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Abstract

In this work, we show that the coexistence of the maximal number of real spectral values of generic single-delay retarded second-order differential equations guarantees the realness of the rightmost spectral value. From a control theory standpoint, this entails that a delayed proportional-derivative (PD) controller can stabilize a delayed second-order differential equation. By assigning the maximum number of negative roots to the corresponding characteristic function (a quasipolynomial), we establish the conditions for asymptotic stability. If the assigned real spectral values are uniformly distributed, we specify a *necessary and sufficient condition* for the rightmost root to be negative, thus guaranteeing the exponential decay rate of the system's solutions. We illustrate the proposed design methodology in the delayed PD control of the damped harmonic oscillator. It is worth mentioning that this work represents a natural continuation of [Amrane et al., 2018] and [Bedouhene et al., 2020], addressing the problem of coexisting real spectral values for linear dynamical systems including delays in their models.

Keywords : Time-delay differential equation, Stability and stabilization, Spectral method, Exponential decay, Partial pole placement, Coexisting-real-root-induced-dominancy

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1 Introduction

The purpose of this paper is to study the asymptotic stability of the solutions of a dynamical system represented by a second-order linear differential equation of the form:

$$y''(t) + a_1 y'(t) + a_0 y(t) = u(t - \tau_0), \quad \tau_0 \geq 0,$$

where τ_0 is a minimum delay due to physical constraints and $u(t)$ is a forcing delayed proportional-derivative controller given by

$$u(t) = -\alpha_1 y'(t - \tau_1) - \alpha_0 y(t - \tau_1), \quad \tau_1 \geq 0.$$

The closed-loop system is represented by a second-order autonomous delay-differential equation (DDE) including five parameters: the parameters of the open-loop system a_1, $a_0 \in \mathbb{R}$, the controller gains α_1, $\alpha_0 \in \mathbb{R}$ and the sum of input and output channel delays $\tau = \tau_0 + \tau_1 \in \mathbb{R}_+^*$, see for instance [2]–[9] for further insights on time-delay systems and their qualitative properties.

If the interest in considering a delayed control law as above lies in its generality and simplicity (tuning only three parameters, including the *delay*) in real-time implementation [10]–[12], its main inconvenient concerns the fact that the closed-loop system becomes *infinite-dimensional* leading, in some cases, to unexpected dynamical behaviors with respect to changes in the system's parameters. Such a controller belongs to the class of *low complexity controllers*. For a more general discussion on such controllers as well as some of their applications, we refer to [13]. In this frame, it should be mentioned that the idea to use the *delay* as a *controller parameter* is not new and there exists several examples in the literature exhibiting its relevance. For instance, at the end of 1970, [10] introduced the so-called "proportional-minus-delay" (PMD) controller and highlighted a few interesting properties. More precisely, similar to the PD controllers, it operates quick responses to input changes, but, surprisingly, it is less sensitive to high-frequency noise, and thus it can offer an alternative to the "classical" PD controllers by conserving its low-complexity character*. Such controllers have been implemented and have demonstrated their interest in various case studies [14] including, among others, the well-known inverted pendulum [11], [12] and, more recently, human balancing [15]. Finally, as [16] points out in a different framework, the use of a delay in the controller led to a broader class of stabilizing second-order systems. In other words, there exists stabilizing delayed PD controller such that the delay-free PD controller does not stabilize the corresponding dynamical system. This idea represents additional motivation for our paper.

In this framework, the method that we propose consists in tuning the control parameters (the "gains" α_1, α_0 and the "delay" τ) in such a way that the solutions of the closed-loop system are asymptotically stable with a guaranteed exponential decay rate.

To achieve this general purpose, several methodologies exist with various objectives. On the one hand, a time-domain approach based on Lyapunov functions has proven its efficiency in estimating solutions decay; see e.g. [17] and the references therein. On the other hand, in frequency-domain, recent studies [18]–[21] have illustrated that the so-called *Multiplicity-Induced-Dominancy* (MID) property can effectively be used to prescribe the decay rate of a DDE solution, [22]. In this approach, one assigns to the characteristic function of the system a single real root s_0 with maximal multiplicity. Then, one shows (under necessary and sufficient conditions) that any other root should have a real part less than s_0, which proves the exponential stability for the solutions of the system. In other words, s_0 represents the *spectral abscissa* of the characteristic function.

The originality of the proposed approach is to assign a maximal number of simple real zeros of the characteristic function rather than a multiple zero. Then, we give necessary and sufficient

*i.e., three tuning parameters

conditions ensuring that the largest assigned root is *negative and dominant*, so that it is nothing but the corresponding spectral abscissa. We call this property: *Coexisting-Real-Root-Induced-Dominancy* (CRRID). The systematic analysis of the CRRID began in [18], [23], where the asymptotic stability of the solutions of an n-order linear differential equation commanded by a delayed proportional-position-controller. Notice that the MID property mentioned above [19], [20] can be seen as the limit of the CRRID property when all the simple real roots tend to the same (real) value. Moreover, it is commonly accepted that multiple roots are sensitive with respect to small parameter changes; see for instance [24] and references therein. Therefore, the CRRID property exhibits an interesting parametric robustness compared to the MID. As a matter of fact, the effectiveness of the CRRID property has already been emphasized through the problem of control of vibrations, see for instance [25]. For other methods/approaches and ideas, we refer to [26] and [27] (see also, [28] for an overview on such topics).

The contribution of this work is twofold. First, we show that the CRRID property can successfully be applied to stabilize second-order linear systems represented by differential equations with a delayed proportional-minus-derivative controller. More precisely, we provide a *necessary and sufficient* condition that guarantees that the assigned real roots are negative and that the largest one is the dominant root of the characteristic function that is, the spectral abscissa of the dynamical system in closed-loop. Hence, we guarantee the exponential decay rate of the solutions of the system. To the best of the authors' knowledge, such a result represents a novelty in the literature. The proof makes use of the Stépán-Hassard formula [3], [29] applied to an appropriate transformation of the characteristic function. Second, we apply our methodology to two case studies, showing that the proposed methodology can be applied to stabilize unstable (second-order) plants and to improve the solution decay rate. Furthermore, in the first case, the controller gains do not guarantee the closed-loop stability of the system in the delay-free case reinforcing thus the argument that the delay has a stabilizing effect (see, e.g., Remark 4.1). In the second case, the decay rate is guaranteed for "small" (controller) gains, and thus less sensitive to high-frequency noise (see, for instance, Remark 4.2).

The remaining of this paper is organized as follows: some preliminary results (functional Vandermonde-type matrices, Stépán-Hassard formula) are presented in Section 2. Next, the main results are derived and commented in Section 3. Two illustrative examples are proposed in Section 4 and some concluding remarks end the paper. The notations are standard.

2 Preliminaries

In this paper, we consider a dynamical system represented by a general second-order linear differential equation with a proportional-derivative-delay term of the form

$$y''(t) + a_1 y'(t) + a_0 y(t) + \alpha_1 y'(t-\tau) + \alpha_0 y(t-\tau) = 0, \tag{1}$$

under appropriate initial conditions belonging to the Banach space of continuous functions $\mathcal{C}([-\tau, 0], \mathbb{R})$. In the Laplace domain, the corresponding characteristic function is a quasipolynomial defined by $\Delta : \mathbb{C} \times \mathbb{R}^{+*} \to \mathbb{C}$:

$$\Delta(s, \tau) = s^2 + a_1 s + a_0 + e^{-\tau s} (\alpha_1 s + \alpha_0), \tag{2}$$

where $(a_1, a_0, \alpha_1, \alpha_0) \in \mathbb{R}^4$, see for instance [16]. The degree of Δ, defined by the sum of the degrees of the polynomials plus the number of delays (one in our case), is four. The following result asserts that the maximal number of real roots counted with multiplicity is four, see [30].

Proposition 1 (Pólya-Szegö bound)**.** *Let Δ be the quasipolynomial given by (2), and $\alpha, \beta \in \mathbb{R}$ be such that $\alpha \leq \beta$. Denote by $M_{\alpha,\beta}$ the number of roots of Δ counted with multiplicities* *contained in the set* $\{s \in \mathbb{C},\, \alpha \leq \Im(s) \leq \beta\}$*. Then,*

$$\frac{\tau(\beta-\alpha)}{2\pi} - 4 \leq M_{\alpha,\beta} \leq \frac{\tau(\beta-\alpha)}{2\pi} + 4.$$

As a consequence of the previous theorem, setting $\alpha = \beta = 0$, we conclude that the degree 4 of Δ is a sharp bound for the number of real roots of the quasipolynomial Δ. Our analysis of the assignment of real roots to the quasipolynomial (2) will rely on the invertibility of a certain matrix, which because of its shape and properties, is called in the sequel *functional Vandermonde-type matrix*, similar to the ones studied in [31], [32]; see Lemma 3 below. This result will be proved with the help of the multivariate functions $G_n : \mathbb{R}^{n+1} \times \mathbb{R}^{+*} \to \mathbb{R}$ defined recursively by $G_0(s_k, \tau) = e^{-\tau s_k}$ and

$$(s_1 - s_{n+1}) G_n(s_1, \ldots, s_{n+1}, \tau) = G_{n-1}(s_2, \ldots, s_{n+1}, \tau) - G_{n-1}(s_1, \ldots, s_n, \tau),$$

for $n \geq 1$. The multivariate functions $G_n(\cdot, \tau)$ are closely related to the functions $F_n(\cdot, \tau)$ introduced and analyzed in [18] and, thus, benefit from similar properties that we present without proof in Lemma 2 below.

Lemma 2 (Properties of the G_n functions)**.** *For any real numbers* $s_1, \ldots, s_{n+1}$ *and* $\tau > 0$*, we have*

1. $G_n(s_1, \ldots, s_{n+1}, \tau) \neq 0$.
2. $\lim_{\tau \to 0} \frac{1}{(-\tau)^n} G_n(s_1, \ldots, s_{n+1}, \tau) = \frac{1}{n!}$.

Lemma 3 (Invertibility of a structured functional Vandermonde-type matrix)**.** *Let* $\mathbf{s} = (s_1, s_2, s_3, s_4)$*, then the matrix*

$$V(\mathbf{s}, \tau) = \begin{pmatrix} s_1 & 1 & s_1 e^{-\tau s_1} & e^{-\tau s_1} \\ s_2 & 1 & s_2 e^{-\tau s_2} & e^{-\tau s_2} \\ s_3 & 1 & s_3 e^{-\tau s_3} & e^{-\tau s_3} \\ s_4 & 1 & s_4 e^{-\tau s_4} & e^{-\tau s_4} \end{pmatrix} \tag{3}$$

is invertible for any $\tau > 0$ *and any distinct real numbers* $s_1, \ldots, s_4$*.*

Proof. We give an explicit factorized form of the determinant $\upsilon(\mathbf{s}, \tau)$ of $V(\mathbf{s}, \tau)$ and then we show that $\upsilon(\mathbf{s}, \tau) \neq 0$ for $\tau > 0$ and $s_i \neq s_j$. Observe that for $1 \leq k \leq 4$, we have $e^{-\tau s_k} = G_0(s_k, \tau)$ and thus $s_k e^{-\tau s_k} = -G_0'(s_k, \tau)$, where the prime denotes the derivative with respect to τ. We start with the structured functional Vandermonde-type matrix V defined by (3) and we denote by L_i, for $1 \leq i \leq 4$, its i-th line. By replacing L_i by $L_i - L_{i+1}$, for $1 \leq i < 4$ and using the properties of the G_n functions we get

$$\upsilon = (s_1 - s_2)(s_2 - s_3)(s_3 - s_4) \begin{vmatrix} 1 & 0 & G_1'(s_1, s_2, \tau) & G_1(s_1, s_2, \tau) \\ 1 & 0 & G_1'(s_2, s_3, \tau) & G_1(s_2, s_3, \tau) \\ 1 & 0 & G_1'(s_3, s_4, \tau) & G_1(s_3, s_4, \tau) \\ s_4 & 1 & -G_0'(s_4, \tau) & G_0(s_4, \tau) \end{vmatrix}.$$

We replace L_i by $L_i - L_{i+1}$ one more time, for $1 \leq i \leq 2$, and factor out $(s_1 - s_3)(s_2 - s_4)$ to obtain

$$\upsilon = (s_1 - s_2)(s_2 - s_3)(s_3 - s_4)(s_1 - s_3)(s_2 - s_4) \begin{vmatrix} 0 & 0 & G_2'(s_1, s_2, s_3, \tau) & G_2(s_1, s_2, s_3, \tau) \\ 0 & 0 & G_2'(s_2, s_3, s_4, \tau) & G_2(s_2, s_3, s_4, \tau) \\ 1 & 0 & G_1'(s_3, s_4, \tau) & G_1(s_3, s_4, \tau) \\ s_4 & 1 & -G_0'(s_4, \tau) & G_0(s_4, \tau) \end{vmatrix}.$$

To show that $\upsilon \neq 0$, it suffices to prove that the latter matrix is invertible. Note that we can express the determinant $\begin{vmatrix} G_2'(s_1,s_2,s_3,\tau) & G_2(s_1,s_2,s_3,\tau) \\ G_2'(s_2,s_3,s_4,\tau) & G_2(s_2,s_3,s_4,\tau) \end{vmatrix}$ via the following quotient derivative

$$-G_2(s_1,s_2,s_3,\tau)^2 \left(\frac{G_2(s_2,s_3,s_4,\tau)}{G_2(s_1,s_2,s_3,\tau)} \right)'.$$

The above expression holds for any $\tau > 0$, thus its vanishing implies that there exists a real constant C (depending on $\mathbf{s}$) such that

$$G_2(s_2,s_3,s_4,\tau) = C G_2(s_1,s_2,s_3,\tau).$$

By continuity, taking $\tau \to 0$ yields that $C = 1$, a contradiction, since

$$G_2(s_2,s_3,s_4,\tau) - G_2(s_1,s_2,s_3,\tau) = (s_4 - s_1) G_3(\mathbf{s},\tau) \neq 0.$$

Therefore, $\upsilon \neq 0$ for any $\tau > 0$ and any distinct numbers s_k, $k = 1, \ldots, 4$. □

To prove the dominancy of the rightmost real spectral value, we will use the Stépán-Hassard formula [3], [29] that we recall in the next theorem, in a particular context.

Theorem 4 (Stépán-Hassard formula)**.** *Let $Q(x,q) = x^n + \sum_{k=0}^{n-1} b_k x^k + e^{-qx} \sum_{k=0}^{m} \beta_k x^k$ be a quasipolynomial function and $0 < \rho_1 \leq \ldots \leq \rho_r$ be the positive zeros (counted with multiplicities) of the real function $R_q(\omega) = \Re(\mathbf{i}^{-n} Q(\mathbf{i}\omega, q))$. For each $1 \leq j \leq r$ such that $Q(\mathbf{i}\rho_j, q) = 0$, assume that the multiplicity of $\mathbf{i}\rho_j$ as a zero of Q is the same as the multiplicity of ρ_j as a zero of $R_q(\omega)$. Then the number of roots of the quasipolynomial $Q(x,q)$ which lie in the half plane $\{\Re(x) > 0\}$, counted with multiplicity, is given by the formula*

$$\mathcal{Z} = \frac{n-K}{2} + \frac{1}{2}(-1)^r \operatorname{sgn} S^{(\kappa)}(0) + \sum_{j=1}^{r} (-1)^{j-1} \operatorname{sgn} S(\rho_j), \tag{4}$$

where K is the number of purely imaginary roots of Δ, counted by multiplicity, κ is the multiplicity of 0 as a root of Q, and $S_q(\omega) = \Im(\mathbf{i}^{-n} Q(\mathbf{i}\omega, q))$.

3 Main results

In this section, we analyze the properties of quasipolynomial functions saturating their number of real zeros, and we derive some results for the asymptotic stability of the solutions of the DDE (1). More precisely, first, in Proposition 5 we give necessary and sufficient conditions on the coefficients of the quasipolynomial (2) for the number of distinct real zeros to be maximal. For the next results, we assume that the real roots are uniformly distributed. Second, in Proposition 7 we give a necessary and sufficient condition that guarantees the negativity of the real zeros of Δ. Third, in Theorem 8 we show that the coexistence of the maximum number of real spectral values implies the CRRID property for the quasipolynomial Δ, i.e. the rightmost real zero is the spectral abscissa. Finally, the combination of these results allows us to prove exponential stability with a guaranteed decay rate for the solutions of (1).

3.1 Assigning real roots of the characteristic function

In this subsection, we show that assigning the maximal number of distinct real roots to the quasipolynomial Δ uniquely determines the coefficients a_k and α_k.

Proposition 5 (Coexisting real roots)**.** *For a given delay $\tau > 0$, the quasipolynomial (2) admits 4 distinct real spectral values $s_1, \dots, s_4$ if, and only if, the real coefficients a_1, a_0 and α_1, α_0 are respectively given by the following functions in τ and $\mathbf{s} = (s_1, \dots, s_4)$:*

$$a_1(\mathbf{s}, \tau) = \frac{-1}{\upsilon(\mathbf{s}, \tau)} \det \begin{pmatrix} s_1^2 & 1 & s_1 e^{-\tau s_1} & e^{-\tau s_1} \\ s_2^2 & 1 & s_2 e^{-\tau s_2} & e^{-\tau s_2} \\ s_3^2 & 1 & s_3 e^{-\tau s_3} & e^{-\tau s_3} \\ s_4^2 & 1 & s_4 e^{-\tau s_4} & e^{-\tau s_4} \end{pmatrix}, \quad a_0(\mathbf{s}, \tau) = \frac{1}{\upsilon(\mathbf{s}, \tau)} \det \begin{pmatrix} s_1^2 & s_1 & s_1 e^{-\tau s_1} & e^{-\tau s_1} \\ s_2^2 & s_2 & s_2 e^{-\tau s_2} & e^{-\tau s_2} \\ s_3^2 & s_3 & s_3 e^{-\tau s_3} & e^{-\tau s_3} \\ s_4^2 & s_4 & s_4 e^{-\tau s_4} & e^{-\tau s_4} \end{pmatrix},$$

and

$$\alpha_1(\mathbf{s}, \tau) = \frac{-1}{\upsilon(\mathbf{s}, \tau)} \det \begin{pmatrix} s_1^2 & s_1 & 1 & e^{-\tau s_1} \\ s_2^2 & s_2 & 1 & e^{-\tau s_2} \\ s_3^2 & s_3 & 1 & e^{-\tau s_3} \\ s_4^2 & s_4 & 1 & e^{-\tau s_4} \end{pmatrix}, \quad \alpha_0(\mathbf{s}, \tau) = \frac{1}{\upsilon(\mathbf{s}, \tau)} \det \begin{pmatrix} s_1^2 & s_1 & 1 & s_1 e^{-\tau s_1} \\ s_2^2 & s_2 & 1 & s_2 e^{-\tau s_2} \\ s_3^2 & s_3 & 1 & s_3 e^{-\tau s_3} \\ s_4^2 & s_4 & 1 & s_4 e^{-\tau s_4} \end{pmatrix},$$

where $\upsilon(\mathbf{s}, \tau)$ is the determinant of the structured functional Vandermonde-type matrix $V(\mathbf{s}, \tau)$, introduced in Lemma 3.

Proof. Assume that (2) admits four distinct real roots $s_1, \dots, s_4$, which entails that the coefficients a_1, a_0 and α_1, α_0 satisfy the linear system

$$a_1 s_k + a_0 + \alpha_1 e^{-\tau s_k} s_k + \alpha_0 e^{-\tau s_k} = -s_k^2,$$

for $1 \leq k \leq 4$. Thanks to the invertibility of matrix $V(\mathbf{s}, \tau)$, as ensured by Lemma 3, one deals with a Cramer system with respect to the coefficients a_1, a_0 and α_1, α_0. So that, we easily compute these coefficients with the standard formulas. □

The following corollary gives the closed-form expression of the coefficients a_1, a_0 and α_1, α_0, in the case where the real zeros are equidisributed.

Corollary 6. *The quasipolynomial Δ admits the equidistanced real roots $s_k = s_1 - (k-1)d$, for $d > 0$ and $1 \leq k \leq 4$, if and only if its coefficients are given by*

$$a_0(s_1, d, \tau) = \frac{6d^2}{(e^{\tau d} - 1)^2} + s_1^2 - d s_1 \left(\frac{e^{\tau d} - 5}{e^{\tau d} - 1} \right), \tag{5}$$

$$a_1(s_1, d, \tau) = d \left(\frac{e^{\tau d} - 5}{e^{\tau d} - 1} \right) - 2 s_1, \tag{6}$$

$$\alpha_0(s_1, d, \tau) = \frac{-2 d e^{\tau s_1}}{(e^{\tau d} - 1)^2} \left(3d - s_1 (1 - e^{-\tau d}) \right), \tag{7}$$

$$\alpha_1(s_1, d, \tau) = -2 d e^{\tau s_1} \frac{e^{-\tau d}}{e^{\tau d} - 1}. \tag{8}$$

Note that the expressions of the coefficients a_1, a_0 and α_1, α_0 in Corollary 6 characterize the existence of four equidistant real roots $s_k = s_1 - (k-1)d$, with $d > 0$ and $k = 1, \cdots, 4$, for the quasipolynomial Δ. In practical applications, notice that a_1 and a_0 are fixed by the model. Hence, one has the solve (6) to determine $\sigma := \tau d$. We inject the expression of σ into (7) and (8) to express α_0 and α_1 as a function of s_1, d, τ and a_1, and in equation (5) to determine the relation verified by d (as a polynomial of degree 2). The suitable solution of d is quickly determined via the positivity constraint of τ and d.

Remark. *Observe that the open-loop system $P(s) = s^2 + a_1 s + a_0$ needs not a priori to be stable when applying the retarded PD controller. Indeed, observe that $a_1(s_1, d, \tau)$ is negative for $\tau < \frac{1}{d} \ln \left(\frac{5d - 2s_1}{d - 2s_1} \right)$, which implies that $P(s)$ has at least one root with a positive real part.*

3.2 Assuring negativity of the real roots

The following proposition gives a necessary and sufficient condition for the coefficients of the quasipolynomial (2) to ensure that the rightmost real root is negative.

Proposition 7. *For a fixed delay $\tau > 0$, assume that the quasipolynomial Δ admits four equidistanced real roots $s_k = s_1 - (k-1)d$, for $d > 0$ and $1 \leq k \leq 4$. Then, s_1 is negative if, and only if, there exits $\tau^* > 0$ such that :*

$$a_1(s_1, d, \tau^*) + s_1 - d = 0. \tag{9}$$

Furthermore, τ^* necessarily satisfies:

$$\tau^* = \frac{1}{d} \ln \left(1 - 4\frac{d}{s_1}\right) > 0.$$

Proof. By a direct calculation using the explicit form of a_1 given in Corollary 6, we obtain

$$a_1(s_1, d, \tau) + s_1 - d = -s_1 - 4d\frac{1}{e^{\tau d} - 1}.$$

Then, $a_1(s_1, d, \tau^*) + s_1 - d = 0$ if, and only if, $\tau^* = \frac{1}{d} \ln \left(1 - 4\frac{d}{s_1}\right)$. Finally, one immediately observes that $\tau^* > 0$ if, and only if, $s_1 < 0$. □

Remark. *Note that Proposition 7 is of importance in stabilization of unstable plants. Indeed,*

- *The existence of the mentioned τ^* represents a guarantee for the negativity of the closed-loop assigned real roots.*
- *Equation (9) is of importance for control design purposes. In particular, if the coefficient a_1 is fixed by the model and one wants to assign equidistanced real roots to stabilize the system, then the distance $d > 0$ needs to be chosen such that $d > a_1 + 2s_1$.*

3.3 The CRRID property

In this subsection, we prove the CRRID property for the quasipolynomial Δ. Namely, we have the following theorem.

Theorem 8. *If the quasipolynomial $\Delta(\cdot, \tau)$ admits the maximal number of four equidistanced real roots $s_k = s_1 - (k-1)d$, for $1 \leq k \leq 4$, then*

$$\forall\, s_0 \in \mathbb{C} \backslash \{s_1\}, \quad \Delta(s_0) = 0 \quad \Longrightarrow \quad \Re(s_0) < s_1.$$

Theorem 8 asserts that s_1 is a strictly dominant root of Δ, i.e. it is the spectral abscissa of the quasipolynomial. Moreover, by carefully studying the proof, we have the stronger result that the pair $(s_1, s_2 = s_1 - d)$ is a system of dominant roots for Δ, i.e.

$$\forall\, s_0 \in \mathbb{C} \backslash \{s_1, s_2\}, \quad \Delta(s_0) = 0 \quad \Longrightarrow \quad \Re(s_0) \leq s_2.$$

Our strategy to prove the dominancy of s_1 for $\Delta(\cdot, \tau)$ relies on the application of the Stépán-Hassard formula [29] on a transformed version of Δ. Let us introduce the following transformation of the complex plane:

$$\begin{aligned} \phi : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto dz + s_1 - d, \end{aligned}$$

which translates the root $s_1 - d$ to the origin and scales the distance between two consecutive real root to 1. We introduce the transformed quasipolynomial

$$Q(z,q) = \frac{1}{d^2}\Delta\left(\phi(z),\tau\right) = z^2 + b_1 z + b_0 + e^{-qz}\left(\beta_1 z + \beta_0\right),$$

where the new delay parameter is $q = \tau d$. The following lemma assert that the normalized quasipolynomial $Q(\cdot,q)$ admits four distinct real roots, gives the explicit form of the coefficients b_1, b_0, β_1, and β_0, and asserts that the problem of proving that s_1 is dominant for Δ is the same as proving that $z = 1$ is dominant for Q.

Lemma 9 (Properties of the normalized quasipolynomial)**.** *The normalized quasipolynomial $Q(z,q)$ admits the real roots $\{1, 0, -1, -2\}$. The coefficients $\{b_1, b_0, \beta_1, \beta_0\}$ of Q are given by*

$$b_0(q) = 2\frac{2e^q + 1}{\left(e^q - 1\right)^2}, \qquad b_1(q) = -\frac{e^q + 3}{e^q - 1},$$
$$\beta_0(q) = -2\frac{2e^q + 1}{\left(e^q - 1\right)^2}, \qquad \beta_1(q) = -\frac{2}{e^q - 1}.$$

The root s_1 of Δ is dominant if, and only if, $z = 1$ is a dominant root of $Q(\cdot,q)$.

Proof. The map $\phi(z)$ is an isomorphism of the complex plane, that maps $\{1, 0, -1, -2\}$ to the real roots $\{s_1, s_2, s_3, s_4\}$ of Δ to. Thus, the coefficients $\{b_1, b_0, \beta_1, \beta_0\}$ of Q satisfy the set of linear equations $Q(z_0) = 0$ for $z_0 \in \{1, 0, -1, -2\}$. Therefore, using Corollary 6 with $s_1 = 1$ and $d = 1$, we deduce that the coefficients $\{b_1, b_0, \beta_1, \beta_0\}$ admit the closed forms given in the statement of the Lemma.

Moreover, the linear transform ϕ is the composition of a translation and a positive scaling (in particular, it does not involve a rotation). Thus, any inequality between the real parts of the roots of Δ is preserved. Hence, the dominancy of 1 for Q corresponds to the dominancy of s_1 for Δ. □

To prove the dominancy of 1 for Q, we apply the Stépán-Hassard formula (see Theorem 4), which relies on the properties of the following parametrized real functions

$$R_q(\omega) = \Re\left(-Q(\mathbf{i}\omega,q)\right) \quad \text{and} \quad S_q(\omega) = \Im\left(-Q(\mathbf{i}\omega,q)\right),$$

given by

$$\begin{aligned} R_q(\omega) &= \omega^2 - b_0(q) - \cos(q\omega)\beta_0(q) - \sin(q\omega)\beta_1(q)\omega, \\ S_q(\omega) &= \sin(q\omega)\beta_0(q) - \cos(q\omega)\omega\beta_1(q) - b_1(q)\omega. \end{aligned} \tag{10}$$

In the remaining part of this section, we compute the different numbers involved in the Stépán-Hassard formula (4).

Number of imaginary roots of Q. Let $\omega \geq 0$ be such that $Q(\mathbf{i}\omega) = 0$, then ω satisfies

$$R_q(\omega) = 0 \quad \text{and} \quad S_q(\omega) = 0.$$

Using simple algebraic manipulations, we obtain that the expressions of $\cos(q\omega)$ and $\sin(q\omega)$ are given by

$$\begin{aligned} \cos(q\omega) &= -\frac{\omega^2(b_1\beta_1 - \beta_0) + b_0\beta_0}{\omega^2\beta_1^2 + \beta_0^2}, \\ \sin(q\omega) &= \frac{\omega(\omega^2\beta_1 - \beta_0\beta_1 + b_1\beta_0)}{\omega^2\beta_1^2 + \beta_0^2}. \end{aligned}$$

The relation $\cos(q\omega)^2 + \sin(q\omega)^2 = 1$, implies that ω is necessarily given by

$$\omega_1 = 0 \quad \text{or} \quad \omega_2 = \pm\sqrt{2b_0 - b_1^2 + \beta_1^2} = \pm\mathbf{i}.$$

Therefore, the only imaginary root of $Q(\cdot, q)$ is $\omega = 0$, which, by definition of Q, is of multiplicity 1. Moreover, we have

$$S_q'(0) = \frac{1}{(e^q - 1)^2}\left(e^{2q} - 4(q-1)e^q - 2q - 5\right),$$

which can be seen positive for any $q > 0$ with the following argument. Denote by $f(q)$ the numerator of the right-hand-side of the above equality. Then, $f''(q) = 4e^q\left(e^q - (q+1)\right)$ is positive for $q > 0$. Thus, $f'(q)$ is increasing and, since $f'(0) = 0$, is positive. Finally, we deduce that $f(q)$ is increasing and, since $f(0) = 0$, we conclude that $f(q)$ is positive for $q > 0$. Hence, in the Stépán-Hassard formula we have $K = 1, \quad \kappa = 1, \quad \text{and} \quad \operatorname{sgn} S_q^{(\kappa)}(0) = 1.$

Locating the real positive roots of R_q. If $\omega > 0$ is a root of (10), then (using the expressions of the coefficients given in Lemma 9) we get

$$\omega^2 + 2\frac{2e^q + 1}{(e^q - 1)^2}\left(\cos(q\omega) - 1\right) + \frac{2}{e^q - 1}\omega\sin(q\omega) = 0.$$

Multiplying the last equation by q^2, reduces the problem to the search of the positive zeros of the parameterized function $F_q : \mathbb{R}^{+*} \to \mathbb{R}$ defined by

$$F_q(\rho) = \rho^2 + A(q)\left(\cos(\rho) - 1\right) + B(q)\rho\sin(\rho) = 0,$$

where $\rho = q\omega > 0$, $A(q) = 2q^2\frac{2e^q+1}{(e^q-1)^2}$, and $B(q) = \frac{2q}{e^q-1}$. The following technical lemma gives the main properties of the functions A and B that we will use the next results.

Lemma 10 (Properties of the A and B functions)**.** *For all $q > 0$, we have*

$$B(q) < 2 \quad \textit{and} \quad 6 > A(q) > 3B(q) > B(q) > 0.$$

The function $\bar{\rho}_q = \left(B(q) + \sqrt{B(q)^2 + 8A(q)}\right)/2$ is strictly decreasing and $\lim_{q\to 0^+}\bar{\rho}_q = 1+\sqrt{13}$.

Proof. Clearly we have $A(q) > 0$ and $B(q) > 0$. Moreover, $B'(q) = -\frac{2}{(e^q-1)^2}\left(e^q(q-1)+1\right) < 0$, thus B is decreasing and $B(q) < \lim_{q\to 0^+} B(q) = 2$. Second, we also obtain $A'(q) < 0$, which yields $A(q) < \lim_{q\to 0^+} A(q) = 6$. Third, we have $A - 3B = \frac{2q}{(e^q-1)^2}\left(2qe^q - 3e^q + q + 3\right)$, the second factor is seen to be positive by differentiating it twice, implying that $A - 3B > 0$. Consider the function $\bar{\rho}_q$ defined in the Lemma and whose expression is $\bar{\rho}_q = \frac{q}{e^q-1}\left(1 + \sqrt{5 + 8e^q}\right)$. To show that $\bar{\rho}_q$ is decreasing, we show that it is the sum of two positive decreasing functions. Namely, we set $\bar{\rho}_q = \bar{\rho}_q^1 + \bar{\rho}_q^2$, with $\bar{\rho}_q^1 = \frac{q}{e^q-1}$ and $\bar{\rho}_q^2 = \frac{q}{e^q-1}\sqrt{5+8e^q}$. First, we obviously have that $\bar{\rho}_q^1$ is decreasing. Second, we have $\frac{\mathrm{d}}{\mathrm{d}q}(\bar{\rho}_q^2) = -e^q\frac{4e^q+9}{(e^q-1)^2\sqrt{8e^q+5}} < 0$, hence $\bar{\rho}_q^2$ is also decreasing. Thus, $\bar{\rho}_q < \lim_{q\to 0^+}\bar{\rho}_q = 1 + \sqrt{13}$ □

The results of Lemma 10 are illustrated by Figure 1.

Lemma 11. *The parametrized transcendental function $F_q(\rho)$ admits at most one real positive zero ρ_q^*, and when it does, it is located within the open interval $\mathtt{I} = \left]0, \frac{3\pi}{2}\right[$.*

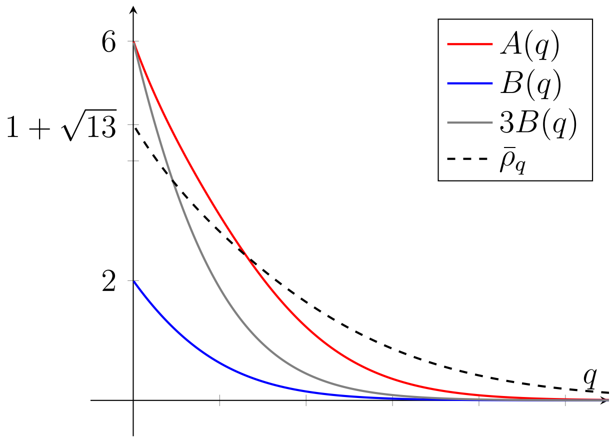

Figure 1: Qualitative behavior of the functions $A(q)$ (red) and $B(q)$ (blue) involved in the definition of F_q. We also illustrate the function $\bar{\rho}_q$ (dashed).

Proof. First, since $A(q) > 0$ and $B(q) > 0$, observe that for $\rho > 0$ we have

$$F_q(\rho) \geq \rho^2 - 2A(q) - B(q)\rho.$$

The right hand side of the above inequality is a polynomial of degree 2 in the variable ρ, whose largest root is $\bar{\rho}_q$ defined in Lemma 10. Since $\lim_{q\to 0}\bar{\rho}_q = 1+\sqrt{13} < \frac{3\pi}{2}$, we deduce that, necessarily, all roots ρ_q^* (if any) of $F_q(\rho)$ must lie within $\mathtt{I} = \left]0, \frac{3}{2}\pi\right[$. Now, by studying the first derivatives of F_q with respect to ρ we show that F_q admits at most one positive root. The third derivative of F_q with respect to ρ is given by:

$$F_q^{(3)}(\rho) = -B(q)\rho\cos(\rho) + (A(q) - 3B(q))\sin(\rho).$$

Depending on whether the function $C(q) = \frac{B(q)}{A(q)-3B(q)}$ is greater than 1, the equation $\tan(\rho) = C(t)\rho$ has either one or two solutions within the interval $\mathtt{I}$. We separate the two cases.

1. If $F_q^{(3)}(\rho)$ has only one zero $\rho_1 \in \mathtt{I}$. Using that $F_q^{(3)}(0) = 0$ and $F_q^{(3)}\left(\frac{3\pi}{2}\right) = 3B(q) - A(q) < 0$, we deduce that $F_q''(\rho)$ is increasing on $]0, \rho_1[$ and decreasing on $\left]\rho_1, \frac{3\pi}{2}\right[$. Observe that $F_q''\left(\frac{3\pi}{2}\right) = 2 + \frac{3\pi}{2}B(q) > 0$ and denote $f_2(q) = F_q''(0) = 2 - A(q) + 2B(q)$.
   (a) If $f_2(q) \geq 0$, then $F_q''(\rho)$ is positive on $\mathtt{I}$. Thus, $F_q'(\rho)$ is increasing and since $F_q'(0) = 0$, we conclude that F_q' is positive. Finally, $F_q(\rho)$ is increasing on $\mathtt{I}$ and, since $F_q(0) = 0$, we conclude that F_q does not admit a zero on $\mathtt{I}$.
   (b) If $f_2(q) < 0$, then $F_q''(\rho)$ admits a zero $\rho_2 \in]0, \rho_1[$. Thus, $F_q'(\rho)$ is decreasing on $]0, \rho_2[$ and increasing on $\left]\rho_2, \frac{3\pi}{2}\right[$. Using that $F_q'(0) = 0$ and $F_q'\left(\frac{3\pi}{2}\right) = A - B + 3\pi > 0$, we conclude that $F_q'(\rho)$ admits a zero $\rho_3 \in \left]\rho_2, \frac{3\pi}{2}\right[$. Therefore, F_q is decreasing on $]0, \rho_3[$ and increasing on $\left]\rho_3, \frac{3\pi}{2}\right[$. Since $F_q(0) = 0$ and $F_q\left(\frac{3\pi}{2}\right) = \frac{9\pi^2}{4} - \frac{3\pi}{2}B(q) - A(q) > 0$, we conclude that F_q admits a unique zero $\rho^* \in \left]\rho_3, \frac{3\pi}{2}\right[$.
2. If $F_q^{(3)}(\rho)$ has two zeros $\rho_1, \rho_2 \in \mathtt{I}$. Then, $F_q''(\rho)$ is decreasing on $]0, \rho_1[$, increasing on $]\rho_1, \rho_2[$, and decreasing on $\left]\rho_2, \frac{3\pi}{2}\right[$. In this case we necessarily have $F_q''(0) < 0$ and, using $F_q''\left(\frac{3\pi}{2}\right) > 0$, we conclude that F_q'' has a unique zero $\rho_3 \in]\rho_1, \rho_2[$. So $F_q'(\rho)$ is decreasing on $]0, \rho_3[$ and increasing on $]\rho_3, 0[$. Since $F_q'(0) = 0$ and $F_q'\left(\frac{3\pi}{2}\right) > 0$, the function F_q' admits a unique zero $\rho_4 \in \left]\rho_3, \frac{3\pi}{2}\right[$. Finally, F_q is decreasing on $]0, \rho_4[$ and increasing on $\left]\rho_4, \frac{3\pi}{2}\right[$. Since, $F_q(0) = 0$ and $F_q\left(\frac{3\pi}{2}\right) > 0$, we conclude that F_q admits a unique zero $\rho^* \in \left]\rho_4, \frac{3\pi}{2}\right[$.

To summarize the case distinction made above, we have proved that for any value of the parameter q, the function $F_q(\rho)$ admits at most a root $\rho^* > 0$. □

Therefore, for a given delay q^*, there exists at most one $\omega^* > 0$, defined by $\rho^* = q^*\omega^*$, such that $R_q(\omega^*) = 0$. We now show that $S_q(\omega^*) > 0$. In the same fashion as for R_q, we introduce the parametrized univariate function $H_q(\rho) = qS_q(\frac{\rho}{q})$, which can be expressed as

$$H_q(\rho) = C(q)\rho + D(q)\rho\cos(\rho) + E(q)\sin(\rho),$$

with $C(q) = \frac{e^q+3}{e^q-1}$, $D(q) = \frac{2}{e^q-1}$, and $E(q) = -2q\frac{2e^q+1}{(e^q-1)^2}$. The following lemma details the properties of the C, D, E functions that we will use in the next results.

Lemma 12 (Properties of the C, D, E functions)**.** *For all $q > 0$, we have*

$$C(q) > D(q) > 0 \quad \textit{and} \quad 2D(q) + E(q) < 0.$$

Proof. It is immediate to obtain $C(q) > 0$ and $D(q) > 0$; moreover, $C(q) - D(q) = \frac{e^q+1}{e^q-1} > 0$. Next, we have $2D(q) + E(q) = -\frac{2}{(e^q-1)^2}\left(2e^q(q-1) + q + 2\right)$, which can be seen to be positive since the second factor is positive (proved using its second derivative). □

The results of Lemma 12 are illustrated by Figure 2.

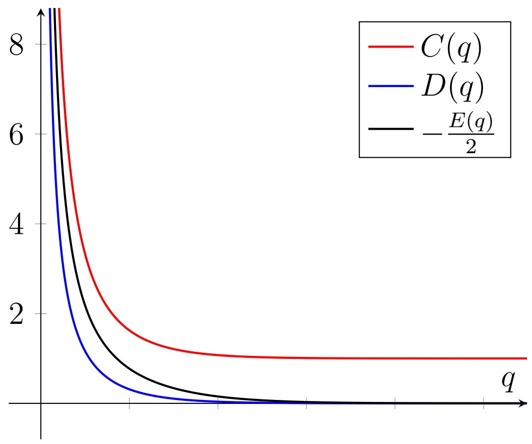

Figure 2: Qualitative behavior of the functions $C(q)$ (red), $D(q)$ (blue), and $E(q)$ (black), involved in the definition of H_q.

Lemma 13. *For any $q > 0$ and $\rho \in \left]0, \frac{3\pi}{2}\right[$, we have $H_q(\rho) > 0$.*

Proof. By a direct calculation, we obtain

$$H_q''(\rho) = -D(q)\rho\cos(\rho) - (2D(q) + E(q))\sin(\rho).$$

The function $J(q) = -\frac{D(q)}{2D(q)+E(q)}$ satisfies $0 < J(q) < 1$, so we conclude that the equation $\tan(\rho) = J(q)\rho$ admits a unique solution in the interval I. Using $H_q''(0) = 0$ and $H_q''\left(\frac{3\pi}{2}\right) = 2D(q) + E(q) < 0$, it follows that H_q' is increasing on $]0, \rho_1[$ and decreasing on $\left]\rho_1, \frac{3\pi}{2}\right[$. Since, $H_q'(0) = C(q) + D(q) + E(q) > 0$ and $H_q'\left(\frac{3\pi}{2}\right) = \frac{3\pi}{2}D(q) + C(q) > 0$, we conclude that H_q' is positive, and H_q is increasing, since $H_q(0) = 0$, it follows that $H_q(\rho) > 0$. □

Hence, in the Stépán-Hassard formula we either have $r = 0$ or $r = 1$ and $\operatorname{sgn} S_q(\rho^*) = 1$.

Applying the Stépán-Hassard formula. The previous preliminary results allow to compute the number of zeros of Q located in the right-half plane.

Theorem 14. *The root* 1 *is dominant for* Q*.*

Proof. We apply the Stépán-Hassard formula (4) to obtain the number of zeros of Q in $\{\Re(z) > 0\}$. If $r = 0$, then

$$\mathscr{Z}_0 = \frac{2-1}{2} + \frac{1}{2}(-1)^0(1) = 1.$$

If $r = 1$, then

$$\mathscr{Z}_1 = \frac{2-1}{2} + \frac{1}{2}(-1)^1(1) + (-1)^0(1) = 1.$$

Thus, in both cases, the quasipolynomial Q admits only one root in the right-half plane. But, we already know that $z = 1$ is a root of Q. Therefore any other root $z_0 \in \mathbb{C}$ of Q satisfies $\Re(z_0) \leq 0$ and we conclude the dominancy of 1 for Q. □

As a consequence of Theorem 14 and of Lemma 9, we conclude that s_1 is dominant for $\Delta(s, \tau)$ and that Δ satisfies the CRRID property.

3.4 Asymptotic stability of the DDE

Collecting the results of the previous subsections give the following theorem.

Theorem 15. *If the quasipolynomial* Δ *given by (2) admits four equidistributed real zeros and (9) is satisfied, then the trivial solution of (1) is asymptotically stable with a decay rate equal to* s_1*.*

4 Illustrative examples

In this section, we illustrate the effectiveness of our approach by providing two examples. First, we show that the CRRID property can be applied via a delayed PD controller to successfully stabilize unstable plants. Second, we show that even the second-order open-loop plant is stable, one is able to improve its solution decay rate.

4.1 Stabilizing two unstable poles

To highlight the utility of using a retarded PD controller, we use the approach developed in the previous section to stabilize an unstable second-order plant free of delays. Consider the open-loop system given by $y''(t) + a_1 y'(t) + a_0 y(t) = 0$ and assume that $a_1 < 0$, which implies that its characteristic function has at least one unstable zero. Following the methodology introduced in the paper, we consider that we use a delayed PD controller $u(t) = -\alpha_1 y'(t-\tau) - \alpha_0 y(t-\tau)$ and that the closed loop characteristic function

$$\Delta(s) = s^2 + a_1 s + a_0 + e^{-\tau s}\left(\alpha_1 s + \alpha_0\right) \tag{11}$$

admits four equidistributed real roots $s_k = s_1 - (k-1)d$, for $1 \leq k \leq 4$ and $d > 0$. Therefore, Δ satisfy the CRRID property, i.e. the largest real root s_1 is the dominant one, and thus the negativity of s_1 implies that the solutions of the closed-loop system are asymptotically stable.

The control parameters s_1, d, and τ cannot be chosen arbitrarily. Indeed, a_1 must be given by Corollary 6 and its negativity implies that the control parameters have to be chosen such that it holds

$$s_1 > \frac{d}{2}\frac{e^{\tau d}-5}{e^{\tau d}-1}.$$

Hence, to guarantee that the spectral abscissa of the closed loop system s_1 is negative, we need to satisfy $\tau d < \ln(5)$. The precise assignability region for s_1 as a function of $\tau > 0$ and $d > 0$ is depicted in Figure 3. For a numerical illustration, we set $a_1 = -2$, $a_0 = 51/8$, $s_1 = -1$, and

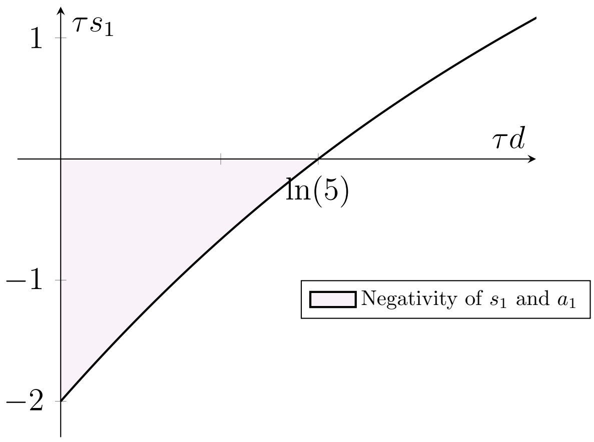

Figure 3: Assignability region for the spectral abscissa of the closed loop system (11) as a function of the distance d and delay τ.

$d = 1$. Then, the formula for a_1 of Corollary 6 implies that necessarily $\tau = \ln\left(\frac{9}{5}\right)$ giving the following values of the controller's gains $\alpha_0 = -3875/648$, $\alpha_1 = -125/162$. First, Figure 4 shows that the roots of the unstable open-loop system (given by $1 \pm \frac{1}{2}\mathbf{i}\sqrt{\frac{43}{2}}$) are located in the right half of the complex plane, whereas the infinite number of roots of the closed-loop (11) system are located in the left half of the complex plane and are dominated by s_1.

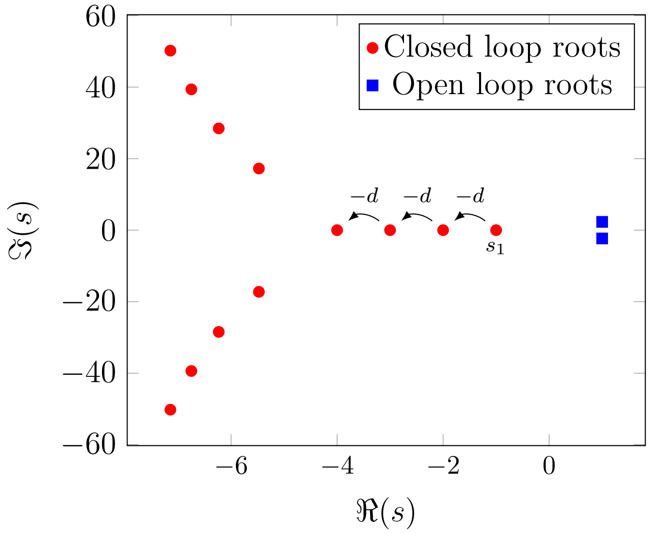

Figure 4: Spectrum ditribution of (red) the closed-loop system (blue) the closed-loop system (11).

Remark. *In the numerical example above, it is easy to observe that the gain α_1 of the proposed controller satisfies the inequality $|\alpha_1| < |a_1|$, and since $a_1 < 0$, the corresponding delay-free*

PD controller $u(t) = -\alpha_0 y(t) - \alpha_1 y'(t)$ does not stabilize the corresponding system. In other words, the presence of the delay in the control law guarantees the stability of the closed-loop system reinforcing thus the stabilizing effect *induced by the* delay *on the system's dynamics. For further discussions on such topics, we refer to [16], [7], and the references therein.*

4.2 Improving decay rate

In this second example, we show that we can improve the decay rate of a stable second-order control system without instantaneous velocity and state observations. We consider the control problem

$$y''(t) + 2\zeta\omega y'(t) + \omega^2 y(t) = u(t), \tag{12}$$

where $\zeta > 0$ is the damping ratio and $\omega > 0$ is the oscillator's natural frequency. In absence of a controller, i.e. $u(t) = 0$, the characteristic polynomial is $\Delta(s) = s^2 + 2\zeta\omega s + \omega^2$, the roots of which are $s_\pm = \omega\left(-\zeta \pm \mathbf{i}\sqrt{1-\zeta^2}\right)$ if $0 < \zeta < 1$, and $s_\pm = \omega\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$ if $\zeta \geq 1$. Hence, the spectral abscissa of Δ is given by

$$\gamma_0 = \begin{cases} -\zeta\omega & \text{if} \quad 0 < \zeta < 1, \\ \omega\left(-\zeta + \sqrt{\zeta^2 - 1}\right) & \text{if} \quad \zeta \geq 1. \end{cases}$$

In particular, we have $\gamma_0 < 0$ and the system is asymptotically stable. For many applications, we need a delay $\tau > 0$ to access a good approximation of the velocity and state of the system. This suggests to use a controller u having a *delayed proportional-minus-derivative* structure, i.e.

$$u(t) = -\alpha_1 y'(t-\tau) - \alpha_0 y(t-\tau), \quad \tau > 0. \tag{13}$$

The closed-loop system defined by (12) and (13) has a characteristic function given by the four-degree quasipolynomial

$$\Delta(s) = s^2 + 2\zeta\omega s + \omega^2 + e^{-\tau s}(\alpha_1 s + \alpha_0). \tag{14}$$

To illustrate the results of Section 3.3, we consider the case where Δ admits equidistributed real roots $s_k = s_1 - (k-1)d$, with $d > 0$ and $1 \leq k \leq 4$. One solves the system of transcendental equations $\Delta(s_k) = 0$ for the control parameters $(\alpha_1, \alpha_0, \tau)$ in terms of the system's physical parameters ζ and ω, as well as the assigned root s_1 and the distance between two consecutive roots d. More precisely, we obtain the following expressions:

$$\begin{aligned} \tau &= \frac{\sigma}{d}, \\ \alpha_1 &= -\frac{1}{2}\left(d - 2s_1 - 2\zeta\omega\right) e^{-\tau(d-s_1)}, \\ \alpha_0 &= -\frac{15}{8}\left(d - 2s_1 - 2\zeta\omega\right)\left(d - \frac{2}{3}s_1 - \frac{2}{5}\zeta\omega\right) e^{-\tau(d-s_1)}, \end{aligned} \tag{15}$$

with

$$\sigma = \ln\left(\frac{5d - 2s_1 - 2\zeta\omega}{d - 2s_1 - 2\zeta\omega}\right).$$

Moreover, we also get the following relation that links the distance d, the spectral value s_1, and the parameters ζ and ω

$$\frac{1}{2}d^2 - 2(\omega\zeta + s_1)d + \frac{2}{3}(3\zeta^2 - 2)\omega^2 + \frac{4}{3}(\zeta\omega s_1) + \frac{1}{3}(2s_1^2) = 0. \quad (16)$$

Now, to achieve asymptotic stability, we need to choose the positive distance d and the negative root s_1 such that the positivity of the delay τ, given by (15), is guaranteed and such that the constraint (16) is satisfied. First, note that $\tau > 0$ corresponds to choosing s_1 and d such that

$$\frac{5d - 2s_1 - 2\zeta\omega}{d - 2s_1 - 2\zeta\omega} = 1 + \frac{4d}{d - 2s_1 - 2\zeta\omega} > 1,$$

which is equivalent to the condition

$$d - 2s_1 - 2\zeta\omega > 0. \quad (17)$$

Second, solving (16) we infer that d is necessarily given by

$$d = 2s_1 + 2\zeta\omega + \frac{2}{3}\sqrt{6s_1^2 + 6\omega^2 + 12\zeta\omega s_1},$$

which is real if,

$$s_1 < \omega\left(-\zeta - \sqrt{\zeta^2 - 1}\right) \quad \text{or} \quad s_1 > \omega\left(-\zeta + \sqrt{\zeta^2 - 1}\right),$$

when $\zeta > 1$ or s_1 is arbitrary when $0 < \zeta \leq 1$. Furthermore, we have $d > 0$ if

$$\begin{aligned} s_1 &> \omega\left(-\zeta - \sqrt{2}\sqrt{1-\zeta^2}\right) \qquad \text{if} \quad 0 < \zeta \leq 1, \\ s_1 &> \omega\left(-\zeta - \sqrt{\zeta^2 - 1}\right) \qquad \text{if} \quad \zeta \geq 1, \end{aligned}$$

Summing up all the conditions to ensure the realness and the positivity of d, we obtain the assignment region for the spectral abscissa s_1 that is depicted in Figure 5. In particular, we observe that we can improve the negative spectral abscissa γ_0 only in the case $\zeta < 1$.

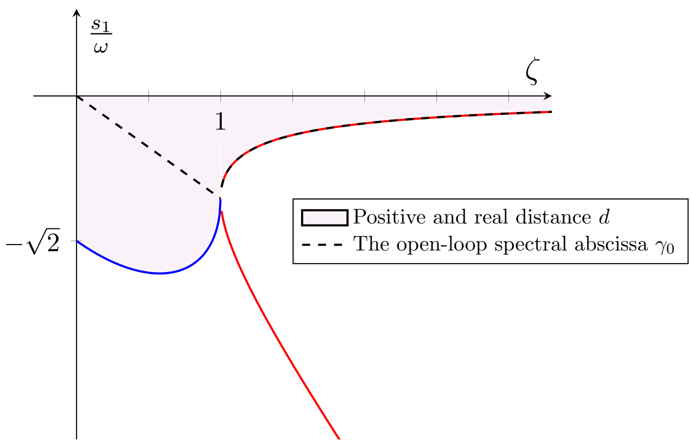

Figure 5: Assignment region for $\frac{s_1}{\omega}$ to ensure the positivity (blue) and realness (red) of the distance d.

It is interesting to compare our approach with the MID one. Adapting the results of [20], [21] to our context, we see that the MID can be applied with a controller of the form (13) only when $0 < \zeta < 1$, in which case the assigned root of maximal multiplicity four is $s_0 = \omega\left(-\zeta - \sqrt{2}\sqrt{1-\zeta^2}\right)$. This case illustrates the property that the MID is the limiting case of the CRRID when the distance d approaches 0. To apply the CRRID, we need to choose the spectral abscissa s_1 in such a way that the corresponding distance d is not zero. Hence, the spectral abscissa given by the CRRID is larger than the one given by the MID. However, the CRRID approach involves choosing the spectral abscissa in an open interval, whereas the MID approach assigns a unique value for the spectral abscissa given in terms of the corresponding delay, illustrating the robustness property of the CRRID. Moreover, the CRRID approach can be applied to systems with $\zeta > 1$, even though the exponential decay rate will be worse than the one of the open-loop system.

For a numerical illustration of our results (dominancy of s_1 and stability), we set $\omega = 1$ and $\zeta = \frac{1}{5}$. In this case, the assigned root s_1 must satisfy the bound $s_1 > -\frac{1}{5} - \frac{4\sqrt{3}}{5}$ to ensure the positivity of the distance d. Let us set $s_1 = -1$, which yields the following values of the other parameters

$$\begin{cases} d = -\frac{8}{5} + \frac{8}{\sqrt{15}} \approx 0.47, \\ \tau = \frac{30\ln\left(5\sqrt{5}-5\sqrt{3}\right)-15\ln(5)}{16\sqrt{15}-48} \approx 1.38, \\ \alpha_1 = -4\,15^{\frac{15}{16\sqrt{15}-48}}\left(5\sqrt{15}-12\right)^{\frac{9-8\sqrt{15}}{8\sqrt{15}-24}} \approx -0.14, \\ \alpha_0 = -\frac{4}{75}\left(10\sqrt{15}-19\right)15^{\frac{16\sqrt{15}-33}{16\sqrt{15}-48}} \\ \qquad \left(5\sqrt{15}-12\right)^{\frac{9-8\sqrt{15}}{8\sqrt{15}-24}} \approx -0.54. \end{cases} \tag{18}$$

In Figure 6, we give the spectrum of the characteristic function (14) with the above parameters and we illustrate the dominancy of the root s_1 with respect to all the other complex roots.

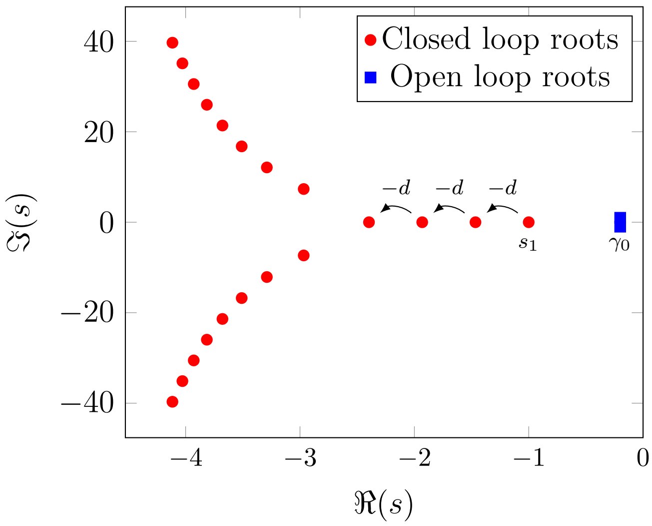

Figure 6: Spectrum distribution of the characteristic function (14) corresponding to the closed-loop system given by (12) and (13), with parameters given in (18)

Remark. *Regarding the control problem* (12)-(13)*, it is crucial to highlight that in the absence of the input delay, that is if* $\tau = 0$*, then employing a standard filtered PD may lead to outcomes featuring dominant poles. Notably, one can strategically place the poles of the finite-dimensional closed-loop system anywhere in the complex plane by employing a "state-feedback plus a second-order observer" type controller. This approach provides further flexibility when performing a finite-dimensional (delay-free) prescribed stabilization. However, the undoubted advantage of the approach proposed through this paper consists of the reduced complexity control that it suggests.*

Remark. *Practical control systems are often subject to operational constraints such as limited control capacity, i.e. limitations on input, state, and output variables. To emphasis the interest* *of using a delayed controller (13) in such scenarios we compare it to a classical PD controller* $v(t) = -\beta_1 y'(t) - \beta_0 y(t)$. *Let us focus on the limitation on the controllers amplitude by defining the following constraint on the gains:*

$$|\beta_1|^2 + |\beta_0|^2 < 1, \tag{19}$$

replacing β_i *by* α_i *in the case of the retarded PD controller* $u(t)$ *given by (13). Using a classical pole placement strategy, we assign the same spectral abscissa* $s_1 = -1$ *and we set the second root to* $s_2 = s_1 - d$, *which yields* $\beta_1 = \frac{8\sqrt{15}}{15} \approx 2.07$ *and* $\beta_0 = \frac{8\sqrt{15}}{15} - \frac{8}{5} \approx 0.47$. *Hence, we observe that the gains of the PD controller* $v(t)$ *do not satisfy the constraints (19). On the other hand, the gains* $\alpha_1 \approx -0.14$ *and* $\alpha_0 \approx -0.54$ *of the delayed version* $u(t)$ *satisfy such a constraint. Thus, similar to the discussion proposed in [33]*[†]*, we illustrate the positive effect of time-delay in constrained control systems. Furthermore, it should be emphasized that the decay rate is guaranteed for"small" (controller) gains, and thus less sensitive to high-frequency noise.*

5 P3δ Software

Partial pole placement via delay action (P3δ) is an intuitive Python software application that facilitates the design of stabilizing feedback laws incorporating time-delays. The software makes use of quasipolynomials properties (in particular, the MID property as studied in [19], [21]) to compute efficient control laws. Based on the results of this work, as well as reference [18], the software has been enhanced to include the CRRID property for design control strategies. P3δ is available for free download at https://cutt.ly/p3delta, where installation instructions, video demonstrations, and the user manual can also be found. Interested readers may also contact directly the software authors.

6 Concluding remarks

This work focused on the stabilization of a generic second-order linear differential equation with a delayed PD controller via the assignment of the maximum number of real roots of the corresponding characteristic function. We provided the expression of the differential equation for which the corresponding quasipolynomial admits the maximal number of real roots (which is, in our case, equal to four). In the case where the real roots are equidistant, we determined a necessary and sufficient condition on those coefficients for the largest real root to be negative. And we showed that this root is always the spectral abscissa of the quasipolynomial. Therefore, we proved exponential stability of the second-order linear differential equation and thus, we provided a partial poles placement alternative to the MID property.

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[†]for controlling the integrator by using two-delay blocks

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