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Timothée Schmoderer[1,2,*], Islam Boussaada[1,2], and Silviu-Iulian Niculescu[1]

[1]Université Paris-Saclay, CNRS, CentraleSupélec, Inria, L2S, 91190 Gif-sur-Yvette, France
[2]IPSA, 94200 Ivry-sur-Seine, France

[*]Corresponding author: timothee.schmoderer@centralesupelec.fr

November 27, 2023

Abstract

Recently, an intriguing property called *coexistent-real-roots-induced-dominancy* (CRRID) has been set and emphasized for some classes of linear time-invariant dynamical systems represented by retarded delay-differential equations. In this paper, we extend such a property to a class of neutral systems, and exploit it in the boundary control of the standard transport equation. Namely, by using the CRRID property, we show that one can arbitrarily and robustly prescribe the exponential decay of the closed-loop transport solution, yielding the prospect of applying the CRRID partial poles placement methodology to hyperbolic PDE's.

Keywords: Transport Equation, Boundary control, Time-delay controller, Neutral equations, PI-controller, Coexistent-real-roots-induced-dominancy, Partial pole placement

1 Introduction

Pole placement is a well-known and classical method for controlling finite-dimensional linear time-invariant (LTI) systems. Under appropriate controllability conditions, it consists in assigning poles of the closed-loop system to specified locations by an appropriate choice of the controller gains guaranteeing the stability of the closed-loop scheme with a prescribed decay rate of the corresponding system's solution, and its construction makes use of the characteristic polynomial degree and the controllability of the system. Its extension for infinite-dimensional linear systems is far from being trivial.

To the best of the authors' knowledge, in the case of systems governed by retarded and/or neutral delay-differential equations (DDEs), there exist only two effective extensions - *continuous pole placement* and *partial pole placement*. If the first method is mainly numeric and based on the continuity of the roots with respect to the system parameters[1], see, e.g., [1], [2], the second one, which is analytic, can be seen as complementary, and, as in the finite-dimensional case, makes use of the degree of characteristic function. As shown in [3], the degree of the

[1]including the controller gains

quasipolynomial[2] is a fundamental property, and it simply represents the maximal admissible multiplicity (and/or coexisting) characteristic real root(s). With the remarks above, the question is: *Does the assignment of such real roots guarantee the closed-loop stability of the closed-loop system?* Due to the infinite-dimensional nature of the closed-loop system, the problem is complicated since assigning a finite number of roots should "govern" the location of the remaining roots. This paper gives an explicit answer to such a question in the particular case of the transport equation, and opens some perspectives in controlling some classes of systems represented by PDEs.

It should be noted that in the single delay case, independently of the order of the DDEs, it was shown[3] in [4] that the real characteristic root with the maximal admissible multiplicity is dominant in the sense that it corresponds to the spectral abscissa of the system. This procedure simply describes the *partial pole placement* method mentioned above.

The aim of this paper is to control a dynamical system, represented by the classical linear transport equation, using a standard PI boundary control law and the partial pole placement methodology mentioned above. We will explore the cases where the coexistence of real roots guarantees the dominance for the rightmost one. More precisely, in frequency-domain, our method consists in assigning a maximal number of simple real roots to the characteristic function of the closed-loop system. Next, we give *necessary and sufficient conditions* to show that the largest characteristic root of the closed-loop system is negative and dominant, so that it corresponds to the spectral abscissa. Such a property is called *Coexisting-Real-Root-Induced-Dominancy* (CRRID). Beyond the generic case, where the number of real roots is equal to the degree of the corresponding characteristic function (quasipolynomial), we will explore also the non-generic case, that is when the number of real roots is smaller than the degree, and such a property is intuitively called *intermediate* CRRID. Although the first remarks on the coexistence of real roots for scalar DDEs goes back to the 50s-60s and the works of Hayes [5] and Wright [6], the systematic analysis of the CRRID began in [7], [8], where the exponential stability of a dynamical system represented by an nth-order linear differential equation controlled by a delayed proportional-controller is provided. Recent insights on the CRRID can be found in [9].

This paper provides the first extension of the CRRID property to neutral equations yielding effective prospects in the control of classes of partial differential equations and, as such, the proposed results represent a novelty in the open literature.

The remaining of the paper is organised as follows: the problem motivating this study is presented in Section 2. In Section 3 we define more rigorously the CRRID property, introduce an appropriate normalization of the quasipolynomial and state some of its properties. The main results are presented in Section 4 and then applied to the boundary control of the transport equation and illustrated through some numerical simulations in Section 5. Some concluding remarks end the paper. Finally, the main results use several technical lemmas that are stated and proved in the Appendix. The notations are standard and/or introduced when first used.

2 Motivating problem

In this paper, we investigate the problem of exponential stabilization of the following scalar conservation law:

$$\frac{\partial \varphi}{\partial t}(t,x) + \lambda \frac{\partial \varphi}{\partial x}(t,x) = 0, \quad t \in [0,+\infty[, \quad x \in]0,L[, \tag{1}$$

[2]given by the sum of the degrees of the involved polynomials plus the number of non-zero delays

[3]in both retarded and neutral cases

where $L > 0$, the value λ is a positive constant and denotes the velocity of propagation, and $\varphi(t, x)$ stands for the system state in time $t \geq 0$ and at position $x \in]0, L[$. Following [10], we control equation (1) by using the boundary condition

$$\varphi(t, 0) = k_p\, \varphi(t, L) + k_i \int_0^t \varphi(u, L)\, \mathrm{d}u, \tag{2}$$

which is a “standard” PI controller, where k_p and k_i are the feedback parameters representing ”proportional” and ”integral” control gains. Applying the Laplace transform to both sides of the boundary condition (2) and multiplying by s we obtain the closed-loop characteristic function

$$\Delta_{\mathrm{T}}(s) = s - e^{-\frac{L}{\lambda}s} \left(k_p s + k_i\right). \tag{3}$$

The asymptotic behavior of the solutions of (1) and (2) is dictated by the location of the rightmost roots of the characteristic quasipolynomial Δ_{T} given in (3) . Notice also that Δ_{T} corresponds to the characteristic function of the following first-order neutral delay-differential equation (NDDE)

$$y'(t) - k_p\, y'\left(t - \frac{L}{\lambda}\right) - k_i\, y\left(t - \frac{L}{\lambda}\right) = 0. \tag{4}$$

Therefore the systematic study of NDDEs is useful to understand the PI boundary control of the transport equation. In the seminal work [10], the stability of the solutions of the closed-loop system given by (1) and (2) is analyzed via the Walton-Marshall stability criterion [11]. However, the analysis of [10], based on the coefficients of k_p and k_i along with the delay $\frac{L}{\lambda}$, does not offer information regarding the decay rate of the solutions[4]. We refer to [12] for further insights on the study of the transport equation (1) and see [13] for a deeper discussion of the spectral properties of NDDEs (4), as well as related stability analysis and control approaches. In [4], the boundary stabilization of the transport equation solutions is conducted by assigning a root of maximal multiplicity three, which defines the decay rate of the solutions; corresponding to the generic multiplicity-induced-dominancy (GMID) property. Similarly, the authors of [14] make use of the intermediate MID property, or IMID for short, achieved by assigning a root of intermediate multiplicity, to explicitly describe the exponential decay rate of the solutions, see also [15]. However, it is commonly accepted, such as suggested in [16], that non-semisimple spectral values are sensitive to small perturbations due to their splitting mechanism, thereby questioning the design robustness relying on the MID property.

The CRRID proposed in the sequel generalizes the MID property as it also prescribes the decay rate of the solutions, albeit with the advantage of satisfying some robustness requirements with respect to parametric uncertainties. This advantage is due to the assignation of simple spectral values.

3 Preliminaries

To figure out the asymptotic behavior of the solutions of (1) and (2) or equivalently (4), we study the more general characteristic quasipolynomial function

$$\Delta(s) = s + a_0 + e^{-\tau s}(\alpha_1 s + \alpha_0), \tag{5}$$

where $(a_0, \alpha_1, \alpha_0) \in \mathbb{R}^3$. The degree of Δ, defined by the sum of the degrees of the involved polynomials plus the number of delays, is three. The following result asserts that the maximal number of real roots counted with multiplicity is three; see [3], [17].

[4]the value of the real part of the rightmost root of (3)

Proposition 1 (Pólya-Szegö bound)**.** *Let Δ be the quasipolynomial given by (5), and $\alpha, \beta \in \mathbb{R}$ be such that $\alpha \leq \beta$. Denote by $M_{\alpha,\beta}$ the number of roots of Δ counted with multiplicities contained in the set $\{s \in \mathbb{C},\, \alpha \leq \Im(s) \leq \beta\}$. Then,*

$$\frac{\tau(\beta - \alpha)}{2\pi} - 3 \leq M_{\alpha,\beta} \leq \frac{\tau(\beta - \alpha)}{2\pi} + 3. \tag{6}$$

As a consequence of the last result, when setting $\alpha = \beta = 0$, we conclude that the degree 3 of Δ is a sharp bound for the number of real roots of the quasipolynomial Δ. By inverting inequality (6), the following corollary follows:

Corollary 2. *If Δ admits three real roots, then any root $s = x + \mathbf{i}\omega$ of Δ with $\omega \neq 0$ satisfies $|\omega| \geq 2\pi/\tau$.*

A rightmost root s^* is characterised by the property

$$\forall\, s_0 \in \mathbb{C}\backslash\{s^*\}, \quad \Delta(s_0) = 0 \quad \Longrightarrow \quad \Re(s_0) \leq \Re(s^*)$$

and is called a *dominant root* of the quasipolynomial Δ. Let us define the main property that we use in our control design.

Definition 3 (CRRID property)**.** *We say that a general quasipolynomial Δ of degree N satisfy the intermediate coexistence-real-root-induced-dominancy (or ICRRID for short) property of corank K if it admits $N-K$, for $0 \leq K \leq N-1$, distinct real roots $s_1 \geq \ldots \geq s_{N-K}$ and s_1 is a dominant root of Δ. Note that if $s_k = \ldots = s_{k+\ell}$, for some $\ell \geq 0$, then s_k should be understood as a root of multiplicity $\ell + 1$. The CRRID of corank 0 is called the Generic CRRID (or GCRRID for short) property.*

To study the CRRID property for the quasipolynomial (5), we introduce the following auxiliary function

$$Q(z) = \frac{1}{s_1 - s_2}\Delta\Big((s_1 - s_2)\, z + s_1\Big), \tag{7}$$

where s_1 and s_2 are two distinct real roots of $\Delta(s)$ satisfying $s_1 > s_2$. Some properties of the function $Q(\cdot)$ are listed in the proposition below.

Proposition 4 (Normalized quasipolynomial properties)**.** *The following statements hold.*

1. *The auxiliary function Q is a quasipolynomial of degree 3: $Q(z) = z + b_0 + e^{-qz}\,(\beta_1 z + \beta_0)$, where the auxiliary delay $q > 0$ and the real coefficients b_0, β_1, β_0 are related to those of Δ via*
$$\begin{cases} q &= \tau(s_1 - s_2), \quad b_0 = \frac{s_1 + a_0}{s_1 - s_2}, \\ \beta_1 &= \alpha_1 e^{-\tau s_1}, \quad \beta_0 = \frac{\alpha_1 s_1 + \alpha_0}{s_1 - s_2} e^{-\tau s_1}. \end{cases}$$

2. *The normalized quasipolynomial Q admits at least two real roots 0 and -1, which implies that*
$$\beta_1 = \left(e^{-q} - 1\right) b_0 - e^{-q} \quad \textit{and} \quad \beta_0 = -b_0. \tag{8}$$
Moreover, if Q admits a third real root ζ then
$$\beta_1 = \frac{\zeta e^{q} + e^{-q\zeta} - \zeta - 1}{(-1 - \zeta)\, e^{-q(\zeta - 1)} + e^{-q\zeta}\zeta + e^{q}}. \tag{9}$$
If $\zeta = 0$ or $\zeta = -1$, then the above expression should be taken as the limit.

3. *The root $z = 0$ is a dominant root for Q if, and only if, s_1 is a dominant root for Δ.*

Proof. The proof of the first item follows from simple computations using definition (7). The mapping $\varphi : z \mapsto (s_1 - s_2)z + s_1$, used to construct Q, clearly satisfy $\varphi(0) = s_1$ and $\varphi(-1) = s_2$. Hence, Q admits two roots: $Q(0) = \Delta(s_1) = 0$ and $Q(-1) = \Delta(s_2) = 0$. The expressions (8) and (9) are thus obtained by solving the set of transcendental equations $Q(0) = Q(-1) = Q(\zeta) = 0$. When $\zeta = 0$ or -1, then Q has a double root and one solves the equations $Q(0) = Q(-1) = Q'(\zeta) = 0$, which yields the same value of the coefficient β_1 as the one obtained by taking the corresponding limit in (9). Finally, the mapping φ is the composition of a positive scaling (since $s_1 > s_2$) and a translation. Hence, inequalities between the real parts of complex numbers are preserved and, since 0 is mapped to s_1, the root 0 is dominant for Q if, and only if, s_1 is dominant for Δ. □

4 Further results on the CRRID property

In this section we establish our main results, we sketch their proofs and we defer the technical details to the Appendix. First, Theorem 5 establishes the GCRRID property for first order neutral equation. As a consequence, if the real roots of the corresponding quasipolynomial are negative, then we obtain exponential stability with a guaranteed decay rate of the solutions. Second, Theorem 6 gives necessary and sufficient conditions on the coefficients of the neutral equation for it to satisfy the ICRRID property of corank 1.

Theorem 5 (GCRRID property for neutral equations)**.** *Assume that the quasipolynomial Δ given by (5) admits three real roots $s_1 \geq s_2 \geq s_3$, then s_1 is a dominant root for Δ.*

Proof. The case $s_1 = s_2 = s_3$ corresponds to the GMID property, which is described and proven for neutral systems in [4]. In this proof, which is based on the continuous dependence of the spectrum distribution with respect to parameters' variations, we deal with the case where at least two real roots are distinct. In this case, using Proposition 4, we consider the normalized quasipolynomial Q and, assuming that it admits three real zeros, we prove that 0 is a dominant root. The main arguments of our proof are described hereafter, and all technicalities are given in the lemmas of the Appendix.

Let us show that the normalized quasipolynomial Q, which admits the real roots $z = 0$, $z = -1$, and $\zeta \leq 0$, satisfies the GCRRID property. First, we prove that Q satisfies the GCRRID property when $\zeta = -2$ (refer to Lemma 8). Second, we infer that if $\zeta \leq 0$, then the only purely imaginary root is zero (refer to Lemma 9). Therefore, if we let ζ vary around $\zeta = -2$, the only way that a non-real root passes in the right-half-plane is through the origin. But the latter fact is impossible since it would imply the existence of four real roots for Q, contradicting its degree. □

Note that the specific instance of roots $z = 0, -1, -2$ that we assign for Q generates equidistributed roots for Δ. In figure 1, we illustrate the argument made in the proof of Theorem 5 by displaying the spectrum distribution of the normalized quasipolynomial Q. The equidistributed scenario is denoted by black squares and, employing a color gradient, we exhibit the spectrum of Q as ζ spans the domain $\zeta \in [-3, 0]$.

Remark. *In the limiting configuration of Theorem 5, which corresponds to a triple real spectral value $s_0 := s_1 = s_2 = s_3$, the GMID is proven in [4] and interestingly the remaining spectrum is completely characterized. Thus, we have*

$$s = s_0 + \frac{\omega}{\tau}\mathbf{i}, \quad \text{where} \quad \tan\left(\frac{\omega}{2}\right) = \frac{\omega}{2}.$$

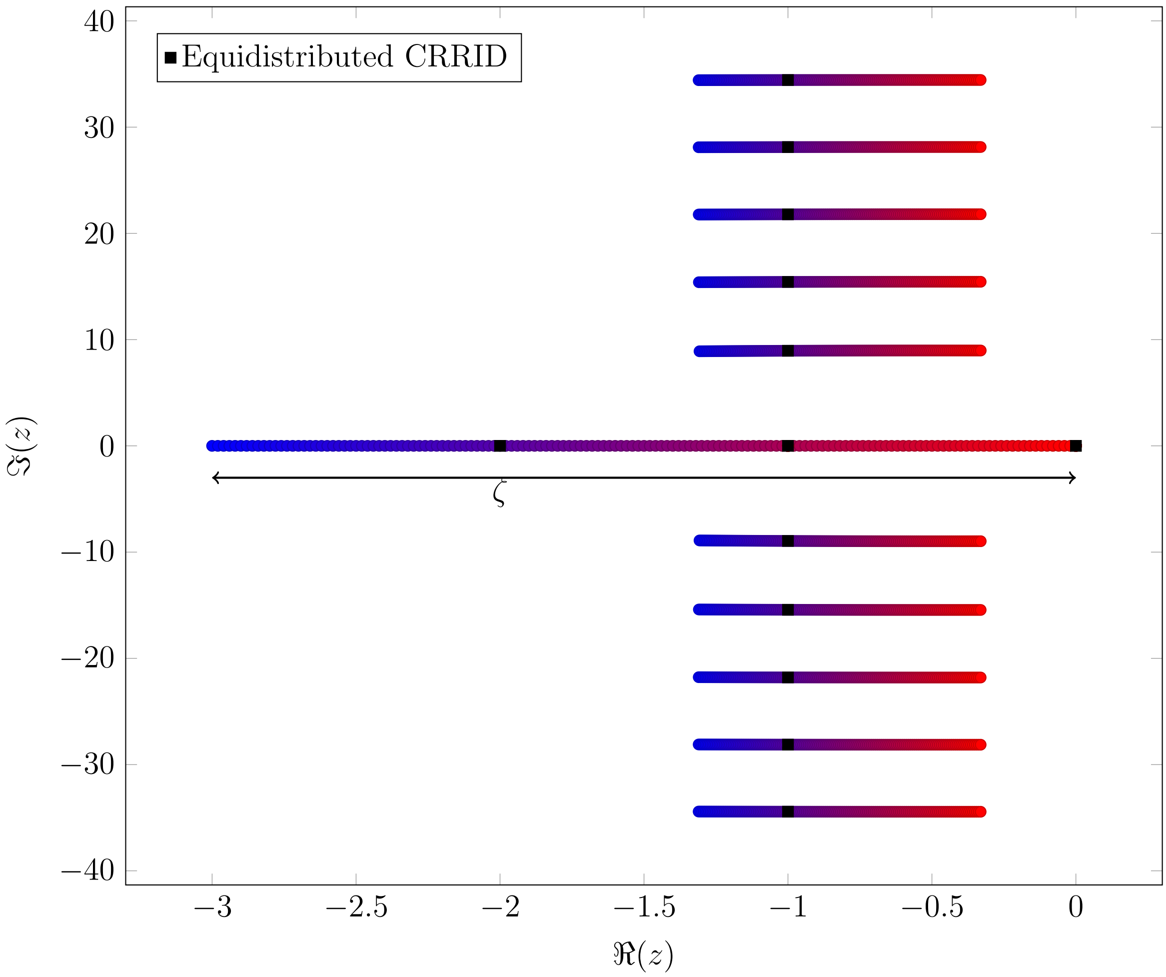

Figure 1: Spectrum distribution of the normalized quasipolynomial Q for $q = 1$ and $\zeta \in [-3, 0]$. The black squares show the spectrum for the case of equidistributed roots $z = 0, -1, -2$. The blue color corresponds to $\zeta \to -3$ and the red one is for $\zeta \to 0$.

In the case of distinct real spectral values, numerical simulations suggest that the remaining spectrum is also located on a single vertical line; see figure 1. However, its characterization and the determination of the imaginary parts remain open questions.

Theorem 6 (ICRRID property for neutral equations)**.** *Assume that the quasipolynomial Δ admits two distinct real roots $s_1 > s_2$. Then, the root s_1 is a dominant root of Δ if, and only if,*

$$\frac{1 - e^{\tau(s_1 - s_2)}}{e^{\tau(s_1 - s_2)}(\tau(s_1 - s_2) - 1) + 1} \leq \frac{s_1 + a_0}{s_1 - s_2} \leq 1.$$

The above theorem gives a necessary and sufficient condition on the auxiliary delay q (of the auxiliary quasipolynomial Q) and on the coefficient a_0 (of the original quasipolynomial Δ) that ensures the dominancy of the root s_1. If in addition s_1 is assumed to be negative, then the exponential decay rate of the solutions is assigned. The conditions obtained in the above theorem are expressed in terms of a_0 because, for application purposes, it is more natural to tune α_1 and α_0 in terms of the dynamics parameter a_0 and the delay τ. It is noteworthy that these conditions may be expressed in terms of other coefficients[5].

[5]Actually, in the proof, we use the version written in terms of α_1

Proof. Since we assume that Δ has two distinct real roots, we use the normalized quasipolynomial $Q(z)$ and we prove the following equivalent statement. The root $z = 0$ is a dominant root of Q if, and only if, $\frac{1-e^q}{e^q(q-1)+1} \leq b_0 \leq 1$, or equivalently, using the first equation of (8), we have

$$-1 \leq \beta_1 \leq \frac{e^q - q - 1}{e^q(q-1)+1}. \tag{10}$$

For neutral equations, a necessary condition for stability, giving the dominancy of 0, is $|\beta_1| \leq 1$, see for instance [13]. To show the dominancy of the root $z = 0$ when β_1 satisfies (10), we distinguish the cases whether there exists a third real root or not.

1. If $0 < \beta_1 \leq \frac{e^q-q-1}{e^q(q-1)+1}$, then owing to Lemma 7, there exists a third real root $\zeta \leq 0$. Hence, the conclusion of Theorem 5 holds and the root $z = 0$ is dominant for Q.
2. If $-1 < \beta_1 \leq 0$, then according to Lemma 9 any root crossing the imaginary axis does so via the origin. In this case, it would imply the existence of a third real root, contradicting Lemma 7.
3. If $\beta_1 = -1$, then $Q(z) = -(z+1)\left(e^{-qz} - 1\right)$, and its roots are analytically given by $z = -1$ and $z = \frac{2k\pi}{q}\mathbf{i}$, for $k \in \mathbb{Z}$, and therefore $z = 0$ is a dominant root of Q.

Conversely, if β_1 does not fulfill (10), then 0 is not a dominant root. Indeed, if $\beta_1 < -1$, then Q admits an infinite number of roots in the right-half-plane; see [13]. If $\beta_1 > \frac{e^q-q-1}{e^q(q-1)+1}$, then owing to Lemma 7, there exists a third real root $\zeta > 0$ and 0 is not dominant. $\square$

Figure 2 displays the assignability region (10) for the coefficient β_1 and the auxiliary delay q of Q such that the dominancy of $z = 0$ is guaranteed.

Remark. *In Theorem 6 it is assumed that the quasipolynomial Δ admits at least two distinct real roots $s_1 > s_2$. The limiting case where $s_1 = s_2$ corresponds to the IMID property and has been studied in [14]. It has been shown that a quasipolynomial satisfying the IMID, necessarily shares its remaining zeros with an appropriate linear combination of Kummer hypergeometric functions, see [18].*

5 Boundary Control of the Transport Equation

We illustrate the GCRRID property in the tuning of the parameters gains of the transport equation (1) with the PI boundary controller (2). To simplify the analysis, consider the case of three equidistributed roots: $s_1, s_1 - \delta, s_1 - 2\delta$, with the distance $\delta > 0$. By solving the set of transcendental equations $\Delta\left(s_1 - (k-1)\delta\right) = 0$, for $1 \leq k \leq 3$, we obtain

$$\tau = \frac{\sigma}{\delta}, \quad k_p = -\frac{s_1}{s_1 - 2\delta} e^{\frac{s_1}{\delta}\sigma}, \quad k_i = \frac{2s_1(s_1-\delta)}{(s_1-2\delta)} e^{\frac{s_1}{\delta}\sigma},$$

where $\sigma = \ln\left(\frac{s_1-2\delta}{s_1}\right)$. Notice that the delay τ is fixed by the system parameters λ and L, and thus it imposes a relation between s_1 and δ:

$$s_1 = \frac{2\delta}{1 - e^{\tau\delta}}. \tag{11}$$

Observe that the condition $\delta > 0$ implies,that $-\frac{2}{\tau} < s_1 < 0$. Setting $\tau = \frac{L}{\lambda} = 1$ and $s_1 = -1$ yields

$$\delta \approx 1.2564, \quad k_p \approx -0.1047, \quad k_i \approx -0.4726. \tag{12}$$

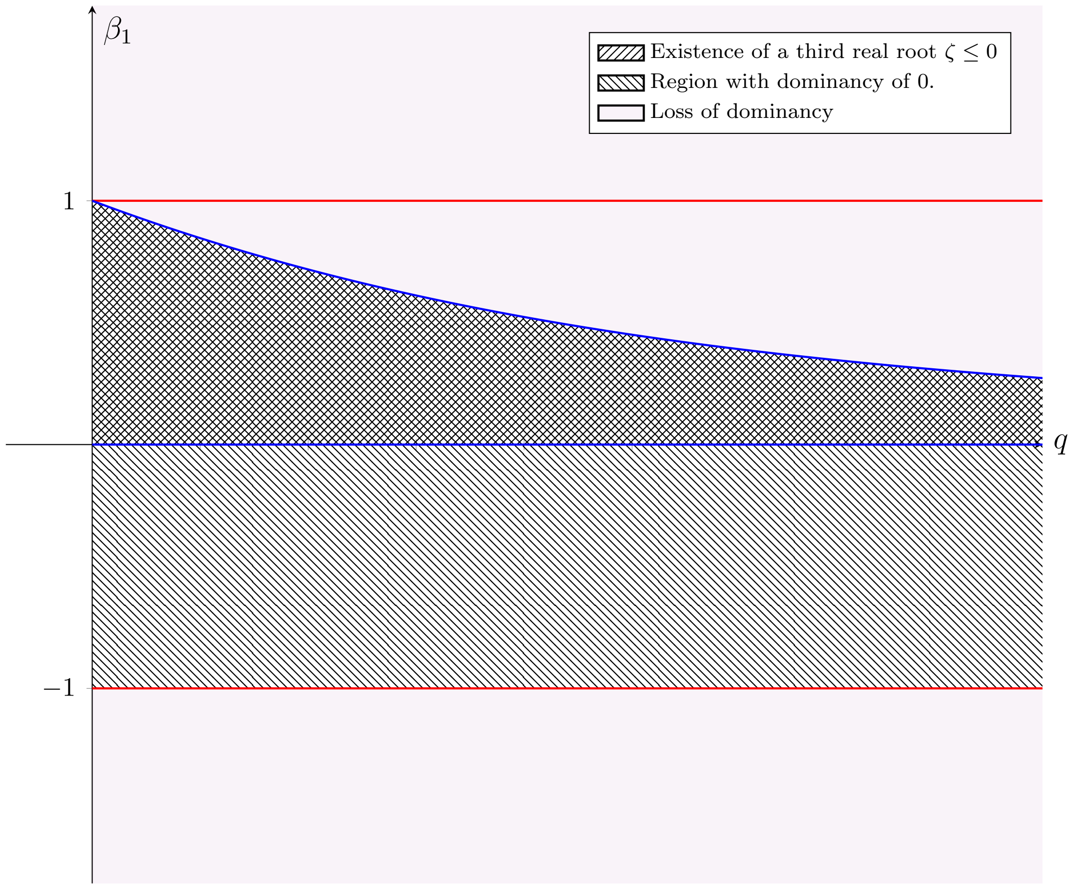

Figure 2: Assignability region (10) for the coefficient β_1 of the normalized quasipolynomial $Q(z)$ to guarantee the dominancy of the root $z = 0$, when it admits at least two real zeros. The red area shows the region where the dominancy of $z = 0$ fails, the red lines at -1 and 1 corresponds to the necessary condition $\beta_1 < 1$, the dashed areas display the region with the dominancy of $z = 0$ due to either the existence of a third negative real root (delimited by blue curves) or to the nonexistence of a root crossing the imaginary axis.

To perform numerical simulations, we adopt the same initial condition $\varphi(0, x) = \sin(2\pi x)$ as the one considered in [4], where the PI boundary control has been considered with the use of the GMID property, since it is worthwhile to compare these two approaches. In the context of the GMID property, the assigned triple root s_0 is uniquely determined by the system parameters, namely $s_0 = -\frac{2}{\tau}$. Similarly, the controller gains are explicitly given by

$$k_p = -e^{-2} \quad \text{and} \quad k_i = -\frac{4e^{-2}}{\tau}. \tag{13}$$

On the one hand, the decay rate associated with the GMID is always better than the decay rate derived via the equidistributed GCRRID, since (11) implies that $s_1 > -\frac{2}{\tau}$. Figure 3 illustrates this fact, in the logarithmic scale, by exhibiting the decay rate of the solution $\varphi(t, x)$ at $x = L$ of (1) with the GCRRID parameters (12) and the GMID parameters (13). After the transitory regime, we observe an order 2, resp. 1, convergence rate for the GMID, resp. for the GCRRID.

On the other hand, due to the fact that the GMID-assigned spectral abscissa s_0 is triple, the GMID property is sensitive to small parametric variation [16]. The GCRRID property is less sensitive since the assigned roots are simple. Figure 4 shows the migration of the spectrum of the characteristic function Δ_{T} when the propagation velocity λ is assumed to be uncertain and thus the delay τ increases from its nominal value 1. In this case, we observe (displayed as hollow diamonds) that the triple root s_0 immediately splits in one real root and two complex conjugate

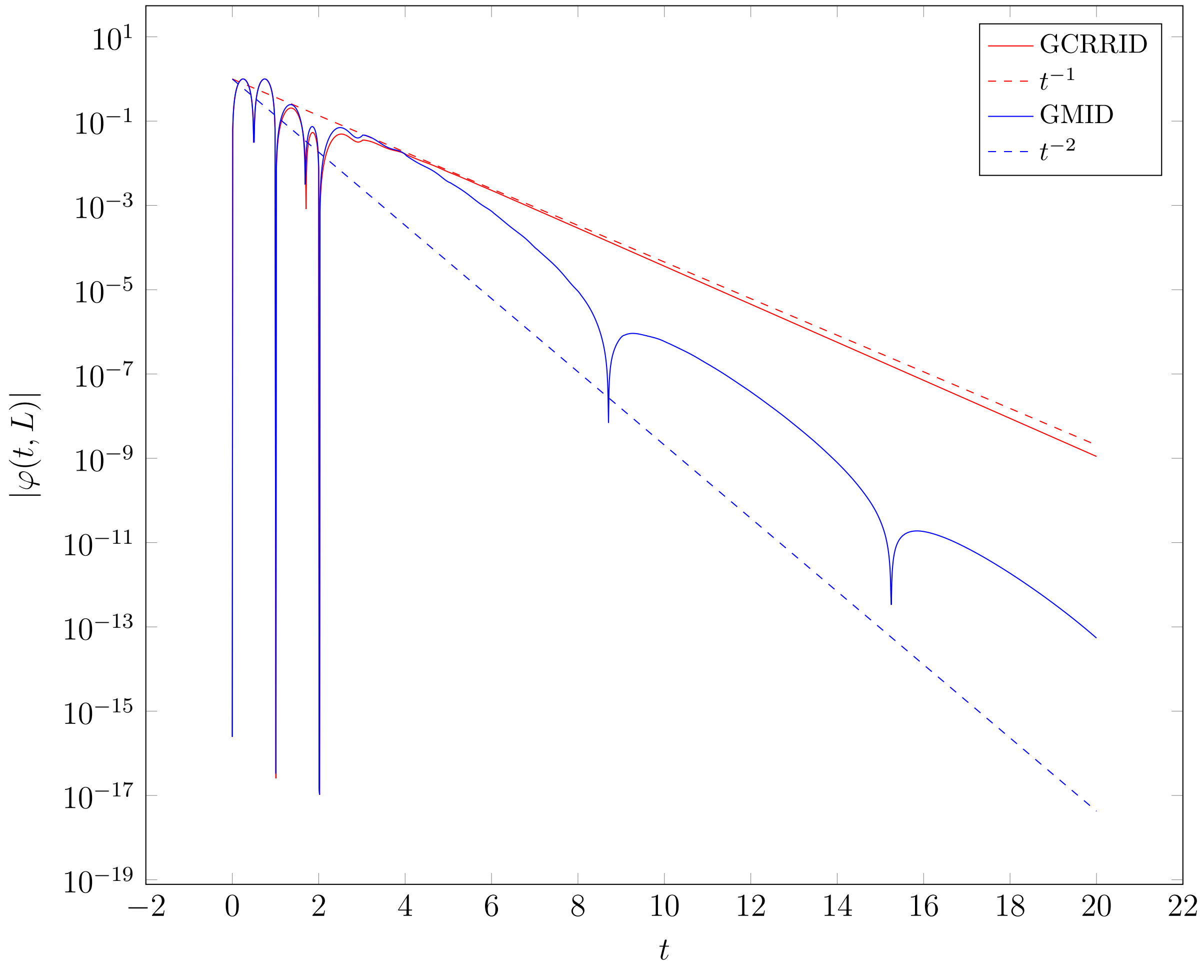

Figure 3: Comparison of the decay rate of the solutions of (1) with the GCRRID and GMID methods.

roots. However, in the GCRRID case (displayed with circles) the perturbation of the delay yields a fusion of the two dominant roots s_1 and s_2 into a double zero, which itself splits in two conjugate complex roots. Moreover, the critical delay (i.e., the delay for which the roots meet the imaginary axis) is larger, hence less sensitive (more robust), in the GCRRID case compared with the GMID case. Indeed, we numerically obtain $\tau^c_{\text{GMID}} \approx 3.12$ and $\tau^c_{\text{GCRRID}} \approx 3.53$.

6 Concluding remarks

In this paper, we prove the CRRID property for first-order neutral delay-differential-equation and apply it to the boundary control of the transport equation. We show the GCRRID: the coexistence of three real roots (not necessarily distinct) implies that the rightmost of them is necessarily the spectral abscissa of the characteristic function and if it is negative it defines the exponential decay rate of the solutions of the closed-loop system. We also state and prove the ICRRID property in the case where only two real roots exist. It is shown that our methodology explicitly sets the exponential decay rate of the solutions and is robust with respect to parametric variations.

Acknowledgement. *The authors are grateful to their colleagues Guilherme Mazanti (Inria Saclay) and Karim L. Trabelsi (IPSA) for careful reading and insightful discussions.*

The following technical lemmas are useful in the proofs of Theorems 5 and 6.

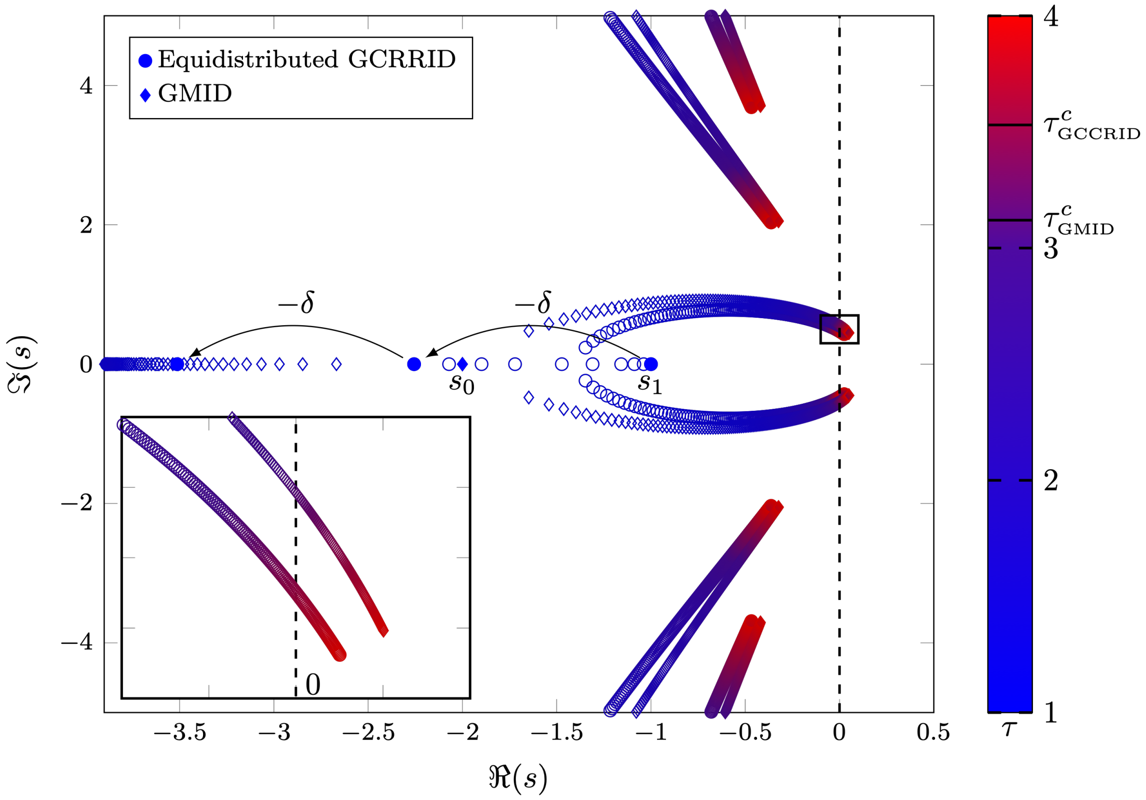

Figure 4: Migration of the spectrum of the characteristic function (3) with parameters given in (12) in the GCRRID case and in (13) in the GMID case with respect to the perturbation of the delay $\tau \in [1, 4]$. The perturbed GMID, resp. GCRRID, case are displayed with hollow diamonds, resp. circles.

Lemma 7 (Properties of β_1)**.** *Suppose that the normalized quasipolynomial Q admits three real roots $0, -1, \zeta$. Then, the map $\zeta \mapsto \beta_1(\zeta)$ is strictly increasing and defines a bijection from $\mathbb{R}$ onto $\mathbb{R}_+^*$. In particular, for $\zeta \leq 0$, we have $0 < \beta_1 \leq \frac{e^q - q - 1}{1 + e^q(q-1)} < 1$.*

Proof. The normalized quasipolynomial Q admits three real roots, thus we use the expression of β_1 given by (9). One can show that the denominator of β_1 vanishes only at $\zeta = 0$ and $\zeta = -1$. Recall that at these points β_1 can be defined by taking the limit (see Proposition 4). To show that $\beta_1(\zeta)$ is strictly increasing, we set the function $f(\zeta) = n'(\zeta)d(\zeta) - n(\zeta)d'(\zeta)$, where n and d are the numerator and denominator of β_1, and we prove that $f(\zeta) > 0$ for $\zeta \in \mathbb{R}\backslash\{-1, 0\}$. We consider the auxiliary function $h(\zeta) = f'(\zeta)e^{q\zeta}$, which admits $h''(\zeta) = 2q^2\left(qe^{-q\zeta} - e^q + 1\right)(1 - e^q)$. The second derivative $h''(\zeta)$ vanishes only for $\bar{\zeta} = \frac{1}{q}\ln\left(\frac{q}{e^q - 1}\right) \in]-1, 0[$ and, since $\lim_{\zeta\to-\infty} h''(\zeta) = -\infty$ and $\lim_{\zeta\to+\infty} h''(\zeta) = 2q^2\left(e^q - 1\right)^2 > 0$, we deduce that $h'(\zeta)$ is decreasing on $]-\infty, \bar{\zeta}[$ and increasing on $]\bar{\zeta}, +\infty[$. In addition, due to $h'(\bar{\zeta}) = \left(e^q - 1\right)^2\left(q + 2\ln(q) - 2\ln(e^q - 1)\right)q < 0$, we deduce that $h'(\zeta)$ admits two roots $-1 < \bar{\zeta}_1 < \bar{\zeta}_2 < 0$. Furthermore, using that $\lim_{\zeta\to\pm\infty} h'(\zeta) = +\infty$, it follows that $h(\zeta)$ is increasing on $]-\infty, \bar{\zeta}_1[$, decreasing on $]\bar{\zeta}_1, \bar{\zeta}_2[$, and increasing on $]\bar{\zeta}_2, +\infty[$. Therefore, $h(\zeta)$ admits a zero $\bar{\zeta}_3 \in]\bar{\zeta}_1, \bar{\zeta}_2[\subset] - 1, 0[$. Next, we infer that $f(\zeta)$ decreases on $]-\infty, -1[\cup]\bar{\zeta}_3, 0[$, increases on $]-1, \bar{\zeta}_3[\cup]0, +\infty[$, and is bounded by below by zero (which is achieved at $\zeta = -1$ and 0). So $f(\zeta)$ is strictly positive on $\mathbb{R}\backslash\{-1, 0\}$ and $\beta_1(\zeta)$ is a strictly increasing function. Since $\lim_{\zeta\to-\infty} \beta_1 = 0$ and $\lim_{\zeta\to+\infty} \beta_1 = +\infty$, we conclude that β_1 defines a bijection from $\mathbb{R}$ to $\mathbb{R}^{+*}$. Finally, if $\zeta \leq 0$, then $\beta_1(\zeta) \leq \beta_1(0) = \frac{e^q - q - 1}{1 + (q-1)e^q} < 1$. □

Lemma 8. *If the normalized quasipolynomial Q admits three real roots $z = 0$, $z = -1$ and $z = -2$, then it satisfies the GCRRID property.*

Proof. To prove that Q does not admit a root in the right half plane, we proceed by contradiction assuming that $z = x + \mathbf{i}\omega$, with $x > 0$, is a root of Q. By extracting the real and imaginary parts of $Q(x + \mathbf{i}\omega) = 0$ we obtain a system of equations, which owing to (8) yields

$$\cos(\omega q) = -\frac{e^{qx}\left(\left(\omega^2 + x^2 - x\beta_0\right)\beta_1 + \beta_0\left(x - \beta_0\right)\right)}{\left(\omega^2 + x^2\right)\beta_1^2 + 2x\beta_0\beta_1 + \beta_0^2},$$
$$\sin(\omega q) = \frac{e^{qx}\omega\beta_0\left(\beta_1 + 1\right)}{\left(\omega^2 + x^2\right)\beta_1^2 + 2x\beta_0\beta_1 + \beta_0^2}.$$

Using the relation $\cos(\omega q)^2 + \sin(\omega q)^2 = 1$, we deduce that ω is necessarily given by $\omega = \pm\sqrt{\left(\left(x\beta_1 + \beta_0\right)^2 - \left(x - \beta_0\right)^2 e^{2qx}\right) / \left(e^{2qx} - \beta_1^2\right)}$.
We set $\Omega = \omega^2$ and prove that $\Omega < \frac{1}{q^2}$ for all $x > 0$ and $q > 0$. To this end, we define $f_q(x) = n(x) - \frac{1}{q^2}d(x)$, with n and d the numerator and denominator of Ω, and show that $f_q(x) < 0$. We have $f_q(0) = (e^{-2q} - 1)q^{-2}$,

$$f_q'(0) = \frac{-4e^{-q}q - 2e^{2q} + (4q + 4)\,e^{q} - 8q^2 - 2}{\left(e^{q} - 1\right)^2 q},$$
$$f_q''(0) = -\frac{2e^{-2q}}{\left(e^{q} - 1\right)^2}\Bigg(8\left(q + \frac{1}{2}\right)^2 e^{2q} - 8\left(q + \frac{3}{4}\right)e^{3q} + 3e^{4q} + 2e^{q} - 1\Bigg),$$

and $f_q'''(x) = -8e^{2qx}q\left(\left(x - \beta_0\right)^2 q^2 + 3\left(x - \beta_0\right)q + \frac{5}{2}\right)$, which is strictly negative for all x (since the discriminant of the quadratic polynomial in the right factor is $-q^2$ and the coefficient in front of x^2 is positive). Hence, $f_q''(x)$ is decreasing with respect to x. Moreover, as $f_q''(0) < 0$ holds[6] for all $q > 0$, it follows that f_q' is also decreasing with respect to x. Since $f_q'(0) < 0$[6], thus f_q is a decreasing function for $x > 0$. Combining this with the fact that $f_q(0) < 0$, we conclude that $f_q(x) < 0$ for all $x > 0$ and $q > 0$.

Finally, we have demonstrated that if $x + \mathbf{i}\omega$ is a root of Q with $x > 0$, then necessarily $|\omega| < \frac{1}{q}$. However, this contradicts corollary 2, thus establishing that Q has no roots in the right half-plane. □

Lemma 9 (Purely imaginary roots of $Q(z)$)**.** *If* $|\beta_1| < 1$*, then* $Q(\mathbf{i}\omega) = 0$ *if, and only if,* $\omega = 0$.

Proof. Suppose that $Q(\mathbf{i}\omega) = 0$, then we obtain

$$\left(\mathbf{i}\omega + b_0\right) + e^{-q\mathbf{i}\omega}\left(\beta_1\mathbf{i}\omega + \beta_0\right) = 0,$$
$$\left|\mathbf{i}\omega + b_0\right|^2 = \left|\beta_1\mathbf{i}\omega + \beta_0\right|^2,$$
$$\omega^2\left(1 - \beta_1^2\right) + b_0^2 - \beta_0^2 = 0.$$

By assumption, $\beta_1^2 < 1$ and, using equation (8), we have $b_0 + \beta_0 = 0$. Thus, we conclude that $Q(\mathbf{i}\omega) = 0$ is satisfied only by $\omega = 0$. □

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