# Design of Quasipolynomial-Based Controllers with Dynamical Parameters - Application to Active Vibration Damping

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**Abstract:** In order to achieve a partial pole placement for linear time-invariant systems including time-delays in their models' representation, a method for the design of *quasipolynomial-based controllers* has been proposed in recent works. The ensuing controllers correspond to some output feedback control laws with constant parameters. It appears that such a controller has a limited number of degrees of freedom limiting the potential performances in closed-loop. To overcome this issue, we propose to modify the previous quasipolynomial-based controller by using *dynamical parameters* in their design. It turns out that the use of dynamical parameters corresponds to linear filtered terms in the control law of the original one. Such a controller is applied to the active vibration damping problem for a piezo-actuated flexible structure.

*Keywords:* Time-delay systems, Control design, MID property, Active vibration damping.

## 1. INTRODUCTION

A common feature in modeling transport and propagation phenomena and processes is the time heterogeneity that can be described by using *delays* in their mathematical representation. Such delays may be constant or time-varying, distributed (or not) over on some appropriate time-intervals, depending (or not) on the state-vector. There are several ways to represent delays in the system's dynamics and, in the sequel, we are focusing on dynamical systems described by delay-differential equations (DDEs; for more insights, see, e.g., Hale and Verduyn Lunel (1993); Michiels and Niculescu (2014)).

In the context of mechanical engineering problems, the effect induced by the presence of time-delays on system's dynamics was emphasized in Stépán (1989) where practical applications were studied, such as the machine tools or robotic systems. For further examples, the reader is referred to Niculescu (2001); Gu et al. (2003); Insperger and Stépán (2011) and the references therein. Furthermore, delays are intrinsically present in practical control systems. Inspired by Hazen's theory of servomechanisms [1] published in the 30s, one of the first approaches to handle second- and third-order systems with delay in the input was proposed by Callender et al. (1936), Hartree et al. (1937). For a historical perspective in the analysis and control of delay systems, we refer to Kolmanovskii and Nosov (1986), Stépán (1989), Michiels and Niculescu (2014).

At the end of the 1970s, the use of the delays in the controller design was introduced in Suh and Bien (1979) where the authors showed that the conventional proportional controller equipped with an appropriate time-delay performs an averaged derivative action and thus it can replace the classical proportional-derivative (PD) controller.

While the pole placement represents a classical well-known control method for finite-dimensional systems, its extension to infinite dimensional systems is far to be well developed and understood. More precisely, several pole placement paradigms exist for time-delay systems, each of them has its own advantages and drawbacks, see for instance, Olbrot (1978); Manitius and Olbrot (1979); Michiels et al. (2002); Brethé and Loiseau (1998). In particular, a recently defined paradigm, called *Partial Pole Placement* (PPP), has shown its effectiveness with respect to the robustness consideration as well as the simplicity of the resulting controller structure. The PPP paradigm is mainly based on two properties called respectively *Multiplicity-induced-dominancy* (MID) and *Coexistent-real-roots-induced-dominancy* (CRRID). As a matter of fact, the MID (respectively the CRRID) property consists of the conditions under which a given multiple zero (respectively a number of real simple zeros) of a quasipolynomial is/are dominant. For instance, in the generic quasipolynomial case, the real root of maximal multiplicity is necessarily the dominant (GMID). However, multiple roots with intermediate admissible multiplicities may be dominant or not. Thanks to this property, a consistent control strategy is proposed in Boussaada et al. (2019); Balogh et al. (2022); Boussaada et al. (2022b), which consists in assigning a root with an intermediate admissible multiplicity once appropriate conditions guaranteeing its dominancy are established. Furthermore, the MID property may be used to tune standard controllers. For instance, in Ma et al. (2022) it is applied to the systematic tuning of the stabilizing PID controller of a first order plant.

[1] more precisely, position control systems

The contribution of this paper is twofold. First, a new control strategy is proposed. Such a strategy uses the quasipolynomial-based (QPB) controller with dynamic parameters and an appropriate tuning method for these parameters. To the best of the authors' knowledge, such a control strategy represents a novelty in the open literature. Second, the effectiveness of the proposed methodology is shown in a practical case study - the active vibration damping of a piezo-actuated flexible structure.

The remaining of the paper is as follows: Some prerequisites, preliminaries as well as the problem statement are briefly presented in Section 2. Section 3 includes the explicit construction of the dynamical QBP controller as well as an appropriate control algorithm. Next, a practical application on active vibration damping is discussed in Section 4 and some concluding remarks end the paper.

## 2. PREREQUISITES AND PROBLEM STATEMENT

Consider the dynamical system described by the delay-differential equation (DDE):

$$y^{(n)}(t)+\sum_{k=0}^{n-1} a_k y^{(k)}(t)+\sum_{k=0}^{m} \alpha_k y^{(k)}(t-\tau)=0, \tag{1}$$

under appropriate initial conditions, where $y(\cdot)$ is the real-valued unknown function, $\tau>0$ is the delay, and $a_0,\dots,a_{n-1}$, $\alpha_0,\dots,\alpha_m$ are real coefficients. When the highest order of derivation appears only in the non-delayed term $y^{(n)}(t)$, the DDE (1) is said to be of *retarded type* if $m<n$, or of *neutral type* if $m=n$. We refer to Stépán (1989); Hale and Verduyn Lunel (1993); Michiels and Niculescu (2014) for a deeper discussions on DDEs and related results and properties. The characteristic function associated to (1) is the quasipolynomial $\Delta:\mathbb{C}\mapsto\mathbb{C}$ defined by

$$\Delta(s):=P_0(s)+P_\tau(s)\,e^{-\tau s}, \tag{2}$$

where $P_0$ and $P_\tau$ are the polynomials with real coefficients given by

$$P_0(s)=s^n+\sum_{k=0}^{n-1} a_k s^k, \quad P_\tau(s)=\sum_{k=0}^{m}\alpha_k s^k, \tag{3}$$

and the degree of $\Delta$ is the integer $\deg(\Delta):=n+m+1$. We say that a characteristic root $s_0$ of $\Delta$ satisfies the *MID property* if **(i)** its *algebraic multiplicity* (denoted by $M(s_0)$) is *larger than one*, and **(ii)** it is *dominant* in the sense that all the characteristic roots $\lambda_\sigma$ of the spectrum are located to the left[2] of $s_0$ in $\mathbb{C}_-$. In other words, $s_0$ is the rightmost root of the spectrum and defines the *spectral abscissa* of the quasipolynomial $\Delta$. In the case $M(s_0)=\deg(\Delta)$, it was shown in Mazanti et al. (2021) (case $m=n-1$) and Boussaada et al. (2022a) (general case $m\le n$) that $s_0$ satisfies the MID property. This "limit" case is also called *generic MID* or GMID for short.

*Remark 2.1.* As noticed in Boussaada et al. (2022a), the GMID does not allow any degree of freedom in assigning $s_0$. In order to allow for some additional degrees of freedom when assigning $s_0$, one can relax such a constraint by forcing the root $s_0$ to have a multiplicity lower than the maximal one, and consider, for instance, the delay as a *free tuning parameter*.

### 2.1 Problem statement

Consider a linear time invariant (LTI) system $\mathscr{S}$ with a scalar control input $u(t)$, a scalar measured output $y(t)$, a scalar disturbance input $w(t)$ and an output of interest $z(t)$. The model of $\mathscr{S}$ based on transfer functions is given by

$$\mathscr{S}\begin{cases} Z(s) &= \dfrac{N_{wz}(s)}{\psi(s)}W(s)+\dfrac{N_{uz}(s)}{\psi(s)}U(s),\\ Y(s) &= \dfrac{N_{wy}(s)}{\psi(s)}W(s)+\dfrac{N_{uy}(s)}{\psi(s)}U(s),\end{cases} \tag{4}$$

where the polynomials, with real coefficients, have the form:

$$N_{ij}(s):=\sum_{k=0}^{n_p} n_{ij_k}\,s^k \quad\text{and}\quad \psi(s):=s^{n_p}+\sum_{k=0}^{n_p-1} a_k\,s^k, \tag{5}$$

where $i\in\{u,w\}$ and $j\in\{y,z\}$ and $n_p$ is the order of the system. For sake of simplicity, $\psi(\cdot)$ is chosen to be a monic polynomial. The control model, given by $\dfrac{N_{uy}(s)}{\psi(s)}$, is assumed to be in its minimal form, such that $N_{uy}(\cdot)$ and $\psi(\cdot)$ are co-prime polynomials. This assumption simply means that the $n_p$ poles of the linear systems are not simplified by the roots of $N_{uy}(\cdot)$. In practice, it means that the dynamics related to these poles are controllable and observable.

The proposed control problem is to *design an output feedback controller in order to assign the rightmost root of the closed-loop system on a desired location in the open left-half part of the complex plane.*

### 2.2 MID property and QPB controller design

To solve the control problem above, we consider *low-complexity controllers* based on quasipolynomials. Such controllers are called *QPB controllers*, and were introduced in Boussaada et al. (2017).

*Definition 2.2.* Let $n_0$, $n_{\tau_0}$, $d_0$, $d_{\tau_0}\in\mathbb{R}$ be such that $\tau\in\mathbb{R}_+$, $d_0\neq 0$ and at least one of the two other numbers $n_0$ and $n_{\tau_0}$ is nonzero. Then, a generic output feedback *QPB controller* is defined by the following continuous-time delay-difference equation:

$$u(t)=-\frac{d_{\tau_0}}{d_0}u(t-\tau)+\frac{n_0}{d_0}y(t)+\frac{n_{\tau_0}}{d_0}y(t-\tau). \tag{6}$$

In the Laplace domain, (6) yields $U(s)=C(s,\tau)Y(s)$, with

$$C(s,\tau):=\frac{n_0+n_{\tau_0}\,e^{-\tau s}}{d_0+d_{\tau_0}\,e^{-\tau s}}. \tag{7}$$

This control law is based on nothing else than an addition of proportional and delayed-proportional terms carrying on the signals $u(t)$ and $y(t)$, which makes it having a *low complexity* feature. As indicated above, the parameters of the controller are the four scalars $n_0$, $n_{\tau_0}$, $d_0$, $d_{\tau_0}$, and the positive time delay $\tau$, giving an amount of 4 independent degrees-of-freedom[3] for the pole assignment problem subject to the constraint that the controller is well-posed. It should be mentioned that the delay $\tau$ is considered here as a *design parameter* of the QPB controller, used with the other gains to assign the rightmost root of the closed-loop system.

Consider now the closed-loop characteristic function $\Delta(s)$ of the system $\mathscr{S}$ with the standard QPB controller $C(s,\tau)$. This latter is written as in (2) where here, $P_0(s):=d_0\,\psi(s)-n_0 N_{uy}(s)$ and $P_\tau(s):=d_{\tau_0}\,\psi(s)-n_{\tau_0}N_{uy}(s)$.

[2] In other words, $\lambda_\sigma$ satisfies the condition $\Re(\lambda_\sigma)\le\Re(s_0)$.

[3] Without any loss of generality, one may assume $d_0=1$.

The underlying idea can be resumed as follows: the QPB controller is designed to assign the closed-loop rightmost root by using the MID property introduced and shortly presented in the previous section. The main result on the MID property is recalled next in order to describe the design method leading to the sought gains $n_0$, $n_{\tau_0}$, $d_0$ and $d_{\tau_0}$, as well as the delay $\tau$ used as a design parameter, in order to achieve the assignment of the rightmost root $s_0 \in \mathbb{C}_-$ while guaranteeing the closed-loop stability.

### 2.3 MID-based partial pole placement

The partial pole placement used throughout this paper is based on a control-oriented MID property as introduced and discussed in Boussaada et al. (2022b), see also Boussaada et al. (2019) and Balogh et al. (2022). Let us first consider the generic quasipolynomial $\Delta(s)$ in (2) with $m \leq n$. The control-oriented MID property's main idea consists in forcing a given negative scalar $s_0$ to be a multiple spectral root of the system's closed-loop characteristic function given by $\Delta(s)$, and leading to some algebraic relations among the controller's parameters. More precisely, when the assigned root reaches a multiplicity at least equal to $n$, this guarantees some integral representation of the corresponding quasipolynomial as emphasized in Boussaada et al. (2016). Next, the controller's parameters are obtained thanks to the parametric conditions reflecting the dominant feature of the multiple spectral root, see for instance Boussaada et al. (2019) and Balogh et al. (2022). The following Theorem from Boussaada et al. (2022b) gives explicitly the integral representation of the quasipolynomial.

*Theorem 2.3.* Let $\tau > 0$, $s_0 \in \mathbb{R}$, and consider the quasipolynomial $\Delta$ from (2)–(3). The number $s_0$ is a root of $\Delta$ with multiplicity at least $n+m$ if, and only if there exists $A \in \mathbb{R}$ such that

$$\Delta(s) = \frac{\tau^m (s-s_0)^{n+m}}{(m-1)!}\int_0^1 t^{m-1}(1-t)^{n-1}(1-At)e^{-t\tau(s-s_0)}\mathrm{d}t. \qquad (8)$$

A helpful technique is to establish *a priori* information on the location of roots of $\Delta$ with real part greater than $s_0$ and, in particular, bounds on their imaginary parts.

A standard first step to do so is to introduce the normalized quasipolynomial $\tilde{\Delta}(\lambda) = \tau^n \Delta(s_0 + \frac{\lambda}{\tau})$, which can be written as $\tilde{\Delta}(\lambda) = \tilde{P}_0(\lambda) + e^{-\lambda}\tilde{P}_\tau(\lambda)$ for some suitable polynomials $\tilde{P}_0$ and $\tilde{P}_\tau$ of degrees $n$ and $m$, respectively. Hence, the problem of studying eventual roots of $\Delta$ with real part greater than $s_0$ reduces to the study of eventual roots of $\tilde{\Delta}$ with positive real part.

A possible strategy to do so is to notice that any root $\lambda$ of $\tilde{\Delta}$ satisfies

$$|\tilde{P}_0(x+i\omega)|^2 e^{2x} = |\tilde{P}_\tau(x+i\omega)|^2,$$

where $x := \Re(\lambda)$ and $\omega := \Im(\lambda)$. In particular, if $\lambda$ has nonnegative real part, then $e^{2x} \geq T_\ell(x)$, where, for $\ell \in \mathbb{N}$, the polynomial $T_\ell$ is the truncation of the Taylor expansion of $e^{2x}$ at order $\ell$, i.e., $T_\ell(x) = \sum_{k=0}^{\ell} \frac{(2x)^\ell}{\ell!}$. Hence, any root $\lambda := x + i\omega$ of $\tilde{\Delta}$ with nonnegative real part satisfies $\mathscr{F}(x,\omega) \geq 0$, where $\mathscr{F}$ is the polynomial given by $\mathscr{F}(x,\omega) := |\tilde{P}_\tau(x+i\omega)|^2 - |\tilde{P}_0(x+i\omega)|^2 T_\ell(x)$. In addition, $\mathscr{F}$ only depends on $\omega$ through $\omega^2$ (which is a consequence of the fact that $\tilde{P}_0$ and $\tilde{P}_\tau$ are polynomials with real coefficients), and one may thus introduce the variable $\Omega = \omega^2$ and define the polynomial $H$ by setting $H(x,\Omega) = F(x,\sqrt{\Omega})$ for $\Omega \geq 0$. Hence, any root $\lambda = x + i\omega$ of $\tilde{\Delta}$ with nonnegative real part satisfies $H(x,\Omega) \geq 0$, where $\Omega = \omega^2$. One can thus establish a bound on the imaginary parts for the roots of $\tilde{\Delta}$ by exploiting the last polynomial inequality. This has been done for some low-order cases in Benarab et al. (2022). In particular, all these works have shown that it is sufficient to bound the absolute value of the imaginary parts of the roots in the right half-plane by $\pi$, as one can in general easily exclude by other arguments Boussaada et al. (2022a) the possibility of having roots in the right-half plane with imaginary part at most $\pi$, thus concluding the proof of dominance of $s_0$.

The procedure described in this subsection is synthesized in Algorithm 1 (see Benarab et al. (2022)), in which one increases the order of the Taylor expansion of $e^{2x}$ until a suitable bound is found.

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**Algorithm 1:** Estimation of a frequency bound for time-delay differential equations with a single delay

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**Input:** $\tilde{\Delta}(\lambda) = \tilde{P}_0(\lambda) + \tilde{P}_\tau(\lambda)e^{-\lambda}$ and `maxOrd`;
  `// Normalized quasipolynomial and Maximal order`
`// Initialization`
`ord` $= 0$; `// ord: order of truncation of the Taylor expansion of` $e^{2x}$`;`
`dominance` $=$ `false`;
**while** *(not* `dominance`*) and (*`ord` $\leq$ `maxOrd`*)* **do**
  Set $\mathscr{F}(x,\omega) = |\tilde{P}_\tau(x+i\omega)|^2 - |\tilde{P}_0(x+i\omega)|^2 T_{\mathtt{ord}}(x)$;
  `//` $T_{\mathtt{ord}}(x)$`: Taylor expansion of` $e^{2x}$ `of order = ord`
  Set $H(x,\Omega) = \mathscr{F}(x,\sqrt{\Omega})$; `// H is a polynomial`
  Set $\Omega_k(x)$ as the $k$-th real root of $H(x,\cdot)$;
  **if** $\sup_{x\geq 0}\max_k \Omega_k(x) \leq \pi^2$ **then**
    `dominance` $=$ `true`;
  `ord` $=$ `ord` $+ 1$;
**Output:** Frequency bound: If `dominance` is `true`, then $|\omega| \leq \pi$ for every root of $\tilde{\Delta}$ with positive real part;

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## 3. MAIN RESULTS

### 3.1 Dynamical QPB controller

Based on the standard QPB controller's structure, recalled in Section 2, we propose to extend the features of such a controller by introducing some dynamical parameters instead of the static ones $n_0$, $n_{\tau_0}$, $d_0$ and $d_{\tau_0}$. The aim is to offer more degrees-of-freedom to handle high-order linear systems, in order to cope with various issues such as the robustness one, arising for example with the spillover phenomenon in the control of flexible structures.

*Definition 3.1.* The output feedback QPB controller with dynamical parameters is defined, in Laplace domain, by

$$D(s,\tau) := \left(N_0(s) + N_{\tau_0}(s)e^{-\tau s}\right)/\left(D_0(s) + D_{\tau_0}(s)e^{-\tau s}\right), \quad (9)$$

where $N_0(s)$, $N_{\tau_0}(s)$, $D_0(s)$, $D_{\tau_0}(s)$ are polynomials in $s$ with finite degree. The total amount of available independent parameters, denoted $N_P$, is given by $N_P := \deg(N_0) + \deg(N_{\tau_0}) + \deg(D_0) + \deg(D_{\tau_0}) + 4$.

*Remark 3.2.* It is important to note that the degrees of these polynomials are assumed to be such that all the following transfer functions remain proper for practical purposes: $F_y(s) := \frac{N_0(s)}{D_0(s)}$, $F_{y_d}(s) := \frac{N_{\tau_0}(s)}{D_0(s)}$ and $F_{u_d}(s) := \frac{D_{\tau_0}(s)}{D_0(s)}$.

*Fact 1.* The closed-loop system $\mathscr{S}$ in (4) with the Dynamical QPB controller in (9), has the same characteristic equation than in (2) where now, $P_0(s) := D_0(s)\,\psi(s) - N_0(s)\,N_{uy}(s)$ and $P_\tau(s) := D_{\tau_0}(s)\,\psi(s) - N_{\tau_0}(s)\,N_{uy}(s)$. Moreover, $\deg(\Delta) = \deg(D_0) + \deg(D_{\tau_0}) + 2n_p + 1$.

This fact shows that the MID property used for the design of the standard QPB controller can also be used for the dynamical case. The main difference relies on the practical implementation of the controller. We shall say few words about that in the next subsection.

### 3.2 Some practical implementation schemes

Let us denote $f_{u_d}(t)$, $f_y(t)$ and $f_{y_d}(t)$ the inverse Laplace transform of $F_{u_d}(s)$, $F_y(s)$ and $F_{y_d}(s)$ respectively. In time domain, the control law derived from (9) reads:

$$u(t) := -f_{u_d}(t) * u(t-\tau) + f_y(t) * y(t) + f_{y_d}(t) * y(t-\tau), \quad (10)$$

where the symbol $*$ stands for the time domain convolution product of causal signals.

*Remark 3.3.* It is worth noticing that the control signal (10) is, with the extended QPB controller, the result of filtered terms carrying on the delayed control signal and the measured output as well as its delayed part, that are all added. In consequence, the complexity of the control law is slightly increased w.r.t. the one from the standard QPB controller, but with the benefit of a greater set of available degrees-of-freedom, *ie* the coefficients of the polynomials introduced in Def. 3.1.

*Remark 3.4.* Note that the filters $F_{u_d}(s)$, $F_y(s)$ and $F_{y_d}(s)$ share the same poles. It can also be interesting to filter each term of the control law (10) with a separate filter, *ie* each one with its own dynamic.

Let us define the following proper linear transfer functions $G_{u_d}(s) := \frac{N_{u_d}(s)}{D_{u_d}(s)}$, $G_y(s) := \frac{N_y(s)}{D_y(s)}$, $G_{y_d}(s) := \frac{N_{y_d}(s)}{D_{y_d}(s)}$. Those transfer functions can be considered as the mathematical models of distinct linear filters. The QPB controller with dynamical parameters is defined by the following control law in the time domain

$$u(t) = -g_{u_d}(s) * u(t-\tau) + g_y(t) * y(t) + g_{y_d}(t) * y(t-\tau) \quad (11)$$

where $g_{u_d}(t)$, $g_y(t)$ and $g_{y_d}(t)$ are the inverse Laplace transforms of the previous transfer functions, *ie* their associated impulse responses. The resulting controller derived from this control law expressed in the the Laplace domain leads to

$$U(s) = D(s,\tau)\,Y(s) \text{ where } D(s,\tau) := \frac{G_y(s) + G_{y_d}(s)\,e^{-\tau s}}{1 + G_{u_d}(s)\,e^{-\tau s}}. \quad (12)$$

This last corresponds to a QPB controller with dynamical parameters as in (9), with $N_0(s) := D_{u_d}(s)N_y(s)D_{y_d}(s)$, $N_{\tau_0}(s) := D_{u_d}(s)N_{y_d}(s)D_y(s)$, $D_0(s) := D_{y_d}(s)D_y(s)D_{u_d}(s)$ and $D_{\tau_0}(s) := N_{u_d}(s)D_y(s)D_{y_d}(s)$.

### 3.3 Obtaining the parameters of the QPB controller

The QPB controller's structure, *ie* the degree of each polynomial composing it in (10), is now assumed to be fixed to handle the considered control problem constraints. Then, thanks to the linear dependency of $\Delta$ w.r.t the control parameters, the construction procedure of these parameters arises from an elimination procedure allowing the resolution of the equation set, stating the multiplicity of the root $s_0$. In other words, given $M(s_0)$, under the *necessary conditions* $N_P \leq M(s_0) \leq \deg(\Delta)$, this procedure consists in solving sequentially the set of equations

$$\Delta^{(k-1)}(s)\Big|_{s=s_0} = 0, \quad (13)$$

for $k = 1$ to $M(s_0)$ in the controller's parameters, where $\Delta^{(j)}(s)$ stands for the $j^{th}$ derivative of $\Delta(s)$ in terms of $s$.

## 4. APPLICATION TO ACTIVE VIBRATION DAMPING

### 4.1 System description and problem statement

The previous results are now applied to the active vibration control problem presented both in Boussaada et al. (2017) and in Tliba et al. (2019). It concerns a lightly-damped beam-like flexible structure with one clamped edge and the other free. This beam is equipped with a piezoelectric rectangular patch used as an actuator and bonded on one side of the beam, near the fixation. Another piezoelectric patch, with the same dimensions and used as a sensor, is collocated to the actuator and bonded on the other side. A sketch of this system is depicted on Fig. 1.

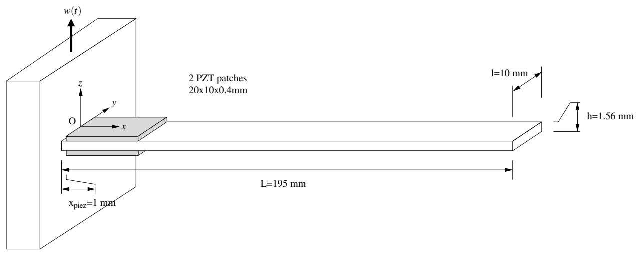


Fig. 1. Piezo-actuated beam with one clamped edge.

Such a system obeys mechanical and electrostatic laws expressed through coupled partial derivative equations, giving place to an *infinite-dimensional system*. The inputs-to-outputs dynamical model of finite dimension is obtained thanks to a *finite element modeling*, whose presentation is out of the scope of this paper. More details are given in Boussaada et al. (2017) and references therein. The obtained model is linear and of finite dimension. It is of order 24 which is enough for an accurate description in the low-frequency range, up to 3500 ($Hz$). For control purposes, a design model is derived from the full order one after a reduction to order 2, containing only the first mode dynamic. Some figures of the frequency responses for both models can be found in Boussaada et al. (2017). These models are given with the form (4), where $Z(s)$ stands here for the Laplace transform of the relative acceleration of the point located at the middle-end of the beam (relatively to the clamped edge's acceleration), $Y(s)$ corresponds to the voltage across the piezoelectric sensor, $U(s)$ is the voltage applied across the piezoelectric actuator and $W(s)$ is the acceleration imposed to the clamped edge of the beam.

The flexible beam is submitted to a shock-like disturbance $w$, applied to its clamped edge. It is represented by a rectangular signal, of magnitude 1 ($m/s^2$) with a pulse-width of $6 \cdot 10^{-3}$ ($s$). In response to this disturbance signal, the control objective consists in damping the dominant vibration mode without degrading the natural damping of the other modes located at higher frequencies. Some pictures of the first three vibration modes of bending type can be found in Boussaada et al. (2017), where the first one has a resonant frequency at roughly 38 $Hz$.

### 4.2 Controller design

The *disturbance rejection problem* considered here is formulated as a *robust performance control problem*, where the controller has to be robust w.r.t. the vibration modes neglected in the design model. To cope with this issue, designers frequently introduce a high-order roll-off filter in serial with the controller, with a cutoff frequency located between the last mode's frequency included in the design model and the first neglected mode's one. The choice of this filter's parameters is generally rather empirical. The fundamental result behind the use of this filter is the *low-gain theorem* recalled in Zhou et al. (1996) pp. 204. Based on the same idea, in this work, the choice is made to use a first-order filter combined with a QPB controller, with a low-frequency unitary gain but with a cutoff frequency let free for the design procedure. More precisely, the dynamical QPB controller in (9) is sought with $D_0(s) := d_0\,(1+\alpha\, s)$, $D_{\tau_0}(s) := d_{\tau_0}$, $N_0(s) := n_0$ and $N_{\tau_0}(s) := n_{\tau_0}$. It has $N_P = 5$ independent parameters.

Using the notations of Sec. 3 and the design model's data, the resulting polynomials of the system's characteristic function $\Delta(s)$ in (2) are:

$$\begin{aligned} P_0(s) &= \alpha\, d_0\, s^3 + \left((1+\alpha\, a_1) d_0 - n_0\, n_{uy_2}\right) s^2 \\ &\quad + \left((a_1+\alpha\, a_0)\, d_0 - n_0\, n_{uy_1}\right) s + d_0\, a_0 - n_0\, b_{uy_0}, \\ P_\tau(s) &= \left(d_{\tau_0} - n_{\tau_0}\, n_{uy_2}\right) s^2 + \left(d_{\tau_0}\, a_1 - n_{\tau_0}\, n_{uy_1}\right) s \\ &\quad + \left(d_{\tau_0}\, a_0 - n_{\tau_0}\, n_{uy_0}\right). \end{aligned}$$

Notice that $\Delta(s)$ is here of retarded type with $m = 2$ and $n = 3$, thanks to the presence of the first-order filter. Let $s_0 \in \mathbb{R}_-$ be the multiple root to be assigned. The total amount of independent parameters to be tuned is $N_P = 5$. Here, we have imposed $\alpha = 1/d_0$. As mentioned in Remark 2.1, the targeted multiplicity $M(s_0)$ has been taken lower than the quasipolynomial's degree, $5 = N_P \leq M(s_0) = 5 < \deg(\Delta(s)) = 6$, in order to offer more possibility to assign $s_0$ while giving enough equations to deal with the number of unknown parameters.

Hence, the controller's parameters and the multiple root are obtained by solving the set of equations given by (13) for $k = 1$ to $M(s_0) = 5$.

### *4.3 Numerical and simulation results*

Given the numerical data of the design model indicated in Boussaada et al. (2017), the numerical values of the dynamical QPB controller are given in Table 1, for $s_0 = -220$. The

Table 1. Numerical results in the case $s_0 = -220$.

| | | |
|---|---|---|
| $n_0 \approx 10182.71$ | $n_{\tau_0} \approx -7611.07$ | $\tau \approx 8.9366 \cdot 10^{-3}$ |
| $d_0 \approx 895.519$ | $d_{\tau_0} \approx -637.158$ | $\alpha \approx 1.1167 \cdot 10^{-3}$ |

choice of this value is driven by twofold. It is selected in the admissible $s_0(\tau)$-curve in Fig. 2, derived from the previous design procedure. Among the admissible values, the one with a modulus close to the open-loop system's pole is preferred, in order to reduce the control's effort in closed-loop. The roll-off filter's cutoff frequency is roughly equal to 142.5 $Hz$, clearly located at the right frequency region as usually set by the specialists of flexible structures' control.

Two simulations have been performed to check the closed-loop performances, each one with the design model first and then the full-order model. The one in Fig. 3a is the time response of the free-end's acceleration to the shock-like disturbance. In addition to the closed-loop stability for both models, one can notice the spectacular enhancement of the closed-loop settling time w.r.t. the open-loop. Fig. 3b shows a comparison of the

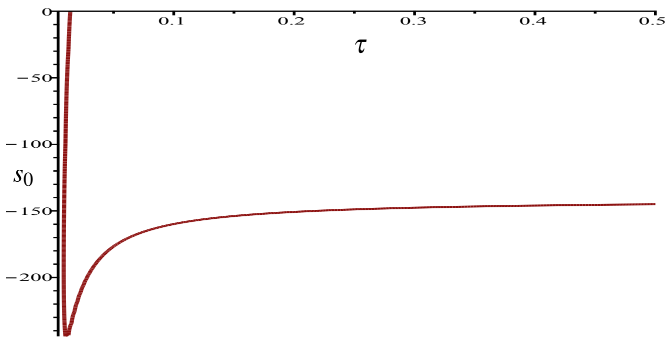


Fig. 2. Admissible $(s_0, \tau)$ pair.

frequency responses for the accelerometric transfer function, *ie* from input $w$ to output $w+z$, in both cases: open-loop *vs* closed-loop. The damping of the first mode's peak of resonance has been successfully achieved, with an attenuation of more than 50 ($dB$). Moreover, this level of damping is maintained with the full-order model. It is worth mentioning that the neglected modes in the design model are remained stable and have also been damped.

To conclude, the problem of disturbance rejection on the controlled output $z(t)$ has been successfully addressed. The achieved closed-loop performances are very close to those in Tliba (2012), obtained for the same system with an optimal $\mathscr{H}_\infty$ controller of finite dimension, designed with regional pole placement constraints and reduced to order 6. However, the structure of these controllers are very different.

## 5. CONCLUSION AND FUTURE WORKS

This work has shown the very promising features of an output-feedback controller based on a basic but triky combination of delayed and filtered terms carrying on the input and output signals: the QPB controller with dynamical parameters where the time-delay is used as a design parameter among the others. The fundamental MID property has been smartly adapted to propose a controller design procedure, with enough degrees-of-freedom, in order to achieve the closed-loop assignment of a given multiple and dominant root. All of this has been illustrated by a realistic but challenging application of active vibration damping, though done in simulation. A further challenge will concern the practical implementation of the QPB controller. This will be the job of subsequent work.

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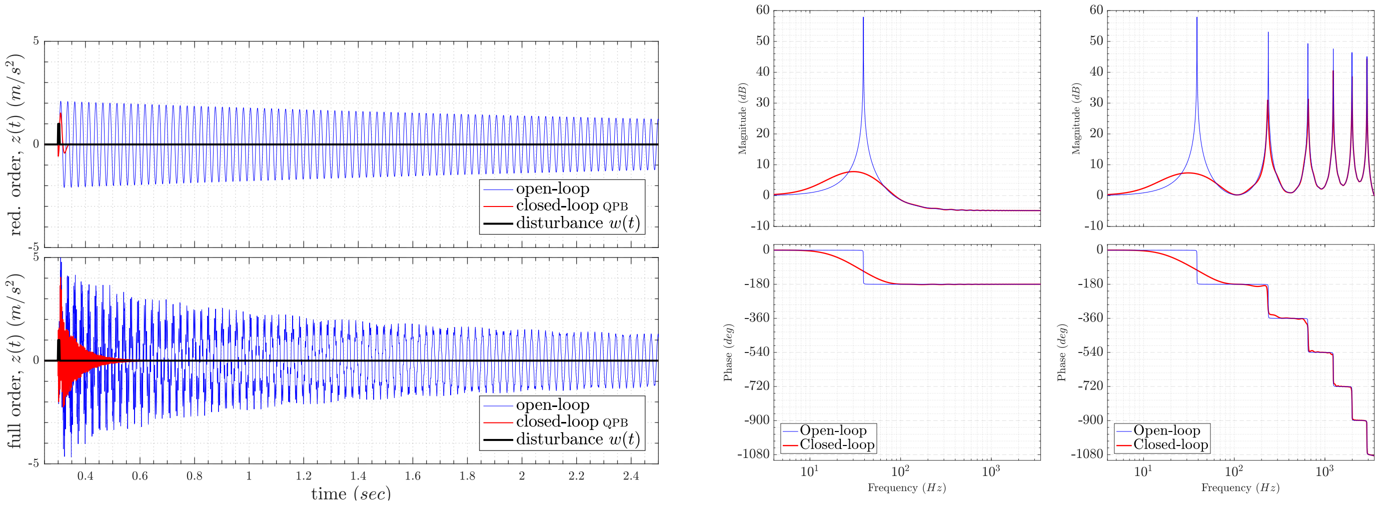


(a) Time responses of the controlled output $z$, for the design model (top) and the full order model (bottom).

(b) Accelerometric frequency responses of the $w$-to-$(w+z)$ transfer function, for the design model (left) and the full order model (right).

Fig. 3. Open (blue) *vs* closed-loop (red) simulations.

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