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Decay Rate Assignment through Multiple Spectral Values in Delay Systems

Islam Boussaada[*][†] Guilherme Mazanti[*] Silviu-Iulian Niculescu[*]
Wim Michiels[‡]

Abstract

This paper focuses on a spectral property for linear time-invariant (LTI) dynamical systems represented by delay-differential equations (DDEs) entitled *multiplicity-induced-dominancy* (MID), which consists, roughly speaking, in the spectral abscissa of the system being defined by a multiple spectral value. More precisely, we focus on the MID property for spectral values with over-order multiplicity, i.e., a multiplicity larger than the order of the DDE. We highlight the fact that a root of over-order multiplicity is necessarily a root of a particular polynomial, called the *elimination-produced-polynomial*, and we address the MID property using a suitable factorization of the corresponding characteristic function involving special functions of Kummer type. Additional results and discussion are provided in the case of the n-th order integrator, in particular on the local optimality of a multiple root. The derived results show how the delay can be further exploited as a *control parameter* and are applied to some problems of stabilization of standard benchmarks with prescribed exponential decay.

Keywords: delay, characteristic function, exponential stability, Kummer functions, hypergeometric functions, Green–Hille transformation, partial pole placement.

1 Introduction

Since Hazen's paper [32] on the theory of servomechanisms in the 1930s, it is commonly accepted that the delays in systems' dynamics are at the origin of dynamics oscillations and instabilities. As a consequence, modeling delays, understanding the effects induced by the delays and controlling delay systems represented a problem of recurring interest during the last century. More precisely, one of the ways to describe time heterogeneity of processes and/or phenomena is to use mathematical models based on delay-differential equations (DDEs). For example, transport and propagation phenomena, signal transmission in communication networks, or age structure in population dynamics are typical classes of processes and/or phenomena where delay can be used to model time heterogeneity leading to DDEs. For further

[*]Université Paris-Saclay, CNRS, CentraleSupélec, Inria, Laboratoire des Signaux et Systèmes, 91190 Gif-sur-Yvette, France (e-mails: {first name.last name}@centralesupelec.fr).

[†]IPSA Paris, 94200 Ivry-sur-Seine, France

[‡]Department of Computer Science, KU Leuven, B-3001 Heverlee, Belgium (e-mail: wim.michiels@cs.kuleuven.be).

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examples, we refer to [27, 36, 39, 42, 49, 58] and the references therein. Finally, for an appropriate definition and related classification of DDEs, the reader is referred to [4, 26, 30, 37, 39].

As pointed out in [63], a delay can induce stability (stabilizing effect) in some cases, and a lack of stability (destabilizing) in other cases. These stability issues have been extensively discussed in the open literature, and there exists a systematic methodological and numerical treatment for the stability analysis of most cases (see, e.g., [23,29,41,49,53,55,64]). To the best of the authors' knowledge, in the control area, the the beneficial effect of the delay in closed-loop appears in the 1970s within the framework of the approximation of the derivative action of PID controllers by a delay-difference operator [65,66]. Moreover, the τ-decomposition method, proposed a decade earlier by [40], enables the computation of the delay intervals guaranteeing asymptotic stability, explicitly showing that, in certain cases, augmentation of the delay leads to stability. In this context, the idea of using the *delay* as a *control parameter* came naturally. For instance, a chain of n integrators can be stabilized by a controller including n delays [38, 54]. If the main advantage of exploiting delays in the controllers is the simplicity of their implementation, their infinite-dimensional character, however, yields some unexpected behaviors of the corresponding closed-loop systems which imposes, as a consequence, some limitations in the choice of the parameters. For an overview of some of the methods and techniques, we refer to [24, 49, 63].

A classical approach in the stability analysis and stabilization of linear time-invariant (LTI) dynamical systems including delays is the application of spectral methods (see, e.g., [49]). The spectrum of a DDE can be characterized as the set of complex roots of its characteristic function, which presents itself under the form of a *quasipolynomial*, i.e., a finite sum of polynomials multiplied by exponentials. These roots are usually referred to as *spectral values* or *characteristic roots* of the system. The analysis of quasipolynomials and, in particular, the location of their roots, is of fundamental importance for the spectral analysis of DDEs, and many works have addressed this question. For instance, the origin of a LTI DDE is exponentially stable if, and only if, the *spectral abscissa*[1] of the system, defined as the supremum of the real parts of the roots of its characteristic function, is negative. We refer the interested reader to [49] and [8, Chapter 3] (where these functions are referred to as *exponential polynomials*). In particular, an important fact about a quasipolynomial is that the multiplicity of any of its roots is upper bounded by some positive integer, known as the *degree* of the quasipolynomial, as stated, for instance, in [60, Part Three, Problem 206.2], [18, 19].

In the case of LTI systems represented by DDEs, recent works have highlighted a particularly interesting spectral property, called *multiplicity-induced-dominancy (MID)*, which consists in conditions on the system's parameters under which a multiple spectral value corresponds to the spectral abscissa [20,62] (see Section 4 below for a more detailed presentation of the MID property). The first analytic proof of this property has been proposed for first-order DDEs in [22], and it relies on an integral representation of the corresponding characteristic function and a contradiction argument. In particular, it appears that a characteristic root of maximal multiplicity (i.e., equal to the degree of the corresponding quasipolynomial) necessarily defines the spectral abscissa of the system.

To the best of the authors' knowledge, such a systematic study of the links between roots of large multiplicity and the spectral abscissa was not sufficiently addressed in the literature until the early work [22], even though some hints in this direction are provided in [58] in the case of

[1] Also called the rightmost characteristic root

low-order systems. Since these works, the case of the assignment of a characteristic root with maximal multiplicity, called *generic MID property*, was recently addressed and completely characterized in [45] (retarded case) and in [10] (unifying retarded and neutral cases) for LTI DDEs including a *single delay* in their models. As discussed in [10,45], this property opens an interesting perspective in control through the so-called *partial pole placement* method, that is, imposing the multiplicity of a characteristic root of the closed-loop system by an appropriate choice of the controller gains guarantees the exponential stability of the closed-loop system with a prescribed decay rate.

The arguments used to prove the generic MID property in [10, 45] are based on some analytical properties of Kummer and Whittaker confluent hypergeometric functions, which cannot be extended straightforwardly to treat the case of *spectral values with intermediate multiplicity*, i.e., of multiplicity strictly smaller than the degree of the quasipolynomial; a fact that represents a drawback of the method. However, as shown in [3], by way of different arguments that exploit the structure of the system, the MID property still holds in some cases with lower multiplicity, but to the best of the authors' knowledge, there does not exist any systematic procedure to treat them.

The aim of this paper is to address these problems and to outline the ideas of a new method that could also encompass the MID with intermediate *over-order* multiplicities, i.e., multiplicities greater than the order of the DDE. More precisely, the contribution of the paper is threefold.

First, we provide conditions under which spectral values with the *lowest* over-order (algebraic) multiplicity are *dominant*, i.e., the have the largest real part among all spectral values. To prove the proposed results, we compute and exploit explicitly the properties of the so-called *elimination-produced polynomial.* To guarantee the dominancy of the multiple root, one makes use of the Green–Hille (integral) transformation introduced by Hille one century ago [33] for characterizing the location of the non-asymptotic zeros of Whittaker hypergeometric functions. It should be noted that these ideas complete the previous approaches based on the properties of Kummer hypergeometric functions to handle *generic MID* in the retarded and neutral cases (see, e.g., [10, 45]). To the best of the authors' knowledge, this a method represents a novelty in the open literature, which was only explored recently by some of the authors in [13] in the context of the MID property for the largest over-order multiplicity strictly smaller than the degree, in which computations turned out to be simpler than in the present setting.

Second, we show that the spectral abscissa function reaches a strict local minimum in the configuration corresponding to a root with the lowest over-order multiplicity. Finally, as a byproduct of the analysis, new insights on MID control of the dynamics of a chain of integrators and of a pendulum are proposed.

The remainder of the paper is organized as follows. Some elementary results on the distribution of the spectrum of DDEs are recalled in Section 2. In particular, a discussion on frequency bounds in the right-half plane for the spectra of dynamical systems represented by DDEs is proposed. Next, some basic properties of Kummer and Whittaker special functions are also presented. A motivating example (controlling the double integrator) is presented in Section 3. The formulation of the problem addressed in the paper is presented in Section 4. The main results are derived in Section 5 and Section 6. In Section 5, the concept of *Elimination-produced polynomial* is introduced and the systematic representation of quasipolynomials in terms of a linear combination of two Kummer functions is carried out.

Then, sufficient conditions for the MID property to hold for the considered intermediate multiplicity are established. In Section 6, a focus on the prescribed stabilization of a chain of integrators of arbitrary length is provided. Next, the link between the proposed partial pole placement and the problem of minimizing the spectral abscissa is investigated. An illustrative example is discussed in Section 7 and novelties and perspectives on the P3δ software are stated in Section 8. Some concluding remarks in Section 9 end the paper.

Notations. Throughout the paper, the following notations are used: $\mathbb{N}^*$, $\mathbb{R}$, $\mathbb{C}$ denote the sets of positive integers, real numbers, and complex numbers, respectively, and we set $\mathbb{N} = \mathbb{N}^* \cup \{0\}$. The set of all integers is denoted by $\mathbb{Z}$ and, for $a, b \in \mathbb{R}$, we denote $[\![a, b]\!] = [a, b] \cap \mathbb{Z}$, with the convention that $[a, b] = \emptyset$ if $a > b$. For a complex number λ, $\Re(\lambda)$ and $\Im(\lambda)$ denote its real and imaginary parts, respectively. The open left and right complex half-planes are the sets $\mathbb{C}_-$ and $\mathbb{C}_+$ defined respectively by $\mathbb{C}_- = \{\lambda \in \mathbb{C} \mid \Re(\lambda) < 0\}$ and $\mathbb{C}_+ = \{\lambda \in \mathbb{C} \mid \Re(\lambda) > 0\}$.

2 Prerequisites

2.1 Spectral Properties of DDEs

Consider the LTI dynamical system described by the DDE

$$y^{(n)}(t) + \sum_{k=0}^{n-1} a_k y^{(k)}(t) + \sum_{k=0}^{m} \alpha_k y^{(k)}(t-\tau) = 0, \tag{1}$$

under appropriate initial conditions, where $y(\cdot)$ is the real-valued unknown function, $\tau > 0$ is the delay, and $a_0, \dots, a_{n-1}, \alpha_0, \dots, \alpha_m$ are real coefficients. The DDE (1) is said to be of *retarded type* if[2] $m < n$, or of *neutral type* if $m = n$. The goal of this section is to provide elementary results on the spectral properties of (1) that will be useful in the sequel, and we refer to [30, 49] for a deeper discussions on DDEs and related results and properties.

Notice that (1) is a particular case of the time-delay system

$$\dot{\xi}(t) + B_\tau \dot{\xi}(t-\tau) = A_0 \xi(t) + A_\tau \xi(t-\tau), \tag{2}$$

where $\xi(t) = (y(t), y'(t), \dots, y^{(n-1)}(t))^T \in \mathbb{R}^n$ is the state vector and A_0, A_τ, $B_\tau \in \mathcal{M}_n(\mathbb{R})$ are real-valued matrices which can be easily constructed from (1).

2.1.1 Characteristic Function and its Properties

The characteristic function associated with (1) is the quasipolynomial $\Delta \colon \mathbb{C} \to \mathbb{C}$ defined by

$$\Delta(\lambda) = P_0(\lambda) + P_\tau(\lambda) \mathrm{e}^{-\lambda\tau}, \tag{3}$$

where P_0 and P_τ are the polynomials with real coefficients given by

$$P_0(\lambda) = \lambda^n + \sum_{k=0}^{n-1} a_k \lambda^k, \quad P_\tau(\lambda) = \sum_{k=0}^{m} \alpha_k \lambda^k. \tag{4}$$

Roots of Δ are usually called *characteristic roots* or *spectral values* of (1), and they are infinite in number, except in the trivial case where Δ reduces to a polynomial. The interest

[2] i.e., the highest order of derivation appears only in the non-delayed term $y^{(n)}(t)$.

in introducing the characteristic function Δ is that the exponential stability of the trivial solution of (1) can be described by the location of the roots of Δ: exponential stability holds true if, and only if, there exists $\gamma > 0$ such that $\Re(\lambda) \leq -\gamma$ for every root λ of Δ (see, e.g., [5, 49]).

The *degree* of the quasipolynomial Δ from (3) is the integer $\deg(\Delta) = n + m + 1$. As discussed in [17], this integer, which is larger than the degrees of the polynomials P_0 ($\deg(P_0) = n$) and P_τ ($\deg(P_\tau) = m$), is nothing but the integer appearing in the Pólya–Szegő bound from [60, Part Three, Problem 206.2], and also corresponds to the maximal multiplicity that a characteristic root of (3)–(4) may have. In addition, a characteristic root reaching this a bound is necessarily real.

Remark 1. *On the imaginary axis, the characteristic roots of the quasipolynomial Δ defined by* (3) *admit a bounded frequency, i.e., a bounded imaginary part. Indeed, any imaginary root $\lambda_0 = i\,\omega_0$ of Δ necessarily satisfies*

$$|P_0(i\,\omega_0)|^2 = |P_\tau(i\,\omega_0)|^2.$$

The function $\mathcal{F}$ defined by $\mathcal{F}(\omega) = |P_0(i\,\omega)|^2 - |P_\tau(i\,\omega)|^2$ is a polynomial on ω with real coefficients, and thus all its positive roots can be bounded in terms of its coefficients (see, for instance, [43]). However, this observation does not provide insights on frequency bounds for other roots, in particular roots in $\mathbb{C}_+$.

Despite the fact that the characteristic function of some DDE has an infinite number of characteristic roots, retarded systems, that is, (1) with $m < n$ or, equivalently, (2) with $B_\tau = 0$, admit finitely many roots on any vertical strip in the complex plane [30, Chapter 1, Lemma 4.1]. Several general results on the location of roots of (3) can be found in the literature, and we refer the interested reader to [9].

The next proposition collects two interesting properties, proofs of which can be found, respectively, in [49] and [57].

Proposition 2. *Consider the LTI system* (1)*, the corresponding system* (2)*, and their characteristic quasipolynomial Δ given by* (3)*–*(4)*.*

1. *If $m < n$ and λ is a characteristic root of system* (2) *with $B_\tau = 0$, then it satisfies*

$$|\lambda| \leq \|A_0 + A_\tau\,\mathrm{e}^{-\tau\lambda}\|, \tag{5}$$

where $\|\cdot\|$ is any induced matrix norm.

2. *If $m = n$ and $\lim\limits_{|\lambda|\to\infty} |P_\tau(\lambda)/P_0(\lambda)| < 1$, then the characteristic equation Δ defined by* (4) *has a finite number of roots in the right half-plane.*

Remark 3. *Inequality* (5)*, combined with the triangular inequality, provides a generic* envelope curve *around the characteristic roots corresponding to system* (2)*. In other words, the equality case in* (5) *defines a curve in the complex plane where all characteristic roots of Δ are located to the left of it. We refer to [52] for further insights on spectral envelopes for retarded time-delay systems with a single delay.*

2.1.2 DDEs Frequency Bound in the Right Half-Plane

In many situations, it is useful to obtain *a priori* information on the location of roots of a given quasipolynomial Δ of the form (3). In this section, we describe a procedure from [6], inferred from ideas used in the analysis of particular cases in [7, 44, 47], that, given a real number λ_0 and a positive real number ω_0, checks whether it is possible for Δ to admit a root λ of Δ with $\Re(\lambda) \geq \lambda_0$ and $|\Im(\lambda)| > \omega_0$. We are interested, more precisely, in the case in which this procedure provides a negative answer, as it implies that any root λ of Δ with $\Re(\lambda) \geq \lambda_0$ necessarily satisfies $|\Im(\lambda)| \leq \omega_0$, providing thus some *a priori* information on the location of such roots.

To do so, we first notice that one may assume, without loss of generality, that $\lambda_0 = 0$ and $\tau = 1$, since one can replace Δ by the normalized quasipolynomial $\tilde{\Delta}(z) = \tau^n \Delta(\lambda_0 + \frac{z}{\tau})$, which can be written as $\tilde{\Delta}(z) = \tilde{P}_0(z) + \mathrm{e}^{-z}\tilde{P}_\tau(z)$ for some suitable polynomials $\tilde{P}_0$ and $\tilde{P}_\tau$ of degrees n and m, respectively. Hence, verifying the presence of roots of Δ with real part larger than or equal to λ_0 reduces to verifying the presence of roots of $\tilde{\Delta}$ with nonnegative real part.

A possible strategy is to follow ideas similar to those proposed in Remark 1, i.e., to notice that any root z of $\tilde{\Delta}$ satisfies

$$|\tilde{P}_0(x + i\omega)|^2 \mathrm{e}^{2x} = |\tilde{P}_\tau(x + i\omega)|^2,$$

where $x = \Re(z)$ and $\omega = \Im(z)$. If $x \geq 0$, then $\mathrm{e}^{2x} \geq T_\ell(x)$, where, for $\ell \in \mathbb{N}$, the polynomial T_ℓ is the truncation of the Taylor expansion of e^{2x} at order ℓ, i.e., $T_\ell(x) = \sum_{k=0}^{\ell} \frac{(2x)^\ell}{\ell!}$. Hence, any root $z = x + i\omega$ of $\tilde{\Delta}$ with nonnegative real part satisfies

$$\mathcal{F}(x, \omega) \geq 0,$$

where $\mathcal{F}$ is the polynomial given by

$$\mathcal{F}(x, \omega) = |\tilde{P}_\tau(x + i\omega)|^2 - |\tilde{P}_0(x + i\omega)|^2 T_\ell(x).$$

In addition, $\mathcal{F}$ only depends on ω through ω^2 (which is a consequence of the fact that $\tilde{P}_0$ and $\tilde{P}_\tau$ are polynomials with real coefficients), and one may thus introduce the variable $\Omega = \omega^2$ and define the polynomial H by setting $H(x, \Omega) = F(x, \sqrt{\Omega})$ for $\Omega \geq 0$. Hence, any root $z = x + i\omega$ of $\tilde{\Delta}$ with nonnegative real part satisfies

$$H(x, \Omega) \geq 0, \tag{6}$$

where $\Omega = \omega^2$. One can thereby establish a bound on the imaginary parts of roots of $\tilde{\Delta}$ by exploiting the polynomial inequality (6). This has been done for some low-order cases in [6, 7, 44, 47], where this a procedure was applied in order to show that any possible root of the normalized quasipolynomial with nonnegative real part has an imaginary part bounded in absolute value by π. Such *a priori* information on the imaginary part was valuable in those references, since, coupled with further arguments, one is able to exclude the possibility of existence of such roots.

The procedure described in this subsection is synthesized in Algorithm 1, adapted from [6], which consists in increasing the order of the Taylor expansion of e^{2x} until a suitable bound is found or a maximal truncation order is reached. We refer the interested reader to [6, 7, 44, 47] for examples of applications of this procedure.

Algorithm 1: Estimation of a frequency bound for time-delay differential equations with a single delay

Input: $\tilde{\Delta}(z) = \tilde{P}_0(z) + \tilde{P}_\tau(z)\,\mathrm{e}^{-z}$; `// Normalized quasipolynomial`
Input: $\omega_0 > 0$; `// Desired frequency bound`
Input: `maxOrd`; `// Maximal truncation order`
`// Initialization`
`ord` $= 0$; `// ord: order of truncation of the Taylor expansion of` e^{2x};
`Bound` $=$ `false`;
while *(not* `Bound`*) and (*`ord` $\leq$ `maxOrd`*)* **do**
  Set $\mathcal{F}(x, \omega) = |\tilde{P}_\tau(x + i\omega)|^2 - |\tilde{P}_0(x + i\omega)|^2 T_{\mathtt{ord}}(x)$;
  `//` $T_{\mathrm{ord}}(x)$: `Taylor expansion of` e^{2x} `of order = ord`
  Set $H(x, \Omega) = \mathcal{F}(x, \sqrt{\Omega})$; `//` H `is a polynomial`
  Set $\Omega_k(x)$ as the k-th real root of $H(x, \cdot)$;
  if $\sup_{x \geq 0} \max_k \Omega_k(x) \leq \omega_0^2$ **then**
    `Bound` $=$ `true`;
  `ord` $=$ `ord` $+ 1$;
Output: Frequency bound: If `Bound` is `true`, then $|\omega| \leq \omega_0$ for every root $z = x + i\omega$ of $\tilde{\Delta}$ with $x \geq 0$;

2.2 Kummer and Whittaker Functions and the Hille Oscillation Theorem

The main ingredient of the partial pole placement method proposed is a particular class of hypergeometric functions, namely, *Kummer confluent hypergeometric functions*, which, for $a, b \in \mathbb{C}$ such that $-b \notin \mathbb{N}$, is the entire function $\Phi(a, b, \cdot) : \mathbb{C} \to \mathbb{C}$ defined by the series

$$\Phi(a, b, z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}, \tag{7}$$

where, for $\alpha \in \mathbb{C}$ and $k \in \mathbb{N}$, $(\alpha)_k$ is the *Pochhammer symbol* for the *ascending factorial*, defined inductively as $(\alpha)_0 = 1$ and $(\alpha)_{k+1} = (\alpha + k)(\alpha)_k$.

The series in (7) converges for every $z \in \mathbb{C}$ and, as presented in [25, 28, 56], the Kummer function satisfies the *Kummer differential equation*, that is,

$$z \frac{\partial^2 \Phi}{\partial z^2}(a, b, z) + (b - z) \frac{\partial \Phi}{\partial z}(a, b, z) - a\Phi(a, b, z) = 0. \tag{8}$$

As emphasized in [25, 28, 56], for every $a, b, z \in \mathbb{C}$ such that $\Re(b) > \Re(a) > 0$, Kummer functions also admit the integral representation

$$\Phi(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b - a)} \int_0^1 \mathrm{e}^{zt} t^{a-1} (1 - t)^{b-a-1} \, \mathrm{d}t, \tag{9}$$

where Γ denotes the Gamma function. This integral representation has been used in [10,12,45] to characterize the spectrum of some DDEs.

Kummer functions exhibit a range of remarkable properties. In particular, they satisfy some recurrence relations, often called *contiguous relations*, which will be exploited in the sequel, see for instance [56].

Lemma 4 ([56, p. 325])**.** *Let $a, b, z \in \mathbb{C}$ with $a \neq b$, $z \neq 0$, and $-b \notin \mathbb{N}$. The following relations hold:*

$$\begin{aligned} \Phi(a, b+1, z) &= \frac{-b\,(a+z)\;\Phi(a,b,z) + ab\,\Phi(a+1,b,z)}{z\,(a-b)}, \\ \Phi(a+1, b+1, z) &= -\frac{-b\,\Phi(a+1,b,z) + b\,\Phi(a,b,z)}{z}. \end{aligned} \tag{10}$$

Kummer functions are strongly related to another interesting class of hypergeometric functions called *Whittaker functions*. In fact, for $k, l \in \mathbb{C}$ with $-2l \notin \mathbb{N}^*$, the *Whittaker function* $\mathcal{M}_{k,l}$ is defined for $z \in \mathbb{C}$ by

$$\mathcal{M}_{k,l}(z) = \mathrm{e}^{-\frac{z}{2}} z^{\frac{1}{2}+l} \Phi(\tfrac{1}{2} + l - k, 1 + 2l, z), \tag{11}$$

(see, e.g., [56]). Note that, if $\frac{1}{2} + l$ is not an integer, then the function $\mathcal{M}_{k,l}$ is a multi-valued complex function with branch point at $z = 0$. In addition, the nontrivial roots of $\mathcal{M}_{k,l}$ coincide with those of $\Phi(\frac{1}{2} + l - k, 1 + 2l, \cdot)$ and $\mathcal{M}_{k,l}$ satisfies the *Whittaker differential equation*

$$\varphi''(z) = \left(\frac{1}{4} - \frac{k}{z} + \frac{l^2 - \frac{1}{4}}{z^2}\right) \varphi(z). \tag{12}$$

Taking into account that $\mathcal{M}_{k,l}$ is a nontrivial solution of the second-order linear differential equation (12), then any nontrivial root of $\mathcal{M}_{k,l}$ is simple.

In the pioneering work by E. Hille [33], some oscillation theorems in the complex domain have been proposed. Among others, Hille studied the distribution of zeros of functions of a complex variable satisfying linear second-order homogeneous differential equations with variable coefficients, as is the case for the Whittaker function $\mathcal{M}_{k,l}$, which satisfies (12). In particular, Hille introduced an integral transformation called *Green–Hille transformation* ensuing from the differential equation and allowing the removal of regions in the complex plane that do not contain complex roots. To illustrate Hille's idea, consider the general homogeneous second-order differential equation

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[K(z)\,\frac{\mathrm{d}\,\varphi}{\mathrm{d}z}(z)\right] + G(z)\varphi(z) = 0, \tag{13}$$

where z is the complex independent variable, and the functions G and K are assumed to be analytic in some region Θ such that K does not vanish in that region.

Equation (13) can be written in Θ as a planar system by introducing the dependent variables $\varphi_1(z) = \varphi(z)$ and $\varphi_2(z) = K(z)\,\frac{d\varphi}{dz}(z)$, and the Green–Hille transformation consists in multiplying the equation for φ_1 by $\overline{\varphi_2(z)}$, the one for φ_2 by $\overline{\varphi_1(z)}$, and integrating z along a path in Θ, which yields

$$\left[\overline{\varphi_1(z)}\,\varphi_2(z)\right]_{z_1}^{z_2} - \int_{z_1}^{z_2} |\varphi_2(z)|^2 \frac{\overline{\mathrm{d}z}}{\overline{K(z)}} + \int_{z_1}^{z_2} |\varphi_1(z)|^2 G(z)\,\mathrm{d}z = 0, \tag{14}$$

where $z_1, z_2 \in \Theta$ and both integrals are taken along the same arbitrary smooth path in Θ, connecting z_1 to z_2.

The following result, which is proved in [12] using the Green–Hille transformation from [33], gives insights on the distribution of the nonasymptotic zeros of Kummer hypergeometric functions with real arguments a and b.

Proposition 5 ([12]). *Let $a, b \in \mathbb{R}$ be such that $b \geq 2$.*

1. *If $b = 2a$, then all nontrivial roots z of $\Phi(a, b, \cdot)$ are purely imaginary;*
2. *If $b > 2a$ (resp., $b < 2a$), then all nontrivial roots z of $\Phi(a, b, \cdot)$ satisfy $\Re(z) > 0$ (resp., $\Re(z) < 0$);*
3. *If $b \neq 2a$, then all nontrivial roots z of $\Phi(a, b, \cdot)$ satisfy*

$$(b - 2a)^2 \Im(z)^2 - (4a(b - a) - 2b)\, \Re(z)^2 > 0.$$

3 Motivating Example: Controlling the Double Integrator

The problem of stabilization of a chain of integrators is considered in [54], where it is shown that a single integrator can be stabilized by a single delay state-feedback. Indeed, a positive gain guarantees the closed-loop stability of the system free of delay, and, by continuity, there exists a (sufficiently small) delay in the output preserving the stability of the closed-loop system. However, the situation is completely different for a chain of integrators of order n when $n > 1$. For instance, consider the time-delay system $y''(t) + \alpha y(t - \tau) = 0$, the characteristic quasipolynomial of which is

$$\Delta(\lambda) = \lambda^2 + \alpha\, \mathrm{e}^{-\tau \lambda}. \tag{15}$$

Note that the degree of Δ is 3. It can be checked that the maximal multiplicity that a root λ_0 of (15) can have is 2, and it is attained if, and only if,

$$\alpha = -4\, \frac{\mathrm{e}^{-2}}{\tau^2},\ \lambda_0 = -\frac{2}{\tau}. \tag{16}$$

The main result from [54] asserts that either n distinct delays or a proportional+delay compensator with $n-1$ distinct delays are sufficient to stabilize a chain including n integrators. Later, in [38], it is shown that this number of terms is also necessary to stabilize the chain of n integrators. Therefore, in our case, either 2 distinct delays or a proportional+delay are necessary and sufficient to stabilize the double integrator. Since (15) contains only a single delay, the result of [38] implies that there exists at least one spectral value for (15) with a positive real part. Consequently, $\lambda_0 = -\frac{2}{\tau}$, while being a multiple root, is not dominant.

Indeed, consider (15)–(16) with $\tau = 1$, that is,

$$\Delta(\lambda) = \lambda^2 - 4\mathrm{e}^{-(\lambda+2)}. \tag{17}$$

As illustrated in Figure 1, the dominancy property is lost since $\lambda_1 \approx 0.557$ is a root of the function (17). This is justified by the sparsity of (17), i.e., the corresponding polynomial P_0, when (15) is written under the form (3), has some null coefficients.

In accordance with the previous observation, let us now consider the problem of stabilization of the double integrator using a delayed PD controller, which, in the frequency domain, gives the closed-loop characteristic function:

$$\Delta(\lambda) = \lambda^2 + (\alpha_1 \lambda + \alpha_0)\, \mathrm{e}^{-\lambda \tau}, \tag{18}$$

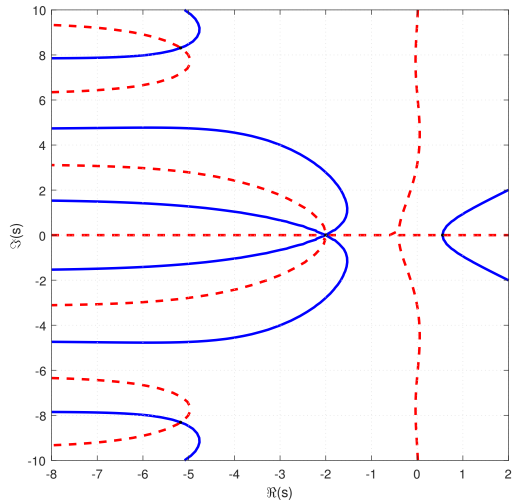

Figure 1: Illustration of the roots of (17): solid blue curves represent the region $\Re(\Delta(\lambda)) = 0$, dashed red curves represent the region $\Im(\Delta(\lambda)) = 0$, so that the roots of Δ correspond to the intersections between solid blue curves and dashed red curves. This quasipolynomial Δ admits a double root at $\lambda_0 = -2$, which is not its rightmost root, since Δ also admits a real root $\lambda_1 \approx 0.557$. The figure was produced using the QPmR toolbox from [68].

which is a quasipolynomial of degree 4. Following [20, Theorem 4.2] it has been shown that for an arbitrary positive delay τ, the quasipolynomial (18) admits a real spectral value at $\lambda = \lambda_\pm$ with algebraic multiplicity 3 if, and only if,

$$\lambda_\pm = \frac{-2 \pm \sqrt{2}}{\tau}, \tag{19}$$

and the system parameters satisfy:

$$\begin{cases} \alpha_0 = \dfrac{6 + 10\,\lambda_\pm\,\tau}{\tau^2}\,\mathrm{e}^{\lambda_\pm \tau}, \\ \alpha_1 = \dfrac{2 + 2\,\lambda_\pm\,\tau}{\tau}\,\mathrm{e}^{\lambda_\pm \tau}. \end{cases} \tag{$\star_\pm$}$$

Furthermore, it has been shown in [20, Theorem 4.2] that the MID property is valid for λ_+, that is, the triple spectral value corresponds to the spectral abscissa only if $(\star_+)$ is satisfied. It has also been emphasized in [20] that the multiple spectral value at λ_- is always dominated by a single real root. Stability wise, it is clear that, independently from the chosen delay $\tau > 0$, the closed-loop solution is always exponentially stable with a decay rate corresponding to $\lambda_+ < 0$.

In what follows, we consider the effect of spectral values of intermediate multiplicities, i.e., multiplicities which are less than the degree of the quasipolynomial. As emphasized in the above examples, in such cases, multiple spectral values are not necessarily dominant and an answer to this question needs a deeper investigation, which is the aim of this paper.

4 Problem Formulation

Consider the DDE (1) and its characteristic function Δ given by (3)–(4). As recalled in Section 2.1, the degree of the characteristic function Δ is $\deg(\Delta) = n + m + 1$.

We say that a characteristic root λ_0 of Δ satisfies the *MID property* if (i) its *algebraic multiplicity* (denoted by $M(\lambda_0)$) is *larger than one*, and (ii) it is *dominant*, meaning that all the characteristic roots λ_σ of Δ satisfy $\Re(\lambda_\sigma) \leq \Re(\lambda_0)$. In other words, λ_0 corresponds to the rightmost root of the spectrum and defines the *spectral abscissa* of the quasipolynomial Δ. In addition, we say that the root λ_0 is *over-order* if its algebraic multiplicity is strictly greater than the order n, which corresponds to the degree of the characteristic function in the delay-free case. In this situation, we refer to the MID property as *over-order MID*. In the case $M(\lambda_0) = \deg(\Delta)$, it was shown in [45] (case $m = n - 1$) and [10] (general case $m \leq n$) that λ_0 satisfies the MID property. This "limit" case corresponding to the maximal possible multiplicity is also called *generic MID* (or GMID for short). Finally, with the notions and notations above, the *lowest over-order (algebraic) multiplicity* corresponds to the case when the intermediate multiplicity of λ_0 is $M(\lambda_0) = n + 1$.

The problem addressed in this paper can be formulated as follows: *finding, on the one hand, conditions on the parameters of the dynamical system* (1) *ensuring that a characteristic root λ_0 has over-order (intermediate) algebraic multiplicity*[3]*, and determining, on the other hand, if such a root satisfies the MID property.* More precisely, and for the sake of brevity, our focus will be to infer *appropriate conditions guaranteeing that λ_0 has multiplicity $M(\lambda_0) =$* $n + 1$, which corresponds to the smallest possible over-order multiplicity, and that it satisfies the *MID property*, in the particular case where $m = n + 1$. It should be mentioned that this configuration has been investigated in [3], in the particular case where the delay-free polynomial P_0 of Δ is real-rooted. The approach proposed hereafter relaxes the the former assumption.

Note that the PD control of the double integrator in the case of intentional delay in the input/output channel, considered as a motivating example in Section 3, corresponds to such a situation. Namely, for the quasipolynomial Δ from (18), we have $\deg(\Delta) = 4$, $n = 2$, and so the only possible intermediate multiplicity is $M(\lambda_0) = 3$, which, in our terminology, coincides with the lowest over-order (algebraic) multiplicity.

5 Main Results

Our first set of main results, presented in Section 5.1, provides a necessary condition for the existence of a root of Δ with the lowest over-order multiplicity, in terms of the so-called *elimination-produced polynomial*. Exploiting this result, necessary and sufficient conditions for the existence of a root with the lowest over-order multiplicity are derived in Section 5.2, in terms of the elimination-produced polynomial and suitable factorizations of Δ. We then exploit the links between quasipolynomials Δ with a root of over-order multiplicity and Kummer confluent hypergeometric functions in Section 5.3, and sufficient conditions under which the MID property for a root of over-order multiplicity are finally provided in Section 5.4.

[3] That is, the multiplicity $M(\lambda_0)$ of λ_0 verifies $n + 1 \leq M(\lambda_0) \leq n + m$.

5.1 Elimination-Produced Polynomial

The aim of this section is to provide an appropriate *necessary condition* for a given real number λ_0 to be a root of multiplicity at least $n+1$ of a given quasipolynomial Δ under the form (3)–(4) with $m = n - 1$. The said necessary condition consists in stating that λ_0 must be a root of a polynomial, known as the *elimination-produced polynomial*.

Let us briefly describe the main ideas underlying the construction of the elimination-produced polynomial. Imposing that a real number λ_0 is a root of Δ of multiplicity at least $n+1$ amounts to imposing that the following conditions hold simultaneously:

$$\Delta(\lambda_0) = \Delta'(\lambda_0) = \cdots = \Delta^{(n)}(\lambda_0) = 0. \tag{20}$$

On the other hand, if we consider, in (3)–(4), that the coefficients $a_0, \ldots, a_{n-1}$ of P_0 are "fixed" and known and the coefficients $\alpha_0, \ldots, \alpha_{n-1}$ of P_τ are "free" and available for choice, then (20) imposes $n+1$ linear equality constraints on the n free parameters $\alpha_0, \ldots, \alpha_{n-1}$. While n of those constraints should be sufficient to determine the values of the n free parameters $\alpha_0, \ldots, \alpha_{n-1}$ in terms of the fixed parameters $a_0, \ldots, a_{n-1}$, the delay τ, and the root λ_0, the additional constraint will express a relation that must be satisfied between $a_0, \ldots, a_{n-1}$, the delay τ, and the root λ_0 in order for the multiplicity $n+1$ to be attained. This relation is precisely the elimination-produced polynomial, as described in our next result.

Proposition 6. *Consider the quasipolynomial Δ from (3)–(4) with $m = n - 1$. If the real number λ_0 is a root of multiplicity at least $n+1$ of Δ, then*

$$\mathbb{P}(\lambda_0, \tau) = 0,$$

where $\mathbb{P}$ is the elimination-produced polynomial*, defined by*

$$\mathbb{P}(\lambda, \tau) = \sum_{k=0}^{n} \binom{n}{k} P_0^{(k)}(\lambda) \tau^{n-k}. \tag{21}$$

Proof. Note that λ_0 is a root of $\Delta(\cdot)$ with multiplicity at least $n+1$ if, and only if, it is a root of $Q\colon \lambda \mapsto \mathrm{e}^{\lambda\tau}\Delta(\lambda)$ with the same multiplicity. In particular, we have $Q^{(n)}(\lambda_0) = 0$. Since $Q(\lambda) = \mathrm{e}^{\lambda\tau} P_0(\lambda) + P_1(\lambda)$ and P_1 is a polynomial of degree $n-1$, we deduce that

$$Q^{(n)}(\lambda) = \mathrm{e}^{\lambda\tau} \sum_{k=0}^{n} \binom{n}{k} P_0^{(k)}(\lambda) \tau^{n-k},$$

yielding the conclusion since $\mathrm{e}^{\lambda_0\tau} \neq 0$. □

Before turning to our next result, we recall the statement of the Hermite–Poulain Theorem on roots of polynomials. Its proof can be found, for instance, in [35, Theorem 7.3.3].

Theorem 7 (Hermite–Poulain)**.** *Let $h(x) = c_0 + c_1 x + \ldots + c_n x^n$ be a real-rooted polynomial. If $f(x)$ is a polynomial with real coefficients, then the polynomial*

$$F(x) = c_0 f(x) + c_1 f'(x) + \ldots + c_n f^{(n)}(x)$$

has at least as many real roots as $f(x)$ has.

We now exploit the Hermite–Poulain Theorem to deduce a link between the number of real roots of $\mathbb{P}(\cdot, \tau)$ and that of $P_0(\cdot)$.

Proposition 8. *Let P_0 be a polynomial of degree n with real coefficients and $\mathbb{P}$ be defined from P_0 as in (21). Then, for every $\tau \in \mathbb{R}$, the polynomial $\lambda \mapsto \mathbb{P}(\lambda, \tau)$ has at least as many real roots as P_0 (counted with their multiplicities).*

Proof. This is an immediate consequence of (21) and the Hermite–Poulain Theorem applied to the real-rooted polynomial h given by $h(x) = (x + \tau)^n = \sum_{k=0}^{n} \binom{n}{k} \tau^{n-k} x^k$ and to $f = P_0$. $\square$

5.2 Necessary and Sufficient Conditions for the Lowest Over-Order Multiplicity

We now provide a characterization of the situations under which a quasipolynomial Δ under the form (3)–(4) with $m = n-1$ admits a root with an over-order multiplicity. Our first result is the following, which provides a factorization of Δ in terms of such a root and an integral expression.

Proposition 9. *Consider the quasipolynomial Δ from (3)–(4) with $m = n - 1$. The real number λ_0 is a root of multiplicity at least $n+1$ of Δ if, and only if, there exists a polynomial p of degree $n - 1$ with $p(0) = 1$ such that*

$$\Delta(\lambda) = \tau(\lambda - \lambda_0)^{n+1} \int_0^1 (1-t)p(t)\mathrm{e}^{-t(\lambda-\lambda_0)\tau}\,\mathrm{d}t. \tag{22}$$

Proof. Fix the delay τ and let $\mathcal{V}$ be the set of all functions Δ of the form $\Delta(\lambda) = P_0(\lambda) + \mathrm{e}^{-\lambda\tau} P_\tau(\lambda)$ with P_0 and P_τ given by (4) and $m = n - 1$, i.e.,

$$\mathcal{V} = \left\{ \Delta\colon \mathbb{C} \to \mathbb{C} \;\middle|\; \begin{array}{l} \exists\, a = (a_0, \ldots, a_{n-1}) \in \mathbb{R}^n,\ \exists\, \alpha = (\alpha_0, \ldots, \alpha_{n-1}) \in \mathbb{R}^n \text{ such that} \\ \qquad\qquad \Delta(\lambda) = \lambda^n + \displaystyle\sum_{k=0}^{n-1} a_n \lambda^k + \mathrm{e}^{-\lambda\tau} \sum_{k=0}^{n-1} \alpha_k \lambda^k \end{array} \right\}.$$

Note that $\mathcal{V}$ is a real vector space with $\dim \mathcal{V} = 2n$, which is a subspace of the space of all entire complex functions, seen as a real vector space. In addition, $\mathcal{V}$ can be canonically identified with $\mathbb{R}^{2n}$ by identifying a quasipolynomial Δ with its coefficients $a_0, \ldots, a_{n-1}, \alpha_0, \ldots, \alpha_{n-1}$.

Let us denote by $\mathcal{V}_{\lambda_0}$ the subset of $\mathcal{V}$ of those functions Δ admitting λ_0 as a root of multiplicity at least $n + 1$, i.e.,

$$\mathcal{V}_{\lambda_0} = \left\{ \Delta \in \mathcal{V} \;\middle|\; \Delta^{(k)}(\lambda_0) = 0 \text{ for all } k \in \{0, \ldots, n\} \right\}.$$

Each equation $\Delta^{(k)}(\lambda_0) = 0$, $k \in \{0, \ldots, n\}$, defines a hyperplane in $\mathcal{V}$, and, when identifying $\mathcal{V}$ with the Euclidean space $\mathbb{R}^{2n}$, the normal vectors to all such hyperplanes are linearly independent. Hence $\mathcal{V}_{\lambda_0}$ is a subspace of $\mathcal{V}$ of codimension $n + 1$, i.e., $\dim \mathcal{V}_{\lambda_0} = n - 1$.

Now, we introduce $\mathcal{W}_{\lambda_0}$ as the space of all functions Δ of the form (22) for some polynomial p of degree $n - 1$ with $p(0) = 1$. The set $\mathcal{W}_{\lambda_0}$ is an affine subspace of the space of all entire complex functions, with $\dim \mathcal{W}_{\lambda_0} = n - 1$.

As a first step, we will prove that $\mathcal{W}_{\lambda_0} \subseteq \mathcal{V}$, i.e., that every function Δ of the form (22) is indeed a quasipolynomial of the form (3)–(4). To do so, we first observe that, by an immediate inductive integration by parts, we have (see also [45, Proposition 2.1])

$$\int_0^1 q(t) e^{-zt} \, dt = \sum_{k=0}^{d} \frac{q^{(k)}(0) - q^{(k)}(1) e^{-z}}{z^{k+1}} \tag{23}$$

for every $z \in \mathbb{C} \setminus \{0\}$, $d \in \mathbb{N}$, and q a polynomial of degree d. Next, let $\Delta \in \mathcal{W}_{\lambda_0}$ and p be a polynomial of degree $n-1$ with $p(0) = 1$ be such that Δ is given by (22). Define $q(t) = (1-t)p(t)$ and notice that $q(1) = 0$. By using (23), we deduce that

$$\begin{aligned} \Delta(\lambda) &= \tau (\lambda - \lambda_0)^{n+1} \sum_{k=0}^{n} \frac{q^{(k)}(0) - q^{(k)}(1) e^{-\tau(\lambda - \lambda_0)}}{\tau^{k+1} (\lambda - \lambda_0)^{k+1}} \\ &= (\lambda - \lambda_0)^n + \sum_{k=0}^{n-1} \frac{q^{(n-k)}(0)}{\tau^{n-k}} (\lambda - \lambda_0)^k - e^{-\tau(\lambda - \lambda_0)} \sum_{k=0}^{n-1} \frac{q^{(n-k)}(1)}{\tau^{n-k}} (\lambda - \lambda_0)^k, \end{aligned} \tag{24}$$

so that $\Delta \in \mathcal{V}$, as required.

We now notice that $\mathcal{W}_{\lambda_0} \subseteq \mathcal{V}_{\lambda_0}$, since, for any Δ given by (22), λ_0 is clearly a root of multiplicity at least $n+1$ of Δ. Moreover, $\mathcal{W}_{\lambda_0}$ and $\mathcal{V}_{\lambda_0}$ are both affine spaces with the same dimension, so that $\mathcal{W}_{\lambda_0} = \mathcal{V}_{\lambda_0}$, yielding the conclusion. □

Remark 10. *Note that* (24) *provides explicit expressions for the polynomials* P_0 *and* P_τ *from* (4) *in terms of the polynomial* q *introduced in the above proof. More precisely, we have*

$$\begin{aligned} P_0(\lambda) &= (\lambda - \lambda_0)^n + \sum_{k=0}^{n-1} \frac{q^{(n-k)}(0)}{\tau^{n-k}} (\lambda - \lambda_0)^k, \\ P_\tau(\lambda) &= -e^{\tau \lambda_0} \sum_{k=0}^{n-1} \frac{q^{(n-k)}(1)}{\tau^{n-k}} (\lambda - \lambda_0)^k. \end{aligned}$$

Since $q(t) = (1-t)p(t)$, *one may also provide similar expressions of* P_0 *and* P_τ *in terms of* p. *Indeed, we have*

$$P_0(\lambda) = (\lambda - \lambda_0)^n + \sum_{k=0}^{n-1} \frac{p^{(n-k)}(0) - (n-k) p^{(n-k-1)}(0)}{\tau^{n-k}} (\lambda - \lambda_0)^k, \tag{25a}$$

$$P_\tau(\lambda) = e^{\tau \lambda_0} \sum_{k=0}^{n-1} \frac{(n-k) p^{(n-k-1)}(1)}{\tau^{n-k}} (\lambda - \lambda_0)^k. \tag{25b}$$

Let us now identify the link between the polynomial p from (22) and the elimination-produced polynomial $\mathbb{P}$ from (21).

Proposition 11. *Consider the quasipolynomial* Δ *from* (3)–(4) *with* $m = n-1$. *The real number* λ_0 *is a root of multiplicity at least* $n+1$ *of* Δ *if, and only if,* $\mathbb{P}(\lambda_0, \tau) = 0$ *and*

$$\Delta(\lambda) = \frac{\tau}{n!} (\lambda - \lambda_0)^{n+1} \int_0^1 \mathbb{P}(\lambda_0, \tau t) e^{-t(\lambda - \lambda_0)\tau} \, dt, \tag{26}$$

where $\mathbb{P}$ *is the elimination-produced polynomial defined in* (21).

Proof. First, assume that $\mathbb{P}(\lambda_0, \tau) = 0$ and that Δ is given by (26). In step with (21), the function $t \mapsto \frac{1}{n!}\mathbb{P}(\lambda_0, \tau t)$ is a polynomial in t of degree n with $\frac{1}{n!}\mathbb{P}(\lambda_0, 0) = \frac{P_0^{(n)}(\lambda_0)}{n!} = 1$ and $\frac{1}{n!}\mathbb{P}(\lambda_0, \tau) = 0$, so that $\frac{1}{n!}\mathbb{P}(\lambda_0, \tau t) = (1-t)p(t)$ for some polynomial p of degree $n-1$ with $p(0) = 1$. Hence, by Proposition 9, λ_0 is a root of multiplicity at least $n+1$ of Δ.

Conversely, assume that λ_0 is a root of multiplicity at least $n+1$ of Δ. Proposition 6 states that $\mathbb{P}(\lambda_0, \tau) = 0$ and, To proceed Proposition 9, there exists a polynomial p of degree $n-1$ with $p(0) = 1$ such that Δ is given by (22). In addition, due to Remark 10, we have

$$P_0(\lambda) = \sum_{k=0}^{n} \frac{q^{(n-k)}(0)}{\tau^{n-k}} (\lambda - \lambda_0)^k, \tag{27}$$

where q is the polynomial defined by $q(t) = (1-t)p(t)$. On the other hand, since P_0 is a polynomial of degree n, it coincides with its Taylor expansion of order n at λ_0, i.e.,

$$P_0(\lambda) = \sum_{k=0}^{n} \frac{P_0^{(k)}(\lambda_0)}{k!} (\lambda - \lambda_0)^k. \tag{28}$$

Using the uniqueness of the Taylor expansion at a given point and combining (27) and (28), we deduce that $P_0^{(k)}(\lambda_0) = \frac{k!}{\tau^{n-k}} q^{(n-k)}(0)$. Hence, by (21), we have

$$\begin{aligned} \mathbb{P}(\lambda_0, \tau t) &= \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{\tau^{n-k}} q^{(n-k)}(0) (\tau t)^{n-k} \\ &= n! \sum_{k=0}^{n} q^{(n-k)}(0) \frac{t^{n-k}}{(n-k)!} = n! q(t), \end{aligned}$$

since the last sum is the Taylor expansion of q at 0 of order n and q is a polynomial of degree n. Consequently, $q(t) = \frac{1}{n!}\mathbb{P}(\lambda_0, \tau t)$ and Δ is given by (26). □

5.3 Some Insights on Linear Combinations of Kummer Functions

Here, our goal is to establish links between a quasipolynomial Δ of the form (3)–(4) with $m = n-1$ admitting a root of multiplicity at least $n+1$ and Kummer confluent hypergeometric functions. A first connection is stated in the next result.

Proposition 12. *Consider the quasipolynomial Δ from (3)–(4) with $m = n-1$. The real number λ_0 is a root of multiplicity at least $n+1$ of Δ if, and only if, $\mathbb{P}(\lambda_0, \tau) = 0$ and*

$$\Delta(\lambda) = (\lambda - \lambda_0)^{n+1} \sum_{k=0}^{n-1} \sigma_k \, \Phi(k+1, \, k+3, \, -\tau\,(\lambda - \lambda_0)) \tag{29}$$

where, for $k \in [\![0, n-1]\!]$, we have

$$\sigma_k = \frac{-\tau}{(k+1)\,(k+2)\,n!} \sum_{j=0}^{n-k-1} \binom{n}{j} P_0^{(j)}(\lambda_0)\, \tau^{n-j}. \tag{30}$$

Proof. Note that, owing to Proposition 11, it suffices to show that, if $\mathbb{P}(\lambda_0, \tau) = 0$, then (26) can be rewritten as (29).

Let $q(t) = \frac{\tau}{n!}\mathbb{P}(\lambda_0, \tau t)$, then $q(1) = 0$ since $\mathbb{P}(\lambda_0, \tau) = 0$. As a result, polynomial q can be factorized as $q(t) = (1-t)p(t)$, where p is a polynomial of degree $n-1$. We write $p(t) = \sum_{k=0}^{n-1} \widetilde{\sigma}_k t^k$ for some real coefficients $\widetilde{\sigma}_0, \dots, \widetilde{\sigma}_{n-1}$, and thus (26) can be rewritten as

$$\Delta(\lambda) = (\lambda - \lambda_0)^{n+1} \sum_{k=0}^{n-1} \widetilde{\sigma}_k \int_0^1 t^k (1-t) \mathrm{e}^{-t(\lambda-\lambda_0)\tau} \, \mathrm{d}t.$$

Thanks to (9), the above equation takes the form (29) after setting $\sigma_k = \frac{\widetilde{\sigma}_k}{(k+1)(k+2)}$ for $k \in [\![0, n-1]\!]$.

In order to conclude, it suffices to compute $\widetilde{\sigma}_k$ for $k \in [\![0, n-1]\!]$. To do that, from (21), we infer that

$$q(t) = \frac{\tau}{n!} \sum_{k=0}^{n} \binom{n}{k} P_0^{(k)}(\lambda_0) \tau^{n-k} t^{n-k},$$

and, since $q(t) = (1-t)p(t)$, we also have

$$q(t) = \widetilde{\sigma}_0 + \sum_{k=1}^{n-1} (\widetilde{\sigma}_k - \widetilde{\sigma}_{k-1}) t^k - \widetilde{\sigma}_{n-1} t^n.$$

Hence, equating the coefficients of monomials of the same degree in the two above expressions of q, we deduce, for every $k \in [\![0, n-1]\!]$, that

$$\widetilde{\sigma}_k = -\frac{\tau}{n!} \sum_{j=0}^{n-1-k} \binom{n}{j} P_0^{(j)}(\lambda_0) \tau^{n-j},$$

which concludes the proof. □

Note that (29) factorizes Δ in terms of a linear combination of n Kummer functions with real coefficients. One can also express Δ as a combination of two Kummer functions if one allows for rational functions as coefficients.

Proposition 13. *Consider the quasipolynomial Δ from (3)–(4) with $m = n-1$. The real number λ_0 is a root of multiplicity at least $n+1$ of Δ if, and only if, $\mathbb{P}(\lambda_0, \tau) = 0$ and*

$$\Delta(\lambda) = \beta(\lambda) \, \Phi(0, \, 1, \, -\tau\,(\lambda - \lambda_0)) + \gamma(\lambda) \, \Phi(1, \, 1, \, -\tau\,(\lambda - \lambda_0)), \tag{31}$$

with

$$\beta(\lambda) = -(\lambda - \lambda_0)^{n+1} \sum_{k=0}^{n-1} \frac{\sigma_k (k+2)!(k+1-\tau(\lambda-\lambda_0))}{(\tau(\lambda-\lambda_0))^{k+2}} \tag{32}$$

and

$$\gamma(\lambda) = (\lambda - \lambda_0)^{n+1} \sum_{k=0}^{n-1} \sigma_k \left(\frac{(k+1)^2(k+2)}{(\tau(\lambda-\lambda_0))^2} + \sum_{r=0}^{k-1} \frac{(k+2)!(k+1-\tau(\lambda-\lambda_0))}{r!(\tau(\lambda-\lambda_0))^{k+2-r}} \right), \tag{33}$$

where $\sigma_0, \dots, \sigma_{n-1}$ are defined as in (30).

Proof. According to Proposition 12, it suffices to show that (29) is equivalent to (31). To do so, thanx to (10), we obtain, for every $k \in [\![0, n-1]\!]$ and $z \in \mathbb{C} \setminus \{0\}$, that

$$\begin{aligned}
\Phi(k+1, k+3, z) &= -\frac{k+2}{z}\Phi(k, k+2, z) + \frac{k+2}{z}\Phi(k+1, k+2, z), \\
\Phi(k, k+2, z) &= \frac{(k+1)(k+z)}{z}\Phi(k, k+1, z) - \frac{k(k+1)}{z}\Phi(k+1, k+1, z), \\
\Phi(k+1, k+2, z) &= -\frac{k+1}{z}\Phi(k, k+1, z) + \frac{k+1}{z}\Phi(k+1, k+1, z).
\end{aligned}$$

Next, we remark that, by (7), for every $a \in \mathbb{C}$ with $-a \notin \mathbb{N}$, we have $\Phi(a, a, z) = \Phi(1, 1, z) = \mathrm{e}^z$. As a result,

$$\Phi(k+1, k+3, z) = -\frac{(k+1)(k+2)(z+k+1)}{z^2}\Phi(k, k+1, z) + \frac{(k+1)^2(k+2)}{z^2}\Phi(1, 1, z). \tag{34}$$

Again, (10) entails, for every $j \in \mathbb{N}^*$,

$$\Phi(j, j+1, z) = -\frac{j}{z}\Phi(j-1, j, z) + \frac{j}{z}\Phi(1, 1, z),$$

wherefrom an immediate inductive argument shows that

$$\Phi(j, j+1, z) = \frac{j!}{(-z)^j}\Phi(0, 1, z) - \left(\sum_{r=0}^{j-1} \frac{j!}{r!(-z)^{j-r}}\right)\Phi(1, 1, z).$$

Combining the above with (34), we deduce that

$$\begin{aligned}
\Phi(k+1, k+3, z) = -\frac{(k+2)!(z+k+1)}{(-z)^{k+2}}\Phi(0, 1, z) + \left(\sum_{r=0}^{k-1} \frac{(k+2)!(z+k+1)}{r!(-z)^{k+2-r}}\right)\Phi(1, 1, z) \\
+ \frac{(k+1)^2(k+2)}{z^2}\Phi(1, 1, z),
\end{aligned}$$

and the conclusion follows by inserting the above formula in (29). □

Beyond the standard contiguous relation, to the best of the authors' knowledge, there does not exist any result describing the distribution of the nonasymptotic zeros of linear combinations of Kummer functions. The next lemma provides a partial step towards that goal, by providing a non-autonomous second-order differential equation admitting a given linear combination of Kummer functions as a solution.

Lemma 14. *Let $\widetilde{\beta}$ and $\widetilde{\gamma}$ be two meromorphic functions. Then, the complex function F defined by*

$$F(z) = \widetilde{\beta}(z)\,\Phi(0, 1, z) + \widetilde{\gamma}(z)\,\Phi(1, 1, z), \tag{35}$$

with $\widetilde{\beta}(z)\widetilde{\gamma}'(z) + \widetilde{\gamma}(z)\left(\widetilde{\beta}(z)\tau - \widetilde{\beta}'(z)\right) \neq 0$ satisfies the following second-order differential equation

$$F''(z) + Q(z)F'(z) + R(z)F(z) = 0, \tag{36}$$

where Q and R are given in (37) *and* (38).

$$Q(z) = \frac{\left(\widetilde{\beta}''(z) + \tau\left(\widetilde{\beta}(z)\tau - 2\widetilde{\beta}'(z)\right)\right)\widetilde{\gamma}(z) - (\widetilde{\gamma}''(z))\widetilde{\beta}(z)}{\widetilde{\beta}(z)\widetilde{\gamma}'(z) + \widetilde{\gamma}(z)\left(\widetilde{\beta}(z)\tau - \widetilde{\beta}'(z)\right)}, \tag{37}$$

$$R(z) = \frac{\left(-\widetilde{\beta}(z)\tau + \widetilde{\beta}'(z)\right)\widetilde{\gamma}''(z) - \widetilde{\gamma}'(z)\left(\widetilde{\beta}''(z) + \tau\left(\widetilde{\beta}(z)\tau - 2\widetilde{\beta}'(z)\right)\right)}{\widetilde{\beta}(z)\widetilde{\gamma}'(z) + \widetilde{\gamma}(z)\left(\widetilde{\beta}(z)\tau - \widetilde{\beta}'(z)\right)}. \tag{38}$$

Lemma 14 may be proved by using that $\frac{\partial\Phi}{\partial z}(a, b, z) = \frac{a}{b}\Phi(a + 1, b + 1, z)$, which follows immediately from (7), and exploiting the contiguous relations from Lemma 4. In what follows, we shall refer to functions F of the form (35) as *Kummer-type functions*.

Note that Whittaker functions are defined in terms of Kummer functions in (11) by applying the multiplicative factor $\mathrm{e}^{-\frac{z}{2}} z^{\frac{1}{2}+l}$, thanks to which the Whittaker differential equation (12) has no first-order term. We now proceed similarly from Kummer-type functions in order to define *Whittaker-type functions*. The next lemma can be shown by straightforward computations.

Lemma 15. *Let $\widetilde{\beta}, \widetilde{\gamma}$ be two meromorphic functions, F be the function defined in* (35)*, and Q and R be given by* (37) *and* (38)*, respectively. Let $\mathcal{Q}$ be a primitive of $\frac{Q}{2}$ and define the function W by*

$$W(z) = \mathrm{e}^{\mathcal{Q}(z)} F(z). \tag{39}$$

Then, W satisfies the second-order differential equation

$$W''(z) + G(z)W(z) = 0, \tag{40}$$

where

$$G(z) = R(z) - \frac{(Q(z))^2}{4} - \frac{1}{2}Q'(z). \tag{41}$$

In the sequel, we refer to functions W of the form (39) as *Whittaker-type functions.*

5.4 MID Validity for the Lowest Over-order Multiplicity

Now, we shall use the results of Section 5.3 relating quasipolynomials with roots of over-order multiplicity and Kummer and Whittaker functions in order to provide sufficient conditions under which the MID property is valid for characteristic roots of multiplicity at least $n + 1$ of Δ.

Theorem 16. *Consider the quasipolynomial Δ from* (3)*–*(4) *with $m = n - 1$, and assume that Δ admits a real root λ_0 of multiplicity at least $n + 1$. Let β and γ be the meromorphic functions defined in* (32) *and* (33)*, respectively, and define the meromorphic functions $\widetilde{\beta}$ and $\widetilde{\gamma}$ by*

$$\widetilde{\beta}(z) = \beta\left(\lambda_0 - \frac{z}{\tau}\right), \quad \widetilde{\gamma}(z) = \gamma\left(\lambda_0 - \frac{z}{\tau}\right).$$

Let F, Q, R, and G be defined by (35)*,* (37)*,* (38)*, and* (41)*, respectively. Assume that, for every $t \in (0, 1)$ and every root z of F in $\mathbb{C}_-$, we have $\Re[zG(tz)] \geq 0$. Then, λ_0 is a dominant root of Δ, i.e., λ_0 satisfies the MID property.*

A result similar to Theorem 16 was already shown in [11, Theorem 10] for the case of roots of multiplicity $n+m$. The proof of the former can be obtained by an easy adaptation of that of the latter, and we detail this argument here for the sake of completeness.

Proof. We deduce from Proposition 13 that

$$\Delta(\lambda) = F(-\tau(\lambda - \lambda_0)). \tag{42}$$

In particular, the result is thereby proved if we show that all roots of the Kummer-type function F have nonnegative real part.

To do so, we consider the Whittaker-type function $W(\cdot)$ defined from F as in (39). Note that the differential equation (40) satisfied by W is of the form (13), with $K(z) = 1$. As a consequence, one can apply Hille's method to (40). By taking $z_1 = 0$ and z_2 equal to a root z_* of $F(\cdot)$ in (14), we obtain:

$$\int_0^{z_*} \left|W'(z)\right|^2 \overline{\mathrm{d}z} = \int_0^{z_*} |W(z)|^2 G(z)\, \mathrm{d}z.$$

We choose as integration path the line segment from 0 to z_*. Hence

$$\overline{z_*} \int_0^1 \left|W'(tz_*)\right|^2 \mathrm{d}t = z_* \int_0^1 |W(tz_*)|^2 G(tz_*)\, \mathrm{d}t.$$

Taking the real part, we get

$$x_* \int_0^1 \left|W'(tz_*)\right|^2 \mathrm{d}t = \int_0^1 |W(tz_*)|^2 \Re\left[z_* G(tz_*)\right] \mathrm{d}t, \tag{43}$$

where $x_* = \Re(z_*)$ and $y_* = \Im(z_*)$.

Assume now, by contradiction, that $F(\cdot)$ admits a root with negative real part, and take z_* in (43) as equal to this root. The left-hand side of (43) is negative, however its right-hand side is nonnegative by assumption, yielding the desired contradiction. Hence, all roots of F have nonnegative real parts, entailing the conclusion thanks to (42). □

6 Prescribed Stabilization of the chain of n integrators

This section focuses on the problem of stabilization of a chain of n integrators with a prescribed exponential decay. In this particular configuration, we are able to establish necessary and sufficient conditions for the MID property to hold for a root with the lowest over-order multiplicity. The section closes on the presentation of a link between multiple roots and a local minimizer for the spectral abscissa.

6.1 MID property for the chain of n integrators

Let us consider the chain of n integrators, i.e., we consider the function Δ from (3)–(4) with $P_0(\lambda) = \lambda^n$. In this case, we have the following characterization of the elimination-produced polynomial.

Proposition 17. *Consider the quasipolynomial* Δ *from* (3)–(4) *in the case* $P_0(\lambda) = \lambda^n$ *and* $m = n - 1$*. The elimination-produced polynomial* $\mathbb{P}$ *defined by* (21) *is given by*

$$\mathbb{P}(\lambda, \tau) = \widehat{\mathbb{P}}(\lambda\tau), \tag{44}$$

where $\widehat{\mathbb{P}}$ *is the polynomial defined by*

$$\widehat{\mathbb{P}}(s) = \sum_{k=0}^{n} k! \binom{n}{k}^2 s^{n-k}. \tag{45}$$

In addition, all roots of $\widehat{\mathbb{P}}$ *are negative real numbers.*

Proof. Since $P_0(\lambda) = \lambda^n$, we have $P_0^{(k)}(\lambda) = \frac{n!}{(n-k)!}\lambda^{n-k}$ and thus, inserting these expressions into (21), we deduce (44)–(45).

Furthermore, since P_0 has n real roots, all equal to 0, it follows from Proposition 8 that the polynomial $\lambda \mapsto \mathbb{P}(\lambda, \tau)$ is real rooted, and hence so is $\widehat{\mathbb{P}}$. Since all coefficients of $\widehat{\mathbb{P}}$ are positive, we infer that all its real roots are necessarily negative, yielding the conclusion. □

Recall that, by Proposition 11, when Δ admits a root of multiplicity at least $n + 1$, the elimination-produced polynomial $\mathbb{P}$ appears in the factorization (26) under the form $\mathbb{P}(\lambda_0, \tau t)$. We now study the behavior of this expression, seen as a function of t on the interval $(0, 1)$.

Proposition 18. *Consider the quasipolynomial* Δ *from* (3)–(4) *in the case* $P_0(\lambda) = \lambda^n$ *and* $m = n - 1$*, let* $\widehat{\mathbb{P}}$ *be defined by* (45)*, and define* $q(t) = \frac{1}{n!}\widehat{\mathbb{P}}(\lambda_0\tau t)$*. Assume, in addition, that* $\lambda_0\tau = c_0$*, where* c_0 *is the rightmost root of* $\widehat{\mathbb{P}}$*. Then, for every* $t \in (0, 1)$*, we have* $q(t) > 0$ *and* $q'(t) < 0$*.*

Proof. By Proposition 17, the polynomial $\widehat{\mathbb{P}}$ is real rooted and all its roots are negative, hence its rightmost root c_0 is negative. In addition, the coefficient of the leading monomial of $\widehat{\mathbb{P}}$ is positive, thus we also have that $\widehat{\mathbb{P}}$ and $\widehat{\mathbb{P}}'$ are positive in $(c_0, +\infty)$, and in particular in $(c_0, 0)$. As $\lambda_0\tau = c_0 < 0$, this entails that q is positive and its derivative is negative in $(0, 1)$, as required. □

Note that the factorization (26) involves an integral of the form

$$F(s) = \int_0^1 f(t) \mathrm{e}^{st} \, \mathrm{d}t, \tag{46}$$

which can be seen as the Laplace transform of a (real) function f with support included in $[0, 1]$. The study of the distribution of the zeros of functions F under the form (46) is related to a wide range of problems related to Physics and Engineering, and goes back to the pioneering works by Hardy [31], Pólya [59], and Titchmarsh [67] in the first decades of the 20th century. In particular, we have the following result from [59] (see also [61, Part Five, Chapter 3, Problem 177, page 66]).

Theorem 19 (G. Pólya, 1918)**.** *Let* f *be a positive and continuously differentiable function defined in the interval* $[0, 1]$ *and satisfying* $f'(t) < 0$ *for every* $t \in [0, 1]$*. Consider the function* $F \colon \mathbb{C} \to \mathbb{C}$ *defined in termes of* f *as in* (46)*. Then, all the zeros of* F *lie in the open right half-plane* $\mathbb{C}_+$*.*

As a consequence of Theorem 19, we infer the following result on the dominance of a root λ_0 of multiplicity $n+1$ in the case of the n-th order integrator.

Theorem 20. *Consider the quasipolynomial* Δ *from* (3)–(4) *in the case* $P_0(\lambda) = \lambda^n$ *and* $m = n-1$, *and let* $\widehat{\mathbb{P}}$ *be the polynomial defined in* (45). *Assume that* Δ *admits a root* λ_0 *of multiplicity* $n+1$ *and that* $\lambda_0\tau$ *is the rightmost root of* $\widehat{\mathbb{P}}$. *Then,* λ_0 *is the rightmost root of* Δ.

Proof. Owing to Propositions 11, 17, and 18, we have the following factorization,

$$\Delta(\lambda) = \tau(\lambda-\lambda_0)^{n+1}\int_0^1 q(t)\mathrm{e}^{-t(\lambda-\lambda_0)\tau}\,\mathrm{d}t, \tag{47}$$

where q is the polynomial defined in the statement of Proposition 18. Since q is positive and q' is negative in $(0,1)$, it follows from Theorem 19 that all roots of $s \mapsto \int_0^1 q(t)\mathrm{e}^{ts}\,\mathrm{d}t$ have positive real parts, and, consequently, all roots of Δ different from λ_0 have real parts strictly less than λ_0, as required. ☐

Remark 21. *Note that, if* λ_0 *is a root of multiplicity* $n+1$ *of* Δ, *then* λ_0 *is one of the* n *roots of the elimination-produced polynomial* $\mathbb{P}(\cdot,\tau)$. *The previous theorem ensures that, by selecting* λ_0 *as the rightmost root of* $\mathbb{P}(\cdot,\tau)$, *it will also be a dominant root of* Δ.

6.2 A Link with an Optimization Problem

An intriguing research question concerns whether and when the assignment of a root satisfying the MID property can be recast in terms of minimizing the spectral abscissa function. We shed light on this question by considering a chain of n integrators with input delay, controlled with static state feedback. We start with a theorem.

Theorem 22. *Consider the characteristic function*

$$\Delta(\lambda;\ \alpha_0,\dots,\alpha_{n-1}) = \lambda^n + (\alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0)\mathrm{e}^{-\lambda\tau}.$$

and the choice of gain parameters $(\alpha_0,\dots,\alpha_{n-1}) = (\alpha_0^*,\dots,\alpha_{n-1}^*)$, *which are determined by* (25) *and assign a rightmost root* λ_0 *of multiplicity* $n+1$. *Then in any direction in the parameter space, taken from* $(\alpha_0^*,\dots,\alpha_{n-1}^*)$, *the spectral abscissa function* $c\colon \mathbb{R}^n \to \mathbb{R}$,

$$(\alpha_0,\dots\alpha_{n-1}) \mapsto c(\alpha_0,\dots,\alpha_{n-1}) = \max_{\lambda\in\mathbb{C}}\{\Re(\lambda) \mid \Delta(\lambda;\ \alpha_0,\dots,\alpha_{n-1}) = 0\},$$

is strictly increasing.

Proof. We denote by $(d_0,\dots,d_{n-1}) \in \mathbb{R}^n,\ (d_0,\dots,d_{n-1}) \neq (0,\dots,0)$, the considered direction and by $\epsilon \in \mathbb{R}_+$ a perturbation parameter that determines the step taken in the considered direction. For fixed $(d_0,\dots,d_{n-1})$, we then analyze the zeros of the perturbed quasi-polynomial

$$H(\lambda;\ \epsilon) = \Delta(\lambda) + \epsilon(d_{n-1}\lambda^{n-1} + \cdots + d_1\lambda + d_0)\mathrm{e}^{-\lambda\tau}$$

as a function of ϵ, around $\epsilon = 0$. Due to the continuity of the spectral abscissa function with respect to parameter ϵ and the property that λ_0 is the unique rightmost characteristic root for $\epsilon = 0$, for sufficiently small $\epsilon > 0$ the spectral abscissa is induced by one of the roots that emerge from the splitting of the $n+1$-th order root λ_0 as ϵ is increased from zero.

We can always write H in the form

$$H(\lambda;\ \epsilon) = \Delta(\lambda) + \epsilon\big[f_0 + f_1(\lambda-\lambda_0) + \cdots + f_{n-1}(\lambda-\lambda_0)^{n-1}\big]\mathrm{e}^{-\lambda\tau},$$

where the n-tuple $(f_0, \ldots, f_{n-1})$ is induced by $(d_0, \ldots, d_{n-1})$. Let $j \in \{0, \ldots, n-1\}$ be the index such that

$$f_0 = \cdots = f_{j-1} = 0,\ f_j \neq 0.$$

Then we can factorize

$$H(\lambda;\ \epsilon) = (\lambda-\lambda_0)^j\left(\hat{H}(\lambda) + \epsilon\big[f_j + f_{j+1}(\lambda-\lambda_0) + \cdots + f_{n-1}(\lambda-\lambda_0)^{n-1-j}\big]\mathrm{e}^{-\lambda\tau}\right),$$

where the entire function $\hat{H}$ is the analytic extension of the function $\lambda \mapsto \frac{H(\lambda;\ 0)}{(\lambda-\lambda_0)^j}$, which has a removable singularity at λ_0. Note that, due to the deflation, $\hat{H}$ has a zero with multiplicity $n+1-j$ at λ_0. The roots of

$$\hat{H}(\lambda) + \epsilon\big[f_j + f_{j+1}(\lambda-\lambda_0) + \cdots + f_{n-1}(\lambda-\lambda_0)^{n-1-j}\big]\mathrm{e}^{-\lambda\tau} = 0 \tag{48}$$

satisfy the complete regular splitting property at $\lambda = \lambda_0$ and $\epsilon = 0$ because $\frac{\partial H}{\partial \epsilon}(\lambda_0,\ 0) = f_j \neq 0$ (see [34, 48]). Consequently, they can be expanded as Puiseux series in powers of ϵ^{n+1-j} as

$$\lambda_i(\epsilon) = \lambda_0 + \left(-(n+1-j)!\frac{\frac{\partial H}{\partial \epsilon}(\lambda_0,\ 0)}{\frac{\partial^{n+1-j} H}{\partial \lambda^{n+1-j}}(\lambda_0,\ 0)}\right)^{\frac{1}{n+1-j}} \mathrm{e}^{i\frac{2\pi i}{n+1-j}}\ \epsilon^{\frac{1}{n+1-j}} + \mathcal{O}\left(\epsilon^{\frac{2}{n+1-j}}\right),$$

$$i = 1, \ldots, n+1-j. \tag{49}$$

The additional j rightmost roots of H are invariant with respect to ϵ, that is,

$$\lambda_i(\epsilon) = \lambda_0,\ i = n+2-j, \ldots, n+1.$$

In what follows we can distinguish between two cases:

- If $j < n-1$, then $n+1-j > 2$ roots split according to (49). It follows that there is a real number $\bar{\epsilon} > 0$ such that

  $$\left(\max_{i\in\{1,\ldots,n+1-j\}} \Re(\lambda_i(\epsilon))\right) > \Re(\lambda_0),\ \forall \epsilon \in (0,\ \bar{\epsilon}),$$

  hence, the spectral abscissa is strictly increasing at $\epsilon = 0$;

- If $j = n-1$, then the form of expansion (49) may be inconclusive as two roots can possibly split along the imaginary axis. To proceed, we employ the factorization (47) and rewrite (48) as

  $$\underbrace{\tau M(\lambda\tau)(\lambda-\lambda_0)^2}_{\hat{H}(\lambda)} + \epsilon f_{n-1}\mathrm{e}^{-\lambda\tau} = 0, \tag{50}$$

  with

  $$M(\lambda) = \int_0^1 q(t)\mathrm{e}^{-t(\lambda-\lambda_0\tau)}\,\mathrm{d}t,$$

which implies $M(\lambda_0\tau) > 0$ and $M'(\lambda_0\tau) < 0$. In case $f_{n+1} < 0$, then the double root of λ_0 of (50) splits according to

$$\lambda_{\pm}(\epsilon) = \lambda_0 \pm \sqrt{\frac{-f_{n-1}\mathrm{e}^{-\lambda_0\tau}}{M(\lambda_0\tau)}}\sqrt{\frac{\epsilon}{\tau}} + \mathcal{O}(\epsilon), \tag{51}$$

whole derivation relies on interpreting (50), after pre-multiplication by τ as an equation in the combined argument $\lambda\tau$. In case $f_{n+1} > 0$, then the double root λ_0 splits according to

$$\lambda_{\pm}(\epsilon) = \lambda_0 \pm \sqrt{\frac{f_{n-1}\mathrm{e}^{-\lambda_0\tau}}{M(\lambda_0\tau)}}\sqrt{\frac{\epsilon}{\tau}}\, i + \frac{f_{n-1}\mathrm{e}^{-\lambda_0\tau}}{2M(\lambda_0\tau)}\left(1 + \frac{M'(\lambda_0\tau)}{M(\lambda_0\tau)}\right)\epsilon + \mathcal{O}\left(\epsilon\right)^{\frac{3}{2}}. \tag{52}$$

The coefficient of ϵ is positive since $M(\lambda_0\tau) > 0$ and

$$1 + \frac{M'(\lambda_0\tau)}{M(\lambda_0\tau)} = 1 - \frac{\int_0^1 tq(t)\,dt}{\int_0^1 q(t)\,dt} > 0,$$

since q is positive on $[0,1]$ and the 'weight' function t in the top integral is smaller than one. Hence, if ϵ is increased from zero, $n-1$ characteristic roots are invariant, while two others split along the imaginary axis but bend towards the open right half plane.

The proof is completed. □

Observe that the (local) growth of the spectral abscissa in a given direction, characterized by (49), (51) and (52), can always be bounded from below by a strictly increasing linear function. On the one hand, this argument is mathematically not sufficient to conclude that $(\alpha_0^*, \ldots, \alpha_{n-1}^*)$ is a strict local minimizer of the spectral abscissa function, given that this function is non-smooth (but numerical experiments for n up to 10 even point to a strict global minimizer). For comparison, the function $s:\ \mathbb{R}^2 \mapsto \mathbb{R}$, defined by

$$s(\alpha_0, \alpha_1) = \sqrt{|\alpha_1 - \alpha_0^2|} - \alpha_0^4 - \alpha_1^4,$$

satisfies (with $\epsilon \geq 0$)

$$s(\epsilon d_0, \epsilon d_1) = \begin{cases} \sqrt{|d_1|}\sqrt{\epsilon} + \mathcal{O}(\epsilon), & d_1 \neq 0, \\ |d_0|\epsilon + \mathcal{O}\left(\epsilon^2\right), & d_1 = 0, \end{cases}$$

but $(0,0)$ is no local minimizer since $s\left(\epsilon, \epsilon^2\right) < 0$ for $\epsilon > 0$.

On the other hand, for the second-order integrator, the assertion of Theorem 22 can be strengthened to a strict global minimum at (α_0^*, α_1^*), by using the additional property that for $n = 2$, the spectral abscissa function is quasi-convex [50, Proposition 1]. This reference also contains a complete numerical characterization of global minima of the spectral abscissa function for all second-order input delay systems, controlled by state feedback. From the characterization using TDS-CONTROL [1], global optima are always related to rightmost characteristic roots with multiplicity larger than one.

7 An Illustrative Example: GMID, Intermediate MID, and the Prescribed Pendulum Stabilization

Let us revisit the classical control problem of the stabilization of the friction-free pendulum [2], whose dynamics are governed by the following second-order differential equation:

$$\ddot{\theta}(t) + \frac{g}{L}\sin(\theta(t)) = u(t), \tag{53}$$

where $\theta(t)$ designates the angular displacement of the pendulum at time t with respect to the stable equilibrium position, L is the pendulum length, g is the gravitational acceleration, and $u(t)$ is the control input, which stems from an applied external torque. We follow in this section the control strategy for this problem proposed in [11].

Consider the standard delayed PD controller

$$u(t) = -k_p\,\theta(t-\tau) - k_d\,\dot{\theta}(t-\tau), \tag{54}$$

with $(k_p, k_d) \in \mathbb{R}^2$. The linear stability of the closed-loop system amounts to the location of the roots of the corresponding quasipolynomial

$$\Delta(\lambda) = \lambda^2 + \frac{g}{L} + (k_d\lambda + k_p)\,e^{-\lambda\tau}. \tag{55}$$

Notice that the degree of the above quasipolynomial is equal to 4. Hence, if one exploits the GMID property proved in [46] (i.e., if we consider a root of (55) with multiplicity equal to the degree of the quasipolynomial), it follows that the only admissible quadruple root is $\lambda_0 = -\sqrt{2g/L}$ which is necessarily the corresponding spectral abscissa and is achieved if the system's parameters (k_p, k_d, τ) satisfy $k_d = -e^{-2}\sqrt{2g/L}, \quad k_p = -5\,e^{-2}g/L, \quad \tau = \sqrt{2L/g}$.

As emphasized in [10], the GMID does not allow any degree of freedom in assigning λ_0, i.e., there is a single possible choice of λ_0 ensuring that it is a root of multiplicity 4. In order to allow for some additional degrees of freedom when assigning λ_0, one can relax such a constraint by forcing the root λ_0 to have a multiplicity lower than the maximal, and consider, for instance, the delay as a *free tuning parameter*. This is the subject of the next result, extracted from [11], which considers a root λ_0 of multiplicity 3. Note that, in this case, the multiplicity 3 is also the lowest over-order multiplicity for (55).

Proposition 23 ([13]). *For any* $0 < \tau < \sqrt{2\,L/g}$*, let*

$$\lambda_0 = \frac{-2 + \sqrt{-\frac{g\,\tau^2}{L} + 2}}{\tau}. \tag{56}$$

The delayed PD controller (54) *with*

$$k_d = \frac{2\,(\tau\lambda_0 + 1)\,\mathrm{e}^{\tau\lambda_0}}{\tau},\ k_p = \frac{2\left(5L\tau\lambda_0 + g\,\tau^2 + 3L\right)\mathrm{e}^{\tau\lambda_0}}{\tau^2 L} \tag{57}$$

and $\lambda_0\,\tau \geq -1$*, locally exponentially stabilizes the dynamical system* (53)*. Namely,* λ_0 *is a root of multiplicity at least* 3 *of* (55) *and it is dominant, i.e., the lowest over-order (intermediate) MID property holds true.*

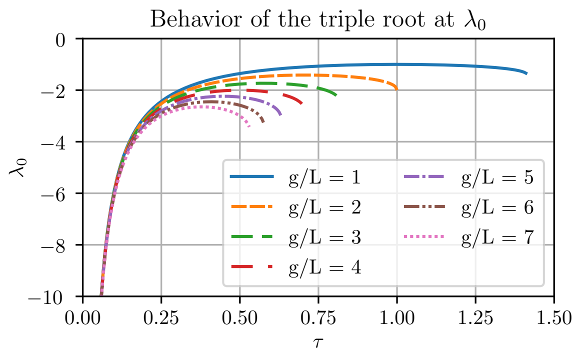

Figure 2: The behavior of the triple root (spectral abscissa) of (55) at $\lambda = \lambda_0$ given by (56) as a function of the tuning ("free") delay parameter $0 < \tau < \sqrt{2\,L/g}$ for $g/L \in \{1, \ldots, 7\}$. Clearly, increasing the ratio g/L decreases the assignment region as well as the delay margin. Figure extracted from [11].

The proof of the above proposition can be found in [13], which is concerned with the over-order MID property with multiplicity $M(\lambda_0) = m + n$. This multiplicity coincides with the lowest over-order multiplicity in the case of (55). It is also a direct consequence of the results of the present paper since, in the pendulum case under consideration, the only intermediate multiplicity corresponds to the lowest over-order multiplicity. It should be mentioned that Proposition 23 can also be proven by using the argument principle as is done in [20].

8 P3δ Software

Partial pole placement via delay action (P3δ) is an intuitive `Python` software [14–16] which enables the design of stabilizing feedback control laws exploiting the delay effect on the closed-loop dynamics. Beyond the MID property, the software relies on another property of quasipolynomial's zeros distribution called CRRID. While the MID has been emphasized through this paper, the CRRID property consists in conditions on the system's parameters guaranteeing the dominance of coexistent real spectral values. When using the MID strategy on P3δ, two options are proposed: the GMID-based design and the control-oriented IMID-based design. The latter exploits the over-order intermediate multiplicity $M = n+1$, offering sufficient freedom in parameters' choice. Notice that both strategies, MID and CRRID, allow to prescribe the exponential decay rate of the closed-loop system solution.

9 Conclusion

This paper discusses the spectral abscissa of linear time-invariant dynamical systems represented by delay-differential equations. It exploits links between spectral values of intermediate admissible multiplicity for a quasipolynomials and the distribution of zeros of linear combinations of Kummer confluent hypergeometric functions. It proposes a delayed control design methodology enabling the closed-loop system's solution to obey a prescribed decay rate, opening perspectives in concrete applications including, among others, vibration control (see, e.g., [21, 51]). In particular, the proposed methodology is illustrated through the stabilization problem of the classical pendulum as well as the stabilization of chains of integrators.

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