On Quasipolynomials Real Roots Coexistence: Effect on Stability of Time-Delay Systems with Perspectives in Partial Pole Placement

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***Abstract*— In this work, which is a natural continuation of [1] and [2], we show, that the coexistence of the maximal number of real spectral values of generic single-delay retarded second-order differential equations guarantees the realness of the rightmost spectral value. This is interpreted from a control theory point of view as, using a delayed PD controller, one can stabilize a second-order delay differential equation by assigning the maximal number of negative roots of the corresponding characteristic function (which is a quasipolynomial). We give a *necessary and sufficient condition* for the rightmost root to be negative and thus guarantee the exponential decay rate of the system solutions. We illustrate the proposed design methodology in the delayed PD control of the harmonic oscillator.**

***Index Terms*— Time-delay equations, coexisting-real-root-induced-dominancy, stability analysis, spectral method, root assignment.**

I. Introduction

The class of dynamical system considered in this work is represented by a second-order linear autonomous differential equation of the form

$$y''(t) + ay'(t) + by(t) = u(t),$$

where $u(t)$ is a forcing delayed proportional-derivative controller given by

$$u(t) = -\alpha y'(t-\tau) - \beta y(t-\tau), \quad \tau > 0.$$

The closed-loop system is represented by a second-order delay-differential equation (DDE) including five parameters: the plant parameters a, $b \in \mathbb{R}$, the controller gains α, $\beta \in \mathbb{R}$ and the delay $\tau \in \mathbb{R}_+$.

The idea to use the delay as a *controller parameter* is not new and there exists several examples in the literature showing its interest. For instance, at the end of 1970, [3] introduced the so-called "proportional-minus-delay" (PMD) controller and showed that such a controller has a few interesting properties. More precisely, similar to the PD controllers, it shows quick responses to input changes, but, surprisingly, it has less sensitivity to high-frequency noise and it, and thus it can offer an alternative to the "classical" PD controllers. Such controllers were applied and showed their interest in various case studies [4] including, among others, the well-known inverted pendulum [5], [6]. Next, [7], [8] (see also [9]) has shown that a chain of n integrators can be stabilized using n delay blocks. Using this terminology, the PMD controller corresponds to a two-delay block. For a deeper discussion on the effects induced by the delays on the systems' dynamics, we refer to [10], [11].

The interest of considering a delayed PD control law of the above form lies in its generality and simplicity; e.g. the term with the delayed derivative can be seen as the use of past friction available information by the system, see for instance [5], [6].

From a control theory point of view, the method that we propose consists in tuning the control parameters (the "gains" α, β and the "delay" τ) such that the solutions of the closed-loop system have a guaranteed exponential decay.

To achieve this general purpose, several methodologies exist. On one hand, a time-domain approach based on Lyapunov functions has proven some efficiency; see e.g. [12] and the references therein. On the other hand, in frequency-domain, recent studies [13]–[17] have illustrated that the so-called *Multiplicity-Induced-Dominancy* (MID) property can effectively be used to prescribe the decay rate of a DDE. In this approach, one assigns to the characteristic function of the system a single real root s_0 with maximal multiplicity. Then, one shows (under necessary and sufficient conditions) that any other root should have a real part less than s_0, which proves the exponential stability for the solutions of the system. In other words, s_0 represents the *spectral abscissa* of the characteristic function. On the opposite side, our approach assigns to the characteristic function of the equation a maximal number of simple real roots. Then, we give necessary and sufficient conditions to show that the largest root is negative and dominant, and thus it is nothing else than the corresponding spectral abscissa. We call this property: *Coexisting-Real-Root-Induced-Dominancy* (CRRID), [1], [14]. Notice that the well studied MID property [16] can be seen as the limit of the CRRID property when all the simple real roots tends to the same value.

In this work, we show that the CRRID property can successfully be applied to stabilize second-order linear differential equations with a proportional-minus-derivative-delay controller. We provide a *necessary and sufficient* condition that guarantees that the assigned roots are negative and that the largest one is the dominant root of the characteristic function. Hence, we guarantee the exponential decay rate of the solutions of the system. We apply our methodology to the control of an oscillator and give a numerical implementation of our results.

The remaining of the paper is organized as follows: some preliminary results and a simple motivating example are

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presented in Section II. Next, the main results are derived in Section III. An illustrative example is proposed in Section IV and some concluding remarks end the paper.

II. Preliminary results and motivating example

In order to fix the ideas, we consider the following control problem:

$$y''(t) + \omega^2 y(t) = u(t), \tag{1}$$

where $\omega > 0$ stands for the oscillator natural frequency, and we consider a controller u having a *proportional-minus-derivative-delay* structure, i.e.

$$u(t) = -a_0 y(t) - k_t y'(t-\tau) - k_p y(t-\tau), \quad \tau > 0. \tag{2}$$

Thus the corresponding closed-loop characteristic function $\Delta : \mathbb{C} \times \mathbb{R}^3 \times \mathbb{R}_+ \mapsto \mathbb{C}$ is given by:

$$\Delta(s; a_0, k_t, k_d, \tau) = s^2 + \omega^2 + a_0 + e^{-\tau s}\left(k_t s + k_p\right), \tag{3}$$

which is a quasipolynomial. In general, a given quasipolynomial admits an infinite number of roots. However, a corollary of a result by Pólya and Szegö [18, Problem 206.2] asserts that the maximal number of real roots (counted with multiplicities) of a quasipolynomial is equal to its degree (defined as the sum of the involved polynomials plus the number of delays). Therefore, since the degree of (3) is $\deg \Delta = 4$, the purpose of our approach consists in *assigning four negative roots* $s_4 < \ldots < s_1$, and *imposing constraints on the parameters*, such that the real root s_1 corresponds to the rightmost root of the characteristic function, that is the *spectral abscissa* of our system (see, e.g., [11] for a deeper discussion of this notion). Finally, if $s_1 < 0$ is the spectral abscissa, the exponential stability of the trivial solution of the closed-loop system follows straightforwardly.

It is important to emphasize that the method based on the *multiplicity-induced-dominancy* property [17] cannot be applied since the plant (oscillator) is without friction and, therefore, it is impossible to have a real characteristic root with multiplicity four.

III. Main results

Consider a general second-order linear differential equation with a proportional-derivative-delay term of the form:

$$y''(t) + a_1 y'(t) + a_0 y(t) + \alpha_1 y'(t-\tau) + \alpha_0 y(t-\tau) = 0, \tag{4}$$

under appropriate initial conditions belonging to the Banach space of continuous functions $\mathcal{C}([-\tau, 0], \mathbb{R})$. In the Laplace domain, the corresponding quasipolynomial characteristic function defined by $\Delta : \mathbb{C} \times \mathbb{R}^4 \times \mathbb{R}_+^* \to \mathbb{C}$ writes

$$\Delta(s; a_0, a_1, \alpha_0, \alpha_1, \tau) = s^2 + a_1 s + a_0 + e^{-\tau s}\left(\alpha_1 s + \alpha_0\right). \tag{5}$$

The study of zeros of Δ plays a central role in the analysis of the asymptotic stability of (4). Indeed, the trivial solution of the DDE (4) is asymptotically stable if, and only if, all the zeros of Δ are in the open left-half plane [11]. The degree of Δ is 4. Thus, due to the Pólya and Szegö bound, the maximal number of real distinct roots is four. The following Theorem 1 gives necessary and sufficient conditions on the coefficients a_k and α_k of (4) in order to Δ to admits 4 distinct real roots as a function of the delay parameter τ.

Proposition 1 (Coexisting real roots): For a given $\tau > 0$, the quasipolynomial (5) admits 4 distinct real values $s_1, \ldots, s_4$ if, and only if, the coefficients a_1, a_0 and α_1, α_0 are respectively given by the following functions in τ and $S = (s_1, \ldots, s_4)$:

$$a_1 = \frac{-1}{\delta(\tau, S)} \det \begin{pmatrix} s_1^2 & 1 & s_1 e^{-\tau s_1} & e^{-\tau s_1} \\ s_2^2 & 1 & s_2 e^{-\tau s_2} & e^{-\tau s_2} \\ s_3^2 & 1 & s_3 e^{-\tau s_3} & e^{-\tau s_3} \\ s_4^2 & 1 & s_4 e^{-\tau s_4} & e^{-\tau s_4} \end{pmatrix},$$

$$a_0 = \frac{1}{\delta(\tau, S)} \det \begin{pmatrix} s_1^2 & s_1 & s_1 e^{-\tau s_1} & e^{-\tau s_1} \\ s_2^2 & s_2 & s_2 e^{-\tau s_2} & e^{-\tau s_2} \\ s_3^2 & s_3 & s_3 e^{-\tau s_3} & e^{-\tau s_3} \\ s_4^2 & s_4 & s_4 e^{-\tau s_4} & e^{-\tau s_4} \end{pmatrix},$$

and

$$\alpha_1 = \frac{-1}{\delta(\tau, S)} \det \begin{pmatrix} s_1^2 & s_1 & 1 & e^{-\tau s_1} \\ s_2^2 & s_2 & 1 & e^{-\tau s_2} \\ s_3^2 & s_3 & 1 & e^{-\tau s_3} \\ s_4^2 & s_4 & 1 & e^{-\tau s_4} \end{pmatrix},$$

$$\alpha_0 = \frac{1}{\delta(\tau, S)} \det \begin{pmatrix} s_1^2 & s_1 & 1 & s_1 e^{-\tau s_1} \\ s_2^2 & s_2 & 1 & s_2 e^{-\tau s_2} \\ s_3^2 & s_3 & 1 & s_3 e^{-\tau s_3} \\ s_4^2 & s_4 & 1 & s_4 e^{-\tau s_4} \end{pmatrix},$$

where $\delta(\tau, S)$ is the determinant of the following matrix

$$V(\tau, S) = \begin{pmatrix} s_1 & 1 & s_1 e^{-\tau s_1} & e^{-\tau s_1} \\ s_2 & 1 & s_2 e^{-\tau s_2} & e^{-\tau s_2} \\ s_3 & 1 & s_3 e^{-\tau s_3} & e^{-\tau s_3} \\ s_4 & 1 & s_4 e^{-\tau s_4} & e^{-\tau s_4} \end{pmatrix}.$$

The proof of Theorem 1 relies on the study of the matrix $V(\tau, S)$, which appears when trying to solve the set of transcendental equations given by $\Delta_(s_i) = 0$, for $1 \leq i \leq 4$. In particular, we show that its determinant $\delta(S, \tau)$ does not vanish for $\tau > 0$ and distinct roots s_i. The following Theorem 2 gives necessary and sufficient conditions for the spectral abscissa of (5) to be negative leading to *necessary and sufficient conditions* for the exponential stability of (4) with decay rate s_1.

Theorem 2 (Main result): For a fixed $\tau > 0$, assume that the characteristic function (5) admits 4 distinct real roots $s_4 < \ldots < s_1$. The following assertions hold:

1) (Negativity) The spectral value s_1 is negative if, and only if, there exists $\tau^* > 0$ such that

$$a_1(\tau^*, S) - s_2 = 0 \tag{6}$$

2) (Dominancy) If $s_0 \in \mathbb{C}$ is a root of Δ, then $\Re(s_0) \leq s_1$, i.e. the spectral value s_1 is the spectral abscissa of the function (5).

The proof of our main theorem relies on the expression of a_1 given in Theorem 1 and also on the following factorisation of the quasipolynomial Δ.

Proposition 3: Assume that the quasipolynomial Δ admits 2 distinct real roots $s_2 < s_1$. Then it can be written under the following factorised form:

$$\Delta(s) = (s - s_1)(s - s_2)\left(1 - \alpha_1 \frac{\mathrm{d}}{\mathrm{d}\tau}\left[(-\tau)^2 F_{\tau,2}(s, s_1, s_2)\right] + \alpha_0(-\tau)^2 F_{\tau,2}(s, s_1, s_2)\right),$$

where the multivariate function $F_{\tau,2}(s, s_1, s_2)$ is given by

$$\int_0^1 \int_0^1 (1 - t_1) e^{-\tau(t_1 s + (1-t_1)(t_2 s_1 + (1-t_2)s_2)}\, \mathrm{d}t_1 \mathrm{d}t_2.$$

The multivariate function $F_{\tau,2}$ has been defined and extensively analysed in [14].

IV. Illustrative example

To show the potential of our approach, we consider the control problem given by (1) and (2). For simplicity, we consider the case of equidistributed roots, which corresponds to $s_k = s_1 - (k-1)d$, with $d > 0$, for $1 \le k \le 4$. One solve the system of transcendental equations for the control parameters (a_0, k_t, k_p) in terms of the system physical parameter ω as well as the assigned root and the distance between two consecutive roots "d". One obtain the following solution:

$$\begin{aligned}
\tau &= \frac{\sigma}{d},\\
a_0 &= \frac{3}{8}d^2 - \frac{3}{2}s_1 d - \omega^2 + \frac{1}{2}s_1^2,\\
k_t &= \frac{-1}{2}(d - 2s_1)e^{-\tau(d - s_1)},\\
k_p &= \frac{5}{8}(d - 2s_1)(3 - 2s_1)e^{-\tau(d - s_1)},
\end{aligned}$$

with

$$\sigma = \ln\left(\frac{5d - 2s_1}{2d - 2s_1}\right).$$

The distance d has to be chosen such that the positivity of the delay τ is guaranteed. To do so, one has to choose d such that

$$\frac{5d - 2s_1}{2d - 2s_1} = 1 + \frac{4d}{d - 2s_1} > 1,$$

which is equivalent to choose d such that $d > 2s_1$. Since $s_1 < 0$, then d can be arbitrarily chosen. For a numerical illustration, set $\omega = 1$, $d = 1$, and $s_1 = -1$, which implies that $\tau = \ln\left(\frac{7}{3}\right) \approx 0.84\ldots$. The obtained positive value of τ allows the spectral distribution illustrated in Figure 1.

Finally, Figure 2 shows the time-domain simulation of equation (1) with initial conditions $y(0) = \frac{1}{2}$ and $y'(0) = 3$ together with the previously described control parameters for (2).

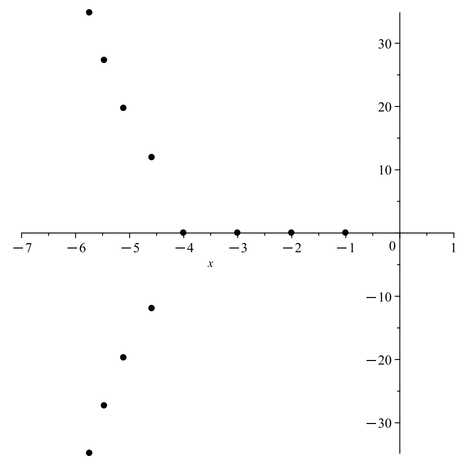

Fig. 1. Spectrum distribution of the closed loop system (1) using a delayed PD controller (2).

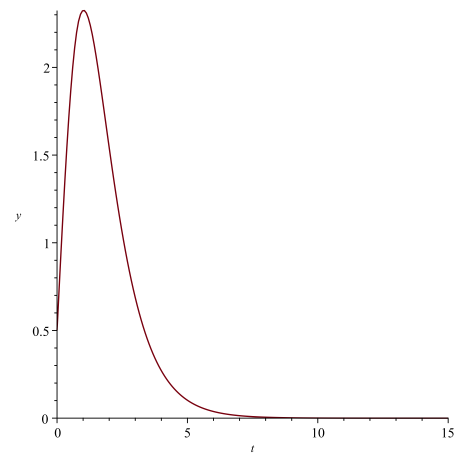

Fig. 2. The closed-loop time-domain simulation of the state variable $y(t)$ in the time-window of 15-seconds. .

V. Concluding remarks

This work focused on the stabilisation of a generic linear differential equation of order 2 with a delayed PD controller via the assignment of the maximum number of real roots of the corresponding characteristic function. We give the expression of the differential equation such that the corresponding quasipolynomial admits the maximal number of real roots (which in our case in four). We give a necessary and sufficient condition on those coefficients in order for the largest real root to be negative. And we show that this root is always the spectral abscissa of the quasipolynomial. Therefore, we proved exponential stability of the second order linear differential equation and thus, we provide an alternative to the MID property.

In future work, we aim to extend these results to systems of higher order and present a detailed proof in a forthcoming publication.

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References

[1] S. Amrane, F. Bedouhene, I. Boussaada, and S.-I. Niculescu, "On qualitative properties of low-degree quasipolynomials: Further remarks on the spectral abscissa and rightmost-roots assignment," *Bulletin math. Soc. Sciences Math. Roumanie*, vol. 61, no. 4, pp. 361–381, 2018.

[2] F. Bedouhene, I. Boussaada, and S.-I. Niculescu, "On pole placement and spectral abscissa characterization for time-delay systems," *IFAC-PapersOnLine*, vol. 52, no. 18, pp. 55–60, 2019.

[3] I. H. Suh and Z. Bien, "Proportional minus delay controller," *IEEE Trans. Automat. Contr.*, vol. 24, no. 2, pp. 370–372, 1979.

[4] I. H. Suh and Z. Bien, "Use of time-delay actions in the controller design," *IEEE Trans. Automat. Contr.*, vol. 25, no. 3, pp. 600–603, 1980.

[5] J. Sieber and B. Krauskopf, "Extending the permissible control loop latency for the controlled inverted pendulum," *Dynamical Systems*, vol. 20, no. 2, pp. 189–199, 2005.

[6] I. Boussaada, I. Morărescu, and S.-I. Niculescu, "Inverted pendulum stabilization: Characterization of codimension-three triple zero bifurcation via multiple delayed proportional gains," *Syst. Control Lett.*, vol. 82, pp. 1–9, 2015.

[7] S.-I. Niculescu and W. Michiels, "Stabilizing a chain of integrators using multiple delays," *IEEE Transactions on Automatic Control*, vol. 49, no. 5, pp. 802–807, 2004.

[8] V. Kharitonov, S.-I. Niculescu, J. Moreno, and W. Michiels, "Static output feedback stabilization: Necessary conditions for multiple delay controllers," *IEEE Trans. Automat. Cont.*, vol. 50, no. 1, pp. 82–86, 2005.

[9] S.-I. Niculescu, W. Michiels, K. Gu, and C. Abdallah, "Delay effects on output feedback control of dynamical systems," *Complex Time-Delay Systems: Theory and Applications*, pp. 63–84, 2010.

[10] S.-I. Niculescu, *Delay Effects on Stability. A Robust Control Approach*. Springer-Verlag: London, 2001.

[11] W. Michiels and S.-I. Niculescu, *Stability and stabilization of time-delay systems*. SIAM: Philadelphia, 2007.

[12] M. Malisoff and F. Mazenc, *Constructions of strict Lyapunov functions*. Springer, 2009.

[13] F. Bedouhene, I. Boussaada, and S.-I. Niculescu, "Real spectral values coexistence and their effect on the stability of time-delay systems: Vandermonde matrices and exponential decay," *C. R. Math. Acad. Sci. Paris*, vol. 358 (9-10), pp. 1011–1032, 2020.

[14] F. Bedouhene, I. Boussaada, and S.-I. Niculescu, "Real spectral values coexistence and their effect on the stability of time-delay systems: Vandermonde matrices and exponential decay," *Comptes Rendus Math.*, vol. 358, no. 9-10, pp. 1011–1032, 2020.

[15] I. Boussaada, G. Mazanti, and S.-I. Niculescu, "The generic multiplicity-induced-dominancy property from retarded to neutral delay-differential equations: When delay-systems characteristics meet the zeros of kummer functions," *C. R. Math. Acad. Sci. Paris*, 2022.

[16] I. Boussaada, S.-I. Niculescu, A. El-Ati, R. Pérez-Ramos, and K. Trabelsi, "Multiplicity-induced-dominancy in parametric second-order delay differential equations: Analysis and application in control design," *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 26, p. 57, 2020.

[17] G. Mazanti, I. Boussaada, and S.-I. Niculescu, "Multiplicity-induced-dominancy for delay-differential equations of retarded type," *J. Differential Equations*, vol. 286, pp. 84–118, 2021, ISSN: 0022-0396.

[18] G. Pólya and G. Szegö, *Problems and Theorems in Analysis: Series · Integral Calculus · Theory of Functions* (Springer Study Edition). Springer: New York, 1972.