Backstepping stabilization of a clamped string with actuation inside the domain

Jeanne Redaud, Jean Auriol

Backstepping stabilization of a clamped string with actuation inside the domain

Jeanne Redaud[1 *] **Jean Auriol**[*] **Silviu-Iulian Niculescu**[1 *]

[*] *Université Paris-Saclay, CNRS, CentraleSupélec, Laboratoire des Signaux et Systèmes, 91190, Gif-sur-Yvette, France.*
[1] *Inria (e-mail: jeanne.redaud@centralesupelec.fr).*

Abstract: In this paper, we consider the stabilization of a clamped string with actuation located inside the domain. Such a model can represent the simplified dynamics of a microendoscope. Inspired by the Port Hamiltonian framework, we use the Riemann invariants of the energy states to reformulate this problem as stabilizing a chain of two coupled hyperbolic subsystems with actuation at the in-between boundary. After applying successive transforms, it is shown to be equivalent to stabilizing a neutral-type delay-differential equation. A suitable controller is derived using the backstepping methodology with a Fredholm integral transform. Some simulations illustrate this approach.

Keywords: Infinite-dimensional system, flexible structures, in-domain actuation, backstepping.

1. INTRODUCTION

In this paper, we consider the stabilization of a wave-like equation with pointwise actuation inside the domain. Stabilizing a one-dimensional wave equation is a well-known problem, and several control strategies have been proposed in the literature. It is well-known that if the damping terms are positive (Cox and Zuazua, 1994), the energy of the system decays exponentially to zero. An adequate control input in this situation can help fasten the convergence. Since the wave equation is conservative, a simple strategy is to add dissipation through the boundary (Lagnese, 1983).

Many other methods, such as spectral methods, multiplier technique, or Lyapunov functionals, have been used to enhance the stability properties of wave equations. For further discussions on these topics, we refer to (Smyshlyaev et al., 2010) (and the references therein). The *backstepping* methodology has more recently proved to be an efficient way of stabilizing infinite-dimensional systems, and in particular wave-like equations and hyperbolic systems (Chen et al., 2022).It can also be used to specify the stability/performance properties of the closed-loop system. For instance, it has been possible to use backstepping transformations to modify the in-domain damping terms of a wave equation (Redaud et al., 2022a) or to handle the case of in-domain velocity recirculation (Jin and Guo, 2022).

However, in most of the examples found in the open literature, the control input is available at one end of the spatial domain (Mounier et al., 1998; Hansen and Zuazua, 1995). This difference is of paramount importance since pointwise in-domain control of the system is then much more difficult due to the intrinsic coupling structure. Interestingly, reformulating two outputs of the interconnected system as a *neutral*-type delay system (Auriol and Di Meglio, 2019), we show that the stabilization of the system is induced by the one of an *integral delay equation* (IDE) with distributed actuation. Modeling the dynamics of the micro-endoscope with in-domain actuation by two interconnected wave-like equation systems, we obtain a relevant case study to apply the approach proposed in (Redaud et al., 2021). However, in this case, the control input acts on both subsystems at the in-between boundary.

Notations Denote the lower (resp. upper) triangular part of the square $\mathcal{T}^- = \{(x,y) \in [0,\ell]^2 |\ y \leq x\}$ (resp. $\mathcal{T}^+ = \{(x,y) \in [0,\ell]^2 |\ x \leq y\}$). For any $\tau > 0$, we denote $D_\tau = H^1([0,\tau],\mathbb{R}^2)$ and the associated norm $\|z_{[t]}\|_{D_\tau} = \left(\int_0^\tau z(t-\theta)^T z(t-\theta)d\theta\right)^{\frac{1}{2}}$. For all $a,b,s \in [0,+\infty)$, define the *characteristic function* $\mathbb{1}_{[a,b]}(s)$, as the function equal to 1 if $s \in [a,b]$, 0 elsewhere.

2. SYSTEM UNDER CONSIDERATION

2.1 Vibrating string model

Consider a long flexible tube of length $\ell > 0$, behaving like a string, clamped at one end ($x = 0$) and free at the other. The actuation is approximated by a discrete stress control action in $x = x_0 \in (0,\ell)$. In this paper, we only consider the case where the transport delays on both sides of the actuator are equal. For constant transport speeds, it corresponds to an actuation in the middle of the beam $x_0 = \frac{\ell}{2}$. For the sake of completeness and in view of these future developments, most of the computations are done in the general case of arbitrary location x_0.

Denote $w(x,t)$ the lateral displacement of the string from a steady-state reference position, with $(x,t) \in [0,\ell] \times [0,+\infty)$. Its dynamics are derived from Hooke's law. The space-dependent strictly positive physical parameters in $C^1([0,\ell],(0,+\infty))$ are $\rho(x)$ the mass density, $c(x)$ the in-domain damping and $E(x)$ the Young's modulus of the string. The displacement $w(x,t)$ satisfies the following PDE

$$\frac{\partial^2 w}{\partial t^2} = \frac{1}{\rho(x)}\frac{\partial}{\partial x}\left(E(x)\frac{\partial w}{\partial x}\right) - c(x)\frac{\partial w}{\partial t}. \quad (1)$$

The two boundary conditions derive from a null speed at the clamped end $\frac{\partial w}{\partial t}|_{x=0}(t) = 0$ and no force at the free one $E(\ell)\frac{\partial w}{\partial x}|_{x=\ell}(t) = 0$. The speed is continuous in $x = x_0$, while there is a discontinuity in force due to the presence of the control input $\frac{\partial w}{\partial x}|_{x_0^-}(t) = \frac{u(t)}{E(x_0)} + \frac{\partial w}{\partial x}|_{x_0^+}(t)$. The initial position of the string is given by $w(x,0) = w_0(x) \in C^1([0,\ell],\mathbb{R})$. Its initial velocity is given by $w_t(x,0) = w_1(x) \in C([0,\ell],\mathbb{R})$. It satisfies appropriate compatibility conditions. Note we could have only required that $(w_0', w_1) \in H^1([0,\ell],\mathbb{R})$. In the rest of the paper, we work in this state space.

2.2 Control objective

Define the strain $X_1(x,t) = \frac{\partial w}{\partial x}(x,t)$ and momentum $X_2(x,t) = \rho(x)\frac{\partial w}{\partial t}(x,t)$ of the string. In this paper, we want to exponentially stabilize the energy of the system. Equivalently, we want the states (X_1, X_2) to be exponentially stable in closed-loop, in the sense of the L^2-norm, i.e. we want to derive a control law such that there exists $C, \nu > 0,\ \forall t > 0$,

$$\|(X_1(t), X_2(t))\|_{L^2} \le Ce^{-\nu t}\|((X_1)_0, (X_2)_0)\|_{L^2}, \quad (2)$$

where $(X_1)_0(x) = w_0'(x)$ and $(X_2)_0(x) = \rho(x)w_1(x)$ correspond to the initial conditions. In the case $c > 0$ under consideration, we can use the control input $u(t)$ to fasten the convergence of the string to its reference position. In a general wave equation stemming from the linearization of an unstable system ($c < 0$), the control input can be used for stabilization purposes.

To design the in-domain actuation, we follow a similar methodology than in (Redaud et al., 2022b).

(1) We use a first change of variables to rewrite the energy states (X_1, X_2) in *Riemann* coordinates. The new states (ξ^+, ξ^-) satisfy transport equations with in-domain couplings. The ones on the diagonal are removed using an exponential change of variables. The new states are denoted $(\bar{\xi}^+, \bar{\xi}^-)$ (Section 2.3).
(2) We use two classical Volterra integral transforms to map $(\bar{\xi}^+, \bar{\xi}^-)$ to a simpler target system (γ^+, γ^-). The in-domain couplings have been moved to the actuated boundary $x = x_0$ (Section 3.1).
(3) Using the method of characteristics, we derive the IDEs satisfied by the boundary states (Section 3.2).
(4) In the case considered in this paper, we can apply the stability results from (Redaud et al., 2022b) under a specific controllability condition. From there, we determine the stabilizing feedback law (Section 3.3).

2.3 Reformulation as interconnected hyperbolic systems

Riemann coordinates Using Riemann coordinates, we can rewrite system (1) as two interconnected systems of heterodirectional coupled hyperbolic equations. This allows us to later apply the backstepping methodology. From now on, we decompose the space domain into two subspaces $\mathcal{I}_1 \doteq [0, x_0)$ and $\mathcal{I}_2 \doteq [x_0, \ell]$. The restriction of the displacement w on $\mathcal{I}_1$ (resp. $\mathcal{I}_2$) is denoted with subscript $\cdot_1$ (resp. $\cdot_2$). Since for all $x \in [0,\ell]$, the matrix $P_H(x) = \begin{pmatrix} 0 & 1/\rho(x) \\ E(x) & 0 \end{pmatrix}$ admits two distinct opposite real eigenvalues $\pm\lambda(x)$, with $\lambda(x) = \sqrt{E(x)/\rho(x)}$, it is diagonalizable. For further simplifications, denote $\Lambda(x) = \mathrm{diag}(-\lambda(x), \lambda(x))$, $\eta(x) = \sqrt{E(x)\rho(x)}$, and define :

$$P(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \frac{1}{\eta(x)} \\ -\eta(x) & 1 \end{pmatrix} \implies P_H(x) = P(x)\Lambda(x)P(x)^{-1}.$$

We now define the *Riemann variables* as follows

$$\forall (x,t) \in [0,\ell]\times[0,+\infty), \begin{pmatrix} \xi^+(x,t) \\ \xi^-(x,t) \end{pmatrix} = P^{-1}(x)\begin{pmatrix} X_1(x,t) \\ X_2(x,t) \end{pmatrix}.$$

It is easy to show that the exponential stability of (X_1, X_2) is equivalent to the one of (ξ^+, ξ^-). For each subsystem $i \in \{1,2\}$, the states ξ_i satisfy two heterodirectional hyperbolic PDEs with in-domain spatially varying couplings

$$\frac{\partial}{\partial t}\begin{pmatrix} \xi_i^+ \\ \xi_i^- \end{pmatrix} + \Lambda(x)\frac{\partial}{\partial x}\begin{pmatrix} \xi_i^+ \\ \xi_i^- \end{pmatrix} = \begin{pmatrix} \sigma^{++} & \sigma^{+-} \\ \sigma^{-+} & \sigma^{--} \end{pmatrix}\begin{pmatrix} \xi_i^+ \\ \xi_i^- \end{pmatrix}, \quad (3)$$

with boundary conditions

$$\begin{aligned} &\xi_1^+(0,t) = \eta(0)^{-1}\xi_1^-(0,t),\ \xi_2^-(\ell,t) = -\eta(\ell)\xi_2^+(\ell,t), \\ &\xi_1^-(x_0,t) = \frac{1}{\sqrt{2}\lambda(x_0)}u(t) + \xi_2^-(x_0,t), \\ &\xi_2^+(x_0,t) = \xi_1^+(x_0,t) - \frac{1}{\sqrt{2}E(x_0)}u(t). \end{aligned} \quad (4)$$

The in-domain couplings $\sigma^{\cdot\cdot}$ are continuous functions that depend on the system parameters and their derivatives. Their expressions come from direct computations and are identical for the two subsystems:

$$\begin{cases} \sigma^{++}(x) = \frac{1}{2}\left(-c(x) + \frac{\lambda(x)}{2}(\frac{\rho'(x)}{\rho(x)} - 3\frac{E'(x)}{E(x)})\right), \\ \sigma^{+-}(x) = \frac{1}{2\eta(x)}\left(c(x) - \frac{\lambda(x)}{2}(\frac{\rho'(x)}{\rho(x)} + \frac{E'(x)}{E(x)})\right), \\ \sigma^{-+}(x) = \frac{\eta(x)}{2}\left(c(x) + \frac{\lambda(x)}{2}(\frac{\rho'(x)}{\rho(x)} + \frac{E'(x)}{E(x)})\right), \\ \sigma^{--}(x) = \frac{1}{2}\left(-c(x) - \frac{\lambda(x)}{2}(3\frac{\rho'(x)}{\rho(x)} - \frac{E'(x)}{E(x)})\right). \end{cases}$$

The initial conditions associated to the interconnected system (3)-(4) are H^1 functions. With the appropriate compatibility conditions, the open-loop system is well-posed (Bastin and Coron, 2016).

Exponential change of variables To rewrite the system as a delay system, we first use an exponential change of variables (Hu et al., 2019) to remove the in-domain couplings σ^{++}, σ^{--}. To ease the notations, define

$$f(x,y) = \sqrt{\frac{E(x)\lambda(x)}{E(y)\lambda(y)}},\ g(x,y) = \sqrt{\frac{\lambda(x)\rho(y)}{\rho(x)\lambda(y)}},$$

$$I_0(x) = \frac{1}{2}\int_0^x \frac{c(s)}{\lambda(s)}\mathrm{d}s,\ I_\ell(x) = \frac{1}{2}\int_x^\ell \frac{c(s)}{\lambda(s)}\mathrm{d}s.$$

Define the new sets of variables $\bar{\xi}_1(t,x)$ and $\bar{\xi}_2(t,x)$ as

$$\begin{pmatrix} \bar{\xi}_1^+ \\ \bar{\xi}_1^- \end{pmatrix} = \begin{pmatrix} f(x,0)e^{I_0(x)} & 0 \\ 0 & g(x,0)e^{-I_0(x)} \end{pmatrix}\begin{pmatrix} \xi_1^+ \\ \xi_1^- \end{pmatrix}, \quad (5)$$

$$\begin{pmatrix} \bar{\xi}_2^+ \\ \bar{\xi}_2^- \end{pmatrix} = \begin{pmatrix} f(x,\ell)e^{-I_\ell(x)} & 0 \\ 0 & g(x,\ell)e^{I_\ell(x)} \end{pmatrix}\begin{pmatrix} \xi_2^+ \\ \xi_2^- \end{pmatrix}. \quad (6)$$

By applying this change of variables to system (3)-(4), the new states satisfy the following equations

$$\frac{\partial}{\partial t}\begin{pmatrix}\bar{\xi}_i^+\\ \bar{\xi}_i^-\end{pmatrix}+\Lambda(x)\frac{\partial}{\partial x}\begin{pmatrix}\bar{\xi}_i^+\\ \bar{\xi}_i^-\end{pmatrix}=\begin{pmatrix}0 & \bar{\sigma}_i^+\\ \bar{\sigma}_i^- & 0\end{pmatrix}\begin{pmatrix}\bar{\xi}_i^+\\ \bar{\xi}_i^-\end{pmatrix}, \quad (7)$$

$$\text{with } \forall x\in\mathcal{I}_1, \begin{cases}\bar{\sigma}_1^+(x)=\dfrac{f(x,0)}{g(x,0)}e^{2I_0(x)}\sigma^{+-}(x),\\ \bar{\sigma}_1^-(x)=\dfrac{g(x,0)}{f(x,0)}e^{-2I_0(x)}\sigma^{-+}(x),\end{cases}$$

$$\forall x\in\mathcal{I}_2, \begin{cases}\bar{\sigma}_2^+(x)=\dfrac{f(x,\ell)}{g(x,\ell)}e^{-2I_\ell(x)}\sigma^{+-}(x),\\ \bar{\sigma}_2^-(x)=\dfrac{g(x,\ell)}{f(x,\ell)}e^{2I_\ell(x)}\sigma^{-+}(x).\end{cases}$$

The boundary conditions are given by

$$\begin{aligned}&\bar{\xi}_1^+(0,t)=\eta(0)^{-1}\bar{\xi}_1^-(0,t),\ \bar{\xi}_2^-(\ell,t)=-\eta(\ell)\bar{\xi}_2^+(\ell,t),\\ &\bar{\xi}_1^-(x_0,t)=-\alpha u^+(t)+q^-\bar{\xi}_2^-(x_0,t),\\ &\bar{\xi}_2^+(x_0,t)=q^+\bar{\xi}_1^+(x_0,t)+u^+(t), \quad (8)\end{aligned}$$

$$\text{with } u^+(t)=-\frac{1}{\sqrt{2}E(x_0)}f(x_0,\ell)e^{-I_\ell(x_0)}u(t), \quad (9)$$

$$q^+=f(0,\ell)e^{-I_0(\ell)},\ q^-=g(\ell,0)e^{-I_0(\ell)}, \quad (10)$$

$$\alpha=\sqrt{\frac{E(\ell)\lambda(\ell)\rho(0)}{\lambda(0)}}e^{I_\ell(x_0)-I_0(x_0)}. \quad (11)$$

Now, since the initial system has been reformulated as two coupled hyperbolic subsystems with actuation at the in-between boundary, we can design a suitable controller inspired by the methodology presented in (Redaud et al., 2022b). Note that, contrary to (Redaud et al., 2022b), the control input simultaneously appears in two boundary conditions. However, this system is said to be underactuated since only one control input is given for both subsystems (Auriol et al., 2019; Auriol and Bresch-Pietri, 2022).

3. CONTROLLER DESIGN

To design the control input, we aim to rewrite the interconnected system as a *delay system*. To do so, we first use an invertible integral transform on each subsystem to move the in-domain couplings at the actuated boundary. We then use the method of characteristics.

3.1 Application of the Backstepping methodology

First Volterra integral transforms Inspired by the backstepping approach (Vazquez et al., 2011), we define the two following Volterra integral operators $\mathcal{K}_i$, $i\in\{1,2\}$

$$\mathcal{K}_1: H^1(\mathcal{I}_1;\mathbb{R}^2)\longrightarrow H^1(\mathcal{I}_1;\mathbb{R}^2) \quad (12)$$

$$\begin{pmatrix}\bar{\xi}_1^+\\ \bar{\xi}_1^-\end{pmatrix}\mapsto\begin{pmatrix}\bar{\xi}_1^+\\ \bar{\xi}_1^-\end{pmatrix}-\int_0^{\cdot}\begin{pmatrix}K_1^{++} & K_1^{+-}\\ K_1^{-+} & K_1^{--}\end{pmatrix}(\cdot,y)\begin{pmatrix}\bar{\xi}_1^+\\ \bar{\xi}_1^-\end{pmatrix}(y)\mathrm{d}y$$

$$\mathcal{K}_2: H^1(\mathcal{I}_2;\mathbb{R}^2)\longrightarrow H^1(\mathcal{I}_2;\mathbb{R}^2) \quad (13)$$

$$\begin{pmatrix}\bar{\xi}_2^+\\ \bar{\xi}_2^-\end{pmatrix}\mapsto\begin{pmatrix}\bar{\xi}_2^+\\ \bar{\xi}_2^-\end{pmatrix}-\int_{\cdot}^{\ell}\begin{pmatrix}K_2^{++} & K_2^{+-}\\ K_2^{-+} & K_2^{--}\end{pmatrix}(\cdot,y)\begin{pmatrix}\bar{\xi}_2^+\\ \bar{\xi}_2^-\end{pmatrix}(y)\mathrm{d}y,$$

where $K_1^{\cdot}$ (resp. $K_2^{\cdot}$) are bounded piecewise continuous functions defined on the lower part of the unit square $\mathcal{T}^-$ (resp. on the upper part $\mathcal{T}^+$). We then introduce the target states $\begin{pmatrix}\gamma_i^+\\ \gamma_i^-\end{pmatrix}=\mathcal{K}_i(\begin{pmatrix}\bar{\xi}_i^+\\ \bar{\xi}_i^-\end{pmatrix})$, $i\in\{1,2\}$.

Kernels equations The kernels satisfy

$$\begin{aligned}\lambda(x)\partial_x K_i^{++}+\lambda(y)\partial_y K_i^{++}&=-\bar{\sigma}_i^-(y)K_i^{+-}-\lambda'(y)K_i^{++},\\ \lambda(x)\partial_x K_i^{+-}-\lambda(y)\partial_y K_i^{+-}&=-\bar{\sigma}_i^+(y)K_i^{++}+\lambda'(y)K_i^{+-},\\ \lambda(x)\partial_x K_i^{-+}-\lambda(y)\partial_y K_i^{-+}&=\bar{\sigma}_i^-(y)K_i^{--}+\lambda'(y)K_i^{-+},\\ \lambda(x)\partial_x K_i^{--}+\lambda(y)\partial_y K_i^{--}&=\bar{\sigma}_i^+(y)K_i^{-+}-\lambda'(y)K_i^{--},\end{aligned}$$

with boundary conditions

$$K_1^{+-}(x,x)=\frac{\bar{\sigma}_1^+(x)}{2\lambda(x)},\ K_1^{-+}(x,x)=-\frac{\bar{\sigma}_1^-(x)}{2\lambda(x)}, \quad (14)$$

$$K_1^{++}(x,0)=\eta(0)K_1^{+-}(x,0), \quad (15)$$

$$K_1^{--}(x,0)=\eta(0)^{-1}K_1^{-+}(x,0), \quad (16)$$

$$K_2^{+-}(x,x)=-\frac{\bar{\sigma}_2^+(x)}{2\lambda(x)},\ K_2^{-+}(x,x)=\frac{\bar{\sigma}_2^-(x)}{2\lambda(x)}, \quad (17)$$

$$K_2^{++}(x,\ell)=-\eta(\ell)K_2^{+-}(x,\ell), \quad (18)$$

$$K_2^{--}(x,\ell)=-\eta(\ell)^{-1}K_2^{-+}(x,\ell). \quad (19)$$

The two sets of equations admit a unique continuous solution on their domain of definition (Coron et al., 2013). The integral transforms $\mathcal{K}_i$, $i\in\{1,2\}$ are both boundedly ivertible operators from $H^1(\mathcal{I}_i;\mathbb{R}^2)$ to $H^1(\mathcal{I}_i;\mathbb{R}^2)$. The inverse operators $\mathcal{L}_i\doteq\mathcal{K}_i^{-1}$ are also Volterra integral operators with the same form.

Equivalent target systems The two Volterra transforms (12)-(13) map system (7)-(8) to

$$\frac{\partial}{\partial t}\begin{pmatrix}\gamma_i^+\\ \gamma_i^-\end{pmatrix}+\Lambda(x)\frac{\partial}{\partial x}\begin{pmatrix}\gamma_i^+\\ \gamma_i^-\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}, \quad (20)$$

with the boundary conditions

$$\gamma_1^+(0,t)=\eta(0)^{-1}\gamma_1^-(0,t),\ \gamma_2^-(\ell,t)=-\eta(\ell)\gamma_2^+(\ell,t), \quad (21)$$

$$\gamma_1^-(x_0,t)=q^-\gamma_2^-(x_0,t)-\alpha u^+(t)+I^-(t), \quad (22)$$

$$\gamma_2^+(x_0,t)=q^+\gamma_1^+(x_0,t)+u^+(t)+I^+(t), \quad (23)$$

where $I^+(t), I^-(t)$ are defined on $[0,+\infty)$ using the inverse kernels. The initial conditions $(\gamma_{i,0}^+,\gamma_{i,0}^-)\in H^1(\mathcal{I}_i;\mathbb{R}^2)$ are obtained from the Volterra transforms of the initial conditions $(\bar{\xi}_{i,0}^+,\bar{\xi}_{i,0}^-)$. The in-domain coupling terms have been replaced by integral terms at the actuated boundary $x=x_0$.

3.2 Reformulation as a delay system

We now reformulate the hyperbolic system as a delay system using the approach proposed in (Auriol and Di Meglio, 2019). More precisely, we show that (20)-(23) is equivalent to an IDE.

Method of characteristics Introduce

$$\begin{cases}y(t)=\gamma_2^+(x_0,t),\\ z(t)=\gamma_1^-(x_0,t)+\alpha\gamma_2^+(x_0,t),\end{cases} \quad (24)$$

with α defined in equation (11). The function z verifies a time-delay equation that does not directly depend on the control input u^+. It somehow corresponds to a flat output of the system. Define $\phi(x)=\int_0^x\frac{1}{\lambda(\nu)}\mathrm{d}\nu$ and $\psi(x)=\int_{x_0}^x\frac{1}{\lambda(\nu)}\mathrm{d}\nu$ two monotically increasing functions and $t_F\doteq 2\max(\phi(x_0),\psi(\ell))$.

Using the method of characteristics, the solution of (20) is given by

$$\gamma_1^+(x,t) = \begin{cases} \gamma_{1,0}^+(\phi^{-1}(\phi(x)-t)), & \text{if } t \leq \phi(x), \\ \gamma_1^+(0, t-\phi(x)), & \text{else}, \end{cases}$$
$$\gamma_1^-(x,t) = \begin{cases} \gamma_{1,0}^-(\phi^{-1}(\phi(x)+t)), & \text{if } t \leq \phi(x_0)-\phi(x), \\ \gamma_1^-(x_0, t-(\phi(x_0)-\phi(x))), & \text{else}, \end{cases}$$
$$\gamma_2^+(x,t) = \begin{cases} \gamma_{2,0}^+(\psi^{-1}(\psi(x)-t)), & \text{if } t \leq \psi(x), \\ \gamma_2^+(x_0, t-\psi(x)), & \text{else}, \end{cases}$$
$$\gamma_2^-(x,t) = \begin{cases} \gamma_{2,0}^-(\psi^{-1}(\psi(x)+t)), & \text{if } t \leq \psi(\ell)-\psi(x), \\ \gamma_2^-(\ell, t-(\psi(\ell)-\psi(x))), & \text{else}. \end{cases}$$

From there, using (21), we have for $t > t_F$,

$$\gamma_1^+(x,t) = \eta(0)^{-1}(z-\alpha y)(t-\phi(x)-\phi(x_0)),$$
$$\gamma_2^-(x,t) = -\eta(\ell)y(t+\psi(x)-2\psi(\ell)).$$

We can also rewrite the integral terms appearing in (22)-(23) in terms of delayed values of (y,z). We can show, by mean of straightforward, yet technical, computations, that in open-loop the outputs (y,z) belong to D_{t_F}.

Reformulation as an integral delay equation Finally, we derive the equations satisfied by functions y, z. For all $t > t_F$, using the boundary condition (23), we have

$$\begin{aligned} y(t) &= u^+(t) + q^+\eta(0)^{-1}(z-\alpha y)(t-2\phi(x_0)) \\ &- q^+ \int_0^{x_0} \eta(0)^{-1} L_1^{++}(x_0,\nu)(z-\alpha y)(t-\phi(\nu)-\phi(x_0)) \\ &\qquad + L_1^{+-}(x_0,\nu)(z-\alpha y)(t+\phi(\nu)-\phi(x_0))\mathrm{d}\nu \\ &+ \int_{x_0}^{\ell} L_2^{++}(x_0,\nu)y(t-\psi(\nu)) \\ &\qquad - \eta(\ell)L_2^{+-}(x_0,\nu)y(t+\psi(\nu)-2\psi(\ell))\mathrm{d}\nu, \quad (25) \\ &\doteq u^+(t) + \mathcal{F}(y,z)(t). \quad (26) \end{aligned}$$

Similarly, from (22), we obtain

$$\begin{aligned} z(t) &= \frac{\alpha q^+}{\eta(0)}(z-\alpha y)(t-2\phi(x_0)) - \eta(\ell)q^- y(t-2\psi(\ell)) \\ &+ \int_0^{x_0} \eta(0)^{-1}[L_1^{-+}(x_0,\nu) - \alpha q^+ L_1^{++}(x_0,\nu)] \\ &\qquad \times (z-\alpha y)(t-\phi(\nu)-\phi(x_0)) \\ &\quad + [L_1^{--}(x_0,\nu) - \alpha q^+ L_1^{+-}(x_0,\nu)] \\ &\qquad \times (z-\alpha y)(t+\phi(\nu)-\phi(x_0))\mathrm{d}\nu \\ &+ \int_{x_0}^{\ell} [\alpha L_2^{++}(x_0,\nu) - q^- L_2^{-+}(x_0,\nu)]y(t-\psi(\nu)) \\ &\quad - \eta(\ell)[\alpha L_2^{+-}(x_0,\nu) - q^- L_2^{--}(x_0,\nu)] \\ &\qquad \times y(t+\psi(\nu)-2\psi(\ell))\mathrm{d}\nu. \quad (27) \end{aligned}$$

By assuming we have access to past values of the functions y, z, we can define for all $t > t_F$ the following new control input $V(t) = u^+(t) + \mathcal{F}(z,y)(t)$. From now on, if $t \leq t_F$, we choose $u^+(t) = 0$. After several changes of variables in the integral terms, the above expression rewrites

$$\begin{aligned} z(t) =& \frac{\alpha q^+}{\eta(0)} z(t-2\phi(x_0)) + \int_0^{2\psi(\ell)} N_V^{\psi}(s)V(t-s)\mathrm{d}s \\ &- \frac{\alpha^2 q^+}{\eta(0)}V(t-2\phi(x_0)) - \eta(\ell)q^- V(t-2\psi(\ell)) \\ &+ \int_0^{2\phi(x_0)} N_z(s)z(t-s) + N_V^{\phi}(s)V(t-s)\mathrm{d}s, \quad (28) \end{aligned}$$

where N_V^{ϕ}, N_z (resp. N_V^{ψ}) are piecewise continuous functions defined on $[0, 2\phi(x_0)]$ (resp. $[0, 2\psi(\ell)]$). In the case $\phi(x_0) = \psi(\ell)$, which corresponds to the case where the transport times on both subsystems are equal, their expression is given in equations (32)-(33). Due to space restriction, we do not give the explicit expression in the general case.

Hence, the output z satisfies an *integral delay equation* with multiple pointwise and distributed delay terms in the actuation. We have the following result

Lemma 1. Consider that there exists a control input $u^+(t)$, such that the functions (y,z) exponentially converge to zero in the sense of the D_{t_F}-norm, i.e there exist $C_1, \nu_1 > 0$ such that

$$\|(y_{[t]}, z_{[t]})\|_{D_{t_F}} \leq C_1 \mathrm{e}^{-\nu_1 t}\|(y_{[t_F]}, z_{[t_F]})\|_{D_{t_F}}, \quad (29)$$

then the original states (X_1, X_2) exponentially converge to zero in the sense of the spatial L^2-norm (2).

Proof. Assume that (29) holds, and let us show that (γ^+, γ^-) exponentially converges to zero in the sense of the spatial L^2-norm. For all $t > t_F$, we have

$$\begin{aligned} \int_0^{x_0} (\gamma_1^+(x,t)^2)\mathrm{d}x &= \int_0^{x_0} \frac{(z-\alpha y)^2}{\eta(0)^2}(t-\phi(x)-\phi(x_0))\mathrm{d}x, \\ &\leq \max(\alpha^2, 1) \max_{[0,x_0]}(\lambda(x))\eta(0)^{-2}\|(y_{[t]}, z_{[t]})\|^2_{D_{t_F}} \\ &\leq \max(\alpha^2, 1) \max_{[0,x_0]}(\lambda(x))\eta(0)^{-2}C_1 e^{-\nu_1 t}\|(y_{[t_F]}, z_{[t_F]})\|^2_{D_{t_F}} \end{aligned}$$

Similar computations can be performed for the other state components γ_2^+, γ_2^- and γ_1^-. Consequently, there exists $C_2 > 0$ such that for all $t > t_F$

$$\|(\gamma^+(t,\cdot), \gamma^-(t,\cdot))\|^2_{L^2} \leq C_2 e^{-\nu_1 t}\|(y_{[t_F]}, z_{[t_F]})\|^2_{D_{t_F}}.$$

Since for $t \leq t_F$ the γ-system is in open-loop (i.e. $u^+ \equiv 0$) and since it is well-posed (Bastin and Coron, 2016), direct but tedious computations give the existence of a constant $C_3 > 0$ such that

$$\|(y_{[t_F]}, z_{[t_F]})\|^2_{D_{t_F}} \leq C_3\|(\gamma^+(0,\cdot), \gamma^-(0,\cdot))\|^2_{L^2}. \quad (30)$$

Injecting (30) in the above equation, we obtain the exponential convergence of state (γ^+, γ^-) to zero in the sense of the spatial L^2-norm. The exponential stability of the initial hyperbolic system (3)-(4) and the states (X_1, X_2) directly follows from the invertibility and boundedness of the different transforms we used. ■

To solve our initial problem, we therefore need to determine an adequate state-feedback controller that exponentially stabilizes the integral delay system (25)-(28).

3.3 Controller design in the case of equal transport delays

First simplification As mentioned in Section 2, we consider in this paper the simplified case for which the leftward and rightward propagation speeds are equal, *i.e* $\phi(x_0) = \psi(\ell) \doteq \tau$. Equation (28) rewrites for $t > 2\tau$

$$\begin{aligned} z(t) =& a_0 z(t-2\tau) + a_1 V(t-2\tau) \quad (31) \\ &+ \int_0^{2\tau} N_z(s)z(t-s) + N_V(s)V(t-s)\mathrm{d}s, \end{aligned}$$

with $a_0 = \frac{\alpha q^+}{\eta(0)}$, $a_1 = -(\frac{\alpha^2 q^+}{\eta(0)} + \eta(\ell)q^-)$ and

$$N_z(s) = \mathbb{1}_{[0,\tau]}(s)\lambda(\phi^{-1}(\tau - s)) \times (L_1^{--} - \alpha q^+ L_1^{+-})(x_0, \phi^{-1}(\tau - s)) + \mathbb{1}_{[\tau,2\tau]}(s)\frac{\lambda(\phi^{-1}(s-\tau))}{\eta(0)}(L_1^{-+} - \alpha q^+ L_1^{++})(x_0, \phi^{-1}(s - \tau)), \tag{32}$$

$$N_V(s) = -\alpha N_z(s) + \mathbb{1}_{[0,\tau]}(s)\lambda(\psi^{-1}(s))(\alpha L_2^{++} - q^- L_2^{-+})(x_0, \psi^{-1}(s)) - \mathbb{1}_{[\tau,2\tau]}(s)\eta(\ell)\lambda(\psi^{-1}(2\tau - s)) \times (q^- L_2^{--} - \alpha L_2^{+-})(x_0, \psi^{-1}(2\tau - s)). \tag{33}$$

System (31) corresponds to an IDE with pointwise and distributed actuation. A control law has successfully been designed in (Redaud et al., 2021, 2022b) to stabilize such a system. The method relied on the introduction of a simple hyperbolic system as a *comparison system*. The control input was then used to cancel the possible destabilizing integral term. To fasten the stabilization, we can also cancel a part of the reflection term $a_0 z(t-2\tau)$ that appears in equation (31). More precisely, for $\bar{a}_0 \in (0, a_0)$, define $\bar{V}(t) = V(t) + \frac{\bar{a}_0}{a_1} z(t)$. Equation (31) rewrites

$$z(t) = (a_0 - \bar{a}_0)z(t - 2\tau) + a_1 \bar{V}(t - 2\tau) + \int_0^{2\tau} \bar{N}_z(s)z(t - s) + N_V(s)\bar{V}(t - s)\mathrm{d}s, \tag{34}$$

with $\bar{N}_z(s) = N_z(s) - \frac{\bar{a}_0}{a_1} N_V(s)$. In what follows, we denote $\tilde{a}_0 = a_0 - \bar{a}_0$. One must be aware that cancelling all the reflection terms could yield to possible robustness issues as emphasized in (Auriol et al., 2023). This is why we introduced the degree of freedom $\bar{a}_0$ in the design.

Controllability assumptions It was shown in (Redaud et al., 2021) that a stabilizing control law for the IDE (34) can be obtained under several conditions that are listed in

Assumption 2. The system parameters must satisfy

- $|\tilde{a}_0| < 1$ and $a_1 \neq 0$,
- For all $s \in \mathbb{C}$, $\text{rank}[F_0(s), F_1(s)] = 1$, where the two holomorphic functions are defined by

$$F_0(s) = 1 - \tilde{a}_0 \mathrm{e}^{-2\tau s} - \int_0^{2\tau} \bar{N}_z(\nu)\mathrm{e}^{-\nu s} d\nu, \tag{35}$$

$$F_1(s) = a_1 \mathrm{e}^{-2\tau s} + \int_0^{2\tau} N_V(\nu)\mathrm{e}^{-\nu s} d\nu. \tag{36}$$

The parameter a_1 is the sum of two strictly negative terms. Moreover, in the case of positive average damping, we have $a_0 = e^{-2I_0(x_0)} < 1$. Consequently, in that case, the first condition is always verified. Otherwise, we can choose $\bar{a}_0$ such that it is the case (even if this may raise some robustness issues). The second requirement can be related to *spectral controllability conditions* (Mounier, 1998). In the case with no damping $c = 0$, it is immediately satisfied since $N_V \equiv 0$, $\bar{N}_z \equiv 0$. More generally, it can be numerically verified using a zero-location algorithm adjusted from the one presented in (Bou Saba et al., 2019). Additional details are given in (Redaud et al., 2022b).

Expression of the control input Under Assumption 2, we have the following result

Theorem 3. There exists two piecewise continuous functions M^1, M^2 defined on $[0, 1]$, such that the state-feedback controller $\bar{V}(t)$ defined for all $t > 2\tau$ by

$$\bar{V}(t) = -\frac{1}{a_1}\int_0^1 M^1(\nu)z(t - \frac{\nu}{\lambda}) + M^2(\nu)y(t - \frac{1-\nu}{\lambda})\mathrm{d}\nu. \tag{37}$$

exponentially stabilizes the integral delay dynamics (34) in the sense of the $D_{2\tau}$-norm. Consequently, it exponentially stabilizes the original (X_1, X_2) in the sense of the spatial L^2-norm.

Proof. From (Redaud et al., 2021), we can prove the existence of functions M^1, M^2, corresponding to kernel boundaries, such that the control input (37) exponentially stabilizes z in the sense of the $D_{2\tau}$-norm. The proof relies on a *comparison system* mapped to an exponentially stable system using a Fredholm integral transform. Moreover, the control input exponentially converges to zero. Consequently, $V(t) = \bar{V}(t) - \frac{\bar{a}_0}{a_1} z(t)$ exponentially converges to zero. The function $y(t)$, defined by equation (25) and that corresponds to $V(t)$, then exponentially converges to zero in the sense of the $D_{2\tau}$-norm. Applying Lemma 1 completes the proof. ■

Note that the expression of the kernels M^1 and M^2 can be found in (Redaud et al., 2021). From the control input $\bar{V}(t)$, it is possible to obtain the adequate control input $u(t)$ stabilizing the initial system. It can be expressed with the energy states using the different transforms. Indeed, by definition $u(t) = -\sqrt{2}E(x_0)f(x_0, \ell)^{-1}e^{I_\ell(x_0)}u^+(t)$ and $u^+(t) = \bar{V}(t) - \frac{\bar{a}_0}{a_1} z(t) - \mathcal{F}(y, z)$, where $\mathcal{F}$ is a pointwise and distributed delay operator defined in (25), using past values of (y, z) over a time $[0, 2\tau]$. Using definition (24), it rewrites with past values of $(\gamma_1^-(x_0, .), \gamma_2^+(x_0, .))$. Therefore, the initial control input $u(t)$ can be computed using the history of the boundary outputs $(\gamma_1^-(x_0, .), \gamma_2^+(x_0, .))$.

4. SIMULATION RESULTS

To illustrate the proposed control strategy, we present some simulation results implemented using Matlab. For the sake of simplicity, we consider a clamped string of length $\ell = 2$m. Its motion satisfies the wave equation (1) with constant coefficients $\rho = 936\text{kg.m}^{-3}, E = 4.14\text{GPa}$, and a constant damping term $c = 1$. This initial system is then naturally stable due to the presence of dissipative terms. The initial string position is $w_0(x) = \frac{2\ell}{\pi}\cos(\frac{\pi}{2\ell}(\ell - x))$. The space domain $[0, \ell]$ is discretized with a mesh of $\texttt{nx} = 200$ points. We simulated system (7)-(8) on a time horizon of 10s using a Godounov Scheme (LeVeque, 2002) ($CFL = 1$). We first compute the values of the different coupling terms. Then, we solve the sets of kernel equations for the two Volterra integral transforms and the Fredholm integral transform, along with the kernels of the inverse transforms. We use a fixed-point algorithm (successive approximation technique) with an error threshold $\epsilon = 10^{-8}$, and store their values in matrices of size $\texttt{nx} \times \texttt{nx}$. All integral terms are approximated using the trapezoidal method. The control input is computed at each time step using (37). We can illustrate the evolution of the virtual output $z(t)$ defined on $[-2\tau, 10]$s. The control input is then applied for $t \geq 0$. We see in Figure 1 that the more we cancel the reflection term, the faster the amplitude of the oscillations decreases (to the price of numerical instabilities).

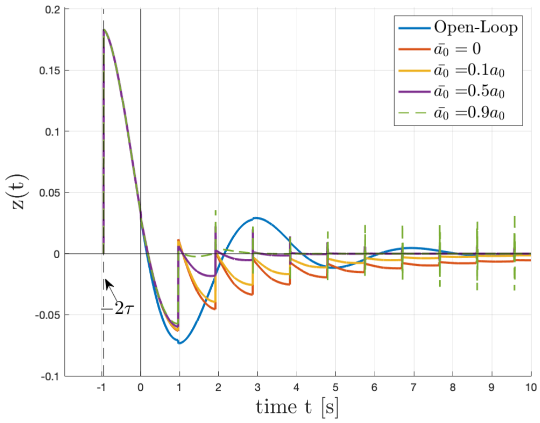

Fig. 1. Evolution of the output $z(t)$ for different $\bar{a}_0$

5. CONCLUSION

In this paper, we designed an exponentially stabilizing control law for a clamped string with actuation located inside the domain. However, we only solved the case where the transport delays on both sides of the actuator were equal. It should be mentioned that classical backstepping control strategies cannot be straightforwardly applied since the actuation is not at the boundary of the domain. We believe the proposed strategy can be extended to the stabilization of strings when the control input is in any arbitrary location x_0. This would result in the following general form of integral delay equations

$$z(t) = b_0 z(t-\tau_0) + c_0 V(t-\tau_0) + c_1 V(t-\tau_1) + \int_0^{\max(\tau_0,\tau_1)} N_V V(t-s) + N_z(s) z(t-s) \mathrm{d}s,$$

with multiple pointwise delays in the actuation. The paper does not address the stabilization of such a class of systems. Moreover, it could be of high interest to consider non-scalar systems, such as Timoshenko beams. Higher dimensions would lead to more intricate computations, which could result in many non-commensurate delays. Finally, to be implemented on real systems, a state estimation is necessary. As shown in (Redaud et al., 2022b), a state observer could be designed using a similar approach.

REFERENCES

Auriol, J. and Bresch-Pietri, D. (2022). Robust state-feedback stabilization of an underactuated network of interconnected n+ m hyperbolic PDE systems. *Automatica*, 136, 110040.

Auriol, J., Bribiesca Argomedo, F., and Di Meglio, F. (2023). Robustification of stabilizing controllers for ODE–PDE–ODE systems: a filtering approach. *Automatica*, 147, 110724.

Auriol, J. and Di Meglio, F. (2019). An explicit mapping from linear first order hyperbolic PDEs to difference systems. *Systems & Control Letters*, 123, 144–150.

Auriol, J., Di Meglio, F., and Bribiesca-Argomedo, F. (2019). Delay robust state feedback stabilization of an underactuated network of two interconnected PDE systems. In *American Control Conference*, 593–599.

Bastin, G. and Coron, J.M. (2016). *Stability and boundary stabilization of 1-D hyperbolic systems*. Springer.

Bou Saba, D., Bribiesca Argomedo, F., Auriol, J., Di Loreto, M., and Di Meglio, F. (2019). Stability analysis for a class of linear 2×2 hyperbolic PDEs using a backstepping transform. *IEEE Transactions on Automatic Control*, 65(7), 2941–2956.

Chen, G., Vazquez, R., and Krstic, M. (2022). Rapid stabilization of timoshenko beam by PDE backstepping. *arXiv preprint arXiv:2207.04746*.

Coron, J.M., Vazquez, R., Krstic, M., and Bastin, G. (2013). Local exponential H^2 stabilization of a 2×2 quasilinear hyperbolic system using backstepping. *SIAM Journal on Control and Optimization*, 51(3), 2005–2035.

Cox, S. and Zuazua, E. (1994). The rate at which energy decays in a damped string. *Communications in partial differential equations*, 19(1-2), 213–243.

Hansen, S. and Zuazua, E. (1995). Exact controllability and stabilization of a vibrating string with an interior point mass. *SIAM Journal on Control and Optimization*, 33(5), 1357–1391.

Hu, L., Di Meglio, F., Vazquez, R., and Krstic, M. (2019). Boundary exponential stabilization of 1-dimensional inhomogeneous quasi-linear hyperbolic systems. *SIAM Journal on Control and Optimization*, 57(2), 963–998. doi:10.1137/15M1012712.

Jin, F.F. and Guo, W. (2022). Boundary stabilization of a 1-D wave equation with multi-point velocity recirculations. *Systems & control letters*, 164, 105230.

Lagnese, J. (1983). Decay of solutions of wave equations in a bounded region with boundary dissipation. *Journal of Differential equations*, 50(2), 163–182.

LeVeque, R.J. (2002). *Finite volume methods for hyperbolic problems*. Cambridge university press.

Mounier, H. (1998). Algebraic interpretations of the spectral controllability of a linear delay system. In *Forum Mathematicum*, volume 10, 39–58. De Gruyter.

Mounier, H., Rudolph, J., Fliess, M., and Rouchon, P. (1998). Tracking control of a vibrating string with an interior mass viewed as delay system. *ESAIM: Control, Optimisation and Calculus of Variations*, 3, 315–321. doi:10.1051/cocv:1998112.

Redaud, J., Auriol, J., and Le Gorrec, Y. (2022a). Distributed damping assignment for a wave equation in the port-hamiltonian framework. In *IFAC CPDE Workshop*.

Redaud, J., Auriol, J., and Niculescu, S.I. (2021). Stabilizing integral delay dynamics and hyperbolic systems using a Fredholm transformation. In *Conference on Decision and Control*. IEEE.

Redaud, J., Auriol, J., and Niculescu, S.I. (2022b). Stabilizing output-feedback control law for hyperbolic systems using a fredholm transformation. *IEEE transactions on automatic control*, 1–16.

Smyshlyaev, A., Cerpa, E., and Krstic, M. (2010). Boundary stabilization of a 1-D wave equation with in-domain antidamping. *SIAM journal on control and optimization*, 48(6), 4014–4031.

Vazquez, R., Krstic, M., and Coron, J.M. (2011). Backstepping boundary stabilization and state estimation of a 2×2 linear hyperbolic system. In *Conference on Decision and Control and European Control Conference*, 4937–4942. IEEE.