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Abstract

This work addresses the event-triggered control (ETC) of discrete-time piecewise affine systems. We propose a method to design a triggering strategy relying on an implicit representation of piecewise affine systems. Thanks to this implicit representation based on ramp functions, we propose a partition-dependent piecewise quadratic functions to define the trigger criterion and use a piecewise quadratic Lyapunov function candidate to derive conditions to certify the global exponential stability of the origin under the ETC strategy. Since the stability conditions can be expressed as linear matrix inequalities constraints, we propose a convex optimization solution to design the triggering function parameters and to compute the Lyapunov function to ensure the closed-loop stability and a reduction on the control updates. The approach is illustrated by numerical examples.

Keywords: piecewise affine systems; event-triggered control; stability

1. Introduction

Many physical applications can be described by piecewise affine systems (PWA) such as nonlinear circuits [1–3], and systems with saturation and deadzones nonlinearities [4]. Moreover, smooth nonlinearities appearing in dynamical systems can be approximated by piecewise affine functions leading to PWA models. In the context of model predictive control (MPC) for linear systems with affine constraints and quadratic costs, the optimal solution can be expressed as a piecewise affine control law [5], leading to a PWA closed-loop system. Neural networks with Rectifier Linear Unit (ReLU) activation functions (i.e. ramp functions) can also be modeled by PWA systems. Moreover, the class of PWA systems encompasses the one of piecewise linear (PWL) systems [6, 7].

PWA systems are in general described by an explicit representation [8]. In this case, the piecewise dynamics is associated to a partition of the state space and the sets of the partition are described by the intersection of halfspaces [9–13] or by cone rays and vertices representations [14, 15]. Implicit representations instead do not have separate description for the sets in the partition and the dynamics defined in each set. Among the implicit representations, we can cite the max-min-plus-

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scaling (MMPS), which is equivalent to other models discussed in [16]. More recently, an implicit representation, based on vector valued ramp functions has been proposed in [17].

Most of the works addressing the stability analysis of discrete-time PWA systems consider an explicit representation and piecewise quadratic (PWQ) Lyapunov functions [10–13], where to each set of the partition a different quadratic function is associated. The main drawback of these methods is the need of enumerating the possible transitions between sets in the partition when evaluating the decrease of the Lyapunov function. To reduce conservatism, a reachability analysis should be performed to eliminate from the analysis transitions that never occur [10]. This drawback can be overcome with implicit representations to describe both the system dynamics and PWQ Lyapunov functions. In [17], we introduced an LMI-based method that avoids the enumeration of the transitions to compute the Lyapunov function parameters.

In the context of networked control systems, bandwidth and energy consumption constraints have motivated the development of event-triggered control (ETC) policies [18, 19]. Differently from time-triggered strategies, ETC updates the control law only when some criterion is satisfied, as means of avoiding unnecessary data transmission [20]. In a continuous-time framework, the triggering rule can be implemented by continuously [21] or periodically [22, 23] monitoring some systems variables. Considering discrete-time models the triggering rule is evaluated at each discrete-time instant, which corresponds in general to a periodic sampling strategy. Regarding linear systems, we can cite in this context the works [24], [25] and [26].

Event triggered control has also been studied in the context of switched and switching affine systems [27, 28]. In these references, the problem is significantly different from the one regarding PWA systems, since the changes on the dynamics is given by an external signal (switching signal), which can also be used for control purposes [29]. For the class of PWA systems studied in the paper, the vector field is continuous and each affine dynamics is defined in a set of the partition. In [30], ETC for PWA discrete-time systems is studied considering an explicit representation and quadratic Lyapunov functions to assess stability with a classical quadratic relative error triggering criterion [21] based only on the control signal. Differently from a time-triggered strategy, when an event-triggered strategy is designed using an explicit representation, the reachability analysis mentioned above becomes rather convoluted. The reason for the increase on the complexity of the reachability analysis is that the successor state also depends on the trigger rule and it is not possible to a priori define in how many instants ahead the control will be effectively updated. In practice, to prove closed-loop stability, we would have to enumerate and verify the decrease of the Lyapunov function for all transitions between two regions, which may lead to conservative results [12]. Moreover, since the dynamics of the system is different in each set of the partition, it is natural to introduce different triggering functions depending on the partition. Two issues arise if an explicit representation is used in this case: a) the set where the current state is has to be determined at each instant and the triggering function associated to each partition evaluated; b) the determination of the parameters of the trigger rule associated to each set of the partition such that the number of control updates is effectively reduced.

To tackle the above issues, this paper addresses the event-triggered control for PWA systems using an implicit representation based on vector-valued ramp functions proposed in [17], which implicitly capture all possible transitions. One of the main aspects of the implicit representation is that no reachability analysis is required. Also considering vector-valued ramp functions, we propose a new PWQ triggering function that implicitly encodes the state partition in the event generator, leading to a partition-dependent weighted relative criterion. In particular, such a function allows to efficiently address the issues a) and b) above. Then, using a piecewise quadratic Lyapunov function candidate, LMI conditions are derived to assess the global exponential stability of the origin of the closed-loop system under the event-triggered control strategy. Based on these conditions, we propose a convex optimization problem to design the triggering function parameters aiming at a reduction of the control updates with respect to a time-triggered implementation. The approach is illustrated by some numerical examples.

The paper is organized as follows. In Section 2, the implicit representation of PWA systems is recalled and the event-triggered control problem is stated. Based on properties of vector-valued ramp functions and PWQ Lyapunov candidate functions, in Section 3 conditions for the exponential stability of the origin of the closed-loop system under the proposed ETC strategy are derived. In Section 4 these conditions are cast as linear matrix inequalities (LMIs) and an optimization problem is proposed to design the triggering function parameters. In Section 5 some numerical examples illustrate the application of the proposed approach. Finally, we present some concluding remarks to summarize the main features of the method.

Notation: Norm $\|M\|$ denotes the largest singular value of M and, for square matrices, $\mathrm{He}\{M\} \triangleq M+M^\top$. $M_{(i,j)}$ denotes the (i,j) entry of matrix M and $v_{(i)}$ represents the i-th element of the vector v. The set of symmetric and symmetric positive (negative) definite matrices of dimension n are denoted, respectively, $\mathbb{S}^n$ and $\mathbb{S}^n_+$ ($\mathbb{S}^n_-$). Define the set of diagonal matrices $\mathbb{D}^n = \{M \in \mathbb{R}^{n\times n} \mid M_{(i,j)} = 0,\ i \neq j\}$ and the sets $\mathscr{D}^n_{\{0,1\}} = \{M \in \mathbb{D}^n \mid M_{(i,i)} \in \{0,1\}\}$ and $\mathscr{D}^n_{[0,1]} = \{M \in \mathbb{D}^n \mid M_{(i,i)} \in [0,1]\}$. $\mathbb{P}^{n\times m} = \{M \in \mathbb{R}^{n\times m} \mid M_{i,j} \geq 0, \forall i,j\}$ denotes a set of matrices with nonnegative elements. For vectors $v,u \in \mathbb{R}^m$, $v \succeq u$ denotes an elementwise inequality, i.e. $v_{(i)} \geq u_{(i)}$ for $i = 1,\ldots m$. I_n and 0_n denote the identity matrix and a square zero matrix of order n, respectively. $0_{n,m}$ is an $n\times m$ matrix of zeros. Given square matrices M_1, M_2, $diag(M_1,M_2)$ denotes a block diagonal matrix composed by these matrices. The indicator function of a set $\mathbf{N} \subseteq \mathbb{N}$ is defined as $\mathsf{I}(k,\mathbf{N}) = 1$ if $k \in \mathbf{N}$, $\mathsf{I}(k,\mathbf{N}) = 0$ if $k \notin \mathbf{N}$. For a set Γ, $Int(\Gamma)$ and $\partial\Gamma$ denotes its interior and its boundary, respectively.

2. Problem Statement

2.1. The PWA system

Consider the discrete-time PWA system in closed-loop with a PWA control law and a polyhedral partition of the state-space given by

$$\begin{array}{rcl} x(k+1) & = & A_j x(k) + a_j + Bu(k) \\ u(k) & = & H_j x(k) + h_j \end{array} \quad \forall x(k) \in \Gamma_j,\ j = 1,\ldots,N_p, \quad k \in \mathbb{N} \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state and $u \in \mathbb{R}^m$ is the control input, $A_j \in \mathbb{R}^{n\times n}$, $a_j \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times m}$, $H_j \in \mathbb{R}^{m\times n}$, $h_j \in \mathbb{R}^m$, and each partition Γ_j is defined as

$$\Gamma_j = \{x \in \mathbb{R}^n \mid D_j x + e_j \succeq 0\}$$

with $\Gamma_j \subset \mathbb{R}^n$, $\cup_{j=1}^{N_p} \Gamma_j = \mathbb{R}^n$, $D_j \in \mathbb{R}^{r_j\times n}$, $e_j \in \mathbb{R}^{r_j}$ and $Int(\Gamma_i) \cap Int(\Gamma_j) = \emptyset$ for all $i \neq j$. We assume that the vector field is continuous, namely, $A_j x + a_j = A_i x + a_i$ for $\{x \in \mathbb{R}^n \mid x \in \partial\Gamma_j \cap \partial\Gamma_i\}$.

The above expressions provide an explicit representation of the PWA system. In this paper we consider instead an implicit representation [17] given by

$$x(k+1) = \bar{F}_1 x(k) + \bar{F}_2 \phi(z(x(k))) + Bu(k) \tag{2a}$$
$$z(x(k)) = \bar{F}_3 x(k) + \bar{F}_4 \phi(z(x(k))) + \bar{f}_5, \tag{2b}$$
$$u(k) = K_1 x(k) + K_2 \phi(z(x(k))) \tag{2c}$$

$\bar{F}_1 \in \mathbb{R}^{n\times n}$, $\bar{F}_2 \in \mathbb{R}^{n\times n_z}$, $B \in \mathbb{R}^{n\times n_u}$, $\bar{F}_3 \in \mathbb{R}^{n_z\times n}$, $\bar{F}_4 \in \mathbb{R}^{n_z\times n_z}$ and $\bar{f}_5 \in \mathbb{R}^{n_z}$, $K_1 \in \mathbb{R}^{m\times n}$, $K_2 \in \mathbb{R}^{m\times n_z}$. $z \in \mathbb{R}^{n_z}$ is the argument of the vector-valued function $\phi : \mathbb{R}^{n_z} \to \mathbb{R}^{n_z}$, defined element-wise by the ramp function $r : \mathbb{R} \to \mathbb{R}$, that is

$$\phi_{(i)}(z) = r(z_{(i)}) = \begin{cases} 0 \text{ if } z_{(i)} < 0 \\ z_{(i)} \text{ if } z_{(i)} \geq 0 \end{cases} \tag{3}$$

for each $i = 1,...,n_z$.

Note that when $\bar{F}_4$ is equal to zero or has an upper (or lower) strict block triangular structure, z (and therefore $\phi(z)$) can be straightforwardly determined from (2b). However, in the general case, (2b) is an implicit equation depending on x as $\tilde{f}(z) := z - \bar{F}_4(\phi(z)) = \bar{F}_3 x + \bar{f}_5$. In this case, the following assumption is made to guarantee the well-posedness of the algebraic loop defined in (2).

Assumption 1. *For all $\zeta \in \mathbb{R}^{n_z}$, there exists a unique solution $z \in \mathbb{R}^{n_z}$ to the implicit equation $z - \bar{F}_4(\phi(z)) = \zeta$.*

The lemma below provides a test to verify whether, for a given $\bar{F}_4$, Assumption 1 holds.

Lemma 1. *[17] If there exists a matrix $X \in \mathbb{D}^{n_z}$ such that*

$$-2X + X\bar{F}_4 + \bar{F}_4^\top X < 0, \tag{4}$$

then the implicit equation $z - \bar{F}_4(\phi(z)) = \zeta$ has a unique solution.

Remark 1. *Note that if $x \in \Gamma_j$ the elements of $\phi(z(x))$ will be either equal to zero or $z_{(i)}(x)$. In this case, we can write that $\phi(z(x)) = \Delta_j z(x)$, with $\Delta_j \in \mathscr{D}^{n_z}_{\{0,1\}}$. Thus, the solution of the algebraic loop for $x \in \Gamma_j$ will be given by*

$$z = (I - \bar{F}_4\Delta_j)^{-1}(\bar{F}_3 x + \bar{f}_5). \tag{5}$$

As a matter of fact, condition (4) ensures that $I - \bar{F}_4\Delta$ is nonsingular $\forall \Delta \in \mathscr{D}^{n_z}_{\{0,1\}}$ [17], [31]. Hence, using (5), the explicit representation in (1) can be obtained from the implicit representation (2) by considering the possible values for $\Delta \in \mathscr{D}^{n_z}_{\{0,1\}}$, as follows:

$$A_j = \bar{F}_1 + \bar{F}_2\Delta_j(I - \bar{F}_4\Delta_j)^{-1}\bar{F}_3, \quad a_j = \bar{F}_2\Delta_j(I - \bar{F}_4\Delta_j)^{-1}\bar{f}_5,$$
$$K_j = K_1 + K_2\Delta_j(I - \bar{F}_4\Delta_j)^{-1}\bar{F}_3, \quad h_j = K_2\Delta_j(I - \bar{F}_4\Delta_j)^{-1}\bar{f}_5.$$

Furthermore, since $\Delta_j(i,i) = 1$ means that $z_{(i)} \geq 0$ and $\Delta_j(i,i) = 0$ means that $z_{(i)} \leq 0$, or equivalently $-z_{(i)} \geq 0$, by defining $\bar{\Delta}_j = (2\Delta_j - I_{n_z})$, the matrices describing the partition Γ_j are recovered as *follows:*

$$D_j = \bar{\Delta}_j(I - \bar{F}_4\Delta_j)^{-1}\bar{F}_3, \quad e_j = \bar{\Delta}_j(I - \bar{F}_4\Delta_j)^{-1}\bar{f}_5.$$

Note that the ramp function is a continuous function and $r(z_{(i)}) = 0$ for $z_{(i)} = 0$. In fact, for $z_{(i)}(x) = 0$, the state x belongs to the boundary of two neighbour regions Γ_j (which will be associated to $\Delta_j(i,i) = 1$) and Γ_r (which will be associated to $\Delta_r(i,i) = 0$).

Remark 2. *PWL systems can be seen as a particular class of PWA systems. This corresponds to the case in which a_j and h_j are equal to 0 in the explicit representation (1), or equivalently, vector $\bar{f}_5 = 0$ in our proposed implicit representation (2).*

2.2. Event-triggered Control Strategy

In this section, we formally state the ETC strategy to be used and the problem we aim to solve. We consider an emulation approach for the event-triggering strategy [19, 20]. In this context, we assume that a stabilizing control law by considering a time-triggered updating of the control signal has been computed, that is (2c) is given. The main objective is therefore the design of a trigger rule aiming at reducing the number of control signal updates, while preserving the closed-loop stability. Hence, the following assumption is considered:

Assumption 2. *Matrices K_1 and K_2 are given and are such that the time-triggered control law* (2c) *ensures the global exponential stability of the origin of the closed-loop PWA system* (2)

Note that the satisfaction of this assumption can be checked by the conditions proposed in [17]. The discussion of methods for the computation of K_1 and K_2 satisfying Assumption 2 is out of the scope of this work. This assumption is satisfied, for instance, when global stabilizing state feedback time-triggered control laws are designed for systems with input saturation [32] or when some MPC strategies are used, which are equivalent to PWA control laws [5]. Furthermore, a method for designing global stabilizing control laws for PWA systems considering the implicit representation, i.e. the synthesis of K_1 and K_2 leading to the satisfaction of Assumption 2, has been recently proposed in [33].

The ETC strategy aims therefore to reduce the number of control updates, with respect to a time-triggered control law (2c), by changing the value of the control signal only when an *event* occurs. The events correspond to a violation of a condition given in terms of a threshold of a function l_t, called the *triggering function*. The triggering function considered here depends on the current value of the state and on its value at the last event.

We thus modify the control signal generated by (2c) by considering the expression below

$$u(k) = u(n_i) = K_1x(n_i) + K_2\phi(z(x(n_i))), \quad \forall k \in [n_i,\, n_{i+1}), i \in \mathbb{N}, \tag{6a}$$
$$z(x(n_i)) = \bar{F}_3x(n_i) + \bar{F}_4\phi(z(x(n_i))) + \bar{f}_5 \tag{6b}$$

where the values $n_i \in \mathbb{N}$, $i \in \mathbb{N}$ indicate the time instants at which an event is triggered. From (6a), the control signal is kept constant between two successive events, namely in the interval $[n_i,\, n_{i+1})$, with the value of u computed at the *trigger instant* n_i.

Let us define the *state degradation*

$$\delta(k) = x(k) - x(n_i), \quad \forall k \in [n_i, n_{i+1}), \tag{7}$$

which measures the difference between the current state and the state at the last event instant [21].

For simplicity of notation, the time dependence is henceforth omitted, that is, $x(k)$ and $\delta(k)$ are simply denoted by x and δ, respectively, and $x(n_i) = x - \delta$. Moreover, $x(k+1)$ is denoted by x^+. Hence, from (2a), (2b) and (6), the closed-loop dynamics under the ETC is described by the following equations:

$$\begin{aligned} x^+ &= (\bar{F}_1 + BK_1)x + \bar{F}_2\phi(z(x)) + BK_2\phi(z(x-\delta)) - BK_1\delta \\ z(x) &= \bar{F}_3 x + \bar{F}_4\phi(z(x)) + \bar{f}_5 \\ z(x-\delta) &= \bar{F}_3 x + \bar{F}_4\phi(z(x-\delta)) + \bar{f}_5 - \bar{F}_3\delta \end{aligned} \tag{8}$$

By defining

$$y(x,\delta) = \begin{bmatrix} z(x) \\ z(x-\delta) \end{bmatrix}, \; F_1 = \bar{F}_1 + BK_1, \; F_2 = \begin{bmatrix} \bar{F}_2 & BK_2 \end{bmatrix}, F_{12} = BK_1$$
$$F_3 = \begin{bmatrix} \bar{F}_3 \\ \bar{F}_3 \end{bmatrix}, \; F_4 = \begin{bmatrix} \bar{F}_4 & 0_{n_z} \\ 0_{n_z} & \bar{F}_4 \end{bmatrix}, \; f_5 = \begin{bmatrix} \bar{f}_5 \\ \bar{f}_5 \end{bmatrix}, \; F_\delta = \begin{bmatrix} 0_{n_z,n} \\ \bar{F}_3 \end{bmatrix},$$

we can write the closed-loop dynamics considering the event-triggered control policy as

$$x^+ = F_1 x + F_2\phi(y(x,\delta)) - F_{12}\delta \tag{9a}$$
$$y(x,\delta) = F_3 x + F_4\phi(y(x,\delta)) + f_5 - F_\delta\delta. \tag{9b}$$

Note that since F_4 has a block diagonal structure depending on $\bar{F}_4$, the well-posedness of the closed-loop ETC system (9) is ensured by Assumption 1.

The generation of an event is based on the evaluation of a *triggering function* $l_t(x,\delta)$ at each instant k. The trigger instants are then obtained from the following triggering rule:

$$n_{i+1} = min\{k > n_i \text{ such that } l_t(x(k),\delta(k)) > 0\}. \tag{10}$$

We consider that $l_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a piecewise quadratic (PWQ) function defined as follows:

$$\begin{aligned} l_t(x,\delta) &= x^\top Q_x x + \phi(z(x))^\top Q_\phi \phi(z(x)) + \delta^\top Q_\delta \delta \\ &= x^\top Q_x x + \phi(y(x,\delta))^\top [I_{n_z} \;\; 0]^\top Q_\phi [I_{n_z} \;\; 0]\phi(y(x,\delta)) + \delta^\top Q_\delta \delta \end{aligned} \tag{11}$$

where $Q_\delta \in \mathbb{S}^n_+$ is a symmetric positive-definite matrix and $Q_x \in \mathbb{S}^n$ and $Q_\phi \in \mathbb{S}^{n_z}$ are symmetric matrices such that

$$x^\top Q_x x + \phi(z(x))^\top Q_\phi \phi(z(x)) \leq 0, \; \forall x \in \mathbb{R}^n. \tag{12}$$

Remark 3. *The use of function $l_t(x,\delta)$ defined in* (11) *leads to a piecewise quadratic weighted relative error threshold trigger criterion. It generalizes the weighted relative error threshold trigger criterion* *based on a quadratic triggering function [21, 34]. Recall that, when the state is in a given partition set* Γ_j *we have that* $\phi(z(x)) = \Delta_j(I-\bar{F}_4\Delta_j)^{-1}(\bar{F}_3x+\bar{f}_5)$, *with the value of* Δ_j *depending on* Γ_j . *Thus the triggering function* (11) *implicitly depends on the set of the partition where the state is at the current time instant* k.

We can now state the problem we aim to solve.

Problem 1. *Design the triggering function matrices* Q_x, Q_ϕ *and* Q_δ *such that the origin of the closed-loop system (9), under the ETC strategy given by* (10), *is globallly exponentially stable.*

To take effective advantage of the ETC strategy, in addition to the stability guarantees, the computation of Q_x, Q_ϕ and Q_δ should be carried out aiming to reduce the number of generated events, i.e. the number of control updates with respect to a time-triggered strategy. We thus propose an optimization procedure to compute these matrices. This optimization problem takes into account the stability conditions presented below in its constraints.

3. Stability analysis under the ETC strategy

In this section, we formulate stability conditions to certify the global exponential stability of the origin of closed-loop piecewise affine systems with the event-triggered strategy described in Section 2.2. For notation simplicity, we may denote $y(x,\delta) \in \mathbb{R}^{2n_z}$ simply by $y \in \mathbb{R}^{n_y}$, with $n_y = 2n_z$.

3.1. Properties of ramp functions

Based on properties of the ramp function, this section states a lemma for the function ϕ. Such lemma will be instrumental for the analysis of the stability of PWA systems using the implict representation (8) and, by consequence, for devising conditions to solve the event-triggered control design Problem 1.

Since ϕ is defined elementwise in terms of a ramp function as given in (3), it inherits the following properties from the ramp function, valid for any vector $y \in \mathbb{R}^{n_y}$ [35], [17]:

$$\phi_{(i)}(y) \geq 0, \tag{13a}$$

$$(\phi_{(i)}(y) - y_{(i)}) \geq 0, \tag{13b}$$

$$\phi_{(i)}(y)(\phi_{(i)}(y) - y_{(i)}) = 0. \tag{13c}$$

$\forall i = 1,\ldots,n_y$.

Remark 4. *In [17], the relations* (13) *were stated using* $\phi(-y)$ *instead of* $(\phi(y)-y)$. *Both forms are equivalent due to the identity* $\phi(-y) = \phi(y) - y$.

Based on properties given in (13), let us define

$$s_1(T,y) := \phi^\top(y)T(\phi(y)-y) \tag{14}$$

$$s_2(M,y) := \begin{bmatrix} \phi(y) \\ \phi(y)-y \\ 1 \end{bmatrix}^\top M \begin{bmatrix} \phi(y) \\ \phi(y)-y \\ 1 \end{bmatrix} \tag{15}$$

and state the following lemma.

Lemma 2. *For any matrix $T \in \mathbb{D}^{n_y}$ and any matrix $M \in \mathbb{P}^{(1+2n_y)\times(1+2n_y)}$, the function $\phi : \mathbb{R}^{n_y} \to \mathbb{R}^{n_y}$, defined elementwise in* (3)*, satisfies the following relations:*

$$s_1(T,y) = 0, \ \forall y \in \mathbb{R}^{n_y}, \tag{16a}$$

$$s_2(M,y) \geq 0, \ \ \forall y \in \mathbb{R}^{n_y}. \tag{16b}$$

Proof. Since the elements of ϕ are ramp functions we have $s_1(T,y) = \sum_{i=1}^{n_y} T_{(i,i)}\phi_{(i)}(y)(\phi_{(i)}(y) - y_{(i)})$ which, using (13c), gives (16a). On the other hand, since $\phi(y)$ verifies elementwise properties (13a) and (13b), $\forall y \in \mathbb{R}^{n_y}$, provided that all elements of M are nonnegative, it directly follows that $s_2(M,y)$ is a nonnegative scalar. □

It should be highlighted that relations (16) apply only to vector valued functions ϕ that are elementwise defined as ramp functions. This is a key difference with respect to sector-bounded relations [36], which applies to a broad class of functions.

3.2. Stability conditions

To evaluate the closed-loop stability of the PWA discrete-time systems under the proposed event-triggered control strategy, we consider the class of continuous piecewise quadratic (PWQ) Lyapunov function candidates $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined as follows

$$V(x) = \begin{bmatrix} x \\ \phi(z(x)) \end{bmatrix}^\top P \begin{bmatrix} x \\ \phi(z(x)) \end{bmatrix}, \tag{17}$$

with $P = \begin{bmatrix} P_1 & P_2 \\ P_2^\top & P_3 \end{bmatrix}$, $P_1 \in \mathbb{S}^n$, $P_2 \in \mathbb{R}^{n\times n_z}$, $P_3 \in \mathbb{S}^{n_z}$.

From the implicit representation (9) and a PWQ Lyapunov function candidate given by (17), we formulate now a condition to certify the global exponential stability of the origin of the closed-loop system with the trigger instants defined by (10) with $l_t(x,\delta)$ given by (11).

To evaluate the variation of the Lyapunov function in one step, namely $V(x^+)-V(x)$, we need to evaluate $\phi(y^+(x,\delta))$. Following (9b), we have $y^+(x,\delta) = F_3x^+ + F_4\phi(y^+(x,\delta)) + f_5 - F_\delta\delta^+$, i.e. $y^+(x,\delta)$ depends on $\delta^+ = \delta(k+1)$. However, note that no dynamics for δ^+ can be explicitly obtained since its values depends on the triggering instants, which can not be defined beforehand. In this case, the following lemma, that relates the values of δ and δ^+ with the values of x and x^+, is proposed to cope with terms presenting δ^+.

For this define

$$s_3(S,\delta,x) := \delta^{+\top}S(\delta^+ - \delta + x - x^+). \tag{18}$$

Lemma 3. *Consider $\delta = \delta(k)$ as defined in* (7)*, and $\delta^+ = \delta(k+1)$, then for any matrix $S \in \mathbb{R}^{n\times n}$ the identity*

$$s_3(S,\delta,x) = 0 \tag{19}$$

is satisfied along the trajectories of system (9) *under the triggering rule* (10)*.*

Proof. Recall from (7) that the state degradation is given by $\delta(k) = x(k) - x(n_i)$. Considering the trajectories of system (9) under the triggering rule (10), at instant $k+1$ we have that either $\delta^+ = 0$ or $(\delta^+ - \delta + x - x^+) = 0$ as detailed in the two cases below:

a) An event is generated. In this case, it follows that $n_{i+1} = k+1$ and $x(n_{i+1}) = x(k+1) = x^+$. Then,

$$\delta(k+1) = \delta^+ = x(k+1) - x(n_{i+1}) = 0$$

and the identity (19) is satisfied.

b) No event is generated. In this case, one has that $\delta(k) = x(k) - x(n_i)$ and $\delta(k+1) = x(k+1) - x(n_i)$. Hence, it follows that

$$\delta(k) - x(k) = x(k) - x(n_i) - x(k) = -x(n_i),$$
$$\delta(k+1) - x(k+1) = x(k+1) - x(n_i) - x(k+1) = -x(n_i),$$

that is $\delta(k) - x(k) = \delta(k+1) - x(k+1)$, thus yielding $\delta^+ - \delta + x - x^+ = 0$, which implies that identity (19) is also satisfied.

□

We can now state the following Lyapunov-based Theorem.

Theorem 1. *Consider the functions V, l_t, s_1, s_2, s_3 as defined in* (17), (11), (14), (15), *and* (18) *respectively. If there exist matrices $P \in \mathbb{S}^{(n+n_z)}$, $Q_\delta \in \mathbb{S}^n_+$, $Q_x \in \mathbb{S}^n$ and $Q_\phi \in \mathbb{S}^{n_z}$, $T_1 \in \mathbb{D}^{n_y}$, $T_2 \in \mathbb{D}^{n_y}$, $T_3 \in \mathbb{D}^{2n_y}$, $M_1 \in \mathbb{P}^{(1+2n_y)\times(1+2n_y)}$, $M_2 \in \mathbb{P}^{(1+2n_y)\times(1+2n_y)}$, $M_3 \in \mathbb{P}^{(1+4n_y)\times(1+4n_y)}$ and $S \in \mathbb{R}^{n\times n}$ and positive scalars $\eta < 1$, ε_1 and ε_2 such that the following inequalities are verified along the trajectories of* (9), *for all $x \in \mathbb{R}^n$,*

$$l_t(x,\delta) \leq 0, \quad \textit{for } \delta = 0 \tag{20a}$$
$$V(x) - \varepsilon_1 x^\top x + s_1(T_1, y) - s_2(M_1, y) \geq 0 \tag{20b}$$
$$-V(x) + \varepsilon_2 x^\top x + s_1(T_2, y) - s_2(M_2, y) \geq 0 \tag{20c}$$
$$(1-\eta)V(x) - V(x^+) + s_1(T_3, \tilde{y}) - s_2(M_3, \tilde{y}) + s_3(S, \delta, x) + l_t(x,\delta) \geq 0 \tag{20d}$$

with $\tilde{y} = [y^\top \ y^{+\top}]^\top$, then the origin of the closed-loop system (9) *under the ETC strategy given by* (10) *is globally exponentially stable.*

Proof. From Lemma 2, if (20c) and (20b) are satisfied, it follows that

$$\varepsilon_1 \|x\|^2 \leq V(x) \leq \varepsilon_2 \|x\|^2. \tag{21}$$

Furthermore, using Lemma 2 and Lemma 3, we have that (20d) implies

$$V(x^+) - (1-\eta)V(x) \leq l_t(x,\delta). \tag{22}$$

along the trajectories of system (9).

The remaining of the proof is carried out considering two cases: $k = n_i$ and $k \in (n_i, n_{i+1})$.

If $k = n_i$, it means that an event occurs at the time instant k. In this case, it follows that $x(n_i) = x(k)$ and, consequently, $\delta = 0$. Since from (20a) $l_t(x,\delta) \leq 0$ for $\delta = 0$, it follows from (22) that $\Delta V(x) \leq -\eta V(x)$ whenever $k = n_i$.

Considering now $k \in (n_i, n_{i+1})$, it means that an event does not occur at the time instant k, which implies from (10) that $l_t(x,\delta) \leq 0$, because otherwise an event would have occurred and the control state would have been updated, leading to the situation analyzed for $k = n_i$. Then, from (22), this leads to $\Delta V(x) \leq -\eta V(x)$ whenever $k \in (n_i, n_{i+1})$.

From both cases, we can conclude that $\Delta V(x) \leq -\eta V(x)$, $\forall k \in \mathbb{N}$, provided that (20d) is verified. This fact along with (21) imply the global exponential stability of the origin □

4. Design of the triggering function

The proposition below shows that the conditions of Theorem 1 can be expressed as a set of LMIs.

Proposition 1. *If there exist matrices* $P = \begin{bmatrix} P_1 & P_2 \\ P_2^\top & P_3 \end{bmatrix} \in \mathbb{S}^{(n+n_z)}$, $T_0 \in \mathbb{D}^{n_y}$, $T_1 \in \mathbb{D}^{n_y}$, $T_2 \in \mathbb{D}^{n_y}$, $T_3 \in \mathbb{D}^{n_y}$, $T_4 \in \mathbb{D}^{n_y}$, $M_0 \in \mathbb{P}^{1+2n_y}$, $M_1 \in \mathbb{P}^{1+2n_y}$, $M_2 \in \mathbb{P}^{1+2n_y}$, $M_3 \in \mathbb{P}^{1+4n_y}$, $L_0 \in \mathbb{R}^{(1+2n+2n_y)\times n_y}$, $L_1 \in \mathbb{R}^{(1+2n+2n_y)\times n_y}$, $L_2 \in \mathbb{R}^{(1+2n+2n_y)\times n_y}$, $L_3 \in \mathbb{R}^{(1+4n+4n_y)\times 2n_y}$, $L_4 \in \mathbb{R}^{(1+4n+4n_y)\times n}$, $S \in \mathbb{R}^{n\times n}$, $Q_\delta \in \mathbb{S}^n_+$, $Q_x \in \mathbb{S}^n$ *and* $Q_\phi \in \mathbb{S}^{n_z}$, *and positive scalars* $\eta < 1$, ε_1 *and* ε_2 *such that*

$$\Pi_0 + He\{\Pi_2(T_0) + L_0 G_1\} - \Theta_1^\top M_0 \Theta_1 \geq 0, \tag{23a}$$

$$\Pi_1(\varepsilon_1) + He\{\Pi_2(T_1) + L_1 G_1\} - \Theta_1^\top M_1 \Theta_1 \geq 0 \tag{23b}$$

$$-\Pi_1(\varepsilon_2) + He\{\Pi_2(T_2) + L_2 G_1\} - \Theta_1^\top M_2 \Theta_1 \geq 0 \tag{23c}$$

$$-\Pi_3 + He\{\Pi_4 + L_3 G_2 + L_4 G_3 + \mathscr{S}\} - \Theta_2^\top M_3 \Theta_2 + \mathscr{Q}_t \geq 0 \tag{23d}$$

with

$$\Pi_0 = \begin{bmatrix} \begin{bmatrix} -Q_x & 0_n & 0_{n,n_y} \\ 0_n & 0_n & 0_{n,n_y} \\ 0_{n_y,n} & 0_{n_y,n} & -\tilde{Q}_\phi \end{bmatrix} & 0_{2n+n_y,n_y+1} \\ 0_{n_y+1,2n+n_y} & 0_{n_y+1} \end{bmatrix}, \quad \Pi_2(T_j) = \begin{bmatrix} 0_{2n,2(n+n_y)+1} \\ \begin{bmatrix} 0_{n_y,2n} & T_j & -T_j & 0_{n_y,1} \end{bmatrix} \\ 0_{n_y+1,2(n+n_y)+1} \end{bmatrix} \quad j = 0,1,2,$$

$$\Pi_1(\varepsilon_i) = \begin{bmatrix} \begin{bmatrix} P_1 - \varepsilon_i I & 0_n & \tilde{P}_2 \\ 0_n & 0_n & 0_{n,n_y} \\ \tilde{P}_2^\top & 0_{n_y,n} & \tilde{P}_3 \end{bmatrix} & 0_{2n+n_y,n_y+1} \\ 0_{n_y+1,2n+n_y} & 0_{n_y+1} \end{bmatrix} \quad i = 1,2,$$

$$\tilde{P}_2 = \begin{bmatrix} P_2 & 0_{n,n_z} \end{bmatrix}, \quad \tilde{P}_3 = diag(P_3,\, 0_{n_z}), \quad \tilde{Q}_\phi = diag(Q_\phi,\, 0_{n_z}),$$

$$G_1 = \begin{bmatrix} -F_3 & F_\delta & -F_4 & I_{n_y} & -f_5 \end{bmatrix}, \quad \Theta_1 = \begin{bmatrix} 0_{2n_y+1,2n} & \begin{bmatrix} I_{n_y} & 0_{n_y,n_y} & 0_{n_y,1} \\ I_{n_y} & -I_{n_y} & 0_{n_y,1} \\ 0_{1,n_y} & 0_{1,n_y} & 1 \end{bmatrix} \end{bmatrix},$$

$$\Pi_3 = \begin{bmatrix} \begin{bmatrix} -(1-\eta)P_1 & 0_n & 0_n & 0_n & -(1-\eta)\tilde{P}_2 & 0_{n,n_y} \\ 0_n & P_1 & 0_n & 0_n & 0_{n,n_y} & \tilde{P}_2 \\ 0_n & 0_n & 0_n & 0_n & 0_{n,n_y} & 0_{n,n_y} \\ 0_n & 0_n & 0_n & 0_n & 0_{n,n_y} & 0_{n,n_y} \\ -(1-\eta)\tilde{P}_2^\top & 0_{n_y,n} & 0_{n_y,n} & 0_{n_y,n} & -(1-\eta)\tilde{P}_3 & 0_{n_y} \\ 0_{n_y,n} & \tilde{P}_2^\top & 0_{n_y,n} & 0_{n_y,n} & 0_{n_y} & \tilde{P}_3 \end{bmatrix} & 0_{4n+2n_y,2n_y+1} \\ 0_{2n_y+1,4n+2n_y} & 0_{2n_y+1} \end{bmatrix},$$

$$\Pi_4 = \begin{bmatrix} 0_{4n,4(n+n_y)+1} \\ \begin{bmatrix} 0_{2n_y,4n} & \begin{bmatrix} T_3 & 0_{n_y} & -T_3 & 0_{n_y} & 0_{n_y,1} \\ 0_{n_y} & T_4 & 0_{n_y} & -T_4 & 0_{n_y,1} \end{bmatrix} \end{bmatrix} \\ 0_{2n_y+1,4(n+n_y)+1} \end{bmatrix},$$

$$G_2 = \begin{bmatrix} -F_3 & 0_{n_y,n} & F_\delta & 0_{n_y,n} & -F_4 & 0_{n_y} & I_{n_y} & 0_{n_y} & -f_5 \\ 0_{n_y,n} & -F_3 & 0_{n_y,n} & F_\delta & 0_{n_y} & -F_4 & 0_{n_y} & I_{n_y} & -f_5 \end{bmatrix},$$
$$G_3 = \begin{bmatrix} -F_1 & I_n & F_{12} & 0_n & -F_2 & 0_{n,n_y} & 0_{n,n_y} & 0_{n,n_y} & 0_{n,1} \end{bmatrix},$$

$$\mathscr{S} = \begin{bmatrix} 0_{3n,4(n+n_y)+1} \\ \begin{bmatrix} S & -S & -S & S & 0_{4n_y+1} \end{bmatrix} \\ 0_{4n_y+1,4(n+n_y)+1} \end{bmatrix}, \quad \Theta_2 = \begin{bmatrix} 0_{4n_y+1,4n} & \begin{bmatrix} I_{n_y} & 0_{n_y} & 0_{n_y} & 0_{n_y} & 0_{n_y,1} \\ 0_{n_y} & I_{n_y} & 0_{n_y} & 0_{n_y} & 0_{n_y,1} \\ I_{n_y} & 0_{n_y} & -I_{n_y} & 0_{n_y} & 0_{n_y,1} \\ 0_{n_y} & I_{n_y} & 0_{n_y} & -I_{n_y} & 0_{n_y,1} \\ 0_{1,n_y} & 0_{1,n_y} & 0_{1,n_y} & 0_{1,n_y} & 1 \end{bmatrix} \end{bmatrix},$$

$$\mathscr{Q}_t = diag(Q_x, 0_n, Q_\delta, 0_n, diag(Q_\phi, 0_{n_z}), 0_{3n_y+1}),$$

then the origin of the closed-loop system (9) *under the ETC strategy given by* (10) *is globally exponentially stable.*

Proof. Define the vectors

$$\xi_1 = \begin{bmatrix} x^\top & \delta^\top & \phi(y)^\top & y^\top & 1 \end{bmatrix}^\top \text{ and } \xi_2 = \begin{bmatrix} x & x^{+\top} & \delta^\top & \delta^{+\top} & \phi(y)^\top & \phi(y^+)^\top & y^\top & y^{+\top} & 1 \end{bmatrix}^\top.$$

If (23a) holds, it follows that

$$\xi_1^\top \left(\Pi_0 + \mathrm{He}\{\Pi_2(T_0) + L_0 G_1\} - \Theta_1^\top M_0 \Theta_1 \right) \xi_1 \geq 0.$$

From the algebraic equation (9b) relating y and x, it follows that $G_1\xi_1 = 0$. From Lemma 2 note that $\xi_1^\top \Pi_2(T_0)\xi_1 = s_1(T_0,y) = 0$. Since all entries of M_0 are non-negative, also from Lemma 2 we have that $\xi_1^\top \Theta_1^\top M_0\Theta_1\xi_1 = s_2(M_0,y) \geq 0$. Hence, since $\xi_1^\top \Pi_0\xi_1 = -x^\top Q_x x - \phi(y)^\top \tilde{Q}_\phi \phi(y) = -x^\top Q_x x - \phi(z(x))^\top Q_\phi \phi(z(x))$ we conclude that condition (20a) from Theorem 1 is verified with $l_t(x,\delta)$ defined in (11).

Following the same reasoning with (23b) and (23c), as $\xi_1^\top \Pi_2(T_1)\xi_1 = s_1(T_1,y) = 0$, $\xi_1^\top \Theta_1^\top M_1\Theta_1\xi_1 = s_2(M_1,y) \geq 0$, $\xi_1^\top \Pi_2(T_2)\xi_1 = s_1(T_2,y) = 0$, $\xi_1^\top \Theta_1^\top M_2\Theta_1\xi_1 = s_2(M_1,y) \geq 0$ and since $\xi_1^\top \Pi_1(\varepsilon_i)\xi_1 = V(x) - \varepsilon_i x^\top x$, $i = 1,2$, we conclude that conditions (20b) and (20c) from Theorem 1 are verified provided that these two LMIs hold.

If (23d) holds, it follows that

$$\xi_2^\top(-\Pi_3 + \mathrm{He}\{\Pi_4 + L_3G_2 + L_4G_3 + \mathscr{S}\} - \Theta_2^\top M_3\Theta_2 + \mathscr{Q}_t)\xi_2 \geq 0.$$

From the definition of G_2 and G_3, we also have that $G_2\xi_2 = 0$ and $G_3\xi_2 = 0$ on the trajectories of system (9). Again from Lemma 2, note that $\xi_2^\top \Pi_4\xi_2 = s_1(T_3,y) + s_1(T_4,y^+) = 0$ and since all entries of M_3 are non-negative, it follows that $\xi_2^\top \Theta_2^\top M_3\Theta_2\xi_2 = s_2(M_3,\tilde{y}) \geq 0$. Moreover, from Lemma 3, $\xi_2^\top \mathscr{S}\xi_2 = s_3(S,\delta,x) = 0$ along the trajectories of system (9). Furthermore, note that $\xi_2^\top \mathscr{Q}_t\xi_2 = l_t(x,\delta)$. Hence, as $\xi_2^\top \Pi_3\xi_2 = V(x^+) - (1-\eta)V(x)$, we can conclude that the condition (20d) from Theorem 1 is verified on the trajectories of the system (9), which concludes the proof.

□

Remark 5. *If $\bar{f}_5 \preceq 0$, the function $V(x)$ has a quadratic upper bound given by $V(x) \leq \varepsilon_2\|x\|^2$, where $\varepsilon_2 = (\|P_1\| + 2\sigma\|P_2\| + \sigma^2\|P_3\|)$ with $\sigma = max_{\Delta \in \mathscr{D}^{n_z}_{\{0,1\}}} \left\|\Delta(I - \bar{F}_4\Delta)^{-1}\bar{F}_3\right\|$(see details in [17]). In this case, LMI (23c) can be dropped from Proposition 1.*

Remark 6. *Note that if Assumption 2 is not verified the conditions in Theorem 1 and Proposition 1 will not be feasible. Indeed, at the instants in which an event is generated, we have $\delta = 0$. Then, at these instants, it follows that (9) becomes (2). Thus condition (20d) implies that $\Delta V < 0$ when x^+ is given by (2), which is not possible if Assumption 2 does not hold.*

Thanks to the proposed formulation, the triggering function matrices Q_δ, Q_x and Q_ϕ can be considered as design parameters, since they appear affinely in the LMIs (23a) and (23d).

We propose below a convex optimization problem to compute the parameters of the triggering function l_t aiming to reduce the trigger activity following an approach similar to the one used in [34]. With this aim, let us define an auxiliary quadratic triggering function $l_q(x,\delta) = \delta^\top Q_\delta\delta + x^\top Q_\sigma x$ with $Q_\delta > 0$, $Q_\sigma < 0$, which is used in the following proposition.

Proposition 2. *If there exist matrices $Q_x \in \mathbb{S}^n$, $Q_\phi \in \mathbb{S}^{n_z}$, $Q_\sigma \in \mathbb{S}^n_-$, $T_a \in \mathbb{D}^{n_y}$, $M_a \in \mathbb{P}^{1+2n_y}$ and $L_a \in \mathbb{R}^{(1+2n+2n_y)\times n_y}$ such that*

$$\Pi_5 + He\{\Pi_6 + L_aG_1\} - \Theta_1^\top M_a\Theta_1 \geq 0, \tag{24}$$

with

$$\Pi_5 = \begin{bmatrix} Q_\sigma - Q_x & 0_n & 0_{n,n_y} & 0_{n,n_y} & 0_{n,1} \\ 0_n & 0_n & 0_{n,n_y} & 0_{n,n_y} & 0_{n,1} \\ 0_{n_y,n} & 0_{n_y,n} & -\tilde{Q}_\phi & 0_{n_y,n_y} & 0_{n_y,1} \\ 0_{n_y,n} & 0_{n_y,n} & 0_{n_y} & 0_{n_y} & 0_{n_y,1} \\ 0_{1,n} & 0_{1,n} & 0_{1,n_y} & 0_{1,n_y} & 0_1 \end{bmatrix},\ \Pi_6 = \begin{bmatrix} 0_{2n,2(n+n_y)+1} \\ \begin{bmatrix} 0_{n_y,2n} & T_a & -T_a & 0_{n_y,1} \end{bmatrix} \\ 0_{n_y+1,2(n+n_y)+1} \end{bmatrix},\ \tilde{Q}_\phi = diag(Q_\phi,\ 0_{n_z}).$$

Then $\{(x,\delta) \in \mathbb{R}^{2n} \mid l_t(x,\delta) \leq 0\} \supseteq \{(x,\delta) \in \mathbb{R}^{2n} \mid l_q(x,\delta) \leq 0\}$.

Proof. Similarly to (23a), the satisfaction of (24) implies that $\forall x, \delta$,

$$x^\top Q_x x + \phi(z(x))^\top Q_\phi \phi(z(x)) \leq x^\top Q_\sigma x. \tag{25}$$

Thus it follows that, $\forall x, \delta,\ l_t(x,\delta) = x^\top Q_x x + \phi^\top Q_\phi \phi + \delta^\top Q_\delta \delta \leq x^\top Q_\sigma x + \delta^\top Q_\delta \delta = l_q(x,\delta)$. In conclusion, if $l_q(x,\delta) \leq 0$ then $l_t(x,\delta) \leq 0$ and the set inclusion of the claim holds. □

Note that matrices Q_x and Q_ϕ are symmetric, but no assumptions on their sign definiteness are made. Thus, provided (25) holds, Q_x and Q_ϕ can be sign-indefinite.

The above lemma implies that if a pair (x,δ) generates an event with l_t it also generates an event using the quadratic triggering function l_q. We can then conclude that the number of events generated with l_t is not larger than the number of events generated by a quadratic triggering function.

To obtain a triggering function that reduces the number of events we aim at enlarging the set $\{(x,\delta) \in \mathbb{R}^{2n} \mid l_t(x,\delta) \leq 0\}$, that is, the set of points $(x,\delta) \in \mathbb{R}^{2n}$ for which an event does not occur. Hence, from Proposition 2, this can be indirectly carried out by enlarging the set $\{(x,\delta) \in \mathbb{R}^{2n} \mid l_q(x,\delta) \leq 0\}$. Recalling that $Q_\delta > 0$ and $Q_\sigma < 0$, the following optimization problem is therefore proposed:

$$\begin{aligned} \text{minimize} \quad & \text{trace}(Q_\delta - Q_\sigma) \\ \text{subject to} \quad & (23),\ (24) \end{aligned} \tag{26}$$

5. Numerical Examples

Example I: Consider the piecewise affine system (2), defined by matrices

$$\bar{F}_1 = \begin{bmatrix} 0.5 & 0.85 \\ -1 & 0.5 \end{bmatrix},\ \bar{F}_2 = \begin{bmatrix} 0.75 & 0.75 \\ 0 & 0 \end{bmatrix},\ B = I_2$$
$$\bar{F}_3 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix},\ \bar{F}_4 = \begin{bmatrix} 0 & -1/3 \\ -1 & 0 \end{bmatrix},\ \bar{f}_5 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$K_1 = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix},\ K_2 = \begin{bmatrix} \alpha & \alpha \\ 0 & 0 \end{bmatrix},$$

with α being a scalar. For $\alpha = -0.1$, the solution to (26), gives the following parameters for the triggering function (11)

$$Q_x = \begin{bmatrix} -2.2359 & -1.0205 \\ -1.0205 & -0.5772 \end{bmatrix}, \quad Q_\phi = \begin{bmatrix} -0.0350 & 0.0636 \\ 0.0636 & 0.0491 \end{bmatrix}, \quad Q_\delta = \begin{bmatrix} 2.7701 & 0.3361 \\ 0.3361 & 1.3508 \end{bmatrix}$$

and the Lyapunov function (17) with

$$P = \begin{bmatrix} 0.0169 & -0.0067 & -2.3250 & -2.3127 \\ -0.0067 & 0.0162 & -2.3127 & 2.3163 \\ -2.3250 & -2.3127 & -4.6369 & 1.5363 \\ -2.3127 & 2.3163 & 1.5363 & 4.6183 \end{bmatrix}.$$

To assess the efficiency of the event-triggered strategy in reducing the control updates we carried out simulations for $k \in [0,50]$ of 100 initial conditions evenly distributed in a unitary circle around the origin. The average number of events over the simulation interval was $n_{\text{avg}} = 27.83$. Note that with a time-trigger policy we would have 51 control updates. The ETC strategy leads therefore to a reduction of 45.5% in the number of control updates. Simulation results for $x_0 = [-0.9989 \;\; 0.0476]$, resulting in 26 events is depicted in Figure 1. Considering the set of event instants given by $N = \{n_0, n_1, n_2, \ldots\}$, the bottom plots shows the indicator function $\mathsf{I}(k,N)$, i.e the value 1 denotes that an event has been generated at the instant k.

We recall that the triggering function $l_t(x,\delta)$ defined in (11) is piecewise quadratic due the presence of the term $\phi(z(x))^\top Q_\phi \phi(z(x))$. In fact, this term allows to adapt the triggering criterion according to the dynamics in each region of the partition. This can be seen as a relevant contribution of the proposed method. To show this, let us consider now a classical quadratic triggering function by setting $Q_\phi = 0$, which corresponds to consider the same triggering criterion in each one of the regions of the partition. In this case, the following parameters were obtained when solving (26)[1]:

$$Q_x = Q_\sigma = \begin{bmatrix} -0.4268 & -0.2126 \\ -0.2126 & -0.1063 \end{bmatrix}, \quad Q_\delta = \begin{bmatrix} 1.0534 & 0.0030 \\ 0.0030 & 1.0141 \end{bmatrix}.$$

We then carried out simulations for 100 initial conditions evenly distributed in a unitary circle around the origin. All initial conditions led to 51 events in this case, meaning that an event happened at every single time instant and therefore the ETC strategy did not reduce the number of the control updates. This example illustrates that the term $\phi(y)^\top Q_\phi \phi(y)$, introducing the piecewise quadratic terms in the triggering function, efficiently helps to reduce the number events.

In the literature, the results in [30] for ETC of PWA systems use an explicit representation, a quadratic Lyapunov function and a relative error triggering function that depends only on the control signal. However, considering the conditions given in [30], no triggering function leading to stability was obtained for the same system, which reinforces that the proposed conditions are less conservative.

Example II: Consider the following discrete-time linear system

$$x^+ = A_p x + B_p u,$$

[1]In this case we can set $Q_x = Q_\sigma < 0$ and drop constraints (23a) and (24)

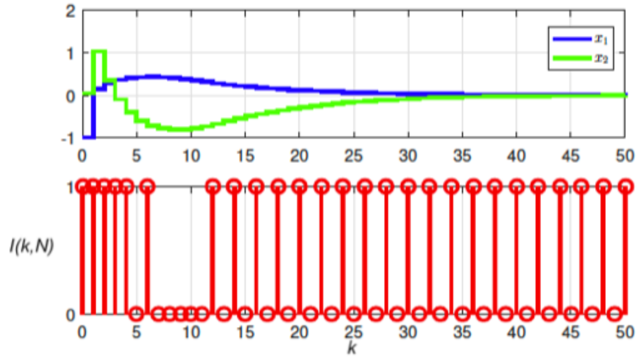

Figure 1: Simulation with $x_0 = [-0.9989 \quad 0.0476]$: 26 events occurred.

with

$$A_p = \begin{bmatrix} 0.7326 & -0.0861 \\ 0.1722 & 0.9909 \end{bmatrix} \text{ and } B_p = \begin{bmatrix} 0.0609 \\ 0.0064 \end{bmatrix}.$$

We consider that the control input u is generated by an explicit MPC law computed as in [5], leading to an explicit PWA representation of the closed-loop system (see Appendix A for the description of the control law). From this explicit representation reported in that paper, we built the following compact implicit representation of the closed-loop system (2) with

$$\bar{F}_1 = A_p,\ \bar{F}_2 = 0_{2,4},\ B = B_p,$$
$$\bar{F}_3 = \begin{bmatrix} \tilde{K}_2 - \tilde{K}_1 \\ \tilde{K}_1 - \tilde{K}_2 \\ -\tilde{K}_1 \\ \tilde{K}_1 \end{bmatrix},\ \bar{F}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix},\ \bar{f}_5 = \begin{bmatrix} -0.6423 \\ -0.6423 \\ -2 \\ -2 \end{bmatrix},$$
$$K_1 = \tilde{K}_1,\ \ K_2 = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix},$$

with $\tilde{K}_1 = \begin{bmatrix} -5.9220 & -6.8883 \end{bmatrix}$ and $\tilde{K}_2 = \begin{bmatrix} -6.4159 & -4.6953 \end{bmatrix}$.

Solving (26), we obtain the following parameters for the triggering function (11)

$$Q_x = \begin{bmatrix} -1.8916 & -1.1546 \\ -1.1546 & -1.1703 \end{bmatrix},\ Q_\delta = \begin{bmatrix} 16.4579 & 18.0110 \\ 18.0110 & 21.9858 \end{bmatrix},$$
$$Q_\phi = \begin{bmatrix} 0.0705 & -0.0135 & -0.0054 & 0.0031 \\ -0.0135 & 0.0680 & -0.0337 & -0.0053 \\ -0.0054 & -0.0337 & 0.0432 & -0.0024 \\ 0.0031 & -0.0053 & -0.0024 & 0.0436 \end{bmatrix}$$

and the PWQ Lyapunov function (17) with

$$P = \begin{bmatrix} 235.7184 & 162.9185 & -13.4113 & 17.7559 & -13.7545 & -7.3717 \\ 162.9185 & 322.5319 & -34.2679 & 16.8953 & -16.4932 & -4.1465 \\ -13.4113 & -34.2679 & 3.2818 & -0.5158 & 2.4523 & 0.3444 \\ 17.7559 & 16.8953 & -0.5158 & -10.4631 & -3.7240 & 0.5601 \\ -13.7545 & -16.4932 & 2.4523 & -3.7240 & 2.8074 & 0.7273 \\ -7.3717 & -4.1465 & 0.3444 & 0.5601 & 0.7273 & -4.0712 \end{bmatrix}.$$

The efficiency of the event-triggered strategy was assessed by simulating 1000 initial conditions generated with uniformly distributed random initial conditions for x_1 and x_2 in the interval $[-1.5 \quad 1.5]$, for $k \in [0 \quad 25]$. The average number of events was $n_{\text{avg}} = 13.50$ thus reducing the number of control updates to almost a half when compared to the number of updates of a time-triggered policy. Figure 4 presents the trajectories for the initial conditions $x_0 = [-1.1804 \ \ 0.4036]$ and $x_0 = [0.7791 \quad -0.7600]$. The simulation results of the of the states and the trigger instants are shown in Figures 2 and 3 where the bottom plot shows the indicator function of the triggering instants. A substantial reduction in the number of control updates can be observed, showing the effectiveness of the proposed ETC strategy. Figure 4 shows the trajectories for both initial conditions along with the state space partition corresponding to the piecewise affine control law.

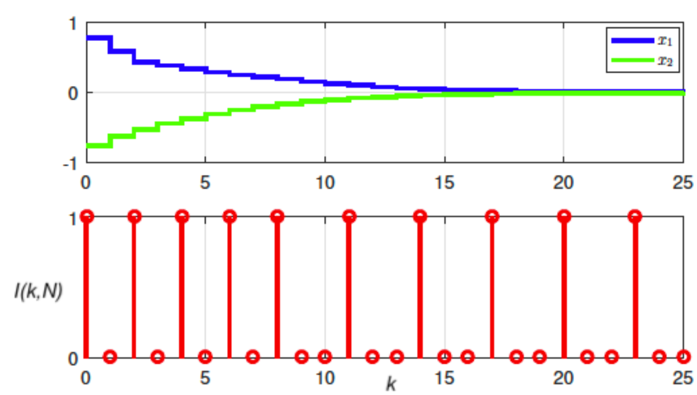

Figure 2: Simulation with $x_0 = [-1.1804 \ \ 0.4036]$: total of 14 events.

6. Conclusions

In this paper an emulation-based design of an ETC strategy for PWA systems was proposed. The adopted approach relies on an implicit representation of the PWA system, which is based on vector valued ramp functions. We proposed the use of a piecewise quadratic triggering function that takes into account the partition information through terms depending also on vector valued ramp functions. In this context, considering a piecewise quadratic Lyapunov candidate functions and properties of the ramp function, we derived conditions to ensure the global exponential stability of the origin under the ETC strategy. These conditions are cast as LMIs and an optimization problem

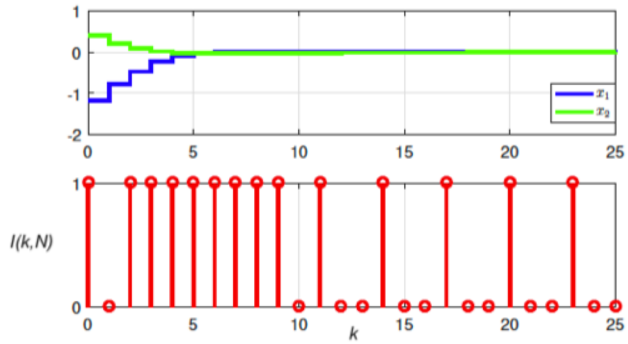

Figure 3: Simulation with $x_0 = [0.7791 \quad -0.7600]$: total of 14 events.

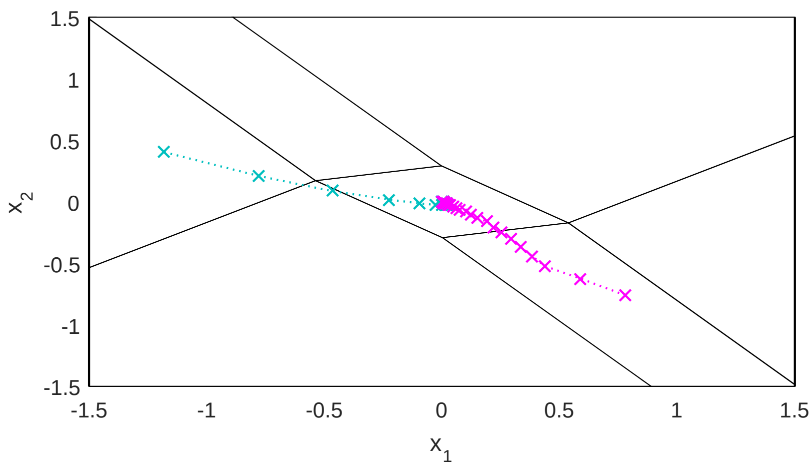

Figure 4: Trajectories from initial conditions $x_0 = [-1.1804 \quad 0.4036]$ (cyan), and $x_0 = [0.7791 \quad -0.7600]$ (magenta)

was formulated to compute the triggering function parameters aiming at stability and reduction of control updates.

Since the transitions are implicitly defined and do not have to be enumerated, the implicit representation makes possible the stability assessment without a preliminary reachability analysis to define the possible transitions between sets in the partition. Moreover, the numerical examples show that it is possible to effectively reduce the number of events using the proposed strategy. The numerical results also illustrate that the proposed PWQ triggering function yields better results than a standard quadratic triggering function.

Appendix A. Control law of Example 2

The explicit MPC law computed in [5] leads to a piecwise affine state feedback, corresponding to seven partitions of the state space, as described in Table A.1.

Table A.1: Explicit MPC law: regions of the partition and associated piecewise linear control laws

Region	Description	$u(k)$
1.	$\begin{bmatrix} -5.9220 & -6.8883 \\ 5.9229 & 6.8883 \\ -1.5379 & 6.8296 \\ 1.5379 & -6.8296 \end{bmatrix} x \leq \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$	$\begin{bmatrix} -5.9220 & -6.8883 \end{bmatrix} x$
2.	$\begin{bmatrix} -6.4159 & -4.6953 \\ -0.0275 & 0.1220 \\ 6.4159 & 4.6953 \end{bmatrix} x \leq \begin{bmatrix} 1.3577 \\ -0.0357 \\ 2.6423 \end{bmatrix}$	$\begin{bmatrix} -6.4159 & -4.6953 \end{bmatrix} x + 0.6423$
3.	$\begin{bmatrix} 6.4159 & 4.6953 \\ 0.0275 & -0.1220 \\ -6.4159 & -4.6953 \end{bmatrix} x \leq \begin{bmatrix} 1.3577 \\ -0.0357 \\ 2.6423 \end{bmatrix}$	$\begin{bmatrix} -6.4159 & -4.6953 \end{bmatrix} x - 0.6423$
4.	$\begin{bmatrix} -3.4155 & 4.6452 \\ 0.1044 & 0.1215 \\ 0.1259 & 0.0922 \end{bmatrix} x \leq \begin{bmatrix} 2.6341 \\ -0.0353 \\ -0.0267 \end{bmatrix}$	2
5.	$\begin{bmatrix} 0.0679 & -0.0924 \\ 0.1259 & 0.0922 \end{bmatrix} x \leq \begin{bmatrix} -0.0524 \\ -0.0519 \end{bmatrix}$	2
6.	$\begin{bmatrix} -0.0679 & 0.0924 \\ -0.1259 & -0.0922 \end{bmatrix} x \leq \begin{bmatrix} -0.0524 \\ -0.0519 \end{bmatrix}$	-2
7.	$\begin{bmatrix} 3.4155 & -4.6452 \\ -0.1044 & -0.1215 \\ -0.1259 & -0.0922 \end{bmatrix} x \leq \begin{bmatrix} 2.6341 \\ -0.0353 \\ -0.0267 \end{bmatrix}$	-2

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