + \frac{E(y)\delta(y)}{\gamma(x)}) \;\; \frac{K+c(y)}{2\gamma(x)} \;\; \frac{\gamma(y)(K-c(y))}{2}\right) \\
\Sigma_{2j}(x,y) = \left(\frac{1}{4}(\frac{E(x)\delta(x)}{\gamma(y)} - \frac{\gamma(x]\delta(y)}{\rho(y)}) \;\; \frac{1}{4}((\lambda\delta_1^3)(x) + (\lambda\delta)(y)) \;\; \frac{c(y)-K}{2\gamma(y)} \;\; -\frac{\gamma(x)(K+c(y))}{2}\right) \\
\Sigma_{3j}(x,y) = \left(\frac{\gamma(x)(c(y)-K)}{2} \;\; \frac{\gamma(y)(c(y)-K)}{2} \;\; \frac{1}{4}((\lambda\delta_1^3)(x) - (\lambda\delta)(y)) \;\; \frac{1}{4}(E(y)(\gamma\delta)(x) + E(x)(\gamma\delta)(y))\right) \\
\Sigma_{4j}(x,y) = \left(\frac{K-c(y)}{2\gamma(y)} \;\; \frac{K-c(y)}{2\gamma(x)} \;\; \frac{1}{4}(\frac{\delta(x)}{\rho(x)\gamma(y)} - \frac{\delta(y)}{\rho(y)\gamma(x)}) \;\; \frac{1}{4}((\lambda\delta_3^1)(x) - (\lambda\delta)(y))\right)
\end{cases} \tag{27}
$$

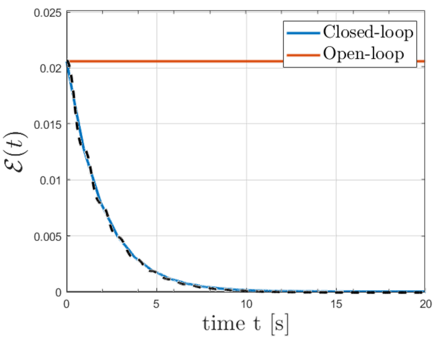

Fig. 3. Evolution of the energy $\mathcal{E}(t)$.

the values of $(\bar{\xi}^+, \bar{\xi}^-)$, we can numerically compute the evolution of the displacement $w(x,t)$ along the string. We see in Figure 4 (bottom) that the oscillations naturally present in open-loop (top) are substantially damped. We

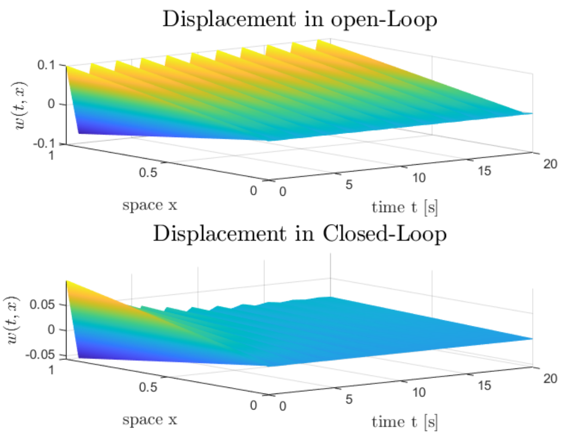

Fig. 4. Evolution of the displacement $w(x,t)$.

see in Figure 5 that the control effort $u(t)$ converges to zero.

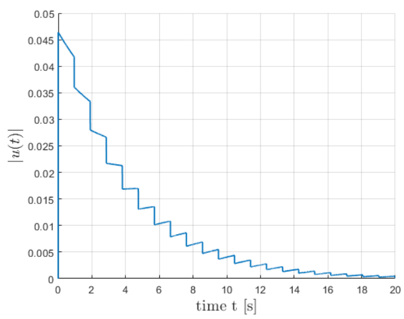

Fig. 5. Evolution of the control effort $|u(t)|$.

Second, we consider the case $c = 0.1$ and we choose as target system (8)-(9) with $K = 5c > 0$. We compare the proposed control law with a simple boundary feedback $X_1(t,1) = -0.1X_2(t,1)$. We represent the evolution of

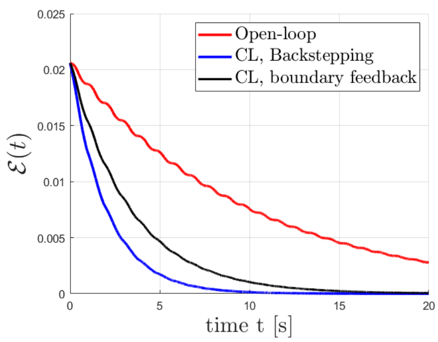

Fig. 6. Evolution of the energy $\mathcal{E}(t)$.

the energy for both the open-loop and closed-loop systems with the two different controllers on Figure 6. As expected, the energy decays faster with the control input we propose (blue). One can verify that the control effort goes to zero.

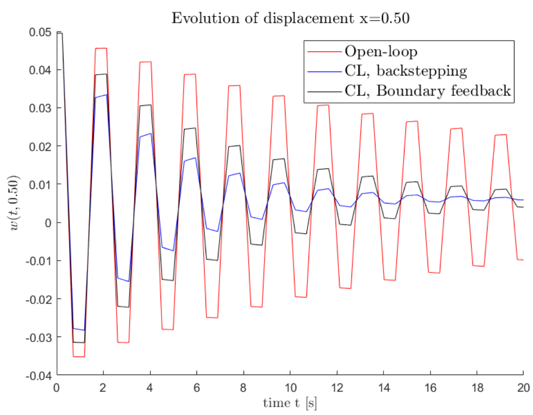

Fig. 7. Displacement $w(0.5,t)$ in open-loop (red), and closed-loop for the proposed controller (blue) and simple boundary feedback (black).

5. CONCLUSION

In this paper, we presented an innovative state-feedback boundary controller inspired by the backstepping methodology that stabilizes a wave equation with in-domain damping with predefined closed-loop properties. We used the Port-Hamiltonian formalism to determine a target system of strictly decreasing energy, whose behavior matches the one of the original system. This combined approach was proposed on a simple 'toy-system' and is a first step towards generalization to more complex PHS. The controller was designed such that the closed-loop initial system can be mapped to this target system with a new form of boundedly invertible transform. Its implementation requires the knowledge of the state on the entire string. This can be done using an observer, similarly to the ones proposed in (Guo and Guo, 2009; Smyshlyaev et al., 2010).

We would like to take advantage of the intrinsic modularity of the Port-Hamiltonian framework to adapt this approach to larger networks of interconnected systems. We believe that Port-Hamiltonian formalism can be of great use to get intuition on adequate target systems in the backstepping approach. Indeed, adjusting energy-shaping techniques to design amenable target systems could improve the performance of the closed-loop system while reducing the control effort (since the target system would take advantage of the natural dissipation of the system). In future works, we will consider systems of higher dimension (as Timoshenko beams) and interconnected systems for which the backstepping approach has already allowed the design of stabilizing control laws (Redaud et al., 2021). We could also extend our result to introduce adaptive time-varying damping with $k(t)$, which would require time-dependent integral transforms.

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