Over-order multiplicities and their application in controlling delay dynamics. On zeros' distribution of linear combinations of Kummer hypergeometric functions

Islam Boussaada, Guilherme Mazanti, Silviu-Iulian Niculescu

Over-order multiplicities and their application in controlling delay dynamics. On zeros' distribution of linear combinations of Kummer hypergeometric functions

Islam Boussaada*† Guilherme Mazanti* Silviu-Iulian Niculescu*

Abstract

A series of recent works have shown that, for a system of linear functional differential equations, a spectral value having a multiplicity exceeding the order of the system tends to correspond to the spectral abscissa of the system, a property called MID for *multiplicity-induced-dominancy*. In particular, when this multiplicity coincides with the degree of the characteristic quasipolynomial, this property is called *generic* MID (GMID), in opposition to the *intermediate* MID (IMID), which corresponds to a multiplicity strictly smaller than the degree. The GMID has been fully characterized for single-delay retarded as well as neutral delay-differential equations thanks to the representation of the corresponding quasipolynomial in terms of a *Kummer hypergeometric function*. However, apart from partial results, in full generality, no result of the literature enables the characterization of the dominance of a spectral value having an intermediate multiplicity, which is essentially due to the lack of existing results among the open literature pertaining to linear combinations of Kummer functions' zeros distribution. In this work, we overcome this difficulty and we further investigate the MID to cover the so-called *over-order MID*, that is, the cases where the multiplicity is larger than the order of the corresponding differential equation.

2020 Mathematics Subject Classification. 33C15, 34M03, 30A10, 34K06, 93D15
Keywords. Confluent hypergeometric functions, Whittaker function, Kummer function, zeros location, stability and stabilization, exponential decay, pole placement, prescribed stabilization

1 Introduction

The common feature of propagation, transport phenomena/processes and population dynamics is their *time heterogeneity*, and there exists a wide variety of mathematical models describing these dynamic behaviors in biology, physics, economics and engineering. One of the simplest ways of capturing the said time heterogeneity of the dynamics is to use time-delay systems represented by delay-differential equations (DDEs) under appropriate initial conditions. For the basic theory of DDEs[1], we refer to [10, 25, 28, 30–32, 36, 37, 39, 41–43, 65, 69]. The classification of the DDEs[2] and fundamental properties of the solutions can be found in [39]. Although delay systems are infinite-dimensional, a first idea to properly understand and analyze their dynamics was to extend methods and techniques from ordinary differential equations (ODEs) to DDEs, leading to some unitary viewpoint of the qualitative properties for both classes of differential equations. Such an angle was adopted since the 1960s, and we refer to [31, 38] for further insights on the underlying methods.

In this spirit, and in the same fashion as in [31] for the presentation of the properties of the solutions of DDEs, we adopt some terminology coming from ODEs. In particular, the *order* of a DDE simply means the order of the highest

*Université Paris-Saclay, CNRS, CentraleSupélec, Inria, Laboratoire des signaux et systèmes, 91190, Gif-sur-Yvette, France. E-mails: Islam.Boussaada@l2s.centralesupelec.fr, Guilherme.Mazanti@l2s.centralesupelec.fr, Silviu.Niculescu@l2s.centralesupelec.fr
†IPSA, 94200, Ivry-sur-Seine, France
[1] initial value problem, existence, uniqueness of the solutions, linear systems, stability theory
[2] in two classes: retarded and neutral DDEs

derivative involved in the equation[3]. In the linear DDE case, the characteristic function is a quasipolynomial (see, e.g., [10]) and, likewise the ODE case, we can introduce the notion of *degree*. However, since a quasipolynomial has an infinite number of (characteristic) roots, the meaning of its degree is slightly different. Indeed, contrarily to the polynomial case, the degree does not determine the number of roots, it is rather related to the number of parameters of the quasipolynomial. By exploiting the Pólya-Szegő bound [59] pertaining to the number of roots of exponential polynomials in horizontal strips, [23] (see also [47]) showed that the maximal admissible multiplicity of a characteristic root is given by the degree of the corresponding quasipolynomial, and, in addition to the first and second-order DDE cases discussed in [58], we can mention a classical control example — stabilizing a second-order inverted pendulum by a delayed position feedback —, where one may have characteristic roots at the origin with a multiplicity larger than the order of closed-loop system[4] (see, for instance, [22, 60, 63] and the references therein). Such multiple zeros are called *over-order multiple (characteristic) roots*, and to the best of the authors' knowledge, excepting some simple cases, there does not exist a systematic analysis of such (multiple) roots, and, in particular, a better comprehension of the way they may affect the dynamics of the corresponding dynamical system is missing.

One of the objectives of this paper is to further investigate roots with over-order multiplicity and, in particular, to explicitly determine conditions when such a root is *dominant*, in the sense that it defines the spectral abscissa of the corresponding dynamical system, i.e., the real part of the rightmost (characteristic) root. The latter property is called *multiplicity-induced-dominancy (MID)*. Although the existence of multiple characteristic roots for first-order DDEs was emphasized since the end of the 1940s, the first systematic study in the parameter-space for the first- and second-order DDEs (both retarded and neutral cases) including a single delay (delay equal to one) can be found in [58]. It should be mentioned that Pinney observed that the characteristic roots with the highest (possible) multiplicity can also exhibit the greatest real part[5], which, in our terminology, corresponds to the MID property. Namely, in order to avoid oscillations with the ambition of getting the maximum damping of all transient solutions, the author emphasized the difficulty of the problem and studied the triple (dominant) roots in the first-order neutral and quintuple (dominant) in the second-order neutral DDE cases, respectively. It should be pointed out that the proposed approach, which is quite close to the so-called D-decomposition method introduced by Neimark [54], cannot be simply extended to more general classes of DDEs. For a better understanding of the complexity of the analysis, we refer to the study of the over-order MID property for first-order neutral DDEs with a single delay in [11], where the authors proved the dominance by exploiting the explicit (frequency) bound for the imaginary part of the unstable (characteristic) roots[6].

For DDEs of arbitrary order, the MID property was proven in the generic case (i.e., the over-order multiplicity is equal to the degree of the quasipolynomial, a situation known as GMID for *generic MID*) in [47] (retarded case) and in [15] (unified treatment of retarded and neutral cases). In fact, the authors of these papers have shown that the GMID problem can be reduced to (i) solving an appropriate linear system of equations, and (ii) deriving an appropriate factorization of the corresponding quasipolynomial. Finally, the dominance is proved by exploiting the properties of Kummer hypergeometric functions. For a good introduction to hypergeometric functions, we refer to [26, 34, 70]. As discussed in [16], it should be mentioned that new interpretations of the Padé approximation of the exponential function based on the location of the zeros of Kummer functions with real parameters have been emphasized in [17]. Finally, for an overview of some of the methods for the characterization of the behavior of the (characteristic) multiple roots of linear DDEs in the parameter space[7], we refer to [56].

In this paper, following several references on the topic [11, 15, 47, 56, 58], we will focus on the analysis of scalar DDEs with a single delay. This is motivated by the fact that, despite its simple appearance, the single-delay case is subtle to analyze and there are still many open problems concerning the MID property in this setting, some of which are the subject of this paper. The situation is much more delicate in the presence of multiple delays and extensions of the single-delay results to such a more general setting are far from being trivial. We mention the recent

[3]In this context, the terminology of differential-difference equations introduced by Bellman and Cooke [10] should be mentioned. The authors made an explicit distinction between the *differential* order, corresponding to the highest derivative order appearing in the equation, and the so-called *difference* order, corresponding to the number of distinct (time) arguments appearing in the (same) equation minus one; for further arguments concerning the terminology, see, for instance, [56] and the references therein.

[4]For instance, in the inverted pendulum case, we may have one triple characteristic root at the origin of the linearization of the dynamical system in closed-loop represented by a second-order DDE.

[5]More precisely, the reader is referred to Chapters 3 (characteristic equations), 4 (first-order DDE) and 5 (second-order DDE) in [58].

[6]Complex characteristic roots located in the right half-plane.

[7]also including the delays (seen as parameters)

results from [35] dealing with the MID property for a first-order scalar DDE with two delays; its generalization to a non-scalar or higher-order setting is an open problem.

The interest for a deeper understanding and an explicit characterization of the MID property for more general classes of dynamical systems represented by DDEs emanates from the control area. As discussed in [55, 64], starting with the 1950s, in engineering applications, delay was commonly associated to instability, oscillations and bad behaviors in dynamical systems, and there is an abundant literature on the elaboration of criteria allowing to guarantee the exponential stability of the systems *independently* of the size of the *delay*. However, some results showed that the delay, as a *control parameter* can be useful for improving the stability and/or the behavior of linear dynamical systems, see, for instance, the so-called *proportional minus delay* controller [66, 67] or the output delay feedback that stabilizes some oscillatory systems [1]. Furthermore, as highlighted by [44][8], increasing the delay is not necessarily associated with instability and there are cases where a larger delay may induce stability even though the property does not necessarily hold for "small" delay values.

As a consequence of the discussion above, understanding the dependence of the characteristic roots with respect to the system parameters is essential. Several methods/techniques grouped together under the title *eigenvalue-based approaches* have been the subject of several contributions, see, for instance, [51] and the references therein. These methods use and exploit the duality between two types of methods for solving eigenvalue problems: one nonlinear in finite dimension and the other linear in infinite dimension and, as a consequence, to better characterize the properties of the *spectral abscissa* with respect to system parameters, and reinforce the interest in using the *delay as a control parameter*.

The continuous dependence of the characteristic roots on the controller parameters represents an interesting property. For instance, the so-called *continuous pole placement* proposed in [50] in the retarded case uses such a continuity property and explicitly uses the fact that, in the retarded case, the number of unstable roots is finite together with an appropriate monitoring of the characteristic roots with a "large" real part. Another extension of the pole placement method is the so-called *partial pole placement* method, which simply employs the degree of the quasipolynomial and the MID property mentioned above. First, the degree of the quasipolynomial gives an upper bound on the number of functional equations to be taken into account in the appropriate tuning of the controller gains. Second, the MID property gives a guarantee for the decay rate of the solutions of the closed-loop system. This control method was discussed in [47] (retarded case) and [15] (retarded and neutral cases) in the case of maximal admissible multiplicity. A few recent results showed that this property holds for other over-order multiplicities and, in particular, in the case of the lowest over-order multiplicity, we refer to [8] (in the case of real-rooted plants) and [18] (by exploiting the frequency bound). To the best of the authors' knowledge, there does not exist an explicit characterization of the partial pole placement in the case of over-order multiplicity. The method proposed in this paper enables the handling of such a problem in its generality.

To summarize, the aim of this paper is to address such MID spectral problems and to propose a new method that could also encompass over-order (algebraic) multiplicities with the guarantee of dominance of the corresponding spectrum. As the GMID property has already been extensively consider in the literature, we will focus here on the case of intermediate multiplicities, known as *intermediate MID* or IMID. More precisely, the contribution of the paper is threefold: first, to further investigate over-order MID cases. Roughly speaking, if the problem of multiplicity can be reduced to solving an appropriate system of functional equations, the dominance requires a deeper analysis and an appropriate understanding of the way the changes in the parameters may affect the spectrum distribution. As shown below, a central role is played by the so-called *elimination-produced function* (see Section 3), and its interest clearly appears in a control setting in terms of degrees of freedom induced by the system structure and the (controller) gains selection. Next, the appropriate tool for proving the dominancy is the Green–Hille (integral) transformation introduced by Hille one century ago [40] for characterizing the location of the non-asymptotic zeros of (degenerate) Whittaker hypergeometric functions. For an introduction to confluent hypergeometric functions, we refer to [26]. A motivating example (first-order neutral DDEs of degree three representing the closed-loop dynamics of a transport equation subject to a stabilizing boundary proportional-integral (PI) action, see Section 2) helps to a better understanding of the existing links between over-order MID (double characteristic root in this case) and Hille's method.

It should be noted that the *elimination-produced function* represents a novelty and generalizes some of the ideas proposed earlier by the authors of this paper. In particular, we recover some of the cases when this function reduces

[8]The corresponding method is known as the τ-*decomposition method*, see, e.g., [51] for a deeper discussion on such topics.

to a polynomial, called *elimination-produced polynomial* (see Section 3). Next, we further exploit the properties of Kummer hypergeometric functions to handle the dominance property for over-order multiplicities leading to a more unified treatment. Second, as a byproduct of the analysis proposed in this paper, the location of zeros of linear combination of Kummer (hypergeometric) functions can be derived, and these results allow exploiting MID ideas into a different frame and more precisely in the case of *contiguous* (degenerate hypergeometric) *functions* (see, for instance, [57] for further details and related definition). Finally, we use the over-order MID to the control of dynamical systems by using the so-called *partial pole placement* method that simply consists in assigning a given multiplicity of a spectral value for the closed-loop system by an appropriate choice of the controller gains with the guarantee of the exponential stability of the closed-loop system solution with a prescribed rate given by this spectral value. To the best of the authors' knowledge, such ideas represent a novelty in the open literature.

Some illustrative case studies show the effectiveness of the method, see for instance [7, 27, 52]. In fact we consider two control problems of two classes of infinite-dimensional systems whose dynamical equations can be represented by DDEs: the control of a transonic flow in a wind tunnel[9] and the control of a first-order unstable plant including a communication delay in the input/output channel by using a "standard" PID control law.

The remaining of the paper is organized as follows: Prerequisites on degenerate hypergeometric functions, a motivating example (controlling transport equation by a PI controller), the problem formulation, and definitions of over-order multiplicities are presented in Section 2. The methodology for the study of the over-order multiplicities as well as some control perspectives are proposed in Section 3. Next, Section 4 includes two illustrative examples and briefly presents the software $P3\delta$ covering the over-order multiplicity. Finally, some concluding remarks end the paper.

Notation. Throughout the paper, the following notations are used: $\mathbb{N}^*$, $\mathbb{R}$, $\mathbb{C}$ denote the sets of positive integers, real numbers, and complex numbers, respectively, and we set $\mathbb{N} = \mathbb{N}^* \cup \{0\}$. For a complex number λ, $\Re(\lambda)$ and $\Im(\lambda)$ denote its real and imaginary parts, respectively. The open left and right complex half-planes are the sets $\mathbb{C}_-$ and $\mathbb{C}_+$, respectively, defined by $\mathbb{C}_- = \{\lambda \in \mathbb{C} \mid \Re(\lambda) < 0\}$ and $\mathbb{C}_+ = \{\lambda \in \mathbb{C} \mid \Re(\lambda) > 0\}$. Given $k, n \in \mathbb{N}$ with $k \leq n$, the binomial coefficient $\binom{n}{k}$ is defined as $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and this notation is extended to $k, n \in \mathbb{Z}$ by setting $\binom{n}{k} = 0$ when $n < 0$, $k < 0$, or $k > n$. For $\alpha \in \mathbb{C}$ and $k \in \mathbb{N}$, $(\alpha)_k$ is the *Pochhammer symbol* for the *ascending factorial*, defined inductively as $(\alpha)_0 = 1$ and $(\alpha)_{k+1} = (\alpha + k)(\alpha)_k$.

2 Prerequisites and problem formulation

This section provides a brief presentation of the definitions and results that shall be of use in the sequel.

2.1 Dynamical systems with delay

Consider the linear time-invariant (LTI) dynamical system described by the following DDE including a single delay:

$$y^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k y^{(k)}(t) + \sum_{k=0}^{m} \beta_k y^{(k)}(t-\tau) = 0, \tag{1}$$

under appropriate initial conditions, where $y(\cdot)$ is the real-valued unknown function, $\tau > 0$ is the delay, and $\alpha_0, \ldots, \alpha_{n-1}, \beta_0, \ldots, \beta_m$ are real coefficients. The DDE (1) is said to be of *retarded type* if $m < n$, or of *neutral type* if $m = n$. It should be mentioned that these classes of dynamical systems (retarded, neutral) depict different properties. For instance, in the linear case, only point spectrum exists in the retarded case. In the neutral case, excepting the point spectrum, the corresponding system posses also essential spectrum, making the analysis more involved. We refer to [10], [39], [51] for a deeper discussions on the classification of DDEs and related results and properties. Notice that (1) represents a particular case of the following matrix form:

$$\dot{\xi}(t) + B_\tau \dot{\xi}(t-\tau) = A_0 \xi(t) + A_\tau \xi(t-\tau) \tag{2}$$

where $\xi = (y(t), y'(t), \ldots, y^{(n-1)}(t))^T \in \mathbb{R}^n$ is the state vector and A_0, A_τ, $B_\tau \in \mathcal{M}_n(\mathbb{R})$ are real-valued matrices which can be easily constructed from (1).

[9] under the assumption that the flow is uniform across every cross section and the tunnel is a one-dimensional tube of varying cross-sectional area, leading to a a coupled model of nonlinear partial differential equations in one space dimension

The characteristic function associated with (1) is the quasipolynomial $\Delta\colon \mathbb{C} \to \mathbb{C}$ defined by

$$\Delta(\lambda) := P_0(\lambda) + \mathrm{e}^{-\lambda\tau} P_\tau(\lambda), \tag{3}$$

where P_0, P_τ are polynomials with real coefficients given by:

$$\begin{cases} P_0(\lambda) := \lambda^n + \displaystyle\sum_{k=0}^{n-1} \alpha_k \lambda^k, \\ P_\tau(\lambda) := \displaystyle\sum_{k=0}^{m} \beta_k \lambda^k. \end{cases} \tag{4}$$

It is well-known that the exponential stability of the trivial solution of (1) is given by the location of the *characteristic roots* of Δ, see, e.g., [10, 51]. For DDEs, we have infinitely many such roots.

As discussed in [23], the *degree* of the quasipolynomial Δ, denoted by $\deg(\Delta)$, is nothing else than the Pólya–Szegő bound [59] and, in our case, $\deg(\Delta) = n + m + 1$ and it is larger than the degree of the polynomials P_0 and P_τ and the order n of the delay-free ODE. As indicated in the Introduction, $\deg(\Delta)$ is nothing else than the number of the *parameters* of the DDE (1), that is, the number of coefficients of the polynomials P_0 and P_τ. Finally, it follows from the Pólya–Szegő bound that the quasipolynomial degree corresponds also to the maximal allowable multiplicity that a characteristic root of (3) may have. To reach such a bound, the characteristic root should be real.

2.2 Degenerate hypergeometric functions and the corresponding contiguous relations

To develop our results, we need to use some properties of classical hypergoemetric functions. The first such a function we introduce is the *Kummer (confluent) hypergeometric function*, which, for $a, b \in \mathbb{C}$ such that $-b \notin \mathbb{N}$, is the entire function $\Phi(a, b, \cdot)\colon \mathbb{C} \to \mathbb{C}$ defined by the series

$$\Phi(a, b, z) := \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}. \tag{5}$$

The series in (5) converges for every $z \in \mathbb{C}$ and, as presented in [26, 34, 57], it satisfies the (second-order) *Kummer differential equation*

$$z\frac{\partial^2 \Phi}{\partial z^2}(a, b, z) + (b - z)\frac{\partial \Phi}{\partial z}(a, b, z) - a\Phi(a, b, z) = 0. \tag{6}$$

As discussed in [26, 34, 57], for every $a, b, z \in \mathbb{C}$ such that $\Re(b) > \Re(a) > 0$, Kummer functions also admit the integral representation

$$\Phi(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 \mathrm{e}^{zt} t^{a-1} (1-t)^{b-a-1} \, \mathrm{d}t, \tag{7}$$

where Γ denotes the Gamma function. This integral representation has been exploited in [47] to characterize the spectrum of some DDEs of retarded type.

Notice that Kummer functions satisfy some recurrence relations often called *contiguous relations*, see for instance [57]. In particular, in our case, the following relations are of interest.

Lemma 1 ([57, p. 325]). *Let a, b, z be three complex numbers with $a \neq b$ and $z \neq 0$. The following relations hold:*

$$\begin{aligned} \Phi(a, b+1, z) &= \frac{-b\,(a+z)\;\Phi(a, b, z) + ab\,\Phi(a+1, b, z)}{z\,(a-b)}, \\ \Phi(a+1, b+1, z) &= -\frac{-b\,\Phi(a+1, b, z) + b\,\Phi(a, b, z)}{z}. \end{aligned} \tag{8}$$

Kummer confluent hypergeometric functions have close links with *Whittaker functions*. More precisely, for $k, l \in \mathbb{C}$ with $-2l \notin \mathbb{N}^*$, the *Whittaker function* $\mathcal{M}_{k,l}$ is the function defined for $z \in \mathbb{C}$ by

$$\mathcal{M}_{k,l}(z) := \mathrm{e}^{-\frac{z}{2}} z^{\frac{1}{2}+l} \Phi(\tfrac{1}{2} + l - k, 1 + 2l, z) \tag{9}$$

(see, e.g., [57]). Note that, if $\frac{1}{2}+l$ is not an integer, the function $\mathcal{M}_{k,l}$ is a multi-valued complex function with branch point at $z=0$. The nontrivial roots of $\mathcal{M}_{k,l}$ coincide with those of $\Phi(\frac{1}{2}+l-k, 1+2l, \cdot)$ and $\mathcal{M}_{k,l}$ satisfies the (second-order) *Whittaker differential equation*

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}z^2}(z) = \left(\frac{1}{4} - \frac{k}{z} + \frac{l^2 - \frac{1}{4}}{z^2}\right)\varphi(z). \tag{10}$$

Since $\mathcal{M}_{k,l}$ is a nontrivial solution of the second-order linear differential equation (10), any nontrivial root of $\mathcal{M}_{k,l}$ is necessarily simple.

2.3 Some insights on linear combinations of two Kummer functions

Notice that, beyond the standard contiguous relations recalled in Lemma 1 and the other contiguous relations from [57, p. 325], to the best of the authors' knowledge, there does not exist any result describing the distribution of the non-asymptotic zeros of linear combinations of Kummer functions.

The next result, which is established and shown in [18], provides a partial step towards our goal, by giving a non-autonomous second-order differential equation having a given linear combination of two Kummer functions as a solution.

Lemma 2. *Let a, b be two complex numbers and α and β two real numbers. Then the complex function F defined by*

$$F(z) := \alpha\,\Phi(a,b,z) + \beta\,\Phi(a,b+1,z), \tag{11}$$

with $z \notin \{0, \frac{\beta(\beta+\alpha)b^2}{((a-b)\alpha-\beta b)\alpha}\}$, satisfies the second-order differential equation

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}z^2}(z) + Q(z)\frac{\mathrm{d}\varphi}{\mathrm{d}z}(z) + R(z)\varphi(z) = 0, \tag{12}$$

where

$$Q(z) := -1 + \frac{b+1}{z} - \frac{\alpha\,(a\alpha - \alpha b - \beta b)}{D(z)}, \tag{13}$$

$$R(z) := -\frac{N(z)}{D(z)}, \tag{14}$$

with

$$N(z) := a\left(\left((a-b)\,\alpha^2 - \alpha b\beta\right)z - \beta b\,(b+1)\,\alpha\right) - a\,b^2\beta^2,$$
$$D(z) := \left((a-b)\,\alpha^2 - \alpha b\beta\right)z - \alpha\,b^2\beta - b^2\beta^2.$$

Lemma 2 can be proved by using a simple property of Φ, namely, $\frac{\partial\Phi}{\partial z}(a,b,z) = \frac{a}{b}\Phi(a+1,b+1,z)$, which follows immediately from (5), and exploiting the contiguous relations from Lemma 1. In the sequel, we shall refer to functions F of the form (59) as *Kummer-type functions*.

Note that Whittaker functions are defined in terms of Kummer functions in (9) by using the multiplicative factor $\mathrm{e}^{-\frac{z}{2}} z^{\frac{1}{2}+l}$, thanks to which the Whittaker differential equation (10) has no first-order term. We now proceed similarly from Kummer-type functions in order to define *Whittaker-type functions*. The next lemma can be shown by straightforward computations.

Lemma 3. *Let a, b be two complex numbers, α, β be two real numbers, F be the function defined in* (11)*, and Q and R be given by* (13) *and* (14)*, respectively.*

Let $\mathcal{Q}$ be a primitive of $\frac{Q}{2}$ and define the function W by

$$W(z) := \mathrm{e}^{\mathcal{Q}(z)}F(z). \tag{15}$$

Then W satisfies the second-order differential equation

$$\frac{\mathrm{d}^2W}{\mathrm{d}z^2}(z) + G(z)W(z) = 0, \tag{16}$$

where

$$G(z) := R(z) - \frac{(Q(z))^2}{4} - \frac{1}{2}\frac{\mathrm{d}Q}{\mathrm{d}z}(z). \tag{17}$$

In the sequel, we refer to functions W of the form (15) as *Whittaker-type functions*.

As discussed previously, beyond the standard contiguous relations, to the best of the authors' knowledge, there does not exist any result describing the distribution of the non-asymptotic zeros of linear combinations of Kummer functions.

2.4 Hille oscillation theorems

In [40], Hille studied the distribution of zeros of functions of a complex variable satisfying linear second-order homogeneous differential equations with variable coefficients, as is the case for the degenerate Whittaker function $\mathcal{M}_{k,l}$, which satisfies (10). Thanks to an integral transformation defined there and called *Green–Hille transformation*, and some further conditions on the behavior of the function, Hille showed how to discard regions in the complex plane from including complex roots. Consider, for instance, the general homogeneous second-order differential equation

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[K(z)\,\frac{\mathrm{d}\varphi}{\mathrm{d}z}(z)\right] + G(z)\varphi(z) = 0, \tag{18}$$

where z is complex and the functions G and K are assumed analytic in some region Ω such that K does not vanish in that region. Equation (18) can be written in Ω as a second-order system on the unknown functions $\varphi_1(z) = \varphi(z)$ and $\varphi_2(z) = K(z)\,\frac{\mathrm{d}\varphi}{\mathrm{d}z}(z)$, and the Green–Hille transformation consists on multiplying the equation on φ_1 by $\overline{\varphi_2(z)}$, that on φ_2 by $\overline{\varphi_1(z)}$, and integrating on z along a path in Ω, which yields

$$\left[\overline{\varphi_1(z)}\,\varphi_2(z)\right]_{z_1}^{z_2} - \int_{z_1}^{z_2} |\varphi_2(z)|^2\,\frac{\overline{\mathrm{d}z}}{\overline{K(z)}} + \int_{z_1}^{z_2} |\varphi_1(z)|^2 G(z)\,\mathrm{d}z = 0, \tag{19}$$

where $z_1, z_2 \in \Omega$ and both integrals are taken along the same arbitrary smooth path in Ω connecting z_1 to z_2.

The following result, which is proved in [17] using the Green–Hille transformation from [40], gives insights on the distribution of the non asymptotic zeros of Kummer hypergeometric functions with real arguments a and b.

Proposition 2.1 ([17]). *Let $a,\ b \in \mathbb{R}$ be such that $b \geq 2$.*

1. *If $b = 2a$, then all nontrivial roots z of $\Phi(a, b, \cdot)$ are purely imaginary.*
2. *If $b > 2a$ (resp., $b < 2a$), then all nontrivial roots z of $\Phi(a, b, \cdot)$ satisfy $\Re(z) > 0$ (resp., $\Re(z) < 0$).*
3. *If $b \neq 2a$, then all nontrivial roots z of $\Phi(a, b, \cdot)$ satisfy*

$$(b - 2a)^2 \Im(z)^2 - (4a(b - a) - 2b)\,\Re(z)^2 > 0.$$

Remark 1. It should be noted that a quasipolynomial admitting a characteristic root with intermediate multiplicity necessarily shares its remaining roots with an appropriate linear combination of Kummer functions. Unfortunately, to the best of the authors' knowledge, there does not exist any general result in the open literature to describe the *distribution* of the *non-asymptotic zeros* of such a function combination.

2.5 Delay systems frequency bound in the right half-plane

Despite the unquestionable interest and insights of the Hille oscillation theorem [40] and the Green–Hille transform in discarding regions in the complex plane from containing zeros of a meromorphic function which is a solution of a given second-order differential equation, such an approach lacks effectiveness from the numerical point of view.

In this preliminary subsection, we provide a numerical-oriented alternative for the Green–Hille transform leading to an effective algorithm that we are explicitly using in our framework. In this sense, we first establish conditions on the system's parameters which guarantee the existence of a multiple root. Second, perform an affine change of variable in the characteristic equation $\Delta(\lambda) = P_0(\lambda) + P_\tau(\lambda)\,\mathrm{e}^{-\lambda\,\tau}$ in order to reduce the corresponding quasipolynomial to a normalized form: $\tilde{\Delta}(z) = \tilde{P}_0(z) + \tilde{P}_\tau(z)\,\mathrm{e}^{-z}$. Next, we derive a bound on the imaginary part of roots of the normalized characteristic function in the complex right half-plane. Lastly, a certification of the dominance of the multiple root is demonstrated. In what follows, Algorithm 1 lists the steps to be followed to reach a suitable frequency bound (see [12, 48]).

Algorithm 1 Estimation of the MID frequency bound in delay-differential equations (DDEs) with single delay

Require: $\tilde{\Delta}(z) = \tilde{P}_0(z) + \tilde{P}_\tau(z)\,\mathrm{e}^{-z}$ // Normalized quasipolynomial
  // Initialization
  ord $= 0$
  // ord: order of truncation of the Taylor expansion of $\mathrm{e}^{2\,x} = \underbrace{\underbrace{1}_{\text{ord}=0} + 2\,x}_{\text{ord}=1} + 2\,x^2 + \frac{4\,x^3}{3} + \cdots$
  dominance $=$ false
  $\exists\, z_0 = x + i\omega \in \mathbb{R}_+^* + i\mathbb{R}_+^*$ s.t. $\tilde{\Delta}(z_0) = 0$
  $|\tilde{P}_0(x+i\omega)|^2\,\mathrm{e}^{2\,x} = |\tilde{P}_\tau(x+i\omega)|^2$
  while $\sim$ dominance **do**
    ord = ord+1
    $F(x,\omega) = |\tilde{P}_\tau(x+i\omega)|^2 - |\tilde{P}_0(x+i\omega)|^2\,\mathrm{T}_{ord}(\mathrm{e}^{2\,x}) > 0$
    // $\mathrm{T}_{ord}(\mathrm{e}^{2\,x})$: Taylor expansion of $\mathrm{e}^{2\,x}$ of order $= ord$
    $\omega^2 = \Omega$
    $H(x,\Omega)$ // The polynomial characterizing the real roots of F
  end while
  $\Omega_k(x)$ // k^{th} real root of H, depend on free parameters
  if $\max_x(\max_{\mathrm{k}}(\Omega_k(x))) < \pi^2$ **then**
    dominance $=$ true
  end if
  return Frequency bound

2.6 Multiplicity-induced-dominancy property in DDEs and related topics

As briefly explained in the Introduction, a characteristic root λ_0 (of Δ) satisfies the *MID property* if

(i) its *algebraic multiplicity* (denoted $M = M(\lambda_0)$) is *larger than one*,

(ii) it is *dominant* in the sense that the remaining characteristic roots λ of the spectrum satisfy the condition $\Re(\lambda) \leq \Re(\lambda_0)$.

Since the maximal allowable multiplicity is defined by the degree of the quasipolynomial Δ (see, e.g., [47]), it is clear that the multiplicity M satisfies the inequalities $2 \leq M \leq \deg(\Delta)$. The case $M = \deg(\Delta)$ is called *generic multiplicity*, and the MID property in this case is called *generic MID* (GMID), while any multiplicity larger than one and smaller than $\deg(\Delta)$ denotes an *intermediate multiplicity*, the MID property being called *intermediate MID* (IMID) in this case. Regarding these intermediate multiplicities, when compared to $\deg(P_0) = n$, there are three sub-cases:

(i.1) *sub-order* multiplicity, if $1 < M < n = \deg(P_0)$,

(i.2) $M = n$,

(i.3) *over-order* multiplicity, if $1 + \deg(P_0) = 1 + n \leq M \leq \deg(\Delta)$.

The names of the notions of sub-order and over-order multiplicities can be easily understood after recalling that $\deg(P_0) = n$ is the order of the DDE (1). The "limit" case when the multiplicity is equal to the order n simply represents an *intermediate multiplicity*.

With these definitions and notations, the smallest over-order multiplicity is given by one added to the degree of the polynomial P_0, and the largest is given by the degree of the characteristic function Δ and it corresponds to the so-called generic multiplicity. Property (ii) above states that λ_0 should be the rightmost root of the spectrum and defines the *spectral abscissa* of the corresponding quasipolynomial Δ (see, for instance, [51] and the references therein).

For a better understanding of the notions above, consider the following simple example. In the case of second-order DDEs of retarded type including one delay with $\deg(P_0) = 2$, $\deg(P_1) = 1$, we have $\deg(\Delta) = 4$. Thus, in this case, the generic multiplicity is four, the intermediate multiplicities are two and three, and the only over-order multiplicities are three and four since the order of the DDE is two. In this configuration, the only sub-order multiplicity is one (simple roots), and this simply means that, in this configuration, there does not exist any sub-order multiple characteristic root.

With the notations and the definitions above, our main objective is to address the *MID property* in the case of *over-order multiplicities*. It should be mentioned that the generic case, i.e., $M(\lambda_0) = \deg(\Delta)$, was already treated and explicit characterizations exist. More precisely, the retarded case $m = n - 1$ was characterized in [47] and a unified treatment of the retarded and neutral cases, where $n - 1 \leq m \leq n$, was provided in [15]. For proving the dominancy, both characterizations [15, 47] use the properties of Kummer and Whittaker hypergeometric functions. In the sequel, we thus focus on over-order intermediate multiplicities.

To the best of the authors' knowledge, in the open literature, the MID with over-order multiplicities has been addressed in two particular configurations, $M(\lambda_0) = n + 1$ and $M(\lambda_0) = n + m$. More precisely, in the "limit" case $M(\lambda_0) = n + 1$, sufficient conditions for their validity have been proposed in [8], where the authors exploited the particular spectrum location of the open-loop plant. Finally, by using a different argument inspired by the Green–Hille transformation, [18] treated the case $M(\lambda_0) = n + m$ corresponding to the other "limit" case. Finally, it should be mentioned that in the case of second-order DDEs of retarded type with a single delay, these two limits cases coincide and they were treated in [24].

2.7 Motivating example: Boundary control of the transport equation

To illustrate the use of the MID property for hyperbolic PDE control purposes, let us revisit the problem of high-volume, multistage continuous production flow through a re-entrant factory control [5, 29], which, in its linear version, amounts to the problem of exponential stabilization of the standard scalar conservation law

$$\partial_t \varphi(t, x) + \eta\, \partial_x \varphi(t, x) = 0,\ t \in [0,\, \infty),\ x \in (0,\, L), \tag{20}$$

that is, a linear transport equation where $\varphi(t, x)$ represents the density or the concentration at position $x \in (0, L)$, with $L > 0$, and in time $t \in [0, +\infty)$ of the corresponding physical quantity of interest. Under the assumption that the diffusion is neglected, the linear mapping $\varphi \mapsto \eta\varphi$, with $\eta > 0$ defines the flux function[10]. Finally, the quantity η represents the so-called advection speed or the velocity of propagation. In view of the boundary control of the dynamics of (20), [29] uses the standard PI controller

$$\varphi(t,\, 0) = k_p\, \varphi(t, L) + k_i \int_0^t \varphi(\nu, L) \mathrm{d}\nu, \tag{21}$$

where k_p and k_i are the feedback parameters representing proportional and integral control gains. Applying the Laplace transform in time to (20)–(21), one gets, after multiplication by λ, the characteristic function [11]

$$\Delta(\lambda) = \lambda - (k_i + k_p \lambda) \mathrm{e}^{-\frac{L}{\eta}\, \lambda}, \tag{22}$$

which is a quasipolynomial of neutral type with the delay given by the ratio between the length and the speed of propagation, L/η.

As discussed in the previous sections, the degree of the quasipolynomial Δ is equal to 3. Recall also that, thanks to the results from [15], the maximal multiplicity can be achieved only by a real root. Furthermore, if such a maximal multiplicity is reached, the GMID property holds and the corresponding triple characteristic root

[10] linear convection in this case

[11] Closed-loop characteristic function of a first-order DDE of neutral type ($m = n = 1$).

$\lambda_0 = -\frac{2\eta}{L}$ is *necessarily* the *spectral abscissa*. Despite the interest of this property from a purely analytic view point, a control implementation based on the GMID lacks of *robustness*, see for instance [49]. As a matter of fact, for the sake of robustness with respect to the model's parametric uncertainties, it appears that it will be more appropriate to relax constraints on the choice of the closed-loop spectral abscissa. This can be carried out using the IMID property by assigning a root with an over-order intermediate multiplicity (in this case, multiplicity two).

Proposition 2.2. *For any λ_0 satisfying $-\eta/L < \lambda_0 < 0$, the PI controller given by*

$$k_i = -\frac{\lambda_0^2 L\,\mathrm{e}^{\frac{L\lambda_0}{\eta}}}{\eta}, \quad k_p = \frac{\mathrm{e}^{\frac{L\lambda_0}{\eta}}\,(L\lambda_0 + \eta)}{\eta} \tag{23}$$

stabilizes the system (20)*. Furthermore, the intermediate MID property holds for the double root λ_0, which gives the exponential decay rate of the closed-loop system.*

Remark 2. Since $\deg(\Delta) = 3$ and the order of the delay-free ODE is one (first-order case), it is easy to observe that the double root λ_0 represents the only possible over-order multiple root and, in this case, the largest over-order and the smallest over-order multiplicities coincide. Furthermore, since we are in the first-order case, there does not exist any sub-order multiple root, and the characteristic roots of order one correspond to the simple roots of the characteristic function.

Before providing the proof of Proposition 2.2, let us consider the generic quasipolynomial $\Delta\colon \mathbb{C} \to \mathbb{C}$ defined by

$$\Delta(\lambda) = \lambda + \alpha_0 + \mathrm{e}^{-\lambda\tau}(\beta_1\lambda + \beta_0). \tag{24}$$

This corresponds to the first-order neutral systems studied in [11]. In this case, $\deg(\Delta) = 3$ and we are interested in the dominance of roots attaining the intermediate multiplicity 2 which, as explained above, represents an over-order multiplicity.

A number $\lambda_0 \in \mathbb{R}$ is a characteristic root of multiplicity at least 2 of Δ if and only if the coefficients β_1 and β_0 satisfy

$$\begin{aligned} \beta_1 &= -\mathrm{e}^{\lambda_0\tau}\left(1 + (\alpha_0 + \lambda_0)\tau\right), \\ \beta_0 &= \mathrm{e}^{\lambda_0\tau}\left((\alpha_0 + \lambda_0)\lambda_0\tau - \alpha_0\right). \end{aligned} \tag{25}$$

We first remark that, under conditions (25), (24) can be rewritten as

$$\Delta(\lambda) = \lambda + \alpha_0 - \mathrm{e}^{-(\lambda-\lambda_0)\tau}\left((\alpha_0 + \lambda_0)(\lambda - \lambda_0)\tau + \lambda + \alpha_0\right).$$

We perform the change of variables in $\mathbb{C}$ corresponding to the new variable $z = \tau(\lambda - \lambda_0)$. More precisely, consider $\widetilde{\Delta}(z) = \tau\Delta(\lambda_0 + \frac{z}{\tau})$, which satisfies

$$\widetilde{\Delta}(z) = z + \alpha - \mathrm{e}^{-z}\left(z(1 + \alpha) + \alpha\right),$$

where $\alpha = \tau(\alpha_0 + \lambda_0)$. Note that λ_0 is a dominant root of (24) if and only if 0 is a dominant root of $\widetilde{\Delta}$.

A straightforward computation shows that

$$\widetilde{\Delta}(z) = z^2 \int_0^1 [1 + \alpha - \alpha(1 - t)]\,\mathrm{e}^{-zt}\mathrm{d}t,$$

and thus, using the integral representation (7) of Kummer functions, we deduce that

$$\widetilde{\Delta}(z) = z^2\left[(1 + \alpha)\Phi(1, 2, -z) - \frac{\alpha}{2}\Phi(1, 3, -z)\right]. \tag{26}$$

Let $F\colon \mathbb{C} \to \mathbb{C}$ be the function defined by

$$F(z) := (1 + \alpha)\Phi(1, 2, z) - \frac{\alpha}{2}\Phi(1, 3, z).$$

Thanks to (26), the root at the origin, $z = 0$, is a dominant root of $\widetilde{\Delta}$ if and only if all roots of F have non-negative real part. To study the sign of the real part of the roots of F, we apply Hille's method as described in Section 2.2.

Notice first that, by Lemma 2, F satisfies the second-order equation (12) with α replaced by $1+\alpha$ and β replaced by $-\frac{\alpha}{2}$. We now perform a transformation similar to that used to obtain Whittaker functions from Kummer functions. More precisely, we multiply F by a function of z in such a way that the second-order ODE satisfied by the product has no first-order term. This can be achieved introducing the Whittaker-type function

$$W(z) := \frac{\mathrm{e}^{-z/2} z^{3/2}}{\sqrt{(1+\alpha)z - \alpha(\alpha+2)}} F(z),$$

which satisfies

$$\frac{\mathrm{d}^2 W}{\mathrm{d}z^2}(z) + G(z) W(z) = 0, \tag{27}$$

where G is given by

$$G(z) := -\frac{(\alpha+1)^2 z^4 - 2\alpha(\alpha+1)(\alpha+2) z^3 + \alpha^2(1+(\alpha+1)^2) z^2 + 2\alpha^3(\alpha+2) z + 3\alpha^2(\alpha+2)^2}{4z^2((\alpha+1)z - \alpha(\alpha+2))^2}. \tag{28}$$

Note that G can be rewritten as

$$\begin{aligned} G(z) = & -\frac{1}{4} - \frac{3}{4z^2} - \frac{3(\alpha+1)^2}{4((\alpha+1)z - \alpha(\alpha+2))^2} \\ & - \frac{\alpha^2+3\alpha+3}{2\alpha(\alpha+2)z} + \frac{(\alpha+1)(\alpha^2+3\alpha+3)}{2\alpha(\alpha+2)((\alpha+1)z - \alpha(\alpha+2))}. \end{aligned}$$

Applying Hille's method to (27), we obtain, by taking in (18) $z_1 = 0$ and z_2 equal to a root z_* of F, that

$$\int_0^{z_*} |W'(z)|^2 \overline{\mathrm{d}z} = \int_0^{z_*} |W(z)|^2 G(z) \mathrm{d}z.$$

We choose as integration path the line segment from 0 to z_*. Hence

$$\overline{z_*} \int_0^1 |W'(tz_*)|^2 \mathrm{d}t = z_* \int_0^1 |W(tz_*)|^2 G(tz_*) \mathrm{d}t.$$

Taking the real part, we get

$$x_* \int_0^1 |W'(tz_*)|^2 \mathrm{d}t = \int_0^1 |W(tz_*)|^2 \Re\left[z_* G(tz_*)\right] \mathrm{d}t,$$

where $x_* = \Re(z_*)$ and $y_* = \Im(z_*)$.

So, a sufficient condition for F to admit only roots with positive real part is that

$$\Re\left[-\frac{z}{4} - \frac{3}{4t^2 z} - \frac{3z(\alpha+1)^2}{4((\alpha+1)tz - \alpha(\alpha+2))^2} - \frac{\alpha^2+3\alpha+3}{2\alpha(\alpha+2)t} + \frac{z(\alpha+1)(\alpha^2+3\alpha+3)}{2\alpha(\alpha+2)((\alpha+1)tz - \alpha(\alpha+2))}\right] \geq 0, \tag{29}$$

for every $t \in (0,1)$ and $z \in \mathbb{C}_-$, where $\alpha = (a+\lambda_0)\tau$. Then λ_0 is a dominant root of Δ, i.e., $\Re(\lambda) \leq \lambda_0$ for every root λ of Δ.

As emphasized in the previous sections, Hille oscillation method provides only sufficient conditions. In the following proof, we exploit instead the algorithmic alternative provided in Section 2.5.

Proof of Proposition 2.2. Using a standard elimination procedure (thanks to the linear dependency of the quasipolynomial with respect to its parameters) allows to show that a real number λ_0 is a root of multiplicity 2 of Δ if and only if the PI gains k_p and k_i satisfy (23). Thus, the characteristic function can be written as

$$\tilde{\Delta}(z) = \tilde{P_0}(z) + \tilde{P_\tau}(z) \mathrm{e}^{-z}, \tag{30}$$

where $\tilde{P}_0(z) = z + \rho$, and $\tilde{P}_\tau(z) = (-\rho - 1)\, z - \rho$, with $\rho = L\lambda_0/\eta$. A necessary condition for the exponential stability is given by the stability of the corresponding delay-difference operator (see, e.g., [39]). Thus, ρ should verify $|\rho + 1| < 1$, that is $\rho \in (-2, 0)$. Next, $\tilde{\Delta}$ in (30) can be written as

$$\tilde{\Delta}(z) = z^2 \int_0^1 q_\rho(t)\, \mathrm{e}^{-tz}\, \mathrm{d}t \quad \text{where} \quad q_\rho(t) = \rho\, t + 1. \tag{31}$$

In our approach, q_ρ should keep a constant sign for $t \in (0, 1)$, which happens if an only if $\rho \geq -1$, and then we concentrate in the interval of interest $\rho \in (-1, 0)$. Next, one follows the steps of Algorithm 1. By setting $z_0 = x_0 + i\omega_0 \in \mathbb{R}_+ + i\,\mathbb{R}_+$ as a root of $\tilde{\Delta}(z)$, we have

$$|\tilde{P}_0(x_0 + i\omega_0)|^2 \mathrm{e}^{2\,x_0} = |\tilde{P}_\tau(x_0 + i\omega_0)|^2.$$

Since $\mathrm{e}^{2\,x} > 1 + 2\,x$ for any $x \in \mathbb{R}_+$, the real-valued function $F_\rho(x, \omega) = \left|\tilde{P}_\tau(x + i\omega)\right|^2 - |\tilde{P}_0(x + i\omega)|^2\ (1 + 2\,x)$ satisfies $F_\rho(x_0, \omega_0) > 0$. In fact, F_ρ is nothing but the following quadratic polynomial in (real) ω

$$F_\rho(x, \omega) = b_\rho(x)\, \omega^2 + c_\rho(x), \tag{32}$$

where $b_\rho = -2\,x + 2\,\rho + \rho^2, c_\rho = -2\,x^3 + \left(\rho^2 - 2\,\rho\right) x^2$. By setting $x^\pm = \frac{\rho^2}{2} \pm \rho$, one easily checks that F_ρ is positive for $x \in (x^+, x^-)$ if $\rho \in (-1, 0)$.

The next step consists in characterizing the frequency bound for potential unstable roots if $\rho \in (-1, 0)$. Since the discriminant of F_ρ defined in (32) is positive, then F_ρ admits two real roots,

$$\omega_\rho^\pm(x) := \mp \frac{\sqrt{-\left(\rho^2 + 2\,\rho - 2\,x\right)\left(\rho^2 - 2\,\rho - 2\,x\right)}\, x}{\rho^2 + 2\,\rho - 2\,x}$$

where ω_ρ^+ denotes the greatest solution. Since $\rho \in (-1, 0)$ and $x > 0$, ω_ρ^+ is upper bounded with respect to ρ by the parameter-free expression $\omega^+(x) = \frac{x\,\sqrt{-4\,x^2 + 3}}{1 + 2\,x}$, which reaches a maximum value at $x^* = \frac{\sqrt{3}}{2}$. Thus, $\omega = \omega_\rho^+(x) \leq \omega^+(x^*) \approx 0.5899 < \pi$. In other words, all unstable solutions z_u of $\tilde{\Delta}$ should satisfy the condition $0 < \Im(z_u) < \pi$.

Finally, by a contradiction argument, one assumes that such an unstable root z_u exists. Then, the integral representation yields $\int_0^1 (\rho\, t + 1)\, \mathrm{e}^{-t\, z_u}\, \mathrm{d}t = 0$, the imaginary part of which is $\int_0^1 t\ (\rho\, t + 1)\, \mathrm{e}^{-t\,x} \sin(\omega\, t)\mathrm{d}t = 0$. Now, the frequency bound $0 < \omega \leq \pi$ of the previous step entails that the function $t \mapsto t\ (\rho\, t + 1)\, \mathrm{e}^{-x\,t} \sin(\omega\, t)$ is strictly positive in $(0, 1)$, thereby contradicting the last equality. This ends the proof. □

As an illustration of Proposition 2.2, Fig. 1 represents a solution φ of (20) under the PI controller (21) with parameters k_i and k_p computed from (23) with the choice $\lambda_0 = -\frac{\eta}{2L}$. The numerical solution was computed using a standard finite difference upwind scheme to discretize (20) and using Simpson's rule to approximate the integral in (21).

2.8 Problem formulation

As emphasized in the Introduction, the aim of this paper is to further exploit the zeros distribution of linear combination of Kummer/Whittaker functions in the control of dynamical systems represented by DDEs by using the partial pole placement. As a matter of fact, our goal is to characterize quasipolynomials with multiple roots as a particular linear combinations of Kummer functions $\sum_{k=0}^{M-1} \zeta_k\, \Phi(a, b+k, -z)$. Next, for a prescribed λ_0, we aim to establish conditions discarding the right half-plane $\{\lambda \in \mathbb{C} \mid \Re(\lambda) > \lambda_0\}$ from containing zeros of such functions. This step allows to show the validity of the MID property with over-order intermediate multiplicities, i.e., to show the dominancy of the multiple root among the set of all the remaining zeros of the considered quasipolynomial.

Consider now the DDE (1) and its characteristic function

$$\Delta \colon \mathbb{C} \to \mathbb{C}, \quad \Delta(\lambda) := P_0(\lambda) + P_\tau(\lambda)\mathrm{e}^{-\lambda\tau},$$

where $\deg(P_0) = n$, $\deg(P_\tau) = m \leq n$. As indicated in [23], $\deg(\Delta) = n + m + 1$.

The problem addressed in this paper can be formulated as follows: *finding conditions on the parameters of the dynamical system (1) such that a (real) characteristic root λ_0 with* intermediate *algebraic multiplicity $n + 1 \leq M(\lambda_0) \leq n + m$ satisfies the MID property*, i.e., it corresponds to the rightmost characteristic root.

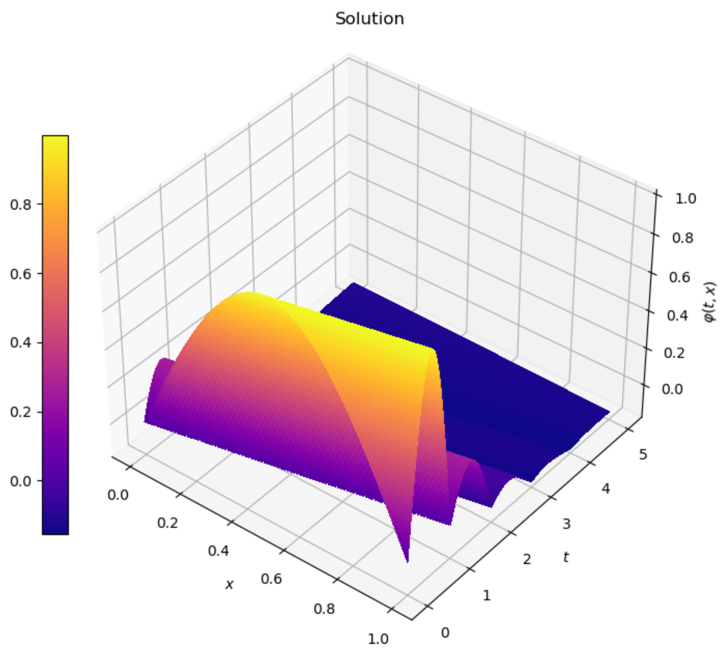

Figure 1: Numerical simulation of a solution of (20)–(21) satisfying (23) with parameters $L = 1$ and $\eta = 1$ and initial condition $\varphi(0, x) = \sin(\pi x)$.

3 Main results

With the definitions, notations and prerequisites above, we are now able to state our main results.

3.1 Necessary and sufficient conditions guaranteeing multiplicity $n + 1 \leq M(\lambda_0) \leq n + m + 1$

Thanks to the preliminary results of Section 2.3, we are now in position to prove the following result, providing a *necessary and sufficient condition* for a given real number λ_0 to be a root of a quasipolynomial Δ with multiplicity between $n + 1$ and $n + m + 1$.

Theorem 3.1. *Let n, m be nonnegative integers with $n \geq m$, $\tau > 0$, $\lambda_0 \in \mathbb{R}$, $M \in [n + 1, n + m + 1]$ be an integer, and consider the quasipolynomial Δ from* (3)–(4)*. The number λ_0 is a root of multiplicity at least M of Δ if and only if there exists a polynomial p of degree $n + m + 1 - M$ with $p(0) = 1$ such that*

$$\Delta(\lambda) = \frac{\tau^{M-n}(\lambda - \lambda_0)^M}{(M - n - 1)!} \int_0^1 t^{M-n-1}(1 - t)^{M-m-1} p(t) \mathrm{e}^{-t\tau(\lambda-\lambda_0)} \mathrm{d}t. \tag{33}$$

Proof. Let $\mathcal{V}$ be the set of all functions Δ of the form $\Delta(\lambda) = P_0(\lambda) + \mathrm{e}^{-\lambda\tau} P_\tau(\lambda)$ with P_0 and P_τ given by (4). Note that $\mathcal{V}$ is is a real affine space with $\dim \mathcal{V} = n + m + 1$, which is an affine subspace of the space of all entire complex functions, seen as a real vector space. In addition, $\mathcal{V}$ can be canonically identified with $\mathbb{R}^{n+m+1}$ by identifying a quasipolynomial Δ with its coefficients $\alpha_0, \dots, \alpha_{n-1}, \beta_0, \dots, \beta_m$.

Let us denote by $\mathcal{V}_{\lambda_0}^M$ the subset of $\mathcal{V}$ of those functions Δ admitting λ_0 as a root of multiplicity at least M, i.e.,

$$\mathcal{V}_{\lambda_0}^M := \left\{ \Delta \in \mathcal{V} \;\middle|\; \Delta^{(k)}(\lambda_0) = 0 \text{ for all } k \in \{0, \dots, M - 1\} \right\}.$$

Each equation $\Delta^{(k)}(\lambda_0) = 0$, $k \in \{0, \dots, M - 1\}$, defines a hyperplane in $\mathcal{V}$ and, when identifying $\mathcal{V}$ with the Euclidean space $\mathbb{R}^{n+m+1}$, the normal vectors to all such hyperplanes are linearly independent. Hence $\mathcal{V}_{\lambda_0}^M$ is a subspace of $\mathcal{V}$ of codimension M, i.e., $\dim \mathcal{V}_{\lambda_0} = n + m + 1 - M$.

Introduce now $\mathcal{W}_{\lambda_0}^M$ as the space of all functions Δ of the form (33) for some polynomial p of degree $n+m+1-M$ with $p(0)=1$. The set $\mathcal{W}_{\lambda_0}^M$ is an affine subspace of the space of all entire complex functions, seen once again as a real vector space, with $\dim \mathcal{W}_{\lambda_0}^M = n+m+1-M$.

As a first step, we will prove that $\mathcal{W}_{\lambda_0}^M \subset \mathcal{V}$, i.e., we show that every function Δ of the form (33) is indeed a quasipolynomial of the form (3)–(4). To do so, we first notice that, by an immediate inductive integration by parts, we have (see also [47, Proposition 2.1])

$$\int_0^1 q(t)\mathrm{e}^{-zt}\mathrm{d}t = \sum_{k=0}^{d} \frac{q^{(k)}(0) - q^{(k)}(1)\mathrm{e}^{-z}}{z^{k+1}} \tag{34}$$

for every $z \in \mathbb{C}\setminus\{0\}$, $d \in \mathbb{N}$, and q a polynomial of degree d. Now, let $\Delta \in \mathcal{W}_{\lambda_0}^M$ and p be a polynomial of degree $n+m+1-M$ with $p(0)=1$ be such that Δ is given by (33). Define $q(t) = t^{M-n-1}(1-t)^{M-m-1}p(t)$ and notice that $q(0) = q'(0) = \cdots = q^{(M-n-2)}(0) = 0$, $q^{(M-n-1)}(0) = (M-n-1)!$, and $q(1) = q'(1) = \cdots = q^{(M-m-2)}(1) = 0$. By using (34), we deduce that

$$\begin{aligned}\Delta(\lambda) &= \frac{\tau^{M-n}(\lambda-\lambda_0)^M}{(M-n-1)!}\sum_{k=0}^{M-1}\frac{q^{(k)}(0) - q^{(k)}(1)\mathrm{e}^{-\tau(\lambda-\lambda_0)}}{\tau^{k+1}(\lambda-\lambda_0)^{k+1}} \\ &= (\lambda-\lambda_0)^n + \sum_{k=0}^{n-1}\frac{q^{(M-k-1)}(0)}{\tau^{n-k}(M-n-1)!}(\lambda-\lambda_0)^k - \mathrm{e}^{-\tau(\lambda-\lambda_0)}\sum_{k=0}^{m}\frac{q^{(M-k-1)}(1)}{\tau^{n-k}(M-n-1)!}(\lambda-\lambda_0)^k,\end{aligned} \tag{35}$$

and thus, as required, $\Delta \in \mathcal{V}$.

We now notice that $\mathcal{W}_{\lambda_0}^M \subset \mathcal{V}_{\lambda_0}^M$, since, by construction, for any Δ given by (33), λ_0 is clearly a root of multiplicity at least M of Δ. Finally, since $\mathcal{W}_{\lambda_0}^M$ and $\mathcal{V}_{\lambda_0}^M$ are both affine spaces with the same dimension, we conclude that $\mathcal{W}_{\lambda_0}^M = \mathcal{V}_{\lambda_0}^M$, yielding the conclusion. $\square$

Remark 3. Note that (35) provides explicit expressions for the polynomials P_0 and P_τ from (4) in terms of the polynomial q introduced in the above proof. More precisely, we have

$$\begin{aligned}P_0(\lambda) &= (\lambda-\lambda_0)^n + \sum_{k=0}^{n-1}\frac{q^{(M-k-1)}(0)}{\tau^{n-k}(M-n-1)!}(\lambda-\lambda_0)^k, \\ P_\tau(\lambda) &= -\mathrm{e}^{\tau\lambda_0}\sum_{k=0}^{m}\frac{q^{(M-k-1)}(1)}{\tau^{n-k}(M-n-1)!}(\lambda-\lambda_0)^k.\end{aligned}$$

Recalling that $q(t) = t^{M-n-1}(1-t)^{M-m-1}p(t)$, one may further express P_0 and P_τ in terms of p. Indeed, we have

$$\begin{aligned}\frac{q^{(M-k-1)}(0)}{(M-n-1)!} &= \sum_{j=0}^{n-k}\binom{M-k-1}{n-k}\binom{n-k}{j}\frac{(-1)^j(M-m-1)!}{(M-m-j-1)!}p^{(n-k-j)}(0), \\ \frac{q^{(M-k-1)}(0)}{(M-n-1)!} &= \sum_{j=0}^{m-k}\binom{M-k-1}{m-k}\binom{m-k}{j}\frac{(-1)^{M-m-1}(M-m-1)!}{(M-n-j-1)!}p^{(m-k-j)}(1),\end{aligned}$$

and thus

$$P_0(\lambda) = (\lambda-\lambda_0)^n + \sum_{k=0}^{n-1}\sum_{j=0}^{n-k}\frac{(-1)^j}{\tau^{n-k}}\binom{M-k-1}{n-k}\binom{n-k}{j}\frac{(M-m-1)!}{(M-m-j-1)!}p^{(n-k-j)}(0)(\lambda-\lambda_0)^k, \tag{36}$$

$$P_\tau(\lambda) = -\mathrm{e}^{\tau\lambda_0}\sum_{k=0}^{m}\sum_{j=0}^{m-k}\frac{(-1)^{M-m-1}}{\tau^{n-k}}\binom{M-k-1}{m-k}\binom{m-k}{j}\frac{(M-m-1)!}{(M-n-j-1)!}p^{(m-k-j)}(1)(\lambda-\lambda_0)^k. \tag{37}$$

Proposition 3.1. *Let n, m be nonnegative integers with $n \geq m$, $\tau > 0$, $\lambda_0 \in \mathbb{R}$, $M \in [n+1, n+m+1]$ be an integer, and consider the quasipolynomial Δ from* (3)–(4)*. The number λ_0 is a root of multiplicity at least M of Δ if and only if there exist real numbers $\pi_0, \pi_1, \dots, \pi_{n+m+1-M}$ with $\sum_{k=0}^{n+m+1-M} \pi_k = 1$ such that*

$$\Delta(\lambda) = \tau^{M-n}(\lambda - \lambda_0)^M \sum_{k=0}^{n+m+1-M} \pi_k \frac{(M-m+k-1)!}{(2M-m-n+k-1)!} \Phi(M-n, 2M-m-n+k, -\tau(\lambda-\lambda_0)). \tag{38}$$

Proof. By Theorem 3.1, λ_0 is a root of multiplicity at least M of Δ if and only if there exists a polynomial p of degree $n+m+1-M$ with $p(0) = 1$ and such that (33) holds. Such a polynomial can be written in a unique manner as

$$p(t) = \sum_{k=0}^{n+m+1-M} \pi_k (1-t)^k \tag{39}$$

for some real numbers $\pi_0, \dots, \pi_{n+m+1-M}$, and the condition $p(0) = 1$ is satisfied if and only if $\sum_{k=0}^{n+m+1-M} \pi_k = 1$. Hence, (33) can be rewritten as

$$\Delta(\lambda) = \frac{\tau^{M-n}(\lambda-\lambda_0)^M}{(M-n-1)!} \sum_{k=0}^{n+m+1-M} \pi_k \int_0^1 t^{M-n-1}(1-t)^{k+M-m-1} \mathrm{e}^{-t\tau(\lambda-\lambda_0)} \mathrm{d}t.$$

The equivalence between this formula and (38) follows from (7), yielding the conclusion. □

Remark 4. It can be observed that the polynomial p involved in the kernel defining Δ and given in (33) can be alternatively written as

$$p(t) = \sum_{k=0}^{n+m+1-M} \sigma_k \, t^k \tag{40}$$

with

$$\sigma_k = (-1)^k \sum_{\ell=k}^{n+m+1-M} \binom{k}{\ell} \pi_\ell \quad \text{for} \quad k \in \{0, \dots, n+m+1-M\}. \tag{41}$$

Furthermore, using (36), one easily recovers the expression of α_k for $k \in \{0, \dots, n-1\}$ as a function of σ_ℓ for $\ell \in \{0, \dots, n+m+1-M\}$ or equivalently, by, using (39), as a function of π_ℓ for $\ell \in \{0, \dots, n+m+1-M\}$.

3.2 Elimination-produced function

In concrete control problems, typically, some of the coefficients of the polynomials P_0 and P_τ from (4) are fixed, corresponding to parameters coming from the physical modeling of the system, while other coefficients can be freely chosen, corresponding to parameters coming, for instance, from an implemented linear feedback controller (see [13] for some illustrations of such applications). In this section, we discuss how one can compute the "free" coefficients of the polynomials (4) in order to impose that a given real number λ_0 is a root of multiplicity at least M of the quasipolynomial Δ from (3).

We will assume in this section that n and m are nonnegative integers with $n \geq m$, $M \in [n+1, n+m+1]$ is an integer representing the desired multiplicity of the root λ_0, and that $M-1$ coefficients of the polynomials P_0 and P_τ from (4) can be freely chosen, the other $n+m+2-M$ coefficients being fixed. In this configuration, which appears in several practical applications (see, e.g., [62], as well as the examples from Section 4 below), the $M-1$ degrees of freedom in the coefficients of the system are typically not enough to ensure the existence of a root of multiplicity M by themselves, and an additional constraint on the root λ_0 and the delay τ must be imposed. This constraint can be formulated as a certain meromorphic function on (λ_0, τ), depending on the known coefficients of P_0 and P_τ, being equal to 0. This meromorphic function will be called *elimination-produced function*, and the main goal of this section is to describe it, first in a general setting, and then in the particular case where the known coefficients are the coefficients of the monomials of highest degree in P_0. For that purpose, we start with some appropriate notations and definitions.

Let n and m be nonnegative integers with $n \geq m$, $D = n + m + 1$, $M \in [n+1, n+m+1]$ be an integer, $I \subset \{1, \ldots, D\}$ be a set of cardinality $M - 1$, and $\overline{I} = \{1, \ldots, D\} \setminus I$.

We define the holomorphic functions $\ell \colon \mathbb{C}^2 \to \mathcal{M}_{1,D}(\mathbb{C})$, $\ell_I \colon \mathbb{C}^2 \to \mathcal{M}_{1,M-1}(\mathbb{C})$, $\ell_{\overline{I}} \colon \mathbb{C}^2 \to \mathcal{M}_{1,D-M+1}(\mathbb{C})$, $A \colon \mathbb{C}^2 \to \mathcal{M}_{M-1,D}(\mathbb{C})$, $A_I \colon \mathbb{C}^2 \to \mathcal{M}_{M-1,M-1}(\mathbb{C})$, $A_{\overline{I}} \colon \mathbb{C}^2 \to \mathcal{M}_{M-1,D-M+1}(\mathbb{C})$, and $b \colon \mathbb{C} \to \mathcal{M}_{M-1,1}(\mathbb{C})$, and the real number b_M as follows.

For $(\lambda_0, \tau) \in \mathbb{C}^2$, we set

$$\ell(\lambda_0, \tau) := \begin{pmatrix} \lambda_0^{n-1} & \cdots & \lambda_0 & 1 & \lambda_0^m \mathrm{e}^{-\lambda_0 \tau} & \cdots & \lambda_0 \mathrm{e}^{-\lambda_0 \tau} & \mathrm{e}^{-\lambda_0 \tau} \end{pmatrix}, \tag{42}$$

$\ell_I(\lambda_0, \tau)$ (respectively, $\ell_{\overline{I}}(\lambda_0, \tau)$) is the row matrix whose entries are the columns of $\frac{\partial^{M-1}}{\partial \lambda_0^{M-1}} \ell(\lambda_0, \tau)$ whose indices belong to I (respectively, to $\overline{I}$),

$$A(\lambda_0, \tau) := \begin{pmatrix} \ell(\lambda_0, \tau) \\ \frac{\partial \ell}{\partial \lambda_0}(\lambda_0, \tau) \\ \vdots \\ \frac{\partial^{M-2} \ell}{\partial \lambda_0^{M-2}}(\lambda_0, \tau) \end{pmatrix}, \tag{43}$$

$A_I(\lambda_0, \tau)$ (respectively, $A_{\overline{I}}(\lambda_0, \tau)$) is the matrix whose columns are the columns of $A(\lambda_0, \tau)$ whose indices belong to I (respectively, to $\overline{I}$), $b(\lambda_0)$ is the column matrix with $M - 1$ rows whose first entry is $-\lambda_0^n$ and such that its $k-$th entry $b_k(\lambda_0)$, for an integer $k \in [2, M-1]$, satisfies $b_k(\lambda_0) = \frac{d}{d\lambda_0} b_{k-1}(\lambda_0)$. We also set

$$b_M := \begin{cases} -n! & \text{if } M = n+1, \\ 0 & \text{otherwise.} \end{cases} \tag{44}$$

The set I introduced above represents the indices of the coefficients of the polynomials P_0 and P_τ from (4) that are assumed to be free, when arranged in the column vector

$$X := \begin{pmatrix} \alpha_{n-1} & \cdots & \alpha_1 & \alpha_0 & \beta_m & \cdots & \beta_1 & \beta_0 \end{pmatrix}^T, \tag{45}$$

and $\overline{I}$ represents the indices of the known coefficients of P_0 and P_τ in this same vector. Note that the order of the coefficients in the vector X corresponds to the order of the entries of the row matrix $\ell(\lambda_0, \tau)$, so that we have the equality $\Delta(\lambda_0) = \lambda_0^n + \ell(\lambda_0, \tau) X$.

Definition 3.1. With the notations above, if $\det A_I(\cdot, \cdot)$ is not the zero function, for $(\lambda_0, \tau) \in \mathbb{C} \times \mathbb{R}_+$ such that $\det A_I(\lambda_0, \tau) \neq 0$ and $\gamma \in \mathbb{R}^{D-M+1}$, we define $\mathbb{P}(\lambda_0, \tau; \gamma) \in \mathbb{C}$ by

$$\mathbb{P}(\lambda_0, \tau; \gamma) := \ell_{\overline{I}}(\lambda_0, \tau)\gamma + \ell_I(\lambda_0, \tau) A_I(\lambda_0, \tau)^{-1} (b(\lambda_0) - A_{\overline{I}}(\lambda_0, \tau)\gamma) - b_M. \tag{46}$$

The function $\mathbb{P}$ is called *elimination-produced function*.

The vector γ in Definition 3.1 represents the numerical values of the known coefficients of P_0 and P_τ from (4): it corresponds to taking, in the above vector X, only the entries corresponding to indices in $\overline{I}$.

Remark 5. Note that $\det A_I(\lambda_0, \tau)$ is a quasipolynomial in the variable λ_0, depending only on τ, n, m, M, and I. In particular, when $\det A_I$ is not identically zero, for every $\gamma \in \mathbb{R}^{D-M+1}$, the function $\mathbb{P}(\cdot, \cdot; \gamma)$ is meromorphic, and can be expressed as a fraction of quasipolynomials, its denominator being $\det A_I(\lambda_0, \tau)$.

Example 3.1. To illustrate Definition 3.1 and notations above, consider the case $n = 2$ and $m = 1$, i.e.,

$$\Delta(\lambda) = \lambda^2 + \alpha_1 \lambda + \alpha_0 + \mathrm{e}^{-\lambda \tau}(\beta_1 \lambda + \beta_0),$$

in which case $D = n + m + 1 = 4$. We are interested here in roots of multiplicity $M = 3$. We assume that α_1 and β_0 are known, and α_0 and β_1 are free. In this case, we have $I = \{2, 3\}$, which correspond to the indices of α_0 and β_1 in the vector $(\alpha_1, \alpha_0, \beta_1, \beta_0)$, and thus $\overline{I} = \{1, 4\}$. We compute

$$\ell(\lambda_0, \tau) = \begin{pmatrix} \lambda_0 & 1 & \lambda_0 \mathrm{e}^{-\lambda_0 \tau} & \mathrm{e}^{-\lambda_0 \tau} \end{pmatrix},$$

$$
\begin{aligned}
\ell_I(\lambda_0, \tau) &= \begin{pmatrix} 0 & (\tau^2\lambda_0 - 2\tau)\mathrm{e}^{-\lambda_0\tau} \end{pmatrix}, \\
\ell_{\overline{I}}(\lambda_0, \tau) &= \begin{pmatrix} 0 & \tau^2\mathrm{e}^{-\lambda_0\tau} \end{pmatrix}, \\
A(\lambda_0, \tau) &= \begin{pmatrix} \lambda_0 & 1 & \lambda_0\mathrm{e}^{-\lambda_0\tau} & \mathrm{e}^{-\lambda_0\tau} \\ 1 & 0 & (1-\lambda_0\tau)\mathrm{e}^{-\lambda_0\tau} & -\tau\mathrm{e}^{-\lambda_0\tau} \end{pmatrix}, \\
A_I(\lambda_0, \tau) &= \begin{pmatrix} 1 & \lambda_0\mathrm{e}^{-\lambda_0\tau} \\ 0 & (1-\lambda_0\tau)\mathrm{e}^{-\lambda_0\tau} \end{pmatrix}, \\
A_{\overline{I}}(\lambda_0, \tau) &= \begin{pmatrix} \lambda_0 & \mathrm{e}^{-\lambda_0\tau} \\ 1 & -\tau\mathrm{e}^{-\lambda_0\tau} \end{pmatrix}, \\
b(\lambda_0) &= \begin{pmatrix} -\lambda_0^2 \\ -2\lambda_0 \end{pmatrix}, \\
b_M &= -2.
\end{aligned}
$$

In particular, we have

$$\det A_I(\lambda_0, \tau) = (1 - \lambda_0\tau)\mathrm{e}^{-\lambda_0\tau},$$

which is not identically zero. Thus, letting $\gamma = (\alpha_1^*, \beta_0^*)$ be the vector containing the numerical values of the fixed parameters α_1 and β_0, the elimination-produced function $\mathbb{P}$ is given by

$$\mathbb{P}(\lambda_0, \tau; (\alpha_1^*, \beta_0^*)) = \frac{\alpha_1^*\lambda_0\tau^2 - 2\alpha_1^*\tau + \beta_0^*\tau^2\mathrm{e}^{-\lambda_0\tau} + 2\lambda_0^2\tau^2 - 2\lambda_0\tau - 2}{\lambda_0\tau - 1}.$$

The main interest of the notation we introduced above, and in particular of the elimination-produced function from Definition 3.1, is that it allows one to express conditions for a real number λ_0 to be a root of multiplicity M of Δ in terms of the free coefficients of P_0 and P_τ, as expressed in the following theorem.

Theorem 3.2. *Let n, m be non-negative integers with $n \geq m$, $\tau > 0$, $M \in [n+1, n+m+1]$ be an integer, and consider the quasipolynomial Δ from* (3)–(4)*.*

Let $I \subset \{1, \dots, D\}$ be a set of cardinality $M-1$ and $\overline{I} = \{1, \dots, D\} \setminus I$. Let X be the parameter vector from (45) *and denote by X_I and $X_{\overline{I}}$ the vectors obtained by keeping in X only the entries with indices in I and $\overline{I}$, respectively. Assume that the values of the parameters in $X_{\overline{I}}$ are known and denote by γ the vector with those values. Consider also the functions ℓ_I, $\ell_{\overline{I}}$, A_I, $A_{\overline{I}}$, b, $\mathbb{P}$, and the real number b_M defined in* (42)–(44)*.*

Let $\lambda_0 \in \mathbb{R}$ be such that $\det A_I(\lambda_0, \tau) \neq 0$. Then λ_0 is a root of multiplicity at least M of Δ if and only if

$$X_I = A_I(\lambda_0, \tau)^{-1}\left(b(\lambda_0) - A_{\overline{I}}(\lambda_0, \tau)\gamma\right) \qquad \textit{and} \qquad \mathbb{P}(\lambda_0, \tau; \gamma) = 0. \tag{47}$$

Proof. The number λ_0 is a root of multiplicity at least M of Δ if and only if $\Delta^{(k)}(\lambda_0) = 0$ for every $k \in \{0, \dots, M-1\}$, and, recalling that $\Delta(\lambda) = \lambda^n + \ell(\lambda, \tau)X$, these M equations can be rewritten as

$$\begin{cases} A(\lambda_0, \tau)X = b(\lambda_0), \\ \dfrac{\partial^{M-1}\ell}{\partial\lambda_0^{M-1}}\ell(\lambda_0, \tau)X = b_M, \end{cases}$$

where ℓ and A are defined in (42) and (43). Splitting X into X_I and $X_{\overline{I}}$ and noticing that $X_{\overline{I}} = \gamma$, the above system is equivalent to

$$\begin{cases} A_I(\lambda_0, \tau)X_I + A_{\overline{I}}(\lambda_0, \tau)\gamma = b(\lambda_0), \\ \ell_I(\lambda_0, \tau)X_I + \ell_{\overline{I}}(\lambda_0, \tau)\gamma = b_M, \end{cases}$$

and, using the assumption $\det A_I(\lambda_0, \tau) \neq 0$, we obtain that the above system is equivalent to (47). □

Remark 6. In the above framework, we have considered that the values of some of the coefficients of the quasipolynomial Δ from (3)–(4) are known. Our approach can also be generalized to the case where, instead of knowing values of the coefficients, we have other information on them, such as the fact that they satisfy some linear dependence relation, written under the form $CX = d$ for some matrix $C \in \mathcal{M}_{D-M+1,D}(\mathbb{C})$ of full rank and some $d \in \mathbb{R}^{D-M+1}$. Indeed, in this case, we decompose the space $\mathbb{R}^D$ into the direct sum $\operatorname{Ker} C \oplus (\operatorname{Ker} C)^\perp$, and we replace the decomposition into indices in I and $\overline{I}$ done in the construction above by projections into $\operatorname{Ker} C$ and $(\operatorname{Ker} C)^\perp$.

3.3 A control-oriented setting

In Section 3.2, I is an arbitrary subset of $\{1, \dots, D\}$ with $M-1$ elements, meaning that we can consider any choice of free parameters of the quasipolynomial Δ from (3)–(4). With the aim of obtaining more precise results in a particular situation, we now consider the case where $I = \{D-M+2, \dots, D\}$, and thus $\overline{I} = \{1, \dots, D-M+1\}$, meaning that the coefficients of highest degree of P_0 are all known, and the $M-1$ free parameters are the coefficients of P_τ and those of lowest degree of P_0 (except in the particular case $n = m$ and $M = n+1$, in which case the coefficient β_m of P_τ is also assumed to be known).

Our first result in this setting is the following characterization of the elimination-produced function.

Proposition 3.2. *Let n, m be nonnegative integers with $n \geq m$, $\tau > 0$, $M \in [n+1, n+m+1]$ be an integer satisfying $M \geq m+2$, and consider the quasipolynomial Δ from* (3)–(4). *Let $\mathbb{P}$ be the elimination-produced function in the case $I = \{D-M+2, \dots, D\}$. Then*

$$\mathbb{P}(\lambda_0, \tau; \gamma) = \sum_{k=0}^{m+1} \binom{m+1}{k} \tau^k P_{F,M}^{(M-1-k)}(\lambda_0), \tag{48}$$

where

$$P_{F,M}(\lambda) = \lambda^n + \sum_{k=1}^{D-M+1} \alpha_{n-k} \lambda^{n-k}. \tag{49}$$

Proof. Note that, with the notations of Section 3.2, we have

$$b(\lambda_0) - A_{\overline{I}}(\lambda_0, \tau)\gamma = \begin{pmatrix} -P_{F,M}(\lambda_0) \\ -P_{F,M}'(\lambda_0) \\ \vdots \\ -P_{F,M}^{(M-2)}(\lambda_0) \end{pmatrix} \tag{50}$$

and

$$\ell_{\overline{I}}(\lambda_0, \tau)\gamma - b_M = P_{F,M}^{(M-1)}(\lambda_0). \tag{51}$$

Let

$$\widetilde{\ell}(\lambda_0, \tau) := \begin{pmatrix} \lambda_0^{M-m-3} & \cdots & \lambda_0 & 1 & \lambda_0^m e^{-\lambda_0 \tau} & \cdots & \lambda_0 e^{-\lambda_0 \tau} & e^{-\lambda_0 \tau} \end{pmatrix}$$

and note that

$$A_I(\lambda_0, \tau) = \begin{pmatrix} \widetilde{\ell}(\lambda_0, \tau) \\ \frac{\partial \widetilde{\ell}}{\partial \lambda_0}(\lambda_0, \tau) \\ \vdots \\ \frac{\partial^{M-2} \widetilde{\ell}}{\partial \lambda_0^{M-2}}(\lambda_0, \tau) \end{pmatrix}$$

The matrix $A_I(\lambda_0, \tau)$ admits the block decomposition

$$A_I(\lambda_0, \tau) = \begin{pmatrix} A_{I,0}(\lambda_0) & A_{I,1}(\lambda_0, \tau) \\ 0 & A_{I,2}(\lambda_0, \tau) \end{pmatrix}, \tag{52}$$

where

$$A_{I,0}(\lambda_0) := \begin{pmatrix} \lambda_0^{M-m-3} & \cdots & \lambda_0 & 1 \\ (M-m-3)\lambda_0^{M-m-4} & \cdots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ (M-m-3)! & \cdots & 0 & 0 \end{pmatrix} \in \mathcal{M}_{M-m-2}(\mathbb{R}),$$

$$A_{I,1}(\lambda_0, \tau) := \begin{pmatrix} \widehat{\ell}(\lambda_0, \tau) \\ \frac{\partial \widehat{\ell}}{\partial \lambda_0}(\lambda_0, \tau) \\ \vdots \\ \frac{\partial^{M-m-3} \widehat{\ell}}{\partial \lambda_0^{M-m-3}}(\lambda_0, \tau) \end{pmatrix} \in \mathcal{M}_{M-m-2, m+1}(\mathbb{R}),$$

$$A_{I,2}(\lambda_0,\tau) := \begin{pmatrix} \frac{\partial^{M-m-2}\widehat{\ell}}{\partial\lambda_0^{M-m-2}}(\lambda_0,\tau) \\ \vdots \\ \frac{\partial^{M-2}\widehat{\ell}}{\partial\lambda_0^{M-2}}(\lambda_0,\tau) \end{pmatrix} \in \mathcal{M}_{m+1,m+1}(\mathbb{R}),$$

and

$$\widehat{\ell}(\lambda_0,\tau) := \begin{pmatrix} \lambda_0^m \mathrm{e}^{-\lambda_0\tau} & \cdots & \lambda_0 \mathrm{e}^{-\lambda_0\tau} & \mathrm{e}^{-\lambda_0\tau} \end{pmatrix} \in \mathcal{M}_{1,m+1}(\mathbb{R}),$$

Clearly, $A_{I,0}(\lambda_0)$ is invertible. Set $\rho(\lambda_0,\tau) := \mathrm{e}^{\lambda_0\tau}\frac{\partial^{M-m-2}\widehat{\ell}}{\partial\lambda_0^{M-m-2}}(\lambda_0,\tau)$ and note that

$$\rho(\lambda_0,\tau) = \begin{pmatrix} \mathfrak{q}_m(\lambda_0,\tau) & \cdots & \mathfrak{q}_1(\lambda_0,\tau) & \mathfrak{q}_0(\lambda_0,\tau) \end{pmatrix}, \tag{53}$$

where, for $j \in \{0,\ldots,m\}$, the function $\lambda_0 \mapsto \mathfrak{q}_j(\lambda_0,\tau)$ is a polynomial of degree j, with coefficients depending on τ, and its term of highest degree is $(-\tau)^{M-m-2}\lambda_0^j$. Thus

$$\begin{pmatrix} \rho(\lambda_0,\tau) \\ \frac{\partial\rho}{\partial\lambda_0}(\lambda_0,\tau) \\ \vdots \\ \frac{\partial^m\rho}{\partial\lambda_0^m}(\lambda_0,\tau) \end{pmatrix} = \begin{pmatrix} \mathfrak{q}_m(\lambda_0,\tau) & \cdots & \mathfrak{q}_2(\lambda_0,\tau) & \mathfrak{q}_1(\lambda_0,\tau) & (-\tau)^{M-m-2} \\ \frac{\partial\mathfrak{q}_m}{\partial\lambda_0}(\lambda_0,\tau) & \cdots & \frac{\partial\mathfrak{q}_2}{\partial\lambda_0}(\lambda_0,\tau) & (-\tau)^{M-m-2} & 0 \\ \frac{\partial^2\mathfrak{q}_m}{\partial\lambda_0^2}(\lambda_0,\tau) & \cdots & 2(-\tau)^{M-m-2} & 0 & 0 \\ \vdots & \iddots & \vdots & \vdots & \vdots \\ m!(-\tau)^{M-m-2} & \cdots & 0 & 0 & 0 \end{pmatrix},$$

and, in particular, since $\tau > 0$, the above matrix is invertible. On the other hand, taking derivatives directly in the definition of ρ, we have

$$\begin{pmatrix} \rho(\lambda_0,\tau) \\ \frac{\partial\rho}{\partial\lambda_0}(\lambda_0,\tau) \\ \vdots \\ \frac{\partial^m\rho}{\partial\lambda_0^m}(\lambda_0,\tau) \end{pmatrix} = e^{\lambda_0\tau} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ \tau & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \binom{k}{k}\tau^k & \binom{k}{k-1}\tau^{k-1} & \ddots & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \binom{m}{m}\tau^m & \binom{m}{m-1}\tau^{m-1} & \cdots & \binom{m}{2}\tau^2 & m\tau & 1 \end{pmatrix} A_{I,2}(\rho,\tau).$$

Hence, $A_{I,2}(\rho,\tau)$ and, thanks to the block decomposition (52), $A_I(\lambda_0,\tau)$ is invertible.

Thanks to (53), we have $\frac{\partial^{m+1}\rho}{\partial\lambda_0^{m+1}}(\lambda_0,\tau) = 0$ and, using the definition of ρ, this implies that

$$\sum_{k=0}^{m+1} \binom{m+1}{k} \tau^k \frac{\partial^{M-k-1}\widehat{\ell}}{\partial\lambda_0^{M-k-1}}(\lambda_0,\tau) = 0,$$

which can be rewritten in matrix form as

$$\begin{pmatrix} \binom{m+1}{m+1}\tau^{m+1} & \binom{m+1}{m}\tau^m & \cdots & \binom{m+1}{1}\tau \end{pmatrix} A_{I,2}(\lambda_0,\tau) = -\frac{\partial^{M-1}\widehat{\ell}}{\partial\lambda_0^{M-1}}(\lambda_0,\tau).$$

Hence

$$\begin{pmatrix} 0 & \cdots & 0 & \binom{m+1}{m+1}\tau^{m+1} & \binom{m+1}{m}\tau^m & \cdots & \binom{m+1}{1}\tau \end{pmatrix} A_I(\lambda_0,\tau) = -\begin{pmatrix} 0 & \frac{\partial^{M-1}\widehat{\ell}}{\partial\lambda_0^{M-1}}(\lambda_0,\tau) \end{pmatrix} = -\ell_I(\lambda_0,\tau),$$

so that

$$\ell_I(\lambda_0,\tau)A_I(\lambda_0,\tau)^{-1} = -\begin{pmatrix} 0 & \cdots & 0 & \binom{m+1}{m+1}\tau^{m+1} & \cdots & \binom{m+1}{1}\tau \end{pmatrix}. \tag{54}$$

Inserting (50), (51), and (54) into (46), we finally deduce (48). □

Remark 7. Note that the elimination-produced function from (48) is a polynomial in (λ_0, τ). We refer to this function as the *elimination-produced polynomial*. Some remarks in the "limit" cases $M = n + m$ (largest over-order multiplicity) and $M = n + 1$ (smallest over-order multiplicity) can be found in [18] and [14], respectively.

Remark 8. Proposition 3.2 requires the additional assumption that $M \geq m + 2$. This is always satisfied if $n > m$, since $M \geq n + 1$, and hence the only case not covered by Proposition 3.2 is the case of neutral systems with the smallest over-order multiplicity, $M = n + 1$. In this case, the conclusion of Remark 7 does not hold. Indeed, consider the case $n = m = 1$, i.e.,

$$\Delta(\lambda) = \lambda + \alpha_0 + e^{-\lambda\tau}(\beta_1\lambda + \beta_0),$$

in which case $D = 3$. The lowest over-order multiplicity is $M = 2$ and, assuming α_0 and β_1 to be known and β_0 to be free (i.e., $I = \{3\}$ and $\overline{I} = \{1, 2\}$), straightforward computations from its definition show that the elimination-produced function is

$$\mathbb{P}(\lambda_0, \tau; \gamma) = 1 + \tau\lambda_0 + \tau\alpha_0 + \beta_1 e^{-\lambda_0\tau},$$

which is not a polynomial in (λ_0, τ).

An interesting fact about the elimination-produced polynomial from Proposition 3.2 is that the elimination-produced polynomial for a root of multiplicity $M + 1$ can be obtained by derivating the elimination-produced polynomial for a root of multiplicity M, as detailed in the next result.

Proposition 3.3. *Let n, m be nonnegative integers with $n \geq m$, $\tau > 0$, $M \in [n + 1, n + m]$ be an integer satisfying $M \geq m + 2$, and consider the quasipolynomial Δ from* (3)–(4)*. Let $\mathbb{P}_M$ (respectively, $\mathbb{P}_{M+1}$) be the elimination-produced polynomial from Proposition 3.2 for a root of multiplicity at least M (respectively, at least $M + 1$). Then*

$$\mathbb{P}_{M+1}(\lambda_0, \tau; \gamma) = \frac{\partial \mathbb{P}_M}{\partial \lambda_0}(\lambda_0, \tau; \gamma).$$

Proof. Let $P_{F,M}$ and $P_{F,M+1}$ be defined as in (49). Then

$$P_{F,M}(\lambda) = P_{F,M+1}(\lambda) + \alpha_{M-m-2}\lambda^{M-m-2}.$$

Hence, for every integer $k \in [0, m + 1]$, we have $M - m - 2 < M - k$, and thus

$$P_{F,M}^{(M-k)}(\lambda) = P_{F,M+1}^{(M-k)}(\lambda).$$

It then follows from (48) that

$$\begin{aligned}\mathbb{P}_{M+1}(\lambda_0, \tau; \gamma) &= \sum_{k=0}^{m+1} \binom{m+1}{k} \tau^k P_{F,M+1}^{(M-k)}(\lambda_0) \\ &= \sum_{k=0}^{m+1} \binom{m+1}{k} \tau^k P_{F,M}^{(M-k)}(\lambda_0) = \frac{\partial \mathbb{P}_M}{\partial \lambda_0}(\lambda_0, \tau; \gamma).\end{aligned}$$

□

We conclude this section by the following result, which provides an explicit expression of the elimination-produced polynomial $\mathbb{P}$ from (48) in terms of the known coefficients $\alpha_{M-m-2}, \dots, \alpha_{n-1}$ of Δ.

Proposition 3.4. *Let n, m be nonnegative integers with $n \geq m$, $\tau > 0$, $M \in [n + 1, n + m + 1]$ be an integer satisfying $M \geq m + 2$, and consider the quasipolynomial Δ from* (3)–(4)*. Let $\mathbb{P}$ be the elimination-produced function from Definition 3.1 in the case $I = \{D - M + 2, \dots, D\}$. Then*

$$\mathbb{P}(\lambda_0, \tau; \gamma) = \sum_{j=0}^{D-M+1} \sum_{\ell=j+M-m-2}^{n} \binom{m+1}{M-\ell-1+j} \frac{\ell!}{j!} \tau^{M-\ell-1+j} \alpha_\ell \lambda_0^j, \tag{55}$$

with the convention $\alpha_n = 1$.

The proof of the above proposition follows by inserting (49) into (48) and standard changes of summation variables in order to group the terms in powers of λ_0.

3.4 Dominance of multiple roots and its consequences for stability

Note that (38) factorizes Δ in terms of a linear combination of $D-M+1$ Kummer functions with real coefficients. However, in the following proposition, an equivalent combination of two Kummer functions with coefficients which are rational functions in the terms of the complex variable λ will be considered.

Proposition 3.5. *Consider the quasipolynomial Δ from* (3)–(4) *with $m \leq n$. The real number λ_0 is a root of multiplicity at least $n+1 \leq M \leq n+m+1$ of Δ if, and only if, $\mathbb{P}(\lambda_0, \tau) = 0$ and*

$$\Delta(\lambda) = \frac{\tau^{M-n}(\lambda-\lambda_0)^M}{(M-n-1)!}\left(\beta(\lambda)\,\Phi(0,\,1,\,-\tau\,(\lambda-\lambda_0)) + \gamma(\lambda)\,\Phi(1,\,1,\,-\tau\,(\lambda-\lambda_0))\right), \tag{56}$$

where

$$\beta(\lambda) := \frac{(M-n-1)!\,P_0(\lambda)}{\tau^{M-n}(\lambda-\lambda_0)^M}, \tag{57}$$

$$\gamma(\lambda) := \frac{(M-n-1)!\,P_\tau(\lambda)}{\tau^{M-n}(\lambda-\lambda_0)^M}, \tag{58}$$

and P_0 and P_τ are given in (36)–(37).

Since for every $a \in \mathbb{C}$ with $-a \notin \mathbb{N}$, we have $\Phi(a,a,z) = \Phi(1,1,z) = \mathrm{e}^z$ and $\Phi(0,1,z) = 1$, thus an immediate representation of the quasipolynomial Δ is given by (56).

Beyond the standard contiguous relation (see for instance [57]), to the best of the authors' knowledge, there does not exist any result describing the distribution of the non-asymptotic zeros of linear combinations of Kummer functions. The next lemma provides a partial step towards that goal, by providing a non-autonomous second-order differential equation admitting a given linear combination of Kummer functions as a solution.

Lemma 4. *Let $\widetilde{\beta}$ and $\widetilde{\gamma}$ be two meromorphic functions. Then, the complex function F defined by*

$$F(z) := \widetilde{\beta}(z)\,\Phi(0,1,z) + \widetilde{\gamma}(z)\,\Phi(1,1,z), \tag{59}$$

with $\widetilde{\beta}\,(z)\,\widetilde{\gamma}'\,(z) + \widetilde{\gamma}\,(z)\left(\widetilde{\beta}\,(z)\,\tau - \widetilde{\beta}'\,(z)\right) \neq 0$ satisfies the following second-order differential equation

$$F''(z) + Q(z)F'(z) + R(z)F(z) = 0, \tag{60}$$

where Q and R are given by

$$Q(z) := \frac{\left(\widetilde{\beta}''\,(z) + \tau\left(\widetilde{\beta}\,(z)\,\tau - 2\,\widetilde{\beta}'\,(z)\right)\right)\widetilde{\gamma}\,(z) - \left(\widetilde{\gamma}''\,(z)\right)\widetilde{\beta}\,(z)}{\widetilde{\beta}\,(z)\,\widetilde{\gamma}'\,(z) + \widetilde{\gamma}\,(z)\left(\widetilde{\beta}\,(z)\,\tau - \widetilde{\beta}'\,(z)\right)}, \tag{61}$$

$$R(z) := \frac{\left(-\widetilde{\beta}\,(z)\,\tau + \widetilde{\beta}'\,(z)\right)\widetilde{\gamma}''\,(z) - \widetilde{\gamma}'\,(z)\left(\widetilde{\beta}''\,(z) + \tau\left(\widetilde{\beta}\,(z)\,\tau - 2\,\widetilde{\beta}'\,(z)\right)\right)}{\widetilde{\beta}\,(z)\,\widetilde{\gamma}'\,(z) + \widetilde{\gamma}\,(z)\left(\widetilde{\beta}\,(z)\,\tau - \widetilde{\beta}'\,(z)\right)}. \tag{62}$$

In what follows, we shall refer to functions F of the form (59) as *Kummer-type functions*. Similarly to what was done in Lemma 3, one can also define a Whittaker-type function W from the Kummer-type function F defined in (59) by the same formula (15), where $\mathcal{Q}$ is a primitive of $\frac{Q}{2}$ for Q given by (61), and W satisfies (16) with G defined from Q and R from (61)–(62) using the same expression (17) as in Lemma 3.

3.5 MID validity for over-order multiplicities

Now, we shall use the results of Section 2.3 relating quasipolynomials with roots of over-order multiplicity and Kummer and Whittaker functions in order to provide sufficient conditions under which the MID property is valid for characteristic roots of multiplicity at least $n+1 \leq M \leq m+n+1$ of Δ.

Theorem 3.3. *Consider the quasipolynomial* Δ *from* (3)–(4) *with* $m \leq n$*, and assume that* Δ *admits a real root* λ_0 *of multiplicity at least* $n+1 \leq M \leq m+n+1$*. Let* β *and* γ *be the meromorphic functions defined in* (57) *and* (58)*, respectively, and define the meromorphic functions* $\widetilde{\beta}$ *and* $\widetilde{\gamma}$ *by*

$$\widetilde{\beta}(z) := \beta\left(\lambda_0 - \frac{z}{\tau}\right), \quad \widetilde{\gamma}(z) := \gamma\left(\lambda_0 - \frac{z}{\tau}\right). \tag{63}$$

Let F*,* Q*,* R*, and* G *be defined by* (59)*,* (61)*,* (62)*, and* (17)*, respectively. Assume that, for every* $t \in (0,1)$ *and every root* z *of* F *in* $\mathbb{C}_-$*, we have* $\Re[zG(tz)] \geq 0$*. Then,* λ_0 *is a dominant root of* Δ*, i.e.,* λ_0 *satisfies the MID property.*

A result similar to Theorem 3.3 was already shown in [18, Theorem 10] for the case of roots of multiplicity $n+m$. The proof of the former can be obtained by an easy adaptation of that of the latter, and we detail this argument here for the sake of completeness.

Proof. We deduce from Proposition 3.5 and Lemma 4 that

$$\Delta(\lambda) = \frac{\tau^{M-n}(\lambda-\lambda_0)^M}{(M-n-1)!} F(-\tau(\lambda-\lambda_0)). \tag{64}$$

Since our objective is to investigate zeros of Δ which are different from λ_0, then we focus on its second factor F. In particular, the result is thereby proved if we show that all roots of the Kummer-type function F have nonnegative real part.

To do so, we consider the Whittaker-type function $W(\cdot)$ defined from F as in (15). Note that the differential equation (16) satisfied by W is of the form (18), with $K(z) = 1$. As a consequence, one can apply Hille's method to (16). By taking $z_1 = 0$ and z_2 equal to a root z_* of $F(\cdot)$ in (19), we obtain

$$\int_0^{z_*} |W'(z)|^2 \overline{\mathrm{d}z} = \int_0^{z_*} |W(z)|^2 G(z) \mathrm{d}z.$$

We choose as integration path the line segment from 0 to z_*. Hence

$$\overline{z_*} \int_0^1 |W'(tz_*)|^2 \mathrm{d}t = z_* \int_0^1 |W(tz_*)|^2 G(tz_*) \mathrm{d}t.$$

Taking the real part, we get

$$x_* \int_0^1 |W'(tz_*)|^2 \mathrm{d}t = \int_0^1 |W(tz_*)|^2 \Re\left[z_* G(tz_*)\right] \mathrm{d}t, \tag{65}$$

where $x_* = \Re(z_*)$ and $y_* = \Im(z_*)$.

Assume now, by contradiction, that $F(\cdot)$ admits a root with negative real part, and take z_* in (65) as equal to this root. The left-hand side of (65) is negative, however its right-hand side is nonnegative by assumption, yielding the desired contradiction. Hence, all roots of F have nonnegative real parts, entailing the conclusion thanks to (64). □

4 Comprehensive illustrative examples with insights on numerics

In this section, we provide some applications of the over-order intermediate MID (IMID)-based design in both retarded and neutral cases: the control of a transonic flow in a wind tunnel as well as the design of the standard PID controller for a prescribed stabilization of unstable delayed plants.

4.1 Control of a transonic flow in a wind tunnel

As a first application of the IMID-based design, let us revisit the regulation problem of the transonic flow in a wind tunnel. Transonic flows in a compressible fluid exhibit complex dynamics, making their analysis challenging since a full model of the flow is governed by a Navier–Stokes equation in a three-dimensional domain and boundary controls for temperature and pressure regulation. A simplified model can be found in [68], assuming that the flow is uniform across every cross section and the tunnel is a one-dimensional tube of varying cross-sectional area, yielding a coupled model of nonlinear partial differential equations in one space dimension.

In order to study the response of the Mach number of the flow to changes in the guide vane angle a further simplified model has been proposed in [6]. Propagation phenomena are modeled through a time delay, leading to the system of delay differential equations

$$\begin{cases} \kappa m'(t) + m(t) = k\vartheta(t-\tau_0), \\ \vartheta''(t) + 2\zeta\omega\vartheta'(t) + \omega^2\vartheta(t) = \omega^2 u(t), \end{cases} \tag{66}$$

where m, ϑ, and u represent, respectively, perturbations of the Mach number of the flow, the guide vane angle, and the input of the guide vane actuator with respect to steady-state values. The parameters κ and k depend on the steady-state operating point and are assumed to be constant as long as m, ϑ, and u remain small, and satisfy $\kappa > 0$ and $k < 0$. The parameters $\zeta \in (0,1)$ and $\omega > 0$ come from the design of the guide vane angle actuator and are thus independent of the operating point. The time-delay τ_0 is assumed to depend only on the temperature of the flow. In the absence of control ($u(t) = 0$), the open-loop system (66) is exponentially stable.

The design of exponentially stabilizing feedback laws for (66) improving its exponential decay rate has been considered, for instance, in [46], in which the author designs a predictor of the state over an interval of length equal to the time-delay, yielding a closed-loop system with finite spectrum. However, the practical implementation of this type of controllers suffers from robustness issues [33, 53], which motivates the research for control laws with reduced implementation complexity.

The goal of this section is to illustrate how the main result of this work can be used to obtain a feedback controller for (66) improving its open-loop characteristics, with a reduced complexity with respect to the controller proposed in [46]. We assume that the parameters $\zeta \in (0,1)$ and $\omega > 0$ are fixed. Here we exploit the control law proposed in [47],

$$u(t) = -\frac{\beta_0\,\kappa}{k\omega^2} m(t-\tau_1) - \frac{\beta_1\,\kappa}{k\omega^2} m'(t-\tau_1) - \frac{\beta_2\,\kappa}{k\omega^2} m''(t-\tau_1), \tag{67}$$

where $\tau_1 > 0$ should be greater than or equal to the time-delay corresponding to measuring m and its first two derivatives. Notice that the tuning of (67) made in [47] relies on the GMID property, making such a design sensitive with respect to parametric uncertainties, and it also assumed that one could choose the parameters ζ and ω.

Substituting the control law (67) into (66), one obtains that the closed-loop characteristic quasipolynomial $\tilde{\Delta}$ is given by

$$\Delta(\lambda) = \frac{\tilde{\Delta}(\lambda)}{\kappa} = \lambda^3 + \left(2\omega\zeta + \frac{1}{\kappa}\right)\lambda^2 + \left(\omega^2 + \frac{2\omega\zeta}{\kappa}\right)\lambda + \frac{\omega^2}{\kappa} + \left(\beta_2\lambda^2 + \beta_1\lambda + \beta_0\right)e^{-\lambda(\tau)}, \tag{68}$$

where $\tau = \tau_0 + \tau_1$ and the division by κ is performed in order to obtain a quasipolynomial under the form (3), for which the polynomial P_0 is monic.

As a consequence of the main results, one gets the following.

Theorem 4.1. *A given complex number λ_0 is a root of multiplicity 4 of the quasipolynomial Δ from (68) if, and only if, λ_0 is a root of the elimination-produced polynomial $\mathbb{P}$, where*

$$\begin{aligned} \mathbb{P}(\lambda) := {}& \kappa\,\lambda^3\tau^3 + \left(2\zeta\kappa\,\tau^3\omega + 9\kappa\,\tau^2 + \tau^3\right)\lambda^2 \\ & + \left(\kappa\,\omega^2\tau^3 + \left(12\zeta\kappa\,\tau^2 + 2\zeta\,\tau^3\right)\omega + 18\kappa\tau + 6\tau^2\right)\lambda \\ & + \left(3\kappa\,\tau^2 + \tau^3\right)\omega^2 + \left(12\zeta\kappa\tau + 6\zeta\,\tau^2\right)\omega + 6\kappa + 6\tau \end{aligned} \tag{69}$$

and the controller's gains satisfy

$$
\begin{aligned}
\frac{\beta_0}{\mathrm{e}^{\lambda_0\tau}} &= \left(\left(2\tau\,\zeta^2-\tau\right)\omega^2+14\omega\zeta+\frac{177}{2\tau}+\frac{7}{\kappa}+\frac{\tau}{2\kappa^2}\right)\lambda_0^2 \\
&+\left(\zeta\tau\,\omega^3+\left(\left(12+\frac{2\tau}{\kappa}\right)\zeta^2+7-\frac{\tau}{\kappa}\right)\omega^2+\left(\frac{\tau}{\kappa^2}+\frac{26}{\kappa}+\frac{162}{\tau}\right)\zeta\omega+\frac{246}{\tau^2}+\frac{81}{\kappa\tau}+\frac{3}{\kappa^2}\right)\lambda_0 \\
&+\left(3+\frac{\tau}{\kappa}\right)\zeta\,\omega^3+\left(\left(\frac{6}{\kappa}+\frac{12}{\tau}\right)\zeta^2+\frac{\tau}{2\kappa^2}+\frac{15}{\kappa}+\frac{87}{2\tau}\right)\omega^2 \\
&+\left(\frac{3}{\kappa^2}+\frac{99}{\kappa\tau}+\frac{180}{\tau^2}\right)\zeta\omega+\frac{87}{\tau^3}+\frac{90}{\kappa\,\tau^2}+\frac{3}{\kappa^2\tau}, \\
\frac{\beta_1}{\mathrm{e}^{\lambda_0\tau}} &= \left(2\omega\tau\zeta+21+\frac{\tau}{\kappa}\right)\lambda_0^2+\left(\left(36+\frac{4\tau}{\kappa}\right)\omega\zeta+2\omega^2\tau+\frac{18}{\kappa}+\frac{66}{\tau}\right)\lambda_0 \\
&+\left(\frac{22}{\kappa}+\frac{48}{\tau}\right)\omega\zeta+\left(11+\frac{3\tau}{\kappa}\right)\omega^2+\frac{24}{\kappa\tau}+\frac{24}{\tau^2}, \\
\frac{\beta_2}{\mathrm{e}^{\lambda_0\tau}} &= \frac{3\tau\lambda_0^2}{2}+\left(2\omega\tau\zeta+6+\frac{\tau}{\kappa}\right)\lambda_0+\left(4+\frac{\tau}{\kappa}\right)\omega\zeta+\frac{\omega^2\tau}{2}+\frac{2}{\kappa}+\frac{3}{\tau},
\end{aligned}
$$

where $\tau=\tau_0+\tau_1$. *Moreover, if the discriminant of* $\mathbb{P}$ *is positive and if, for every* $t\in(0,1)$ *and every root* z *of* F *in* $\mathbb{C}_-$, *we have* $\Re[zG(tz)]\geq 0$, *where* F *is defined by* (57), (58), (59), *and* (63) *and* G *is defined by* (17), (57), (58), (61), (62), *and* (63), *then* $\lambda_0<0$, λ_0 *is a strictly dominant root of* Δ, *and the trivial solution of* (66) *with the control law* (67) *is exponentially stable with exponential decay* $-\lambda_0$.

Proof. The result is a direct consequence of Proposition 3.5 and Theorem 3.3. The realness of λ_0 is guaranteed by the positivity of the discriminant of $\mathbb{P}$, which is a cubic polynomial in λ. The negativity of λ_0 is ensured by the fact that $\mathbb{P}$ is Hurwitz for any $(\tau,\,\kappa,\,\omega,\,\zeta)\in\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+$. □

Remark 9. As illustrated through the above example, despite the sufficient conditions established in Theorem 4.1, the use of Hille oscillation theorems [40] and the Green–Hille transform lacks the explicit character when one is dealing with a parametric study. As a matter of fact, one should further investigate the parametric semi-algebraic problem $\Re[zG(tz)]\geq 0$, which is of huge complexity when a large number of parameters is involved and left free. This deeper analysis has been done in some situations, including for a time-delay model for vibration control problems encountered in oilwell drilling [7] and more general systems in which the multiplicity of the assigned root differs from the maximal one by 1 [18]. In the sequel, we illustrate another possible approach based on an a priori bound using the algorithmic alternative described in Section 2.5, which gives rise to more explicit conditions.

4.2 PID design of unstable low-order plants

PID control is the most popular control technology and dominates industrial control systems [61] because it represents one of the simplest control laws with a small number of control parameters giving satisfactory behaviors for closed-loop systems for large classes of industrial processes. Despite the fact that the design of such a controller is nowadays well-mastered to meet the performance requirements in stabilization and tracking of finite-dimensional industrial processes, to the best of the authors' knowledge, there is no systematical method in tuning a PID controller for infinite-dimensional systems. Recently, in [45], the MID property has been proposed as a method in tuning such controllers. Nonetheless, the proposed solution in [45] lacks freedom on the prescribed stabilization of the delayed first-order case study. By this section, we first revisit the considered problem and results from [45], then we improve such a solution by allowing some additional freedom when assigning the closed-loop decay rate.

Consider the feedback system depicted in Figure 2, in which $K(\lambda)$ represents a finite-dimensional linear time-invariant (LTI) controller, $Q_\tau(\lambda)$ denotes the plant containing a constant but uncertain delay τ, with a transfer function given by

$$
Q_\tau(\lambda)=\frac{1}{\lambda-\kappa}\mathrm{e}^{-\tau\lambda},\qquad \tau\geq 0, \tag{71}
$$

where $\kappa\geq 0$. The controller K of interest is the standard PID controller, i.e., $K(\lambda)=K_{PID}(\lambda)$ where

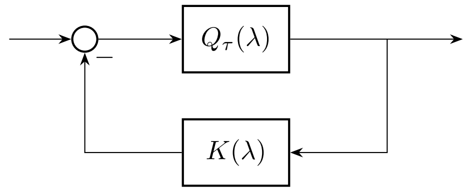

Figure 2: A tracking control system.

$$K_{PID}(\lambda) = k_p + \frac{k_i}{\lambda} + k_d\lambda. \tag{72}$$

Thus, the closed-loop characteristic quasipolynomial is found as

$$\begin{cases} \Delta(\lambda) = P_0(\lambda) + P_\tau(\lambda)\mathrm{e}^{-\lambda\tau} \\ P_0(\lambda) = \lambda^2 - \lambda\, p, \quad P_\tau(\lambda) = k_d\lambda^2 + k_p\,\lambda + k_i. \end{cases} \tag{73}$$

The main result of [45] is stated as follows.

Theorem 4.2 ([45]). *Let P_τ and K_{PID} be given respectively by* (71) *and* (72)*. Then the following statements are true.*

i) For arbitrary real parameters k_p, k_i, k_d and arbitrary positive delay τ, the multiplicity of any root of the quasipolynomial $\Delta(\lambda)$ is less than or equal to 5*.*

ii) The quasipolynomial Δ admits a multiple real root at

$$\lambda_\pm = \frac{\tau\,\kappa - 6 \pm \sqrt{\tau^2\kappa^2 + 12}}{2\tau} \tag{74}$$

with algebraic multiplicity at least 4 *if and only if*

$$\begin{cases} k_d = \dfrac{(4 + 2\,\tau\,\lambda_\pm - \tau\,\kappa)\,\mathrm{e}^{\tau\,s_\pm}}{2}, \\ k_p = -\dfrac{\left(\left(8\,\tau + \tau^2\lambda_\pm\right)\kappa - 18 - 12\,\tau\,\lambda_\pm\right)\mathrm{e}^{\tau\,\lambda_\pm}}{\tau}, \\ k_i = \dfrac{\left((\tau\lambda_\pm + 3)\,\tau^2\kappa^2 - (12\tau\lambda_\pm + 60)\,\tau\kappa + 108 + 84\,\tau\,\lambda_\pm\right)\mathrm{e}^{\tau\,\lambda_\pm}}{2\tau^2}. \end{cases} \tag{75}$$

iii) If (75) *is satisfied, then, under the condition $\tau < \bar{\tau}_{PID} = 2/\kappa$, $\lambda = \lambda_+$ is the rightmost root of the quasipolynomial $\Delta(\lambda)$ with multiplicity equal to* 4*.*

As mentioned earlier, under the assumption that the delay τ and the unstable pole κ are fixed, the assignable quadruple root (the spectral abscissa) λ_+ given in (74) is fully characterized, i.e., it does not allow any degree of freedom in the choice of the spectral abscissa. In order to enable some additional freedom when assigning it, one can relax such a constraint by forcing an over-order multiplicity which is lower than four, which, in our case, gives exactly the only option of multiplicity three.

Theorem 4.3. *Consider the quasipolynomial Δ given in* (73) *and $\lambda_0 \in \mathbb{R}$. Then the following statements are true.*

i) The quasipolynomial Δ admits a triple root at λ_0 if and only if the gains $(k_d,\, k_p,\, k_i)$ satisfy

$$\begin{cases} k_d = \dfrac{\left(\lambda_0\,(-\lambda_0 + \kappa)\,\tau^2 + (2\kappa - 4\lambda_0)\,\tau - 2\right)\mathrm{e}^{\tau\lambda_0}}{2}, \\ k_p = \left(\lambda_0^2\,(\kappa - \lambda_0)\,\tau^2 - \lambda_0\,(\kappa - 3\lambda_0)\,\tau + \kappa\right)\mathrm{e}^{\tau\lambda_0}, \\ k_i = \dfrac{\lambda_0^3\tau\,((\kappa - \lambda_0)\,\tau - 2)\,\mathrm{e}^{\tau\lambda_0}}{2}. \end{cases} \tag{76}$$

ii) Furthermore, if

$$\begin{cases} \kappa \in \left(0, \dfrac{1}{\tau}\right), \\ \lambda_0 \in \left[\dfrac{\kappa}{2} - \dfrac{2}{\tau} + \dfrac{\sqrt{\kappa^2\tau^2+8}}{2\tau}, 0\right), \end{cases} \tag{77}$$

then λ_0 corresponds to the spectral abscissa of (73).

Proof. An integration by part shows that, under (76), the quasipolynomial Δ reads

$$\Delta(\lambda) = \tau\ (\lambda - \lambda_0)^3 \int_0^1 \mathrm{e}^{-\tau(\lambda-\lambda_0)t} \left(\frac{1}{2}(\lambda_0 - \kappa)\lambda_0\tau^2t^2 + (2\lambda_0 - \kappa)\tau\, t + 1\right) \mathrm{d}t. \tag{78}$$

Thus, the corresponding kernel polynomial (39) is given by

$$p(t) = \frac{\lambda_0\left(\lambda_0 - \kappa\right)\tau^2t^2}{2} + \left(2\lambda_0 - \kappa\right)\tau t + 1, \tag{79}$$

which is necessarily real-rooted since its discriminant (w.r.t. the variable t) is given by $\delta = \left((\kappa - \lambda_0)^2 + \lambda_0^2\right)\tau^2 \geq 0$. Clearly, the leading coefficient of the polynomial p is positive. Furthermore, the corresponding zeros are given by

$$t_{\pm} = \frac{\kappa - 2\lambda_0 \pm \sqrt{(\kappa-\lambda_0)^2 + \lambda_0^2}}{\lambda_0\left(\lambda_0 - \kappa\right)\tau}. \tag{80}$$

Since, $\kappa \geq 0$ and the closed-loop multiple zero λ_0 is intended to correspond to the exponential decay of (78), then it needs to satisfy $\lambda_0 < 0$, hence the denominator of $t_{\pm}$ given in (80) is positive, which yields that $0 < t_- < t_+$. Hence, it suffices to set $t_- \geq 1$ to guarantee the sign constancy of the kernel polynomial p given in (79). This assumption is equivalent to

$$\sqrt{\kappa^2 - 2\lambda_0\kappa + 2\lambda_0^2} \leq -\lambda_0^2\tau + \left(\tau\kappa - 2\right)\lambda_0 + \kappa. \tag{81}$$

An immediate necessary condition for the last inequality to hold is to select λ_0 such that

$$\lambda_0 \in \left[\frac{\kappa}{2} - \frac{1}{\tau} - \frac{\sqrt{\kappa^2\tau^2+4}}{2\tau}, \frac{\kappa}{2} - \frac{1}{\tau} + \frac{\sqrt{\kappa^2\tau^2+4}}{2\tau}\right]. \tag{82}$$

Furthermore, by squaring both sides of inequality (81), one gets

$$\lambda_0\left(\kappa - \lambda_0\right)\left(-\lambda_0^2\tau^2 + \tau\left(\tau\kappa - 4\right)\lambda_0 + 2\tau\kappa - 2\right) \geq 0.$$

Again, since $\lambda_0 < 0$ and $\kappa \geq 0$, the third factor of the left-hand side of the above inequality has to be nonpositive, which enables us to choose λ_0 such that

$$\lambda_0 \in \mathbb{R}_-^* \setminus \left(\frac{\kappa}{2} - \frac{2}{\tau} - \frac{\sqrt{\kappa^2\tau^2+8}}{2\tau}, \frac{\kappa}{2} - \frac{2}{\tau} + \frac{\sqrt{\kappa^2\tau^2+8}}{2\tau}\right). \tag{83}$$

Taking into account the intervals (82) and (83), one concludes that the sign constancy (positive) of the kernel polynomial p given by (79) is guaranteed by

$$\lambda_0 \in \left[\frac{\kappa}{2} - \frac{2}{\tau} + \frac{\sqrt{\kappa^2\tau^2+8}}{2\tau}, \frac{\kappa}{2} - \frac{1}{\tau} + \frac{\sqrt{\kappa^2\tau^2+4}}{2\tau}\right].$$

Finally, since $\lambda_0 < 0$ and the corresponding lower-bound is strictly increasing with respect to κ and non-negative for $\kappa\,\tau \geq 1$ and the corresponding upper-bound is positive, then necessarily $\kappa\,\tau < 1$. As illustrated in Figure 3, the appropriate choice of λ_0 is such that

$$\lambda_0 \in \left[\frac{\kappa}{2} - \frac{2}{\tau} + \frac{\sqrt{\kappa^2\tau^2+8}}{2\tau}, 0\right). \tag{84}$$

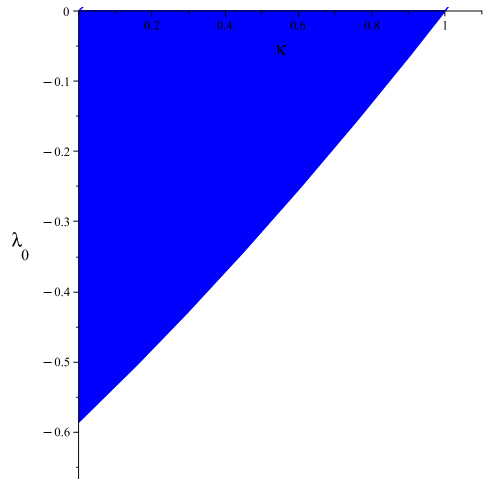

Figure 3: Region in the parameters $(\kappa,\ \lambda_0)$ plane described in (84) where the sign constancy of (79) is guaranteed. The delay τ is taken unitary.

Next, one needs first to write Δ, or, equivalently, the corresponding integral representation given in (78), into a normalized form. This is done using the standard affine change of variables [24, 47]

$$\lambda = \frac{s}{\tau} + \lambda_0 \tag{85}$$

and the new parametrization $\kappa\,\tau = \xi$ allowing to write (73) satisfying (76) as

$$\begin{cases} \hat{\Delta}(s) = \tau^2 \Delta(\lambda = \dfrac{s}{\tau} + \lambda_0) = \hat{P}_0(s) + \hat{P}_\tau(s)\mathrm{e}^{-s}, \\ \hat{P}_0(s) = s^2 + (2\lambda_0 - \xi)\, s + \lambda_0\,(-\xi + \lambda_0)\,, \\ P_\tau(s) = \left(-\dfrac{\lambda_0^2}{2} + \dfrac{(\xi-4)\,\lambda_0}{2} + \xi - 1\right) s^2 + \left(-\lambda_0^2 + \dfrac{(2\xi-4)\,\lambda_0}{2} + \xi\right) s + \lambda_0\,(\xi - \lambda_0)\,, \end{cases} \tag{86}$$

or, equivalently,

$$\hat{\Delta}(s) = \tau\, s^3 \int_0^1 \mathrm{e}^{-s\,t} \left(1 - \frac{\lambda_0\,(\xi-\lambda_0)\,t^2}{2} + \frac{(4\lambda_0 - 2\xi)\,t}{2}\right) \mathrm{d}t. \tag{87}$$

Obviously, showing that λ_0 is the spectral abscissa of (73) with control parameters satisfying (76) amounts to showing that 0 is the spectral abscissa of (86) or, equivalently, (87). Thanks to the algorithm from [12, 48] also recalled in Section 2.5 as Algorithm 1, one is now able to investigate the frequency bound for potential spectral values with positive real parts for (86).

To do so, let us assume that there exists $s_0 = x_0 + i\omega_0 \in \mathbb{C}_+$ such that $\hat{\Delta}(s_0) = 0$, then necessarily

$$|\hat{P}_0(x_0 + i\,\omega_0)|^2\, \mathrm{e}^{2\,x_0} = |\hat{P}_\tau(x_0 + i\,\omega_0)|^2. \tag{88}$$

Consider then the family of auxiliary functions

$$F_n(x,\ \omega) = |\hat{P}_\tau(x + i\,\omega)|^2 - \mathcal{T}_n\left(\mathrm{e}^{2\,x}\right)\, |\hat{P}_0(x + i\,\omega)|^2,$$

where $\mathcal{T}_n$ designates the n-th order Taylor approximation. Since $\mathrm{e}^{2\,x} > \mathcal{T}_n\left(\mathrm{e}^{2\,x}\right) \geq 1$ for any $x > 0$, we then have $F_n(x_0, \omega_0) > 0$. In particular, one has

$$F_0(x,\ \omega) = |\hat{P}_\tau(x + i\,\omega)|^2 - |\hat{P}_0(x + i\,\omega)|^2 > 0.$$

The idea then amounts to show that, if $s_0 = x_0 + i\,\omega_0$ is a root of $\hat{\Delta}$ such that $x_0 > 0$ and $\omega_0 \geq \pi$, then $F(x_0, \omega_0) \leq 0$, thus resulting a contradiction. Toward this end, we set $\omega = \sqrt{\Omega}$ and establish the explicit expression

$$\begin{aligned}F_0(x,\, \sqrt{\Omega}) = &\frac{(\lambda_0+2)\,(-\lambda_0+\xi-2)\;\mu(\lambda_0,\,\xi)}{4}\,\Omega^2 + \left(\frac{(\lambda_0+2)\,(-\lambda_0+\xi-2)\;\mu(\lambda_0,\,\xi)}{2}\,x^2 + \nu(\lambda_0,\,\xi)\,x\right)\,\Omega \\ &+ \frac{(\lambda_0+2)\,(-\lambda_0+\xi-2)\;\mu(\lambda_0,\,\xi)}{2}\,x^4 + \nu(\lambda_0,\,\xi)\,x^3 \\ &+ \left(2\,\lambda_0\,(\xi-\lambda_0)\left(-\lambda_0^2+(\xi-4)\,\lambda_0+2\xi\right)\right)\,x^2 + 2\lambda_0^2\,(\xi-\lambda_0)^2\,x\end{aligned}$$

with

$$\begin{aligned}\mu(\lambda_0,\,\xi) &:= \left(-\lambda_0^2+(\xi-4)\,\lambda_0+2\xi\right),\\ \nu(\lambda_0,\,\xi) &:= \left(\lambda_0^4+(-2\xi+6)\,\lambda_0^3+\left(\xi^2-9\xi+10\right)\lambda_0^2+\left(3\xi^2-10\xi\right)\lambda_0+2\xi^2\right).\end{aligned}$$

Seen as a quadratic polynomial in Ω, the function F_0 is positive only between its real roots (if they exist) since (84) yields that the leading coefficient of F_0 is negative. Let us investigate the existence of positive solutions for F_0.

Fix $x > 0$. The discriminant of F_0 in Ω is given by

$$\begin{aligned}D(x) = &-\Big(\lambda_0^8+(-4\xi+12)\,\lambda_0^7+\left(6\xi^2-42\xi+48\right)\lambda_0^6+\left(-4\xi^3+54\xi^2-144\xi+72\right)\lambda_0^5\\ &+\left(\xi^4-30\xi^3+155\xi^2-180\xi+28\right)\lambda_0^4+\left(6\xi^4-70\xi^3+152\xi^2-56\xi\right)\lambda_0^3\\ &+\left(11\xi^4-48\xi^3+20\xi^2\right)\lambda_0^2+\left(4\xi^4+8\xi^3\right)\lambda_0-4\xi^4\Big)x^2\\ &-2\,(\xi-\lambda_0)^2\,\lambda_0^2\,(-\lambda_0+\xi-2)\,(\lambda_0+2)\left(-\lambda_0^2+(\xi-4)\,\lambda_0+2\xi\right)x,\end{aligned}$$

which is nothing but a real-rooted second-order polynomial in x.

Notice that, if the discriminant D is negative for any positive x in some sub-region of $0 < \xi < 1$ and λ_0 satisfies (84), then F_0 cannot be positive. In that case, it is clear that the MID property holds.

However, if D is positive, then F_0 is positive only in the interval $(\Omega_-,\, \Omega_+)$, where

$$\begin{aligned}\Omega_\pm(x) := &-x^2 - \frac{2\left(\lambda_0^4+(-2\xi+6)\,\lambda_0^3+\left(\xi^2-9\xi+10\right)\lambda_0^2+\left(3\xi^2-10\xi\right)\lambda_0+2\xi^2\right)x}{(-\lambda_0+\xi-2)\,(\lambda_0+2)\,(-\lambda_0^2+(\xi-4)\,\lambda_0+2\xi)}\\ &\mp \frac{\sqrt{D(x)}}{(-\lambda_0+\xi-2)\,(\lambda_0+2)\,(-\lambda_0^2+(\xi-4)\,\lambda_0+2\xi)}.\end{aligned}$$

By using some tedious but simple algebraic estimates, one obtains that

$$\Omega_+(x) < \Gamma(x) = \left(2+\sqrt{2}\right)\,x - x^2. \tag{89}$$

Interestingly, Γ is a second-order polynomial in x, which is positive only in $x \in (0, 2+\sqrt{2})$ and reaches its maximum $\Gamma_{\max} = \frac{3}{2} + \sqrt{2}$ at $x = 1 + \frac{\sqrt{2}}{2}$.

Clearly, in this case $\Omega = \omega^2 < \pi^2$, which allows to conclude that the imaginary part of the integral factor in (87) satisfies

$$\int_0^1 \mathrm{e}^{-x\,t}\,\sin(\omega\,t)\,\left(1 - \frac{\lambda_0\,(\xi-\lambda_0)\,t^2}{2} + \frac{(4\lambda_0-2\xi)\,t}{2}\right)\mathrm{d}t \neq 0 \quad \text{for} \quad (x,\omega) \in \mathbb{R}_+^* \times \mathbb{R}_+^*,$$

which ends the proof. □

4.3 P3δ software

The authors developed recently an intuitive `Python` software called Partial pole placement via delay action (P3δ) [19–21]. P3δ (`https://cutt.ly/p3delta`) enables the design of delayed feedback control laws rendering the closed-loop dynamics stable with a prescribed exponential decay rate. P3δ methodology relies on two properties of quasipolynomial's zeros distribution: (i) the MID and (ii) the CRRID [4, 9], for *coexisting-real-root-induced-dominancy*.

While the MID has been highlighted through this paper, the CRRID property consists in conditions on the system's parameters guaranteeing the dominance of coexistent real spectral values. When using the MID strategy on P3δ, two options are proposed: the GMID-based design and the control-oriented IMID-based design. The first option relies on the fact that a root of maximal multiplicity $M(\lambda_0) = n + m + 1$ is necessarily dominant and the latter exploits the over-order intermediate multiplicity $M(\lambda_0) = n + 1$, offering sufficient freedom in the choice of parameters. In future software developments, the authors will integrate the result of this work into new P3δ functionality, allowing the users to exploit the IMID-based design with intermediate multiplicities $n + 1 \leq M(\lambda_0) \leq n + m + 1$, yielding further freedom for control purposes.

5 Concluding remarks

The MID property defines an intriguing link between multiple spectral values and the spectral abscissa of a given plant opening promising prospects in prescribed stabilization of both finite and infinite-dimensional dynamical systems. While the GMID property ($M(\lambda_0) = m + n + 1$) has been fully characterized in [15, 47] thanks to the hypergeometric representation of the corresponding quasipolynomial, beyond some partial results [8, 18], the over-order intermediate MID remained an open question. In this work, the over-order intermediate MID is investigated in depth. Thanks to a Hille oscillation theorem in the complex domain, we first provide a unified proof for the IMID to hold when $n + 1 \leq M(\lambda_0) \leq m + n$ by exploiting the representation of the quasipolynomial as a linear combination of contiguous Kummer hypergeometric functions. Second, for the sake of the effectiveness of the MID-based design, an algorithmic method relying on an a priori bound on the frequency of complex roots with positive real part is described. Both strategies have been illustrated through concrete control applications in the retarded as well as in the neutral cases: prescribed regulation of the Mach number in a wind-tunnel and the systematic PID control design for unstable first-order plants with input delay. Both examples, as well as some recent works such as [2, 3, 35], show the true potential of the proposed partial pole placement not only for ODEs and DDEs with a single delay but also for DDEs with several delays and some classes of PDEs.

Acknowledgments

The authors warmly thank their colleague KARIM L. TRABELSI (IPSA Paris) for careful reading of the manuscript and for valuable comments. IB is grateful to JEAN MARIE STRELCYN (Université Paris 13 Nord) for the long term and continuous support and discussions sharing remarkable historical facts through suggested keynote literature on topics such as the hypergeometric functions, their wonderful properties and their implication in qualitative properties of dynamical systems. Last but not least, the authors wish thank the editor and reviewers for their remarks and comments that helped us to improve the overall quality of the paper.

References

[1] C. T. Abdallah, P. Dorato, J. Benitez-Read, and R. Byrne. Delayed positive feedback can stabilize oscillatory systems. In *Proceedings of the 1993 American Control Conference*, pages 3106–310, San Francisco, USA, 1993.

[2] K. Ammari, I. Boussaada, S.-I. Niculescu, and S. Tliba. Multiplicity manifolds as an opening to prescribe exponential decay : auto-regressive boundary feedback in wave equation stabilization. preprint, 2023.

[3] K. Ammari, I. Boussaada, S.-I. Niculescu, and S. Tliba. Prescribing transport equation solution's decay via multiplicity manifold and autoregressive boundary control. Preprint, 2023.

[4] S. Amrane, F. Bedouhene, I. Boussaada, and S.-I. Niculescu. On qualitative properties of low-degree quasipolynomials: further remarks on the spectral abscissa and rightmost-roots assignment. *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)*, 61(109)(4):361–381, 2018.

[5] D. Armbruster, D. E. Marthaler, C. Ringhofer, K. Kempf, and T.-C. Jo. A continuum model for a re-entrant factory. *Operations research*, 54(5):933–950, 2006.

[6] E. S. Armstrong and J. S. Tripp. An application of multivariable design techniques to the control of the National Transonic Facility. Technical Paper 1887, NASA, August 1981.

[7] J. Auriol, I. Boussaada, R. J. Shor, H. Mounier, and S.-I. Niculescu. Comparing advanced control strategies to eliminate stick-slip oscillations in drillstrings. *IEEE Access*, 10:10949–10969, 2022.

[8] T. Balogh, I. Boussaada, T. Insperger, and S.-I. Niculescu. Conditions for stabilizability of time-delay systems with real-rooted plant. *Internat. J. Robust Nonlinear Control*, 32(6):3206–3224, 2022.

[9] F. Bedouhene, I. Boussaada, and S.-I. Niculescu. Real spectral values coexistence and their effect on the stability of time-delay systems: Vandermonde matrices and exponential decay. *C. R. Math. Acad. Sci. Paris*, 358(9-10):1011–1032, 2020.

[10] R. Bellman and K. L. Cooke. *Differential-difference equations*. Academic Press, New York-London, 1963.

[11] A. Benarab, I. Boussaada, S.-I. Niculescu, and K. Trabelsi. Multiplicity-induced-dominancy for delay systems: Comprehensive examples in the scalar neutral case. *European J. Contr.*, 2023.

[12] A. Benarab, I. Boussaada, K. Trabelsi, and C. Bonnet. Multiplicity-induced-dominancy property for second-order neutral differential equations with application in oscillation damping. *Eur. J. Control*, 69:Paper No. 100721, 13, 2023.

[13] S. P. Bhattacharyya, H. Chapellat, and L. H. Keel. Robust control: the parametric approach. In *Advances in control education 1994*, pages 49–52. Elsevier, 1995.

[14] I. Boussaada, G. Mazanti, S. Niculescu, and W. Michiels. Decay rate assignment through multiple spectral values in delay systems. Technical report, L2S, July 2023.

[15] I. Boussaada, G. Mazanti, and S.-I. Niculescu. The generic multiplicity-induced-dominancy property from retarded to neutral delay-differential equations: When delay-systems characteristics meet the zeros of Kummer functions. *C. R. Math. Acad. Sci. Paris*, 360:349–369, 2022.

[16] I. Boussaada, G. Mazanti, and S.-I. Niculescu. Padé approximation and hypergeometric functions: A missing link with the spectrum of delay-differential equations. *IFAC-PapersOnLine*, 55(30):206–211, 2022. 25th International Symposium on Mathematical Theory of Networks and Systems (MTNS 2022).

[17] I. Boussaada, G. Mazanti, and S.-I. Niculescu. Some remarks on the location of non-asymptotic zeros of Whittaker and Kummer hypergeometric functions. *Bull. Sci. Math.*, 174:Paper No. 103093, 2022.

[18] I. Boussaada, G. Mazanti, S.-I. Niculescu, and A. Benarab. MID property for delay systems: Insights on spectral values with intermediate multiplicity. In *61st IEEE Conference on Decision and Control (CDC 2022)*, pages 6881–6888, 2022.

[19] I. Boussaada, G. Mazanti, S.-I. Niculescu, A. Hammoumou, T. Millet, J. Raj, and J. Huynh. New features of P3δ software. Insights and demos. *IFAC-PapersOnLine*, 55(36):246–251, 2022. 17th IFAC Workshop on Time Delay Systems (TDS 2022).

[20] I. Boussaada, G. Mazanti, S.-I. Niculescu, J. Huynh, F. Sim, and M. Thomas. Partial pole placement via delay action: A Python software for delayed feedback stabilizing design. In *2020 24th International Conference on System Theory, Control and Computing (ICSTCC)*, pages 196–201, 2020.

[21] I. Boussaada, G. Mazanti, S.-I. Niculescu, A. Leclerc, J. Raj, and M. Perraudin. New features of P3δ software: Partial pole placement via delay action. *IFAC-PapersOnLine*, 54(18):215–221, 2021. 16th IFAC Workshop on Time Delay Systems (IFAC TDS 2021).

[22] I. Boussaada, I. Morarescu, and S. Niculescu. Inverted pendulum stabilization: Characterization of codimension-three triple zero bifurcation via multiple delayed proportional gains. *Syst. Control Lett.*, 82:1–9, 2015.

[23] I. Boussaada and S. Niculescu. Characterizing the codimension of zero singularities for time-delay systems. *Acta Appl. Math.*, 145(1):47–88, 2016.

[24] I. Boussaada, S.-I. Niculescu, A. El-Ati, R. Pérez-Ramos, and K. Trabelsi. Multiplicity-induced-dominancy in parametric second-order delay differential equations: analysis and application in control design. *ESAIM Control Optim. Calc. Var.*, 26:Paper No. 57, 34, 2020.

[25] D. Breda, editor. *Controlling delayed dynamics—advances in theory, methods and applications*, volume 604 of *CISM International Centre for Mechanical Sciences. Courses and Lectures*. Springer, Cham, 2023.

[26] H. Buchholz. *The confluent hypergeometric function with special emphasis on its applications*, volume 15 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, 1969.

[27] J. J. Castillo-Zamora, I. Boussaada, A. Benarab, and J. Escareno. Time-delay control of quadrotor unmanned aerial vehicles: a multiplicity-induced-dominancy-based approach. *Journal of Vibration and Control*, 29(11-12):2593–2608, 2023.

[28] C. Corduneanu, Y. Li, and M. Mahdavi. *Functional differential equations*. Pure and Applied Mathematics (Hoboken). John Wiley & Sons, Inc., Hoboken, NJ, 2016. Advances and applications.

[29] J. Coron and S. Tamasoiu. Feedback stabilization for a scalar conservation law with PID boundary control. *Chin. Ann. Math., B*, 36(5):763–776, 2015.

[30] O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel, and H.-O. Walther. *Delay equations: Functional-, complex-, and nonlinear analysis*, volume 110 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1995.

[31] R. Driver. *Ordinary and delay differential equations*. Springer-Verlag, New York, 1977.

[32] L. E. Els'golts' and S. B. Norkin. *Introduction to the theory and application of the theory of differential equations with deviating argument*. Academic Press: New York, 1973.

[33] K. Engelborghs, M. Dambrine, and D. Roose. Limitations of a class of stabilization methods for delay systems. *IEEE Transactions on Automatic Control*, 46(2):336–339, 2001.

[34] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. Tricomi. *Higher transcendental functions. Vol. I.* Robert E. Krieger Publishing Co., Inc., Melbourne, Fla., 1981.

[35] S. Fueyo, G. Mazanti, I. Boussaada, Y. Chitour, and S.-I. Niculescu. On the pole placement of scalar linear delay systems with two delays. *IMA J. Math. Control Inform*, 40(1):81–105, 2023.

[36] K. Gu, V. Kharitonov, and J. Chen. *Stability of time-delay systems*. Control Engineering. Birkhäuser Boston, Inc., Boston, MA, 2003.

[37] A. Halanay. *Differential equations: stability, oscillations, time lags*. Academic Press, New York, 1966.

[38] A. Halanay and J. Yorke. Some new results and problems in the theory of delay-differential equations. *SIAM Review*, 13(1):55–80, 1971.

[39] J. K. Hale and S. M. Verduyn Lunel. *Introduction to functional-differential equations*, volume 99 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1993.

[40] E. Hille. Oscillation theorems in the complex domain. *Trans. Am. Math. Soc*, 23(4):350–385, 1922.

[41] T. Insperger and G. Stépán. *Semi-discretization for time-delay systems*, volume 178 of *Applied Mathematical Sciences*. Springer, New York, 2011. Stability and engineering applications.

[42] V. B. Kolmanovskii and V. R. Nosov. *Stability of functional differential equations*. Academic Press: New York, 1986.

[43] N. N. Krasovskiĭ. *Stability of motion. Applications of Lyapunov's second method to differential systems and equations with delay*. Stanford University Press, Stanford, Calif., 1963. Translated by J. L. Brenner.

[44] M. S. Lee and C. S. Hsu. On the τ-decomposition for stability analysis for retarded dynamical systems. *SIAM J. Contr.*, 7(2):242–259, 1969.

[45] D. Ma, I. Boussaada, J. Chen, C. Bonnet, S.-I. Niculescu, and J. Chen. PID control design for first-order delay systems via MID pole placement: Performance vs. robustness. *Automatica*, 137:110102, 2022.

[46] A. Z. Manitius. Feedback controllers for a wind tunnel model involving a delay: analytical design and numerical simulation. *IEEE Trans. Automat. Control*, 29(12):1058–1068, 1984.

[47] G. Mazanti, I. Boussaada, and S.-I. Niculescu. Multiplicity-induced-dominancy for delay-differential equations of retarded type. *J. Differential Equations*, 286:84–118, 2021.

[48] G. Mazanti, I. Boussaada, S.-I. Niculescu, and Y. Chitour. Effects of roots of maximal multiplicity on the stability of some classes of delay differential-algebraic systems: The lossless propagation case. In *Proceeding of 24th International Symposium on Mathematical Theory of Networks and Systems (MTNS 2021)*, IFAC-PapersOnLine, Cambridge, United Kingdom, 2021.

[49] W. Michiels, I. Boussaada, and S.-I. Niculescu. An explicit formula for the splitting of multiple eigenvalues for nonlinear eigenvalue problems and connections with the linearization for the delay eigenvalue problem. *SIAM Journal on Matrix Analysis and Applications*, 38(2):599–620, 2017.

[50] W. Michiels, K. Engelborghs, P. Vansevenant, and D. Roose. The continuous pole placement method for delay equations. *Automatica*, 38(5):747–761, 2002.

[51] W. Michiels and S. Niculescu. *Stability, control, and computation for time-delay systems: An eigenvalue-based approach*, volume 27 of *Advances in Design and Control*. Soc. Ind. Appl. Math, Philadelphia, PA, second edition, 2014.

[52] C. A. Molnar, T. Balogh, I. Boussaada, and T. Insperger. Calculation of the critical delay for the double inverted pendulum. *Journal of Vibration and Control*, 27(3-4):1 – 9, 2020.

[53] S. Mondie and W. Michiels. Finite spectrum assignment of unstable time-delay systems with a safe implementation. *IEEE Transactions on Automatic Control*, 48(12):2207–2212, 2003.

[54] Y. I. Neĭmark. The structure of the D-decomposition of the space of quasipolynomials and the diagrams of Vyšnegradskiĭ and Nyquist. *Doklady Akad. Nauk SSSR (N.S.)*, 60:1503–1506, 1948.

[55] S. Niculescu. *Delay effects on stability: A robust control approach*, volume 269 of *Lect. Notes Control. Inf. Sci.* Springer-Verlag London Ltd., London, 2001.

[56] S.-I. Niculescu, I. Boussaada, X.-G. Li, G. Mazanti, and C.-F. Méndez-Barrios. Stability, delays and multiple characteristic roots in dynamical systems: A guided tour. *IFAC-PapersOnLine*, 54(18):222–239, 2021.

[57] F. Olver, D. Lozier, R. Boisvert, and C. Clark, editors. *NIST Handbook of Mathematical Functions*. U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010.

[58] E. Pinney. *Ordinary difference-differential equations*. Univ. California Press, 1958.

[59] G. Pólya and G. Szegő. *Problems and theorems in analysis. I.* Classics in Mathematics. Springer-Verlag, Berlin, 1998.

[60] A. Ramírez, S. Mondié, R. Garrido, and R. Sipahi. Design of proportional-integral-retarded (PIR) controllers for second-order LTI systems. *IEEE Trans. Automat. Control*, 61(6):1688–1693, 2016.

[61] T. Samad. A survey on industry impact and challenges thereof. *IEEE Control Systems Magazine*, 37(1):17–18, 2017.

[62] A. P. Seyranian and A. A. Mailybaev. *Multiparameter stability theory with mechanical applications*, volume 13. World Scientific, 2003.

[63] J. Sieber and B. Krauskopf. Bifurcation analysis of an inverted pendulum with delayed feedback control near a triple-zero eigenvalue singularity. *Nonlinearity*, 17:85–103, 2004.

[64] R. Sipahi, S. Niculescu, C. Abdallah, W. Michiels, and K. Gu. Stability and stabilization of systems with time delay: limitations and opportunities. *IEEE Control Syst. Mag.*, 31(1):38–65, 2011.

[65] G. Stépán. *Retarded dynamical systems: stability and characteristic functions*, volume 210 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989.

[66] I. H. Suh and Z. Bien. Proportional minus delay controller. *IEEE Transactions on Automatic Control*, 24(2):370–372, 1979.

[67] I. H. Suh and Z. Bien. Use of time-delay actions in the controller design. *IEEE Transactions on Automatic Control*, 25(3):600–603, 1980.

[68] J. S. Tripp. Development of a distributed parameter mathematical model for simulation of cryogenic wind tunnels. Technical Paper 2177, NASA, September 1983.

[69] E. Witrant, E. Fridman, O. Sename, and L. Dugard, editors. *Recent results on time-delay systems. Analysis and Control*, volume 5 of *Advances in Delays and Dynamics*. Springer, Cham, 2016. Selected revised papers from the 11th International Federation of Automatic Control (IFAC) Workshop on Time-Delay Systems held in Grenoble, February 4–6, 2013.

[70] M. Yoshida. *Fuchsian differential equations*. Springer, 1987.