In domain dissipation assignment of boundary controlled Port-Hamiltonian systems using backstepping

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Abstract

In this paper, we develop a systematic approach to stabilize a general class of hyperbolic systems while assigning them a specified closed-loop behavior with a clear energy interpretation. More precisely, we address in-domain dissipation assignment for boundary-controlled Port Hamiltonian systems. The controller is designed so that the closed-loop system behaves like a target system with a specified energy decay rate. The PHS framework is used to take advantage of the natural physical properties of the system to define well-posed, exponentially stable, and easily parametrizable target system candidates, thus resulting in modular controllers. Under some generic structural assumptions, we rewrite the considered Port Hamiltonian system in the Riemann coordinates. The control approach is then based on the backstepping methodology. We combine classical Volterra transformations with an innovative time-affine transform to map the original system to the desired target system. The proposed approach is applied to two test cases: a clamped string and a clamped Timoshenko beam. Both are illustrated in numerical simulations.

Keywords: Backstepping-based control design, hyperbolic PDE systems, distributed parameter systems, Port-Hamiltonian Systems

1. Introduction

In recent years, Port-Hamiltonian Systems (PHS) formulations have emerged as a powerful framework for the modeling and control of distributed parameter systems [1, 2, 3, 4, 5]. This formalism is particularly well suited to describe the dynamics of large-scale (multi)physics systems, such as those arising in fluid mechanics [6], heat transfer [7], and structural mechanics [8]. The Port-Hamiltonian approach makes it possible to highlight and take advantage of the physical properties of the considered systems through a well-defined geometric structure. It has been successfully applied to a wide range of engineering applications, including the optimization of systems performances [9], systems stability analysis [10, 11], and the design of controllers for complex systems [12, 13]. Regarding linear hyperbolic PDEs, the Port-Hamiltonian framework can be used to prove the existence of solutions or to parametrize all the boundary conditions guaranteeing the well-posedness of the associated boundary control system [3, 14, 13]. It can also be used to design boundary controllers, exploiting the system's physical properties efficiently. For instance, the energy shaping method via control by interconnection was developed for boundary-controlled port Hamiltonian systems [13, 15]. It allows the modification of the closed-loop energy function and is, therefore, a design method with a clear physical interpretation.

In parallel, the backstepping methodology [16] has proven to be a powerful design method for boundary feedback control of interconnected PDE systems [17, 18, 19]. Based on invertible state transformations (usually Volterra integral transforms), it consists of mapping the original system into a simpler form (called *target system*) amenable to analysis, control, and observer design [20]. One of the main difficulties with the backstepping method lies in finding a suitable target system. It should be simple enough to allow the design of the control law. Still, in the meantime, one must prove the existence of a transformation mapping the original system to this target system. The choice of the target system directly impacts the closed-loop performance. The general question of reachable target systems is still an open problem. In the case of hyperbolic PDE systems, finite-time stable target systems are usually chosen [21, 22] as they often present a simple structure with amenable properties [23]. However, this choice of target systems corresponds to a specific performance criterion (here, finite-time stability), thereby shadowing the robustness properties of the corresponding closed-loop systems [24]. In [25], the authors introduced tuning parameters in the design for two coupled hyperbolic equations, thus guaranteeing potential trade-offs between different specifications (namely delay-robustness and convergence rate). However, these parameters have a limited range of action since they only affect the system's boundary conditions. Recent contribu-

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tions focused on the backstepping exponential stabilization of a potentially unstable Timoshenko beam with an arbitrary decay rate [26], thereby introducing a class of parameterized controllers. These results are currently being extended to multi-layer beams [27]. Similar questions have been raised for parabolic systems with the design of prescribed-time controllers [28, 29, 30], requiring the introduction of non-trivial target systems to achieve the desired specific closed-loop behavior. All these examples illustrate how adding tuning terms in the target systems allows for better closed-loop properties, such as arbitrarily large exponential decay rates or better robustness margins. More general target systems (and thus additional degrees of freedom) could be obtained by preserving dissipative in-domain couplings. This would require precise knowledge of the influence of the different parameters of the system in terms of stability properties. In this context, the Port-Hamiltonian approach could be advantageously used to define well-posed, exponentially stable target system candidates. A first attempt to take advantage of Port Hamiltonian formulations and their associated geometric structure for backstepping control design can be found in [31, 32]. Recently, the backstepping method was successfully combined with PHS on simple test cases (wave equation and Timoshenko beam) to design state-feedback controllers that assign the distributed damping of the closed-loop system, thus determining the decay rate of the solutions while reducing the associated control effort [33, 34].

In this paper, we develop a systematic framework to stabilize a general class of hyperbolic PDE systems while assigning a specified closed-loop behavior with a clear energy interpretation. In this context, we take advantage of the PHS theory, which corresponds to a multi-physical and modular energy-based representation that considers the system's natural physical properties. These natural physical properties can then be advantageously used to define well-posed, exponentially stable, and easily parametrizable target system candidates. Therefore, the proposed approach will allow parameterizing controllers by employing physically inspired tuning parameters (seen as degrees of freedom). In the near future, these controllers could be tuned to fulfill a given set of performance specifications best. Such a methodology was initially developed for simple low-dimensional systems in [33, 34]. Its extension to the general case requires introducing a new kind of transformation.

Our approach is the following. The original hyperbolic system is first expressed in the Port Hamiltonian framework to emphasize its inherent physical properties. In particular, we can easily express the evolution of the associated energy. Then, we can define the desired target system as a copy of the original PHS with modulated internal dissipation (in-domain damping), which implies a specific energy decay of the original system in closed-loop. In the meantime, we express these two systems in the Riemann coordinates (as balance laws) since this framework is required to apply the backstepping method. More precisely, in this set of coordinates, we can combine successive backstepping transformations [35] with an original time-affine transformation to map the original system to the target system, thus fulfilling our control objective. To illustrate our methodology, we apply it to two test cases: distributed damping assignment for a wave equation and the stabilization of a Timoshenko beam. An interesting by-product of our analysis is that we can design controllers that can map a system of linear coupled balance laws [36] to any target system with the same structure but with arbitrary in-domain coupling terms. This allows for a more generic class of target system compared to existing ones in the literature [35, 37, 22].

This article is organized as follows. We first present the general class of Port-Hamiltonian Systems (PHS) under consideration in Section 2. We use this framework to define the desired class of target systems. Under some structural assumptions, we give the different steps to rewrite the initial and target PHS in Riemann coordinates. Next, in Section 3, we present the backstepping-controller design. We define successive invertible transformations to map the initial hyperbolic PDE system to the desired target system. It is then possible to define the associated controller. We then apply the proposed control strategy for in-domain damping assignment for two low-dimensional systems. In the first test case, presented in Section 4, we consider a clamped string modeled by a wave equation with space-varying coefficients and indefinite damping. The second test case, presented in Section 5, is a clamped Timoshenko beam. In both test cases, numerical simulations illustrate our results. Some concluding remarks and perspectives end this paper (Section 6).

Notations

For any $n \in \mathbb{N}^*$, for any compact set $\mathbb{K}$, We denote $C^1(\mathbb{K};\mathbb{R}^n)$ the space of real differentiable functions defined on $\mathbb{K}$ with values in $\mathbb{R}^n$ and a continuous derivative, $C^1(\mathbb{K};\mathbb{R}^n)^+$ its subset of strictly positive functions, and $C^1_{pw}(\mathbb{K};\mathbb{R}^n)$ the set of piecewise differentiable functions with continuous derivative. We denote D_n^+, the set of diagonal matrices in $\mathbb{R}^{n\times n}$ with positive coefficients. The notation I_n stands for the $n \times n$ identity matrix (if the dimensions are not ambiguous, the subindex will be omitted). Following classical notations from the Port-Hamiltonian framework, we define the Hilbert space $\chi_n = H^1([0,1];\mathbb{R}^n)$ equipped with the inner-product

$$< u, v >_{\chi_n} = \frac{1}{2}\int_0^1 u(z)^\top \mathcal{H}(z) v(z) \mathrm{d}z,$$

with $\mathcal{H} \in D_n^+$. Note that the χ_n-norm is equivalent to the classical L^2-norm. Let $\tau > 0$ be a positive fixed time delay. We denote $D_\tau[t] = H^1([-\tau,0],\mathbb{R})$ the Banach space of H^1 real-valued functions mapping the interval $[-\tau,0]$ into $\mathbb{R}$. For any function $\phi : [-\tau,\infty) \mapsto \mathbb{R}$, we define its associated partial trajectory $\phi_{[t]} \in D_\tau$ by $\phi_{[t]}(\theta) = \phi(t+\theta), -\tau \leq \theta \leq 0$. The associated norm is

given by $\|\phi_{[t]}\|_\tau = \left(\int_{-\tau}^0 \phi(t+\theta)^2 \mathrm{d}\theta\right)^{\frac{1}{2}}$.
The unit square $[0,1]^2$ is denoted $\mathcal{S}$. Its lower (resp. upper) triangular part is denoted $\mathcal{T}^- = \{(x,y) \in [0,1]^2 | \ 0 \leq y \leq x\}$ (resp. $\mathcal{T}^+$). For any $\lambda, \mu, a > 0$, we denote $\mathcal{T}_\lambda^+ = \{(x,y) \in [0,1]\times[0,\frac{1}{\lambda}] \mid 0 \leq y \leq \frac{x}{\lambda}\}$, $\mathcal{T}_\mu^- = \{(x,y) \in [0,1]\times[0,\frac{1}{\mu}] \mid 0 \leq y \leq \frac{1}{\mu}(1-x)\}$, and $\mathcal{P}_{a,\lambda}^+ = \{(x,y) \in [0,1]\times[0,a+\frac{1}{\lambda}] \mid \frac{x}{\lambda} \leq y \leq a + \frac{x}{\lambda})\}$, $\mathcal{P}_{a,\mu}^- = \{(x,y) \in [0,1]\times[0,a+\frac{1}{\mu}] \mid \frac{1-x}{\mu} \leq y \leq a + \frac{1-x}{\mu})\}$ two parallelogram domains. We use the standard notation for the Kronecker symbol $\delta_{ij} \doteq 1$ if $i=j$, 0 else. We extend this definition with $\delta_{i \asymp j} \doteq 1$ if $i \asymp j$, 0, where $\asymp$ denote any order relation.

2. Problem under consideration and control objective

As explained in the introduction, the main objective of this paper is to stabilize a general class of hyperbolic systems while assigning a specified closed-loop behavior with a clear energy interpretation. More precisely, we aim to introduce degrees of freedom in the design to obtain a class of easily parametrizable closed-loop systems. The controller will be designed such that it is possible to map the closed-loop system to this desired target system using invertible transformations, thus guaranteeing equivalent properties. We take advantage of the PHS framework to develop this class of attainable (exponentially stable) systems, which can help identify naturally dissipative terms. Consequently, this implies that the desired target system we want to reach has to be expressed in this framework. In this paper, we also chose to express the original hyperbolic system in the PHS framework, as this can give us a clear energy interpretation of the open-loop behavior. However, we emphasize that our methodology could be applied to original systems that do not necessarily adhere to the PHS paradigm as long as they can be expressed in the so-called Riemann coordinates. Indeed, to develop our methodology, we need to rewrite the PHS systems in the Riemann coordinates and actually show that it is possible to design a control law such that we can map any controlled hyperbolic PDE system in the Riemann coordinates to any arbitrary hyperbolic system with an analogous structure but modified in-domain coupling terms. In the rest of this section, we first introduce the system under consideration in the PHS framework before giving its expression in the Riemann coordinates. We then properly state our control objective and introduce the class of target systems we want to reach.

2.1. Original system(s) under consideration

2.1.1. Hyperbolic system in the PHS framework

In this paper, we consider the boundary control of Port-Hamiltonian systems [38] defined on a one-dimensional domain $z \in [0,1]$ by

$$\frac{\partial x}{\partial t} = P_1 \frac{\partial}{\partial z}(\mathcal{H}(z)x(t,z)) + (P_0 - \Pi_0)\mathcal{H}(z)x(t,z), \quad (1)$$

where $x(t,z) \in \chi_{2n}$ ($n \in \mathbb{N}\backslash\{0\}$) is the vector of energy variables defined on $[0,+\infty)\times[0,1]$, $\mathcal{H}$ is a symmetric and Lipschitz continuous coercive matrix-valued function defined on $[0,1]$, P_1 is a full rank matrix such that $P_1 = P_1^\top \in \mathbb{R}^{2n\times 2n}$, the matrix P_0 verifies $P_0 = -P_0^\top \in \mathbb{R}^{2n\times 2n}$ and $\Pi_0 \in \mathbb{R}^{2n\times 2n}$ verifies $\Pi_0 = \Pi_0^\top \in \mathbb{R}^{2n\times 2n}$. The boundary inputs/ouputs are defined by

$$u_\partial(t) = W_B \begin{pmatrix} \mathcal{H}(1)x(t,1) \\ \mathcal{H}(0)x(t,0) \end{pmatrix}, \quad (2)$$

$$y_\partial(t) = W_C \begin{pmatrix} \mathcal{H}(1)x(t,1) \\ \mathcal{H}(0)x(t,0), \end{pmatrix} \quad (3)$$

where $W_B, W_C \in \mathbb{R}^{2n\times 4n}$ with

$$W_B = \begin{pmatrix} \frac{1}{\sqrt{2}}(\Xi^- + \Xi^+ P_1) & \frac{1}{\sqrt{2}}(\Xi^- - \Xi^+ P_1) \end{pmatrix}, \quad (4)$$

$$W_C = \begin{pmatrix} \frac{1}{\sqrt{2}}(\Xi^+ + \Xi^- P_1) & \frac{1}{\sqrt{2}}(\Xi^+ - \Xi^- P_1) \end{pmatrix}, \quad (5)$$

and Ξ^+ and Ξ^- in $\mathbb{R}^{2n\times 2n}$ satisfy

$$\Xi^{-\top}\Xi^+ + \Xi^{+\top}\Xi^- = 0, \text{ and } \Xi^{-\top}\Xi^- + \Xi^{+\top}\Xi^+ = I_{2n}. \quad (6)$$

For all $t \geq 0$, we define the total energy of the system $\mathcal{E}(x)$ as

$$\mathcal{E}(x(t)) = \frac{1}{2}\int_0^1 \left(x^\top(t,z)\mathcal{H}(z)x(t,z)\right) dz = \|x(t,\cdot)\|_\chi^2.$$

We have [38]

$$\frac{\partial \mathcal{E}(x(t))}{\partial t} = y_\partial^\top(t) u_\partial(t) - \int_0^1 x^\top(t,z)\Pi_0 x(t,z) dz.$$

It has been shown in [2] that if Π_0 is semi-definite positive, the system (1-3) defines a boundary control system [39]. Under some conditions on Π_0 (corresponding to some damping in velocity coordinates), it could be shown that the open-loop system is asymptotically or exponentially stable [38]. However, we emphasize that in this paper, we only assume $\Pi_0 = \Pi_0^T$. Therefore, the system we consider may be undamped or anti-damped and unstable. In this study, we make the following assumption:
Assumption 1. *One boundary is fully actuated, i.e.*

$$u_\partial(t) = \begin{pmatrix} u_0(t) \\ 0 \end{pmatrix}$$

where u_0 is the control input, and the matrix W_B is block diagonal or block anti-diagonal, the different blocks being of dimension $n \times 2n$.

Assumption 1 means that one boundary is fully actuated and the other is set to zero, implying the original system can be written into the hyperbolic PDE form of Section 3

(Riemann coordinates). Note that this assumption is verified for numerous physical applications as clamped/actuated strings (described in Section 2.1.3 and Section 4) or clamped-actuated Timoshenko beams (described in Section 5). Without any loss of generality, we consider that the actuated boundary is in $z = 1$.

2.1.2. Hyperbolic system in the Riemann coordinates

Here, we reformulate the system (1)-(3) in the Riemann coordinates. Indeed, one of the key ingredients of the methodology we present in this paper is the backstepping approach that has mostly been developed for systems of balance laws with diagonal velocity matrices [36, 22, 37]. Therefore, it is of specific interest to have an alternate representation of the system (1)-(3) in this framework. The matrix P_1 being full rank and $\mathcal{H}(z)$ coercive, $P_1\mathcal{H}(z)$ is diagonalizable, i.e., there exist a matrix-valued function $Q_1(z) \in \mathbb{R}^{2n\times 2n}$, a diagonal matrix-valued function $\Lambda(z)$ defined on $[0,1]$, such that

$$\forall\ z \in [0,1],\ P_1\mathcal{H}(z) = Q_1(z)\Lambda(z)Q_1^{-1}(z).$$

For all $t > 0$, we first define a new set of variables $\zeta(t,\cdot) = Q_1^{-1}(\cdot)x(t,\cdot) \in \chi_{2n}$. It satisfies a set of transport PDEs with in-domain coupling terms

$$\frac{\partial \zeta}{\partial t}(t,z) + \Lambda(z)\frac{\partial \zeta}{\partial z}(t,z) = \Sigma(z)\zeta(t,z), \tag{7}$$

$$u_\partial(t) = W_B \begin{pmatrix} \mathcal{H}(1)Q_1(1)\zeta(t,1) \\ \mathcal{H}(0)Q_1(0)\zeta(t,0) \end{pmatrix}, \tag{8}$$

with $\Sigma(z) = [Q_1^{-1}(z)(P_1\frac{\partial \mathcal{H}}{\partial z} + (P_0 - \Pi_0))\mathcal{H}(z) - \frac{\partial Q_1^{-1}(z)}{\partial z}]Q_1(z)$. When applying the backstepping approach, the matrix Σ will be required to have zero diagonal terms. To fulfill this constraint, we use an exponential change of variables [40]. For all $z \in [0,1]$, define $A(z) \in D_{2n}^+$ with diagonal terms given by

$$A_{ii}(z) = e^{I_i(z)},\ \text{with}\ I_i(z) = -\int_0^z \frac{\Sigma_{ii}(s)}{\Lambda_{ii}(s)}\mathrm{d}s. \tag{9}$$

The matrix-valued function A is invertible, and we can define a new state variable by $\xi(t,z) = A(z)\zeta(t,z)$. It satisfies

$$\frac{\partial \xi}{\partial t}(t,z) + \Lambda(z)\frac{\partial \xi}{\partial z}(t,z) = \sigma(z)\xi(t,z), \tag{10}$$

$$u_\partial(t) = W_B \begin{pmatrix} \mathcal{H}(1)Q_1(1)A^{-1}(1)\xi(t,1) \\ \mathcal{H}(0)Q_1(0)\xi(t,0) \end{pmatrix}, \tag{11}$$

with $\sigma_{ij}(z) = [A(z)\Sigma(z)A^{-1}(z)]_{ij}$ for $i \neq j$, and 0 else.

2.1.3. Example of a clamped-string (wave equation)

We now illustrate the previous concepts with an example to familiarize the reader with the previous concepts. Consider a vibrating string clamped at the first end ($z = 0$) and actuated at the other end ($z = 1$). We denote the vertical position of the string at point z and time $t > 0$ as $w(t,z)$. It satisfies

$$\rho(z)\frac{\partial^2 w}{\partial t^2}(t,z) = \frac{\partial}{\partial z}\left(E(z)\frac{\partial w}{\partial x}(t,z)\right) - \kappa(z)\frac{\partial w}{\partial t}(t,z), \tag{12}$$

with $\rho(z), E(z) \in C^1([0,1];\mathbb{R})^+$ being the mass density and Young's modulus, which are here space-dependent. The term $\kappa(z) \in C^0([0,1])$ corresponds to a damping term. Its sign impacts the global behavior of the system. For instance, a negative κ corresponds to a destabilizing anti-damping action. Initially, the position of the string is given by $w(z,0) = w_0(z) \in C^1([0,1];\mathbb{R})$, and its speed by $\frac{\partial w}{\partial t}|_{t=0}(z) = w_1(z)$, with $w_0(0) = 0 = w_1(0)$. We first rewrite the model as a Port Hamiltonian system. The energy state variables $x = [x_1, x_2]^\top \in \chi_2$ are defined by

$$x_1(t,z) = \frac{\partial w}{\partial z}(t,z), \quad x_2(t,z) = \rho(z)\frac{\partial w}{\partial t}(t,z), \tag{13}$$

where $x_1(t,z)$ (resp. $x_2(t,z)$) corresponds to the strain (resp. to the momentum). The state (x_1,x_2) satisfies

$$\frac{\partial}{\partial t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial}{\partial z}\left(\frac{1}{\rho(z)}\cdot\right) \\ \frac{\partial}{\partial z}\left(E(z)\cdot\right) & -c(z) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{14}$$

with $c(z) = \frac{\kappa(z)}{\rho(z)}$. The first end of the string is clamped while the other end is actuated such that

$$x_2(0,t) = 0, \quad E(1)x_1(1,t) = u_0(t), \tag{15}$$

where $u_0(t)$ is the control input. The Hamiltonian density is given for all $z \in [0,1]$ by

$$\mathcal{H}(z) = \operatorname{diag}(E(z), \frac{1}{\rho(z)}) \in D_2^+.$$

We have $P_0 = 0, P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\Pi_0 = \begin{pmatrix} 0 & 0 \\ 0 & \kappa \end{pmatrix}$. The boundary conditions rewrite as in equation (2) with $W_B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. It can be checked that W_B verifies equation (4) with

$$\Xi^- = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ and } \Xi^+ = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This system verifies Assumption 1.

We now rewrite the PHS system (14)-(15) in the Riemann coordinates. Let us define $\Lambda(z) = \begin{pmatrix} \lambda(z) & 0 \\ 0 & -\lambda(z) \end{pmatrix}$, where $\lambda(z) = \sqrt{E(z)/\rho(z)} \in \mathbb{R}^+$. To simplify the notations, we introduce the following functions $r(z) = \sqrt{E(z)\rho(z)}$ and $\delta(z) = \left(\frac{\rho'}{\rho} + \frac{E'}{E}\right)(z)$, $\delta_1^3(z) = \left(3\frac{\rho'}{\rho} - \frac{E'}{E}\right)(z)$, $\delta_3^1(z) =$

$\left(\frac{\rho'}{\rho} - 3\frac{E'}{E}\right)(z)$. Since the matrix $P_1\mathcal{H}(z)$ admits two opposite eigenvalues $\pm\lambda(z)$, we have

$$P_1\mathcal{H}(z) = Q_1(z)\Lambda(z)Q_1(z)^{-1},$$

where $Q_1(z) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\frac{1}{\sqrt{\rho(z)E(z)}} \\ \sqrt{\rho(z)E(z)} & 1 \end{pmatrix}$. The new state variables $\zeta = [\zeta^+, \zeta^-]^\top = Q_1^{-1}[x_1, x_2]^\top$ satisfy equation (7) with in-domain spatially varying continuous coupling terms defined by

$$\begin{cases} \Sigma_{11}(z) = \frac{1}{2}\left(-c(z) + \frac{\lambda}{2}\delta_3^1(z)\right), \\ \Sigma_{12}(z) = \frac{1}{2\rho(z)}\left(\frac{c(z)}{\lambda(z)} - \frac{1}{2}\delta(z)\right), \\ \Sigma_{21}(z) = \frac{E(z)}{2}\left(\frac{c(z)}{\lambda(z)} + \frac{1}{2}\delta(z)\right), \\ \Sigma_{22}(z) = \frac{1}{2}\left(-c(z) - \frac{\lambda}{2}\delta_1^3(z)\right), \end{cases}$$

The diagonal coupling terms Σ_{11} and Σ_{22} can be suppressed by introducing the following exponential change of coordinates

$$[\xi^+(t,z), \xi^-(t,z)]^\top = A(z)[\zeta^+(t,z), \zeta^-(t,z)]^\top,$$

with $A(z) = \mathrm{diag}(f(z)e^{I_c(z)}, g(z)e^{-I_c(z)})$, where $\forall z \in [0,1]$,

$$f(z) = \sqrt{\frac{E(z)\lambda(z)}{E(0)\lambda(0)}},\ g(z) = \sqrt{\frac{\lambda(z)\rho(0)}{\rho(z)\lambda(0)}}$$

and $I_c(z) = \int_0^z \frac{c(s)}{2\lambda(s)}ds$. Henceforth, the new variables $\xi = [\xi^+, \xi^-]$ satisfy the hyperbolic PDEs (10)-(11), where the scalar in-domain coupling terms are defined by

$$\begin{cases} \sigma_{11}(z) = \sigma_{22}(z) = 0, \\ \sigma_{12}(z) = \frac{1}{2r(0)}e^{2I_c(z)}(c(z) - \frac{\lambda(z)}{2}\delta(z)), \\ \sigma_{21}(z) = \frac{r(0)}{2}e^{-2I_c(z)}(c(z) + \frac{\lambda(z)}{2}\delta(z)), \end{cases}$$

Finally, we have the following boundary conditions $\xi^+(t,0) = \frac{1}{r(0)}\xi^-(t,0)$ and $\xi^-(t,1) = -r(1)e^{-2I_c(1)}\xi^+(t,1) + u(t)$, where the control input is given by

$$u(t) = g(1)e^{-I_c(1)}\frac{\sqrt{2}}{\lambda(1)}u_0(t).$$

2.2. Control objective

2.2.1. Target system in the PHS framework

This paper aims to stabilize the system (1) while guaranteeing a specific closed-loop behavior with a clear energy interpretation. We want to impose a specific decay rate on the energy function $\mathcal{E}$. The expected outcome is to obtain a class of easily parametrizable closed-loop systems with tuning parameters (degrees of freedom). Then, the associated controllers could be tuned in future works to best fulfill a given set of performance specifications. The desired target systems correspond to the original system (1) but with modified *in-domain damping* terms. More precisely, we want to obtain the following closed-loop behavior

$$\frac{\partial \bar{x}}{\partial t} = P_1\frac{\partial}{\partial z}\left(\mathcal{H}\bar{x}(t,z)\right) + \left(\bar{P}_0 - \bar{\Pi}_0\right)\left(\mathcal{H}\bar{x}(t,z)\right), \quad (16)$$

where $\bar{x} \in \chi_{2n}$ and $\bar{P}_0 = -\bar{P}_0^\top \in \mathbb{R}^{2n\times 2n}$, $\bar{\Pi}_0 \in \mathbb{R}^{2n\times 2n}$ satisfying $\bar{\Pi}_0 + \bar{\Pi}_0^\top \geq 0$. The boundary conditions are given by

$$W_{\bar{B}}\begin{pmatrix}\mathcal{H}(1)x(t,1)\\ \mathcal{H}(0)x(t,0)\end{pmatrix} = 0_{2n},\ \text{with } W_{\bar{B}}\begin{pmatrix}0 & I_{2n}\\ I_{2n} & 0\end{pmatrix}W_{\bar{B}}^\top \geq 0. \quad (17)$$

where $W_{\bar{B}} \in \mathbb{R}^{2n\times 4n}$ has a structure analogous to the one of W_B (i.e., it is block diagonal or bloc anti-diagonal due to Assumption 1). Note that for the sake of simplicity, $\bar{\Pi}_0$ has been chosen constant but it could have been chosen as a function of z, i.e. $\bar{\Pi}_0(z)$ is a real matrix-valued function satisfying $\bar{\Pi}_0(z) + \bar{\Pi}_0^\top(z) \geq 0,\ \forall z \in [0,1]$ and chosen such that the target system is exponentially stable. The energy function associated to this target system satisfies

$$\frac{d\mathcal{E}(\bar{x}(t))}{dt} = -\int_0^1 \bar{x}^\top(t,z)\bar{\Pi}_0\bar{x}(t,z)dz \leq 0. \quad (18)$$

Therefore, in closed-loop, the energy decay is determined by the matrix $\bar{\Pi}_0$. Since $\bar{\Pi}_0$ is chosen as positive definite, the system is exponentially stable. Yet, in most examples, not all the states need to be damped. Following the approach proposed in [41], the associated exponential decay rate of the energy of the target system can be computed using a Lyapunov analysis. This could be a way of choosing the desired target matrix $\bar{\Pi}_0$

2.2.2. Target system in the Riemann coordinates

Similarly to what we have done above, we can rewrite the target system (16)-(17) in the Riemann coordinates, which will be more amenable to apply backstepping transformations. More precisely, we define $\bar{\zeta}(t,z) = Q_1^{-1}(z)\bar{x}(t,z)$, where the matrix Q_1 is defined in Section 2.1.2. This new state satisfies the set of PDEs

$$\frac{\partial \bar{\zeta}}{\partial t}(t,z) + \Lambda(z)\frac{\partial \bar{\zeta}}{\partial z}(t,z) = \bar{\Sigma}(z)\bar{\zeta}(t,z), \quad (19)$$

$$0_{2n} = W_{\bar{B}}\begin{pmatrix}\mathcal{H}(1)Q_1(1)\bar{\zeta}(t,1)\\ \mathcal{H}(0)Q_1(0)\bar{\zeta}(t,0)\end{pmatrix}, \quad (20)$$

with $\bar{\Sigma}(z) = [Q_1^{-1}(z)(P_1\frac{\partial \mathcal{H}}{\partial z} + (\bar{P}_0 - \bar{\Pi}_0)\mathcal{H}(z)) + \frac{\partial Q_1^{-1}(z)}{\partial z}]Q_1(z)$. For all $z \in [0,1]$, define $\bar{A}(z) \in D_{2n}^+$ with

$$\bar{A}_{ii}(z) = e^{\bar{I}_i(z)},\ \text{with } \bar{I}_i(z) = -\int_0^z \frac{\bar{\Sigma}_{ii}(s)}{\Lambda_{ii}(s)}\mathrm{d}s. \quad (21)$$

The matrix-valued function is invertible, and we can define the new state variable by $\bar{\xi}(t,z) = \bar{A}(z)\bar{\zeta}(t,z)$

$$\frac{\partial \bar{\xi}}{\partial t}(t,z) + \Lambda(z)\frac{\partial \bar{\xi}}{\partial z}(t,z) = \bar{\Sigma}(z)\bar{\xi}(t,z), \quad (22)$$

$$0_{2n} = W_{\bar{B}} \begin{pmatrix} \mathcal{H}(1)Q_1(1)\bar{A}^{-1}(1)\bar{\xi}(t,1) \\ \mathcal{H}(0)Q_1(0)\bar{\xi}(t,0) \end{pmatrix}, \qquad (23)$$

with $\bar{\sigma}_{ij}(z) = [\bar{A}(z)\bar{\Sigma}(z)\bar{A}^{-1}(z)]_{ij}$ for $i \neq j$, and 0 else. We can now proceed further with the control design. For the sake of simplicity, we now assume that the Hamiltonian $\mathcal{H}$ is constant (the physical parameters are space-independent). The following approach can be adapted to the space-dependent case to the price of more technical computations.

2.2.3. Example of a clamped-string (wave equation)

Consider the wave equation example introduced in equation (12). For this system, our objective is to impose a specific decay rate to the energy of the system $\mathcal{E}$, using *a distributed damping assignment*. More precisely, we want to make the dynamics of x equivalent to the dynamics of $\bar{x} = [\bar{x}_1, \bar{x}_2]^\top$ satisfying (16)-(17) with $\bar{P}_0 = 0$ and $\bar{\Pi}_0(z) = \begin{pmatrix} 0 & 0 \\ 0 & -K(z) \end{pmatrix}$, with a new strictly positive space-varying damping term $K(z) > 0$. Consequently, the energy of the closed-loop system will exponentially decrease with a decay rate given by K.
We can then rewrite the system $\bar{x}$ in the Riemann coordinates. Using the same notations as the ones given in Section 2.1.3, we define $[\bar{\zeta}^+, \bar{\zeta}^-]^\top = Q^{-1}[\bar{x}_1, \bar{x}_2]^\top$. Next, we use an exponential variable change to suppress the diagonal in-domain couplings. Define for all $z \in [0,1]$,

$$\bar{A}(z) = \mathrm{diag}(f(z)e^{I_K(z)}, g(z)e^{-I_K(z)}),\ I_K(z) = \int_0^z \frac{K(s)}{2\lambda(s)} ds.$$

The new state $[\bar{\xi}^+, \bar{\xi}^-]^\top = \bar{A}[\bar{\zeta}^+, \bar{\zeta}^-]^\top$ satisfies hetero-directional transport equations of form (22)-(23) with in-domain couplings defined by

$$\begin{cases} \bar{\sigma}_{11}(z) = \bar{\sigma}_{22}(z) = 0, \\ \bar{\sigma}_{12}(z) = \frac{1}{2r(0)} e^{2I_K(z)}(K(z) - \frac{\lambda(z)}{2}\delta(z)), \\ \bar{\sigma}_{21}(z) = \frac{r(0)}{2} e^{-2I_K(z)}(K(z) + \frac{\lambda(z)}{2}\delta(z)). \end{cases}$$

The objective is now to design a control law such that the closed-loop system can be mapped to this desired target system.

3. Mapping of an hyperbolic $\ell + m$ PDE system to an arbitrary target system

As explained in Section 2.2, the objective of this paper is to map the original Port-Hamiltonian system (1)-(2) to the target Port-Hamiltonian system (16)-(17), thus ensuring closed-loop stability with a specific decay rate for the energy of the system. The parameter $\bar{\Pi}_0$ in equation(16) can therefore be seen as a degree of freedom reflecting the closed-loop system's dissipativity. This objective somehow corresponds to in-domain damping assignment [33, 34]. In this section, we show how we can use the backstepping methodology to map the original system (1)-(2) to the target system (16)-(17). As explained before, we consider the equivalent problem of mapping the system (10)-(11) (that corresponds to (1)-(2) in the Riemann coordinates) to the target system (22)-(23) (that corresponds to (16)-(17) in the Riemann coordinates). Although the original hyperbolic system we consider in Section 2 is expressed in the Port Hamiltonian framework, the result we have in this paper does not actually require this original system to adhere to the PHS paradigm. More precisely, we show that for any arbitrary system of linear balance laws (with boundary actuation), it is possible to design a controller such that we can map the closed-loop system to any target system with a similar structure but whose source terms can be arbitrarily chosen. This result is in itself a significant breakthrough as it gives the possibility of parametrizable target systems for the backstepping approach. As emphasized in Section 2, the PHS framework can indicate appropriate ones among these parametrizable target systems.

More precisely, we consider the following $(\ell+m)\times(\ell+m)$ hyperbolic PDE system [36]

$$\frac{\partial \xi^+}{\partial t} + \Lambda^+ \frac{\partial \xi^+}{\partial z} = \sigma^{++}(z)\xi^+ + \sigma^{+-}(z)\xi^-, \qquad (24)$$

$$\frac{\partial \xi^-}{\partial t} - \Lambda^- \frac{\partial \xi^-}{\partial z} = \sigma^{-+}(z)\xi^+ + \sigma^{--}(z)\xi^-, \qquad (25)$$

where ξ^+ is a vector of dimension $\ell > 0$ and ξ^- a vector of dimension $m > 0$. The velocity matrices are defined by $\Lambda = \mathrm{diag}(\Lambda^+, -\Lambda^-)$ with $\Lambda^+ = \mathrm{diag}(\lambda_1, ..., \lambda_\ell)$, $\Lambda^- = \mathrm{diag}(\mu_1, ...\mu_m)$ with $\lambda_1 > ... > \lambda_\ell > 0$, $\mu_1 > ... > \mu_m > 0$. The in-domain coupling $\sigma^{\cdot\cdot}$ are continuously differentiable functions. For all $1 \leq i \leq \ell$ and all $1 \leq k \leq m$, we assume $\sigma^{++}_{ii}(\cdot) = 0$, and $\sigma^{--}_{kk}(\cdot) = 0$. We have the boundary conditions

$$\xi^+(t,0) = Q_0\xi^-(t,0),\ \xi^-(t,1) = R_1\xi^+(t,1) + u(t), \quad (26)$$

with coupling matrices $Q_0 \in \mathbb{R}^{m\times\ell}$, $R_1 \in \mathbb{R}^{\ell\times m}$, and where $u(t)$ is the control input. An attentive reader would have noticed that the system (24)-(26) is more generic than the system (10)-(11) introduced in Section 2.1.2, since here ℓ and m are not necessarily equal. Indeed, the system (10)-(11) was obtained from the Port Hamiltonian system (1) and consequently the number of rightward propagating states was equal to the number of leftward propagating states, i.e. we had $\ell = m = n$ that is $\ell + m = 2n$. Again, this emphasizes the generality of the results we give in this Section. The initial condition of the system is denoted $(\xi_0^+,\ \xi_0^-) \in \chi_{\ell+m}$ and satisfies the compatibility conditions $\xi_0^+(0) = Q_0\xi_0^-(0)$, $\xi_0^-(1) = R_1\xi_0^+(1)$. Therefore, the open-loop system is well-posed [36]. We chose to state the well-posedness of the open-loop system in the state space $\chi_{\ell+m}$ (therefore requiring appropriate compatibility conditions) to avoid dealing with the weak formulation. However, we believe all the results we present in the rest of the paper can be extended to L^2 functions (and in that case, no compatibility condition would be required). Since the control operator is admissible, the closed-loop system remains well-posed for a continuous feedback law. Moreover, any stabilizing control law can be dynamically modified to satisfy the compatibility conditions in closed-loop while guaranteeing exponential stability [40]. The system (24)-(26) admits a unique solution, whose state is denoted $\xi = [\xi^{+\top}, \xi^{-\top}]^\top \in \chi_{m+\ell}$ and defined for $(t,z) \in [0,+\infty) \times [0,1]$. System (24)-(26) is schematically represented in Figure 1.

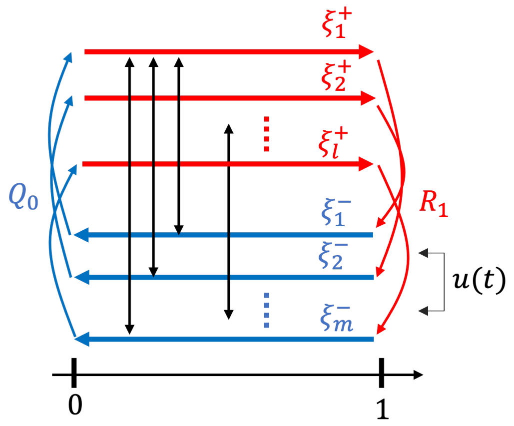

Figure 1: Schematic representation of an hyperbolic PDE system

3.1. Control objective

Our control objective is to find a state-feedback control law $u(t)$ such that the system (24)-(26) behaves in closed-loop as the following target system

$$\frac{\partial \bar{\xi}^+}{\partial t} + \Lambda^+ \frac{\partial \bar{\xi}^+}{\partial z} = \bar{\sigma}^{++}(z)\bar{\xi}^+ + \bar{\sigma}^{+-}(z)\bar{\xi}^-, \quad (27)$$

$$\frac{\partial \bar{\xi}^-}{\partial t} - \Lambda^- \frac{\partial \bar{\xi}^-}{\partial z} = \bar{\sigma}^{-+}(z)\bar{\xi}^+ + \bar{\sigma}^{--}(z)\bar{\xi}^-, \quad (28)$$

with boundary conditions

$$\bar{\xi}^+(t,0) = Q_0 \bar{\xi}^-(t,0), \;\; \bar{\xi}^-(t,1) = \bar{R}_1 \bar{\xi}^+(t,1), \quad (29)$$

where $\bar{\sigma}^{\cdot\cdot}$ are arbitrary continuously differentiable functions (the matrices $\bar{\sigma}^{++}, \bar{\sigma}^{--}$ have no term on their diagonal) and $\bar{R}_1$ is a constant matrix. In other words, we want to design a feedback controller such that the system (24)-(26) can be mapped to the target system (22)-(23) by means of appropriate transformations that will guarantee that the two systems share the same stability properties. Again, we emphasize that system (22)-(23) is a particular case of system (27)-(29). In that sense, the PHS approach provides adequate $\bar{\sigma}^{\cdot\cdot}$ and $\bar{R}_1$ to guarantee the exponential stability of the target system while imposing the energy decay rate in (18). Note that the boundary couplings at the unactuated boundary $z = 0$ and the velocity matrices are not modified.

3.2. Control strategy

We describe below the three steps we follow to fulfill our control objective:

1. In Section 3.3, we introduce a classical backstepping Volterra transform [35] ($\mathcal{K}$) to map the system ξ to a simpler target system γ for which most of the in-domain coupling terms have been moved at the actuated boundary;

2. Similarly, in Section 3.4, we use a second classical backstepping Volterra transform $\bar{\mathcal{K}}$ of the same form, to map the target system $\bar{\xi}$ to the system $\bar{\gamma}$, for which most of the in-domain coupling terms have been moved at the boundary $z = 1$.

3. Finally, in Section 3.5, we use a specific invertible time-affine transform $\mathcal{F}$ to map the system γ to the system $\bar{\gamma}$. This transformation is defined after a specific critical transport time t^*.

Composing the different transforms, it becomes straightforward to design the corresponding feedback law in Section 3.6 and map the system ξ to the system $\bar{\xi}$. Although it should be possible to write a single transformation encompassing the three transformations we propose, the resulting kernel equations become highly involved, and we did not manage to prove the existence of such a transformation in the general case. Introducing two intermediate systems (38)-(41) and (46)-(49) helped us reaching the ultimate control objective stated in the previous section. A schematic representation of the control strategy is given in Figure 2.

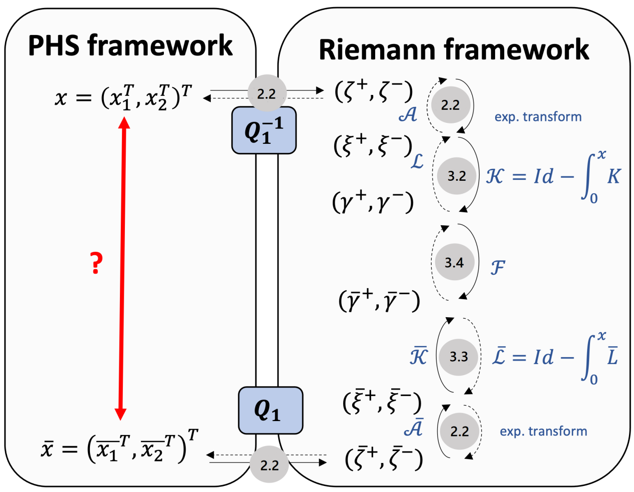

Figure 2: Overall strategy

3.3. First Volterra integral transform: simplify the ξ-system

Inspired by [35, 22], we use a classical Volterra transform to map the ξ-system (24)-(26) to a simpler system. More precisely, we for all $t>0$ and $z\in[0,1]$, we define the state $\gamma(t,z)$ by

$$\gamma(t,z)=\xi(t,z)-\int_0^z K(z,y)\xi(t,y)\mathrm{d}y, \tag{30}$$

where the kernels $K=\begin{pmatrix} K^{++} & K^{+-} \\ K^{-+} & K^{--}\end{pmatrix}$ belong to the space $C^1_{pw}(\mathcal{T}^-;\mathbb{R}^{(\ell+m)\times(\ell+m)})$. They satisfy the following partial differential equations

$$\begin{aligned}
\Lambda^+\frac{\partial K^{++}}{\partial z}+\frac{\partial K^{++}}{\partial y}\Lambda^+ &= -K^{++}\sigma^{++}(y)-K^{+-}\sigma^{-+}(y),\\
\Lambda^+\frac{\partial K^{+-}}{\partial z}-\frac{\partial K^{+-}}{\partial y}\Lambda^- &= -K^{++}\sigma^{+-}(y)-K^{+-}\sigma^{--}(y),\\
\Lambda^-\frac{\partial K^{-+}}{\partial z}-\frac{\partial K^{-+}}{\partial y}\Lambda^+ &= K^{-+}\sigma^{++}(y)+K^{--}\sigma^{-+}(y),\\
\Lambda^-\frac{\partial K^{--}}{\partial z}+\frac{\partial K^{--}}{\partial y}\Lambda^- &= K^{-+}\sigma^{+-}(y)+K^{--}\sigma^{--}(y).
\end{aligned} \tag{31}$$

with the boundary conditions,

$$\begin{aligned}
\Lambda^+K^{++}(z,z)-K^{++}(z,z)\Lambda^+ &= \sigma^{++}(z),\\
\Lambda^+K^{+-}(z,z)+K^{+-}(z,z)\Lambda^- &= \sigma^{+-}(z),\\
\Lambda^-K^{-+}(z,z)+K^{-+}(z,z)\Lambda^+ &= -\sigma^{-+}(z),\\
\Lambda^-K^{--}(z,z)-K^{--}(z,z)\Lambda^- &= -\sigma^{--}(z).
\end{aligned} \tag{32}$$

We add the following boundary condition for $i\le j$,

$$K^{--}_{ij}(z,0)=\left(K^{-+}(z,0)\Lambda^+Q_0(\Lambda^-)^{-1}\right)_{ij}. \tag{33}$$

If the boundary coupling matrix Q_0 is invertible, we can choose

$$K^{++}_{ij}(z,0)=\left(K^{+-}(z,0)\Lambda^-(\Lambda^+Q_0)^{-1}\right)_{ij},\ i\le j, \tag{34}$$

otherwise, we set $K^{++}_{ij}(z,0)=0,\ i\le j$. Finally, we impose arbitrary conditions in $z=1$ for $K^{\pm\pm}_{ij}$ when $i>j$. For instance, we can choose

$$K^{++}_{ij}(1,y)=\frac{\sigma^{++}_{ij}(1)}{\lambda_i-\lambda_j},\ K^{--}_{ij}(1,y)=\frac{\sigma^{--}_{ij}(1)}{\mu_j-\mu_i}. \tag{35}$$

The well-posedness of the kernel equation (31)-(35) is assessed in the following lemma:

Lemma 1. *The system defined by* (31)-(35) *admits a unique solution in* $C^1_{pw}(\mathcal{T}^-;\mathbb{R}^{(\ell+m)\times(\ell+m)})$. *Therefore, the Volterra integral transform defined by* (30) *is boundedly invertible. More precisely, for all* $t>0$ *and all* $z\in[0,1]$, *we have*

$$\xi(t,z)=\gamma(t,z)-\int_0^z L(z,y)\gamma(t,y))\mathrm{d}y, \tag{36}$$

where the kernel $L\in C^1_{pw}(\mathcal{T}^-;\mathbb{R}^{(\ell+m)\times(\ell+m)})$ *satisfies*

$$L(z,y)=-K(z,y)+\int_y^z K(z,s)L(s,y)ds. \tag{37}$$

Proof. The proof can be adjusted from [35, Theorem A.1]. The regularity of the kernels is a consequence of the regularity of the coupling terms. ■

Differentiating (30) with respect to time and space, and injecting therein the dynamics (24)-(26), we can show that the state $\gamma=[\gamma^{+\top},\gamma^{-\top}]^\top$ satisfies

$$\frac{\partial\gamma^+}{\partial t}(t,z)+\Lambda^+\frac{\partial\gamma^+}{\partial z}(t,z)=\Gamma^+(z)\gamma^-(t,0), \tag{38}$$

$$\frac{\partial\gamma^-}{\partial t}(t,z)-\Lambda^-\frac{\partial\gamma^-}{\partial z}(t,z)=\Gamma^-(z)\gamma^-(t,0), \tag{39}$$

with the boundary conditions

$$\gamma^+(t,0)=Q_0\gamma^-(t,0), \tag{40}$$

$$\gamma^-(t,1)=R_1\gamma^+(t,1)+u(t)+I_1(t). \tag{41}$$

The integral term $I_1(t)$ is given by

$$\begin{aligned}
I_1(t)=\int_0^1 &(L^{-+}(1,y)-R_1L^{++}(1,y))\gamma^+(t,y) \\
&+(L^{--}(1,y)-R_1L^{+-}(1,y))\gamma^-(t,y)dy,
\end{aligned} \tag{42}$$

while the coupling matrix-valued functions are defined by

$$\Gamma^+_{ij}(z)=\begin{cases}0,\ \text{for}\ \ j\le i\ \text{if}\ \ Q_0\ \text{invertible}\\ \left[K^{+-}(z,0)\Lambda^--K^{++}(z,0)\Lambda^+Q_0\right]_{ij},\text{else},\end{cases} \tag{43}$$

$$\Gamma^-_{ij}(z)=\begin{cases}0,\ \text{if}\ \ j\le i,\\ \left[K^{--}(z,0)\Lambda^--K^{-+}(z,0)\Lambda^+Q_0\right]_{ij},\text{else}.\end{cases} \tag{44}$$

Note that the matrix Γ^- is strictly lower triangular. The advantage of the system (38)-(41) relies on the triangular structure of the matrix Γ^-, as it creates a cascade structure in the system.

Remark 1. *In most contributions dealing with the backstepping stabilization of* $\ell+m$ *systems, the boundary conditions for the kernels* K^{++}_{ij} *(equation* (34)*) are usually set to zero or to arbitrary values [35]. However, having* Q_0 *invertible and choosing the kernel boundary condition as in equation* (34) *effectively allows for simplifying the target system as it induces a triangular structure for the matrix* Γ^+. *This type of kernel boundary condition is analogous to what has been done in [40] in the case of two equations.*

3.4. Second Volterra Integral transform: simplify the $\bar{\xi}$-system

Similarly to what we have done for the ξ-system, we use a Volterra integral transform of the second kind to map the target system (27)-(29) to a simpler system (46)-(49). More precisely, for all $t>0$ and all $z\in[0,1]$, we define the state $\bar{\gamma}$ as

$$\bar{\gamma}(t,z)=\bar{\xi}(t,z)-\int_0^z\bar{K}(z,y)\bar{\xi}(t,y)\mathrm{d}y. \tag{45}$$

where the kernels $\bar{K}=\begin{bmatrix}\bar{K}^{++} & \bar{K}^{+-}\\ \bar{K}^{-+} & \bar{K}^{--}\end{bmatrix}$ belong to the space $C^1_{pw}(\mathcal{T}^-;\mathbb{R}^{(\ell+m)\times(\ell+m})$ satisfy analogous equations to the ones given by (31)-(33) (except that the terms $\sigma^{\cdot\cdot}$ are replaced by functions $\bar{\sigma}^{\cdot\cdot}$ in equations (31)-(32),(35)). The transformation (45) is boundedly invertible, and the kernels of the associated inverse transform are denoted $\bar{L}$. Differentiating (45) with respect to time and space, and since $[\bar{\xi}^{+\top}, \bar{\xi}^{-\top}]^\top \in C^1([0,+\infty); H^1([0,1], \mathbb{R}^{\ell\times m}))$ is the unique solution of (27)-(29), we can show that the new state $\bar{\gamma} = [\bar{\gamma}^{+\top}, \bar{\gamma}^{-\top}]^\top$ satisfies

$$\frac{\partial \bar{\gamma}^+}{\partial t}(t,z) + \Lambda^+ \frac{\partial \bar{\gamma}^+}{\partial z}(t,z) = \bar{\Gamma}^+(z)\bar{\gamma}^-(t,0), \tag{46}$$

$$\frac{\partial \bar{\gamma}^-}{\partial t}(t,z) - \Lambda^- \frac{\partial \bar{\gamma}^-}{\partial z}(t,z) = \bar{\Gamma}^-(z)\bar{\gamma}^-(t,0), \tag{47}$$

with boundary conditions

$$\bar{\gamma}^+(t,0) = Q_0\bar{\gamma}^-(t,0), \tag{48}$$

$$\bar{\gamma}^-(t,1) = \bar{R}_1\bar{\gamma}^+(t,1) + I_2(t). \tag{49}$$

The integral term $I_2(t)$ given by

$$I_2(t) = \int_0^1 (\bar{L}^{-+}(1,y) - \bar{R}_1\bar{L}^{++}(1,y))\bar{\gamma}^+(t,y) + (\bar{L}^{--}(1,y) - \bar{R}_1\bar{L}^{+-}(1,y))\bar{\gamma}^-(t,y)dy.$$

The matrix $\bar{\Gamma}^+ \in C^1_{pw}([0,1], \mathbb{R}^{\ell\times m})$ and the matrix $\bar{\Gamma}^- \in C^1_{pw}([0,1], \mathbb{R}^{m\times m})$ are defined by

$$\bar{\Gamma}^+_{ij}(z) = \begin{cases} 0, \text{ for } j \le i, \text{ if } Q_0 \text{ invertible} \\ \left[\bar{K}^{+-}(z,0)\Lambda^- - \bar{K}^{++}(z,0)\Lambda^+Q_0\right]_{ij}, \text{else,} \end{cases} \tag{50}$$

$$\bar{\Gamma}^-_{ij}(z) = \begin{cases} 0, \text{ if } j \le i, \\ \left[\bar{K}^{--}(z,0)\Lambda^- - \bar{K}^{-+}(z,0)\Lambda^+Q_0\right]_{ij}, \text{else.} \end{cases} \tag{51}$$

Note that the matrix $\bar{\Gamma}^-$ is also strictly lower triangular.

3.5. Time-space affine change of variable

In this section, we now aim at mapping the system (38)-(41) to the system (46)-(49). Define for all $z \in [0,1]$, $t \ge t^* \doteq m\tau > 0$, the time-affine change of variables by

$$\bar{\gamma}^-_i(t,z) = \gamma^-_i(t,z) + \int_0^{\frac{1-z}{\mu_i}} \left(\sum_{j=1}^{i-1} F^-_{ij}(z,y)\gamma^-_j(t-y,0) + \sum_{j=2}^{i-1} H^-_{ij}(z,y)\bar{\gamma}^-_j(t-y,0) \right) \mathrm{d}y, \ 1 \le i \le m, \tag{52}$$

$$\bar{\gamma}^+_i(t,z) = \gamma^+_i(t,z) + \int_0^{\frac{z}{\lambda_i}} \left(\sum_{j=1}^{m} F^+_{ij}(z,y)\gamma^-_j(t-y,0) + \sum_{j=2}^{m} H^+_{ij}(z,y)\bar{\gamma}^-_j(t-y,0)\mathrm{d}y + \int_{\frac{z}{\lambda_i}}^{\frac{1}{\mu_m}+\frac{z}{\lambda_i}} \sum_{j=1}^{m-1} M^+_{ij}(z,y)\gamma^-_j(t-y,0) + \sum_{j=2}^{m-1} N^+_{ij}(z,y)\bar{\gamma}^-_j(t-y,0) \right) \mathrm{d}y. \ 1 \le i \le \ell, \tag{53}$$

Notice that this transformation requires past values of the boundary state $\bar{\gamma}(.,0)$. Therefore, it is only defined for $t > t^*$ so we can guarantee that the terms $(t-y)$ that appear in the different integrals are always positive. Also note that the first component of the leftward convecting state is not modified: $\forall z \in [0,1]$, $\bar{\gamma}^-_1(t,z) = \gamma^-_1(t,z)$. For all $1 \le i \le \ell$, $1 \le j \le m$ (resp. $2 \le j \le m$), F^+_{ij}, (resp. H^+_{ij}) is a real-valued function in $C^1_{pw}(\mathcal{T}^+_{\lambda_i}; \mathbb{R})$, and for all $1 \le i \le \ell$, $1 \le j \le m-1$ (resp. $2 \le j \le m-1$), M^+_{ij} (resp. N^+_{ij}) is a real-valued function defined in $C^1_{pw}(\mathcal{P}^+_{\frac{1}{\mu_m},\lambda_i}; \mathbb{R})$. For all $1 \le i \le m$, $1 \le j \le i-1$ (resp. $2 \le j \le i-1$), the function F^-_{ij} (resp. H^-_{ij}) is defined in $C^1_{pw}(\mathcal{T}^-_{\mu_i}; \mathbb{R})$. The different kernels are defined as follows

$$\underline{\text{for } \ 1 \le i \le \ell, \ \forall (z,y) \in \mathcal{T}^+_{\lambda_i},} \tag{54}$$

$$F^+_{ij}(z,y) = \delta_{j1}\bar{\Gamma}^+_{i1}(z - \lambda_i y) - \Gamma^+_{ij}(z - \lambda_i y), \ (1 \le j \le m),$$

$$H^+_{ij}(z,y) = \bar{\Gamma}^+_{ij}(z - \lambda_i y), \ (2 \le j \le m),$$

$$\underline{\text{for } \ 2 \le i \le m, \ \forall (z,y) \in \mathcal{T}^-_{\mu_i},} \tag{55}$$

$$F^-_{ij}(z,y) = \delta_{j1}\bar{\Gamma}^-_{i1}(z + \mu_i y) - \Gamma^-_{ij}(z + \mu_i y), \ (1 \le j \le i-1),$$

$$H^-_{ij}(z,y) = \bar{\Gamma}^-_{ij}(z + \mu_i y), \ (2 \le j \le i-1),$$

$$\underline{\text{for } \ 1 \le i \le \ell, \ \forall (z,y) \in \mathcal{P}^+_{\frac{1}{\mu_m},\lambda_i}, \ 1 \le j \le m-1} \tag{56}$$

$$M^+_{ij}(z,y) = \sum_{k=j+1}^{m} \mathbb{1}_{[\frac{z}{\lambda_i}, \frac{1}{\mu_k}+\frac{z}{\lambda_i}]}(y)(Q_0)_{ik}F^-(0, y - \frac{z}{\lambda_i}),$$

$$N^+_{ij}(z,y) = \sum_{k=j+1}^{m} \mathbb{1}_{[\frac{z}{\lambda_i}, \frac{1}{\mu_k}+\frac{z}{\lambda_i}]}(y)(Q_0)_{ik}H^-(0, y - \frac{z}{\lambda_i}).$$

The right-hand side of the proposed time-affine transformation (52)-(53) depends on $\bar{\gamma}^-$. However, due to the strict triangular structure of the transformation, the states $\bar{\gamma}^-_i$ and $\bar{\gamma}^+_i$ are properly defined. With straightforward but tedious computations, we could get rid of the $\bar{\gamma}(.,0)$ dependency in (52)-(53). More precisely, we have the following lemma

Lemma 2. *For all $i \le m$, there exist piecewise continuous functions $\mathscr{F}^-_{ij}$ $(1 \le j < i)$, $\mathscr{M}^-_{ij}$ $(1 \le j \le i-2)$ respectively defined on $\mathcal{T}^-_{\mu_i}$ and $\mathcal{P}^-_{b_{ij},\mu_i}$ (with $b_{ij} = \sum_{k=j+1}^{i-1} \frac{1}{\mu_k}$); and $\mathscr{F}^+_{ij}$ $(1 \le j \le m)$, $\mathscr{M}^+_{ij}$, $(1 \le j \le m-1)$, for $i \le \ell$, respectively defined on $\mathcal{T}^+_{\lambda_i}$ and $\mathcal{P}^+_{\tau,\lambda_i}$, such that the transformation (52)-(53) can be rewritten as*

$$\bar{\gamma}^-_i(t,z) = \gamma^-_i(t,z) + \int_0^{\frac{1-z}{\mu_i}} \sum_{j=1}^{i-1} \mathscr{F}^-_{ij}(z,y)\gamma^-_j(t-y,0)\mathrm{d}y + \sum_{j=1}^{i-2} \int_{\frac{1-z}{\mu_i}}^{b_{ij}+\frac{1-z}{\mu_i}} \mathscr{M}^-_{ij}(z,y)\gamma^-_j(t-y,0)\mathrm{d}y, \tag{57}$$

$$\bar{\gamma}^+_i(t,z) = \gamma^+_i(t,z) + \int_0^{\frac{z}{\lambda_i}} \sum_{j=1}^{m} \mathscr{F}^+_{ij}(z,y)\gamma^-_j(t-y,0)\mathrm{d}y$$

$$+\sum_{j=1}^{m-1}\int_{\frac{z}{\lambda_i}}^{A_j+\frac{z}{\lambda_i}}\mathscr{M}_{ij}^{+}(z,y)\gamma_j^{-}(t-y,0)\mathrm{d}y, \quad (58)$$

where $A_j=\frac{1}{\mu_m}+\sum_{k=j+1}^{m}\frac{1}{\mu_k}$.

PROOF. The proof is straightforward and is omitted. ■

Differentiating (52)-(53) with respect to time and space and integrating by parts, we can easily verify that, for $t \geq t^*$, the transformation (52)-(53) maps the solution of (38)-(41) to the solution of (46)-(49).

Let us now introduce the space $\chi \doteq H^1([0,1],\mathbb{R}^{2n}) \times \mathcal{D}_\tau$, where $\mathcal{D}_\tau \doteq D_{\tau_1}\times D_{\tau_2}\times ... \times D_{\tau_m}$. The delays τ_i are defined as $\tau_i=(m-i+1)\tau$. We emphasize that although the transformation (52)-(53) can be defined on χ, we actually consider here a specific case, where the function in $\mathcal{D}_\tau[t]$ corresponds to delayed values of (a subpart of) the boundary state at time t. This property will be crucial to show that (38)-(41) and (46)-(49) have equivalent stability properties. The following theorem states the invertibility of the transformation (52)-(53)

Theorem 1. *The transformation* (52)-(53) *is boundedly invertible on* χ.

PROOF. The invertibility of the transformation (52)-(53) can be straightforwardly shown by induction componentwise. Consider the proposition $\mathscr{P}_i$, defined for $1\leq i\leq m$ by $\mathscr{P}_i$: " For all $t\in[m\tau,+\infty)$, $\bar{\gamma}_i^{-}(t,.)$ can be expressed as a causal function of $\gamma_j^{-}(t,.)\in H^1([0,1],\mathbb{R}^{l+m})$, $\gamma_j^{-}(.,0)\in D_{\tau_j}$ $(1\leq j\leq i)$ and $\bar{\gamma}_{j'}^{-}(.,0)\in D_{\tau_{j'}}$ $(1\leq j'<i)$." Since $\gamma_1^{-}(t,z)=\bar{\gamma}_1^{-}(t,z)$, proposition $\mathscr{P}_1$ is true. Next, assume $\mathscr{P}_j$ is satisfied for $j<i$, and let us prove that $\mathscr{P}_i$ is true. By induction, the terms in (52) are well defined. In particular, we have

$$\begin{aligned}&\bar{\gamma}_i^{-}(t,0)=\gamma_i^{-}(t,0)\quad \text{known on } [t-\tau_i,t]\\ &+\int_0^{\frac{1}{\mu_i}}\underbrace{\sum_{j=1}^{i-1}F_{ij}^{-}(0,y)\gamma_j^{-}(t-y,0)+\sum_{j=2}^{i-1}H_{ij}^{-}(0,y)\bar{\gamma}_j^{-}(t-y,0)}_{\text{known on } [t-\tau_j+\frac{1}{\mu_i},t]\subset[t-\tau_i,t]}\mathrm{d}y,\end{aligned}$$

such that $\mathscr{P}_i$ is satisfied. The inverse transform is immediately given by

$$\begin{aligned}\gamma_i^{-}(t,z)=\bar{\gamma}_i^{-}(t,z)+\int_0^{\frac{1-z}{\mu_i}}\sum_{j=1}^{i-1}\bar{F}_{ij}^{-}(z,y)\bar{\gamma}_j^{-}(t-y,0)\\ +\sum_{j=2}^{i-1}\bar{H}_{ij}^{-}(z,y)\gamma_j^{-}(t-y,0)\mathrm{d}y, \quad (59)\end{aligned}$$

with $\bar{H}_{ij}^{-}(z,y)=-F_{ij}^{-}(z,y)$ and $\bar{F}_{ij}^{-}(z,y)=-\delta_{j1}F_{i1}^{-}(z,y)-\delta_{j>1}H_{ij}^{-}(z,y)$.
Next, the transform (53) is a well-defined affine transform. Since $\bar{\gamma}^{-}$ can be expressed as a function of γ^{-}, its invertibility is straightforward. ■

3.6. Stabilizing control law

We can finally design the adequate control input $u(t)$ such that (24)-(26) and (27)-(29) share equivalent stability properties. For all $t>2t^\star$, consider the control law

$$u(t)=\bar{R}_1\bar{\gamma}^{+}(t,1)-R_1\gamma^{+}(t,1)-I_1(t)+I_2(t). \quad (60)$$

By convention, we set $u(t)=0$ if $t\leq 2t^\star$. We choose the bound $2t^\star$ to avoid useless case distinctions in the different proofs (the bound $t^\star$ is necessary to define the time-affine transformation (52)-(53) properly). It is possible to express the control law (60) as a function of the original state ξ using the different transforms. Let us now prove that the closed-loop system ξ and the target system $\bar{\xi}$ have equivalent asymptotic stability properties. We first have the following lemma

Lemma 3. *There exists two constants* κ_0 *and* κ_1, *such that for any* $t>t^*$, *and* $0<r<\min(\frac{1}{\mu_1},\frac{1}{2m\|\Gamma^{-}\|_\infty^2})$,

$$\kappa_0\|\gamma^{-}_{[t+r]}(\cdot,0)\|_r\leq\|\gamma^{-}(t,.)\|_{L^2}\leq\kappa_1\|\gamma^{-}_{[t+\frac{1}{\mu_m}]}(\cdot,0)\|_{\frac{1}{\mu_m}}. \quad (61)$$

The exponential stability of $\gamma^{-}(t,\cdot)$ *in the sense of the* L^2-*norm is equivalent to the exponential stability of* $\gamma^{-}_{[t]}(.,0)$ *in the sense of the* $\mathcal{D}_{\frac{1}{\mu_m}}-$*norm.*

PROOF. The proof of this lemma is inspired by [23]. The right-hand side of inequality (61) can be obtained by rewriting $\gamma^{-}(t,z)$ as future values of $\gamma^{-}(\cdot,0)$. In what follows, C will be an overloaded constant. Using the methods of characteristics, we have, for $t>t^*$,

$$\begin{aligned}\|\gamma^{-}(t,.)\|_{L^2}^2&=\int_0^1\sum_{i=1}^{m}\gamma_i^{-}(t,\nu)^2\mathrm{d}\nu \text{ by definition}\\ &=\sum_{i=1}^{m}\int_0^1(\gamma_i^{-}(t+\frac{\nu}{\mu_i},0)\\ &+\int_0^{\frac{\nu}{\mu_i}}\sum_{k=1}^{i-1}\Gamma_{ik}^{-}(\nu-\mu_i s)\gamma_k^{-}(t+s,0)\mathrm{d}s)^2\mathrm{d}\nu\\ &\leq 2(\sum_{i=1}^{m}\int_0^1\gamma_i^{-2}(t+\frac{\nu}{\mu_i},0)\mathrm{d}\nu\\ &+\sum_{i=1}^{m}\int_0^1\int_0^{\frac{\nu}{\mu_i}}\sum_{k=1}^{i-1}\Gamma_{ik}^{-2}(\nu-\mu_i s)\gamma_k^{-2}(t+s,0)\mathrm{d}s\mathrm{d}\nu)\\ &\leq 2(\mu_1\int_0^{\frac{1}{\mu_m}}\sum_{i=1}^{m}\gamma_i^{-2}(t+s,0)\mathrm{d}s\\ &+m\int_0^{\frac{1}{\mu_i}}\sum_{k=1}^{m}\|\Gamma^{-}\|_\infty^2\gamma_k^{-2}(t+s,0)\mathrm{d}s)\\ &\leq C\|\gamma^{-}_{[t+\frac{1}{\mu_m}]}(.,0)\|^2_{\frac{1}{\mu_m}} \text{ using a time translation.}\end{aligned}$$

Similarly, we can obtain the left-hand side of inequality (61)

$$\int_0^r\sum_{i=1}^{m}\gamma_i^{-}(t+s,0)^2\mathrm{d}s=\int_0^r\sum_{i=1}^{m}(\gamma_i^{-}(t,\mu_i s)$$

$$
\begin{aligned}
&-\int_0^s \sum_{k=1}^{i-1} \Gamma_{ik}^-(\mu_i(s-\nu))\gamma_k^-(t+\nu,0)\mathrm{d}\nu)^2 \mathrm{d}s \\
&\leq 2\sum_{i=1}^m \int_0^{\mu_i r} \frac{1}{\mu_i}\gamma_i^{-2}(t,s)\mathrm{d}s \\
&+\sum_{i=1}^m \int_0^r \int_0^s \|\Gamma^-\|_\infty^2 \sum_{k=1}^{i-1<m} \gamma_k^{-2}(t+\nu,0)\mathrm{d}\nu\mathrm{d}s \\
&\leq 2\sum_{i=1}^m \int_0^1 \frac{1}{\mu_i}\gamma_i^{-2}(t,s)\mathrm{d}s \\
&+\sum_{i=1}^m \int_0^r \int_0^r \|\Gamma^-\|_\infty^2 \sum_{k=1}^{i-1<m} \gamma_k^{-2}(t+\nu,0)\mathrm{d}\nu\mathrm{d}s \\
&\leq \frac{2}{\mu_m}\|\gamma^-(t,.)\|_{L^2}^2 + 2mr\|\Gamma^-\|_\infty^2 \|\gamma_{[t+r]}^-(.,0)\|_r^2 \\
&\implies 0 < \frac{\mu_m}{2}(1-2mr\|\Gamma^-\|_\infty^2)\|\gamma_{[t+r]}^-(.,0)\|_r^2 \\
&\qquad\qquad \leq \|\gamma^-(t,.)\|_{L^2}^2 \text{ since } r < \frac{1}{2m\|\Gamma^-\|_\infty^2}
\end{aligned}
$$

Let us now show that the exponential stability of $\|\gamma^-(t,\cdot)\|$ in the sense of the L^2-norm is equivalent to the exponential stability of $\|\gamma_{[t]}^-(.,0)\|$ in the sense of the $\mathcal{D}_{\frac{1}{\mu_m}}-$norm. Let us consider first that $\gamma^-(.,0)$ is exponentially stable in the sense of the $\mathcal{D}_{\frac{1}{\mu_m}}-$norm. By definition, for any $\eta > 0$, there exists $C_0 > 0, \nu > 0$ such that for all $t > \max\{t^*, \eta\}$,

$$
\|\gamma_{[t]}^-(.,0)\|_{\frac{1}{\mu_m}} \leq C_0 e^{-\nu t}\|\gamma_{[\eta]}^-(.,0)\|_{\frac{1}{\mu_m}}.
$$

Then, for all $t > t^*$,

$$
\begin{aligned}
\|\gamma^-(t,.)\|_{L^2} &\leq \kappa_1 \|\gamma_{[t+\frac{1}{\mu_m}]}^-(\cdot,0)\|_{\frac{1}{\mu_m}} \text{ by (61)} \\
&\leq C_0 e^{-\frac{\nu}{\mu_m}}\kappa_1 e^{-\nu t}\|\gamma_{[\eta]}^-(.,0)\|_{\frac{1}{\mu_m}}.
\end{aligned}
$$

We can decompose the term $\|\gamma_{[\eta]}^-(.,0)\|_{\frac{1}{\mu_m}}$ using a finite number of terms defined on intervals of length r (where r is defined in the statement Lemma 3). Define $n_r = \max_{k\in\mathbb{N}}(kr \leq \frac{1}{\mu_m})$. We have

$$
\begin{aligned}
\|\gamma_{[\eta]}^-(.,0)\|_{\frac{1}{\mu_m}}^2 &= \sum_{i=1}^m \int_{-\frac{1}{\mu_m}}^0 \gamma_i^{-2}(\eta+s,0)\mathrm{d}s \\
&\leq \sum_{i=1}^m \sum_{k=0}^{n_r} \int_{-(k+1)r}^{-kr} \gamma_i^{-2}(\eta+s,0)\mathrm{d}s \\
&= \sum_{i=1}^m \sum_{k=0}^{n_r} \int_{-r}^{0} \gamma_i^{-2}(\eta+s-kr,0)\mathrm{d}s \\
&= \sum_{k=0}^{n_r} \|\gamma_{[\eta-kr]}^-(.,0)\|_r^2 \\
&\leq \sum_{k=0}^{n_r} \frac{1}{\kappa_0^2}\|\gamma^-(\eta-(k+1)r,.)\|_{L^2}^2
\end{aligned}
\tag{62}
$$

Choosing η such that $n_r + 1 \leq \eta < 2t^\star$ and since the system (38)-(41) is well-posed, there exists $\kappa > 0$, such that $\|\gamma^-(\eta-(k+1)r,.)\|_{L^2} \leq \kappa\|\gamma_0^-\|_{L^2}$. We finally obtain

$$
\|\gamma^-(t,.)\|_{L^2} \leq C_0\frac{\kappa_1}{\kappa_0^2}\kappa(n_r+1)e^{-\frac{\nu}{\mu_m}}e^{-\nu t}\|\gamma_0^-\|_{L^2}^2.
$$

This implies the exponential stability of γ^- in the sense of the L^2-norm. We can use the same arguments to prove that the exponential stability of γ^- in the sense of the L^2-norm implies the exponential stability of $\gamma^-(.,0)$ in the sense of the $\mathcal{D}_{\frac{1}{\mu_m}}-$norm. ■

We can finally show the equivalence between the original system ξ and the target system $\bar{\xi}$

Theorem 2. *Let choose the functions $\bar{\sigma}^{\pm\pm}$ and $\bar{\sigma}^{\mp\pm}$ such that for any initial conditions $(\bar{\xi}_0^+(z), \bar{\xi}_0^-(z)) \in H^1([0,1], \mathbb{R}^{(\ell+m)})$ satisfying the appropriate compatibility conditions, the solutions of* (27)-(29) *are exponentially stable in the sense of the L^2-norm. Then, for any initial conditions $(\xi_0^+(z), \xi_0^-(z)) \in H^1([0,1],\mathbb{R}^{(l+m)})$ satisfying the appropriate compatibility conditions, the solutions of* (24)-(26) *with the control input defined by* (60) *are also exponentially stable in the sense of the L^2-norm.*

Proof. We know that the L^2-exponential stability of the system (38)-(41) is equivalent to the L^2-exponential stability of the system (24)-(26) [35]. Similarly, we know that the L^2-exponential stability of the system (46)-(49) is equivalent to the L^2-exponential stability of the system (27)-(29). Therefore we only need to show that the L^2-exponential stability of the system (38)-(41) is equivalent to the L^2-exponential stability of the system (46)-(49). Let us assume first that the system (38)-(41) is L^2-exponentially stable. Consequently, there exists $C_0, \nu > 0$, such that for all $t \geq 2t^\star$,

$$
\|(\gamma^+(t,.),\gamma^-(t,.))\|_{L^2} \leq C_0 e^{-\nu t}\|(\gamma^+(\eta,.),\gamma^-(\eta,.))\|_{L^2},
$$

where η is such that $n_r + 1 \leq \eta < 2t^\star$. Using transformation (57), we have for all $t > 2t^\star$

$$
\begin{aligned}
&\int_0^1 \sum_{i=1}^m \bar{\gamma}_i^{-2}(t,s)\mathrm{d}s \\
&= \int_0^1 \sum_{i=1}^m \Bigg(\gamma_i^-(t,s) + \int_0^{\frac{1-s}{\mu_i}} \sum_{j=1}^{i-1} \mathscr{F}_{ij}^-(s,y)\gamma_j^-(t-y,0)\mathrm{d}y \\
&\quad + \sum_{j=1}^{i-2}\int_{\frac{1-s}{\mu_i}}^{b_{ij}+\frac{1-s}{\mu_i}} \mathscr{M}_{ij}^-(s,y)\gamma_j^-(t-y,0)\mathrm{d}y\Bigg)^2 \mathrm{d}s, \\
&\leq C(\int_0^1 \sum_{i=1}^m \gamma_i^{-2}(t,s)\mathrm{d}s + m\|\mathscr{F}^-\|_\infty^2 \|\gamma_{[t]}^-(.,0)\|_{\frac{1}{\mu_m}}^2 \\
&\quad + m\|\mathscr{M}^-\|_\infty^2\|\gamma_{[t]}^-(.,0)\|_{\frac{m}{\mu_m}}^2),
\end{aligned}
\tag{63}
$$

where C is a constant. Using inequality (62), there exists $k_1 > 0$ such that

$$
\|\gamma_{[t]}^-(.,0)\|_{\frac{1}{\mu_m}} \leq k_1 \mathrm{e}^{-\nu t}\|(\gamma^+(\eta,.),\gamma^-(\eta,.))\|_{L^2}.
$$

Similarly, we can show that there exists $k_2 > 0$ such that

$$\|\gamma^-_{[t]}(.,0)\|_{\frac{m}{\mu_m}} \leq k_2 \mathrm{e}^{-\nu t}\|(\gamma^+(\eta,.),\gamma^-(\eta,.))\|_{L^2}.$$

Injecting in equation (63), there exists a constant $C_1 > 0$ such that for all $t > 2t^\star$, we have

$$\|\bar{\gamma}^-(t,.)\|_{L^2} \leq C_1 e^{-\nu t}\|(\gamma^+(\eta,.),\gamma^-(\eta,.))\|_{L^2}.$$

Similarly, using transformation (53), we can show the existence of a constant C_2 such that for all $t > 0$

$$\|\bar{\gamma}^+(t,.)\|_{L^2} \leq C_2 e^{-\nu t}\|(\gamma^+(\eta,.),\gamma^-(\eta,.))\|_{L^2}.$$

We now need to bound $\|(\gamma^+(\eta,.),\gamma^-(\eta,.))\|_{L^2}$ by a term that depends on $\|(\bar{\gamma}^+(0,.),\bar{\gamma}^-(0,.))\|_{L^2}$. Due to transformation (59), we have

$$\int_0^1 \sum_{i=1}^m \gamma_i^{-2}(\eta,s)\mathrm{d}s \leq C\left(\int_0^1 \sum_{i=1}^m \bar{\gamma}_i^{-2}(\eta,s)\mathrm{d}s + m\|\bar{\mathscr{F}}^-\|^2_\infty \right.$$
$$\left. \|\bar{\gamma}^-_{[\eta]}(.,0)\|^2_{\frac{1}{\mu_m}} + m\|\bar{\mathscr{M}}^-\|^2_\infty \|\bar{\gamma}^-_{[\eta]}(.,0)\|^2_{\frac{m}{\mu_m}}\right). \tag{64}$$

Therefore, adjusting inequality (62) and using the well-posedness of the system (46)-(49), we obtain

$$\|(\gamma^-(\eta,.))\|_{L^2} \leq C_3\|(\bar{\gamma}^+(0,.),\bar{\gamma}^-(0,.))\|_{L^2},$$

where C_3 is a positive constant. Similar computations give

$$\|(\gamma^+(\eta,.))\|_{L^2} \leq C_4\|(\bar{\gamma}^+(0,.),\bar{\gamma}^-(0,.))\|_{L^2},$$

where C_4 is a positive constant. Consequently, there exists $C_5 > 0$ such that for all $t > 2t^\star$

$$\|(\bar{\gamma}^+(t,.),\bar{\gamma}^-(t,.))\|_{L^2} \leq C_5 e^{-\nu t}\|(\bar{\gamma}^+(0,.),\bar{\gamma}^-(0,.))\|_{L^2}.$$

This last inequality still holds when $t \leq 2t^\star$ due to the well-posedness of the system (46)-(49). Consequently, the system (46)-(49) is exponentially stable. The converse can be easily proved using the inverse transformation (59). ■

Remark 2. *The control law requires the knowledge of distributed values of the state ξ. It is, therefore, necessary to design a state observer that can be coupled with the proposed state feedback controller to obtain an output-feedback control law. Well-posed backstepping observers have already been designed in the literature for hyperbolic systems (see [37, 25] for instance). The resulting closed-loop system would be well-posed, as the control input would remain continuous. The closed-loop exponential stability would be guaranteed due to the input-to-state stability of the original* linear *hyperbolic system [42]. Obviously, the observer proposed in [37] is not the only possible observer, and we could add degrees of freedom while designing the observer, adjusting the approach presented in the current paper.*

Remark 3. *The proposed time-affine transformation* (52)-(53) *is only defined for $t > t^* = m\tau$. This time $t^\star$ corresponds to propagation of the "information" at the boundary. Since the control input is set to zero for $t \leq 2t^\star$, this may deteriorate the transient properties of the system. Following [37], the delay $t^\star$ may be reduced to a smaller value such as* $\max\{\frac{1}{\mu_i}\} = \frac{1}{\mu_m}$, *to the price of a more intricated transformation.*

Remark 4. *So far, we have not discussed the robustness properties of the proposed control law* (60). *It has been shown in [43] that canceling the boundary coupling terms in equation* (60) *could lead to vanishing robustness margins. Thus, depending on the matrices R_1 and $\bar{R}_1$, the closed-loop system may not be robust to parameter uncertainties or input delays. Although the complete robustness analysis of the closed-loop system is out of the scope of the paper, we can mention several methods to guarantee the existence of robustness margins:*

- *Choose the desired target system such that the conditions of [23, Theorem 5] are satisfied. This means that we can only cancel a small part of the boundary reflection terms (in the Riemann coordinates). However, this condition needs to be expressed in the PHS framework.*
- *Low-pass filter the control law with an appropriate filter, as shown in [44]. With this solution, the closed-loop performance may not be exactly the same as the ones of the desired target system (at least for the high frequencies), but the control transfer function will be strictly proper, thus guaranteeing the existence of robustness margins [44, 39].*

4. Application to the distributed damping assignment for a wave equation

In this section, we apply our strategy to perform distributed damping assignment for a clamped string (wave equation). The results we give below are inspired by [33]. We consider the vibrating string introduced in Section 2.1.3 and Section 2.2.3 and its expression in the Riemann coordinates.

4.1. Control design

We can apply the methodology presented in Section 3 to design a stabilizing control law that guarantees the desired damping assignment. Note that in this low-dimensional case ($\ell = m = 1$), there is no need for the time-affine transformation (52)-(53), and the approach given in Section 3 is a bit superfluous, though the idea behind remains the same. In particular, we have $t^\star = 0$. Indeed, we can directly map the ξ-system (24)-(26) to the system (22)-(23) using the backstepping transformation

$$\begin{pmatrix}\bar{\xi}^+\\ \bar{\xi}^-\end{pmatrix} = \begin{pmatrix}\xi^+\\ \xi^-\end{pmatrix} - \int_0^z \begin{pmatrix}K^{++} & K^{+-}\\ K^{-+} & K^{--}\end{pmatrix}(z,y)\begin{pmatrix}\xi^+\\ \xi^-\end{pmatrix}(y)dy,$$

where the kernels $K^{\pm\mp}$ are uniquely defined on $\mathcal{T}^-$ as the solution of the following set of equations

$$\lambda(z)\frac{\partial K^{++}}{\partial z} + \frac{\partial \lambda(y)K^{++}}{\partial y} = \bar{\sigma}^{+-}(z)K^{-+} - \sigma^{-+}(y)K^{+-},$$
$$\lambda(z)\frac{\partial K^{+-}}{\partial z} - \frac{\partial \lambda(y)K^{+-}}{\partial y} = \bar{\sigma}^{+-}(z)K^{--} - \sigma^{+-}(y)K^{++},$$
$$\lambda(z)\frac{\partial K^{-+}}{\partial z} - \frac{\partial \lambda(y)K^{-+}}{\partial y} = \sigma^{-+}(y)K^{--} - \bar{\sigma}^{-+}(z)K^{++},$$
$$\lambda(z)\frac{\partial K^{--}}{\partial z} + \frac{\partial \lambda(y)K^{--}}{\partial y} = \sigma^{+-}(y)K^{-+} - \bar{\sigma}^{-+}(z)K^{+-},$$

with boundary conditions

$$K^{+-}(z,z) = \frac{\sigma^{+-} - \bar{\sigma}^{+-}}{2\lambda(z)},\ K^{++}(z,0) = q_0^{-1}K^{+-}(z,0),$$
$$K^{-+}(z,z) = \frac{\bar{\sigma}^{-+} - \sigma^{-+}}{2\lambda(z)},\ K^{--}(z,0) = q_0 K^{-+}(z,0).$$

The well-posedness of the kernel equations is proved in [45]. The set of above equations admits a unique piecewise continuous solution on $\mathcal{T}^-$. The control input is then given by

$$u_0(t) = \frac{\lambda(1)}{\sqrt{2}g(1)e^{-I_c(1)}}\Big[(a_1 - r_1)\xi^+(1,t) + \int_0^1 (K^{-+}(1,y) - a_1K^{++}(1,y))\xi^+(y) + (K^{--}(1,y) - a_1K^{+-}(1,y))\xi^-(y)\mathrm{d}y\Big]. \tag{65}$$

Remark 5. *In this specific low-dimensional case, we could have directly mapped the PHS system* (14) *to the desired target system* $\bar{x}$ *using the transformation*

$$\begin{pmatrix}\bar{x}_1\\ \bar{x}_2\end{pmatrix} = C(z)\begin{pmatrix}x_1\\ x_2\end{pmatrix} - \int_0^z \begin{pmatrix}N^{++} & N^{+-}\\ N^{-+} & N^{--}\end{pmatrix}(z,y)\begin{pmatrix}x_1\\ x_2\end{pmatrix}(y)dy.$$

with, for all $z \in [0,1]$, $I(z) = I_c(z) - I_K(z)$ *and*

$$C(z) = \begin{pmatrix}\cosh(I(z)) & -\frac{1}{\gamma(x)}\sinh(I(z))\\ -\gamma(z)\sinh(I(z)) & \cosh(I(z))\end{pmatrix}.$$

4.2. Simulation results

In this section, we illustrate the performances of the proposed control approach with Matlab simulations. We consider a soft PVC string of length 1m, with constant physical parameters $\rho = 1.35\times10^3$kg.m^{-3}, $E = 0.9$ GPa. Its initial position is $w_0(z) = 0.1\sin(\frac{2x}{\pi})$, and no speed. We simulated system (24)-(26) on a time horizon of 20s using a Godunov Scheme [46] (with a Courant–Friedrichs–Lewy condition $CFL = 0.99$). The space domain $[0,1]$ is discretized with a mesh of 200 points. Beforehand, the kernels $K^{\pm\mp}$ are computed offline using a fixed-point algorithm. The control input is computed at each time step using (65). We consider a case where the string is naturally slightly damped ($c = 0.1$). We want to artificially assign a higher damping coefficient $K = 5c > 0$ to the closed-loop system using the proposed control input. From the values of $[\bar{\xi}^+, \bar{\xi}^-]$, we can numerically compute the evolution of the displacement $w(t,z)$ along the string. As illustrated in Figure 3, the oscillations naturally present in open-loop (top) are substantially damped in closed-loop (bottom). The string is stabilized around a stable position. We represent the energy evolution for both the open-loop and closed-loop systems in Figure 4. With the proposed control input (blue), the energy of the closed-loop system decays faster than the natural open-loop decay (red). It follows the reference energy decay of an open-loop system with in-domain damping K (dotted red). We represented in Figure 5 the evolution of the full-state feedback (65). As expected, the control effort goes to zero.

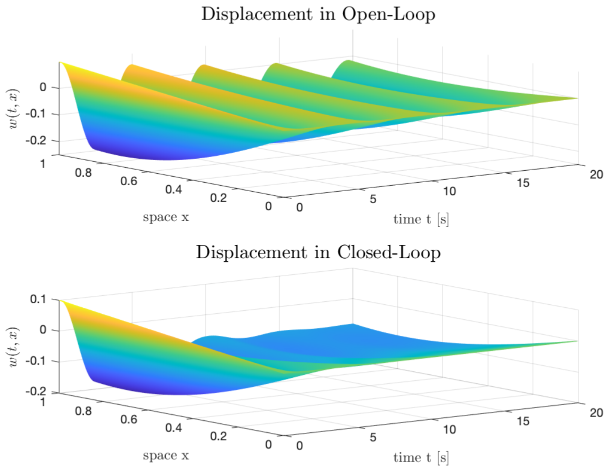

Figure 3: 3D-displacement $w(t,z)$ in open-loop (top), and closed-loop (bottom) using (65).

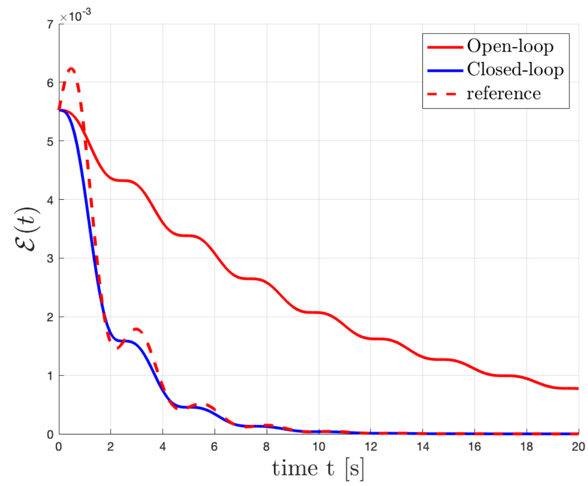

Figure 4: Evolution of the energy $\mathcal{E}(t)$.

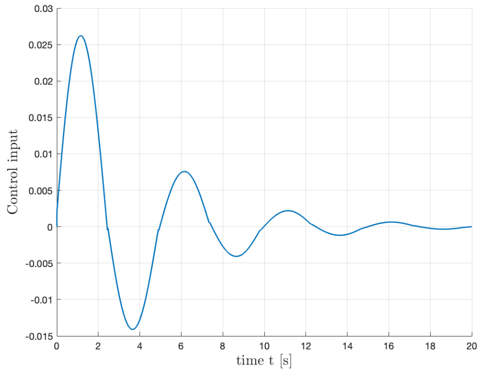

Figure 5: Evolution of the control effort $u(t)$.

525 ## 5. Application to the distributed damping assignment for a Timoshenko Beam

In this section, we apply our strategy to perform distributed damping assignment for a Timoshenko beam. This model is usually used to represent compliant mechanical structures such as cantilevers or flexible endoscopes [47]. It takes into account shear deformation and rotational inertia of the structure. This second test case is more generic than the one presented in Section 4, since the system under consideration is of larger dimension. The results from this section are inspired by [34].

5.1. Reformulation in the Port-Hamiltonian framework

We consider the clamped actuated Timoshenko beam model proposed in [48]. We denote $w(t,z)$ (resp. $\phi(t,z)$) the transverse normalized displacement (resp. rotation angle) defined on $[0,\infty)\times[0,1]$. From the balance equations on the momenta, they satisfy

$$\rho\frac{\partial^2 w}{\partial t^2}(t,z) = \frac{\partial}{\partial z}\left(K_s\left(\frac{\partial w}{\partial z}(t,z)-\phi(t,z)\right)\right), \quad (66)$$

$$I_\rho\frac{\partial^2 \phi}{\partial t^2}(t,z) = \frac{\partial}{\partial z}\left(EI\frac{\partial \phi}{\partial z}\right)+K_s\left(\frac{\partial w}{\partial z}-\phi\right). \quad (67)$$

For the sake of simplicity, we assume that all physical parameters (mass per length unit ρ, rotary moment of inertia of a cross-section I_ρ, Young's modulus of elasticity E, moment of inertia I and shear modulus K_s) are constant. The initial position of the beam is given by $w(z,0)=w_0(z)\in C^1([0,1])$, $\phi(z,0)=\phi_0(z)\in C^1([0,1])$. Following the Port-Hamiltonian framework, we define the energy state variables $x=[x_1^\top, x_2^\top]^\top$ by

$$x_1(t,z)=\begin{pmatrix}\frac{\partial w}{\partial x}(t,z)-\phi(t,z)\\ \frac{\partial \phi}{\partial x}(t,z)\end{pmatrix},\; x_2(t,z)=\begin{pmatrix}\rho\frac{\partial w}{\partial t}(t,z)\\ I_\rho\frac{\partial \phi}{\partial t}(t,z)\end{pmatrix},$$

where $x_1(t,z)$ represents shear and angular displacements, while $x_2(t,z)$ represents momentum and angular momentum. The state x belongs to $\chi_{2+2}=\chi_4$. The Hamiltonian density matrix $\mathcal{H}$ is given by

$$\mathcal{H}=\operatorname{diag}\left(K_s, EI, \frac{1}{\rho}, \frac{1}{I_\rho}\right)\in D_4^+. \quad (68)$$

No movement is allowed at the clamped end, while the opposite end is fully actuated, such that

$$x_2(t,0)=0_{\mathbb{R}^2},\quad \begin{pmatrix}K_s & 0\\ 0 & EI\end{pmatrix}x_1(t,1)=u_0(t)\in\mathbb{R}^2, \quad (69)$$

where $u_0(t)$ is the control input. The system (66)-(69) fits in the general framework presented in Section 2 and verifies Assumption 1. As in Section 4, our objective is to impose a specific decay rate to the energy of the system $\mathcal{E}$, using *a distributed damping assignment*. The target system is defined by equations (16)-(17) where $\bar{P}_0=P_0$ and $\bar{\Pi}_0$ is given by

$$\bar{\Pi}_0=\operatorname{diag}(0,0,c_3\rho,c_4 I_\rho).$$

More precisely, we want the closed-loop system to behave as if there were arbitrary distributed damping terms $c_3, c_4>0$. In that case, due to equation (18), the energy of the system decays as

$$\frac{d\mathcal{E}}{dt}=-\int_0^1\left([\frac{c_3}{\rho},\frac{c_4}{I_\rho}]\bar{x}_2^2(t,z)\right)\mathrm{d}z. \quad (70)$$

5.2. Riemann invariants

To apply the results of Section 3, we need to rewrite the system (66) in the Riemann coordinates. In what follows, we define the transport velocities $\lambda=\sqrt{\frac{K_s}{\rho}}>0$, $\mu=\sqrt{\frac{EI}{I_\rho}}>0$ and the matrices $\Lambda^+=\Lambda^-\doteq\Lambda=\operatorname{diag}(\lambda,\mu)$. We assume (without any loss of generality) that $\lambda>\mu$. We also define $R=\operatorname{diag}(\frac{\lambda}{K_s},\frac{1}{\mu I_\rho})\in D_2^+$, and $\alpha\doteq\frac{c_3}{2\lambda}$, $\beta\doteq\frac{c_4}{2\mu}$. The matrix $P_1\mathcal{H}\in\mathbb{R}^{4\times4}$ is invertible with four distinct real eigenvalues $\{\pm\lambda,\pm\mu\}$. It is diagonalizable, such that we have $P_1\mathcal{H}=Q\operatorname{diag}(-\Lambda,\Lambda)Q^{-1}$ with

$$Q=\frac{1}{\sqrt{2}}\begin{pmatrix}-R & I_2\\ I_2 & R^{-1}\end{pmatrix}\in\mathbb{R}^{4\times4}. \quad (71)$$

Original system

Let us define the new state $\xi=Q^{-1}x$, decomposed into $\xi=[\xi^{+\top},\xi^{-\top}]^\top\in\chi_4$. It verifies system (24)-(26) with boundary coupling terms $Q_0=-R^{-1}$ and $R_1=R$, while the in-domain coupling matrices are given by

$$\sigma^{++}=\frac{1}{2}\begin{pmatrix}0 & \frac{K_s}{\lambda I_\rho}\\ -\lambda & 0\end{pmatrix},\; \sigma^{+-}=\frac{1}{2}\begin{pmatrix}0 & \frac{\mu K_s}{\lambda}\\ K_s & 0\end{pmatrix},$$

$$\sigma^{-+}=\frac{1}{2}\begin{pmatrix}0 & -\frac{1}{I_\rho}\\ -\frac{\lambda}{\mu I_\rho} & 0\end{pmatrix},\; \sigma^{--}=\frac{1}{2}\begin{pmatrix}0 & -\mu\\ \frac{K_s}{\mu I_\rho} & 0\end{pmatrix}.$$

The control input is given by $u(t)=\sqrt{2}u_0(t)$.

Target System

We perform the same change of variables to rewrite the target system (16)-(17) in the Riemann coordinates. We define first $\bar{\zeta} = Q^{-1}\bar{X}$, where $\bar{\zeta} = [\bar{\zeta}^{+\top}, \bar{\zeta}^{-\top}]^\top \in \chi_4$. This new state verifies system (27)-(29), where the coupling terms are defined by

$$\bar{\sigma}^{++} = \frac{1}{2}\begin{pmatrix} -c_3 & \frac{K_s}{\lambda I_\rho} \\ -\lambda & -c_4 \end{pmatrix}, \; \bar{\sigma}^{+-} = \frac{1}{2}\begin{pmatrix} -\frac{K_s}{\lambda}c_3 & \frac{K\mu}{\lambda} \\ K_s & -\mu I_\rho c_4 \end{pmatrix},$$

$$\bar{\sigma}^{-+} = \frac{1}{2}\begin{pmatrix} -\frac{\lambda}{K_s}c_3 & -\frac{1}{I_\rho} \\ -\frac{\lambda}{\mu I_\rho} & -\frac{c_4}{\mu I_\rho} \end{pmatrix}, \; \bar{\sigma}^{--} = \frac{1}{2}\begin{pmatrix} -c_3 & -\mu \\ \frac{K_s}{\mu I_\rho} & -c_4 \end{pmatrix}.$$

For all $z \in [0,1]$, we define $\bar{A}(z) = \text{diag}(e^{\alpha z}, e^{\beta z}, e^{-\alpha z}, e^{-\beta z})$. To remove the diagonal terms of $\bar{\sigma}^{++}$ and $\bar{\sigma}^{--}$, we now consider the state $\bar{\xi} = \left(\bar{\xi}^{+\top}, \bar{\xi}^{-\top}\right)^\top = \bar{A}\bar{\zeta}$. It verifies system (27)-(29), with the boundary coupling

$$\bar{R}_1 = \text{diag}(e^{-2\alpha}, e^{-2\beta})R.$$

The space-dependant in-domain coupling terms are defined by

$$\bar{\sigma}^{++}(z) = \tfrac{1}{2}\begin{pmatrix} 0 & \frac{K_s}{\lambda I_\rho}e^{(\alpha-\beta)z} \\ -\lambda e^{-(\alpha-\beta)z} & 0 \end{pmatrix},$$

$$\bar{\sigma}^{+-}(z) = \tfrac{1}{2}\begin{pmatrix} -\frac{K_s}{\lambda}c_3 e^{2\alpha z} & \frac{K_s\mu}{\lambda}e^{(\alpha+\beta)z} \\ K_s e^{(\alpha+\beta)z} & -\mu I_\rho c_4 e^{2\beta z} \end{pmatrix},$$

$$\bar{\sigma}^{-+}(z) = \tfrac{1}{2}\begin{pmatrix} -\frac{\lambda}{K_s}c_3 e^{-2\alpha z} & -\frac{1}{I_\rho}e^{-(\alpha+\beta)z} \\ -\frac{\lambda}{\mu I_\rho}e^{-(\alpha+\beta)z} & -\frac{c_4}{\mu I_\rho}e^{-2\beta z} \end{pmatrix},$$

$$\bar{\sigma}^{--}(z) = \tfrac{1}{2}\begin{pmatrix} 0 & -\mu e^{-(\alpha-\beta)z} \\ \frac{K_s}{\mu I_\rho}e^{(\alpha-\beta)z} & 0 \end{pmatrix}.$$

5.3. Control design

We can now apply the methodology presented in Section 3 to design a stabilizing control law that guarantees the desired damping assignment.

Following the approach presented in Section 3.5, we define the transformation $\mathcal{F} : \chi_4 \to \chi_4$ such that $\bar{\gamma} = \mathcal{F}(\gamma)$, by (52)-(53). The resulting control input reads

$$u_0(t) = \frac{1}{\sqrt{2}}u(t) \tag{72}$$

with $u(t)$ defined by (60). It could be rewritten with distributed values of the original states x using the different transforms. More details can be found in [34].

5.4. Simulation results

We now illustrate the performance of our control strategy with simulation results. We consider a Timoshenko beam with physical parameters given in Table 1. It is initially at rest at a position $w_0(z) = 0.1\sin(\frac{\pi}{2}z)$, $\phi_0(z) = 0$. Using the above control strategy, we want the closed-loop beam to behave as (16)-(17) with $c_3 = 0.5$, $c_4 = 0.8$SI.

We simulate system (24)-(26) on the time interval $[0, 15]$s using a Godunov Scheme [46] ($CFL = 1$ and $\text{d}x = 0.02$m).

Param.	Description	Value
E	elastic modulus	1kg $\text{m}^{-1}\text{s}^{-2}$
I	second moment of area	0.5m^4
I_ρ	rotary moment of inertia	0.9kg m
K_s	shear modulus	1.2kg m s^{-2}
ρ	linear density	0.9 kg m^{-1}

Table 1: Numerical values for simulation

As illustrated in Figure 6, the position of the open-loop system oscillates around an equilibrium. Its energy is approximately constant due to the absence of dissipative terms. It is represented in Figure 8 (red).

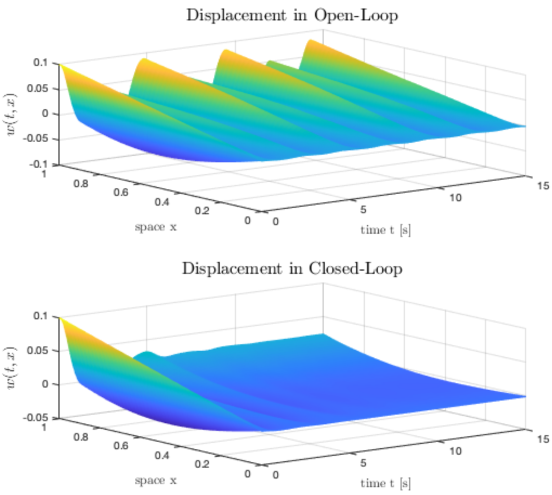

Figure 6: 3D-displacement $w(t, z)$ in open-loop (top) and closed-loop (bottom) using (72).

The evolution of the control effort is represented in Figure 7. Note that here it only equals zero for $t < t^\star$. After the control input is applied, the position of the closed-loop system converges quickly to an equilibrium. As illustrated in Figure 8 (blue), its energy decreases at the same rate as the target system (dotted red). Since the control law (72) requires the computation of state $\gamma, \bar{\gamma}$ at each time step, it is, therefore, more computationally expensive than traditional PI controllers. However, we believe that our approach could yield potential interesting trade-offs between performance and control effort. This will be the purpose of future research works.

6. Conclusion

In this paper, we developed a systematic framework to stabilize a general class of hyperbolic systems while assigning a specified closed-loop behavior with a clear energy interpretation. We took advantage of the PHS framework to emphasize the system's natural physical properties and

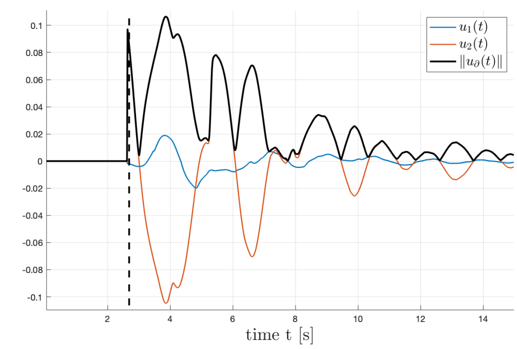

Figure 7: Evolution of the control effort $u_\partial(t)$

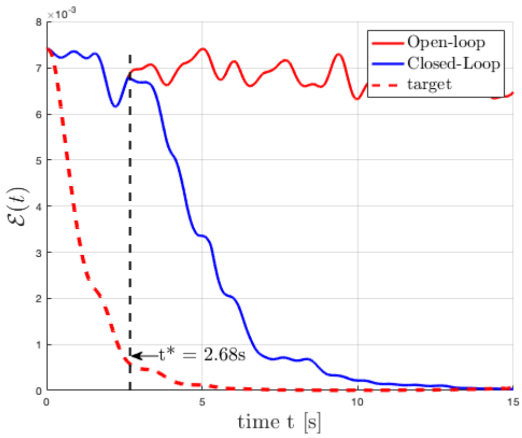

Figure 8: Evolution of the energy $\mathcal{E}(t)$ in open-loop and closed-loop

identify the naturally dissipative terms. This framework was then used to introduce degrees of freedom in the control design to obtain a class of easily parametrizable, exponentially stable closed-loop systems with a specified energy decay. The associated controllers were obtained using the backstepping methodology. Under generic structural assumptions, we could first rewrite the original Port Hamiltonian system in the Riemann coordinates. Then, we combined classical Volterra transformations with an innovative time-affine transform to map the original system to the desired target system. We finally illustrated the proposed approach on two application cases: clamped string with space-varying coefficients and a clamped Timoshenko beam. Interestingly, a by-product of this paper is to give a general class of reachable target systems for balance laws equations. Introducing degrees of freedom in the control design to obtain modular controllers is crucial to guarantee potential trade-offs between different specifications (namely delay-robustness and convergence rate). For instance, preserving naturally dissipative terms while imposing a specific energy decay should allow a reduced control effort compared to traditional approaches. If the qualitative effect of the tuning parameters can be understood in the PHS framework, we believe it is crucial to assess and quantify the performance of the resulting controllers with respect to a given set of specifications. This set of performance criteria should be defined in terms of practically relevant properties for industrial applications, e.g., sensitivity, robustness margins, smoothness of the state, or convergence rate. Such a complete performance analysis has to be developed. It will be the purpose of our future work. We also aim to generalize our approach to more complex interconnected systems.

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