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Abstract

This paper provides a new representation of some Kummer functions with integer coefficients in terms of iterated integrals of an exponential kernel. As a consequence of the existing links between Kummer, Whittaker, and modified Bessel functions, the latter also take advantage of such an iterated integral representation. This novel representation sheds some light on the links between properties of iterated integrals and the ones of those hypergeometric functions.

Keywords. Iterated integrals, hypergeometric functions, Kummer functions, Whittaker functions, modified Bessel functions.
MSC 2020. 33C15, 30E20, 33C90, 30C15.

Contents

1 Introduction

The interest in understanding and exploiting properties of *iterated integrals* (also called *repeated integrals*) has a long history in mathematical analysis owing to their use, among others, in the calculus of variations and in dynamical systems [22]. For instance, in integral calculus, Fubini's [19] and Tonelli's theorems are fundamental results used to integrate functions with several variables or to integrate

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a function defined by an integral depending on a parameter. Iterated integrals served in computing surfaces and volumes [19] and in deriving numerical solutions for systems of linear ordinary differential equations, such as Picard's procedure which is at the origin of many iterative methods for integrating classes of partial differential equations and stochastic dynamical systems, see for instance [24].

From an algebraic geometry point of view, iterated integrals play an interesting role in relating analysis on manifolds (or differentiable spaces) to the homology of the corresponding path spaces, see [14]. As a matter of fact, Lie groups and Lie algebras define an appropriate framework relating iterated integrals to the fundamental group of a manifold, see for instance [20].

Iterated integrals also find interest in number theory and have been exploited in the study of properties of some transcendental functions such as the well-known Riemann ζ function, see for instance [32]. In fact, a year after Apéry's proof for the irrationality of $\zeta(3)$, Beukers provided in [5] a shorter and more elegant proof relying on iterated integrals, the shape of which has been motivated by Apéry's formulas, see for instance [4].

More recently, iterated integrals have appeared in *exponential polynomial*[1] interpolation problems, see for instance [2, 12]. While, in [2], such iterated integrals have been revealed in the context of some Lagrange-type interpolation problems [11], their emergence in [12] is related to Hermite-type interpolation problems, yielding nice factorization formulas for a class of transcendental functions of exponential polynomial type. Interestingly, both findings have been exploited in setting a methodology called *partial pole placement* [10] for controlling infinite-dimensional dynamical systems described by classes of linear functional and partial differential equations, see for instance [1, 6, 26].

On another note, *confluent hypergeometric functions*, such as Kummer, Whittaker, Tricomi, or Coulomb functions, can be characterized as solutions of some classes of nonautonomous second-order differential equations (with respect to the complex independent variable) which are said to be *degenerate* or *confluent*, since two of their regular singularities merge into an irregular singularity. In particular, the Kummer differential equation admits Kummer and Tricomi functions as solutions, while Whittaker and Coulomb functions are solutions of different degenerate differential equations, even though they can be expressed, for instance, in terms of Kummer confluent hypergeometric functions. We refer the interested reader to [23, Chapter 13] for a collection of classical results on these special functions. Notice also that such confluent hypergeometric functions are closely connected to further special functions such as Bessel functions and that, in particular, Laguerre and Hermite polynomials can be explicitly written in terms of Kummer hypergeometric functions. Confluent hypergeometric functions also appear in other contexts, such as in rational approximation theory [7, 16, 25, 29], and they have been extensively studied in the literature, with a wide range of results providing their asymptotic properties and the distribution of their zeros. For further discussions on such topics we refer the readers to [8, 13, 15, 30, 31], [17, Chapter VI].

If the application of hypergeometric functions to the qualitative analysis of some classes of PDEs is well-known (see, for instance, [13, Chapter I, Section 4] for the case of wave equations), it has been recently emphasized in [6, 21] that the distribution of zeros of such confluent hypergeometric functions is closely related to the location of the spectrum of linear functional differential equations of retarded and neutral types. More precisely, it has been shown in [21] that the exponential polynomial characterizing the spectrum of a linear retarded delay differential equation with a spectral value of maximal multiplicity [11] can be factorized in terms of a Kummer hypergeometric function with positive integers as indices (see also [6]). In addition, a linear delay differential equation having a spectral value of multiplicity larger than the order of the equation necessarily shares its remaining spectrum with an appropriate linear combination of contiguous Kummer functions [9]. Thanks to these links, this class of confluent hypergeometric functions has been exploited for control purposes, as detailed in [6, 9, 11, 21].

Although the investigation of integral representations of hypergeometric functions is an old question, see for instance [18, 27, 28], the contribution of this note is threefold. First, it endeavors in

[1]Also called quasiopolynomials, exponential polynomials are analytic functions which are written as $\Delta(z) = \sum_{k=0}^{K} P_k(z)\, \mathrm{e}^{-z\tau_k}$ where P_k are polynomials with real coefficients in the complex variable z, τ_k are nonnegative numbers, and K is a nonnegative integer. The distribution of the zeros of exponential polynomials define the asymptotic behavior of solutions of some linear functional differential equations, as detailed, for instance, in [3].

shedding some light on a missing link relating iterated integrals of exponential kernels to some classes of confluent hypergeometric functions. Second, thanks to the various qualitative existing results on such hypergeometric functions, we derive some properties of iterated integrals. To the best of the authors' knowledge, such properties are new and have not been addressed in the open literature. Finally, an explicit characterization of the nontrivial roots of such iterated integrals of exponential kernels is explicitly proposed.

The remaining of the paper is organized as follows: Section 2 presents standard results on confluent hypergeometric functions which will be used in the sequel. In Section 3, we introduce the family of iterated integrals of interest. Section 4 is dedicated to the main results of the present paper.

2 Confluent hypergeometric functions and some of their properties

In this section, we provide the definitions of the confluent hypergeometric functions to be used in this paper, namely Kummer, Whittaker, and modified Bessel functions. We also present some of their standard properties that will be used in the sequel.

2.1 Kummer functions

Let us denote by $\mathbb{N}$ the set of nonnegative integers. For $a, b \in \mathbb{C}$ such that $-b \notin \mathbb{N}$, the *Kummer (confluent hypergeometric) function* is the entire function $\Phi(a, b, \cdot)\colon \mathbb{C} \to \mathbb{C}$ defined by the series

$$\Phi(a, b, z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}, \tag{1}$$

where, for $\alpha \in \mathbb{C}$ and $k \in \mathbb{N}$, $(\alpha)_k$ is the *Pochhammer symbol* for the *ascending factorial*, defined inductively as $(\alpha)_0 = 1$ and $(\alpha)_{k+1} = (\alpha + k)(\alpha)_k$ for $k \in \mathbb{N}$. The series in (1) converges for every $z \in \mathbb{C}$ and one immediately verifies that

$$\frac{\partial \Phi}{\partial z}(a, b, z) = \frac{a}{b} \Phi(a + 1, b + 1, z). \tag{2}$$

In addition, as presented in [13, 17, 23], $\Phi(a, b, \cdot)$ satisfies the (second-order) *Kummer differential equation*

$$z \frac{\partial^2 \Phi}{\partial z^2}(a, b, z) + (b - z) \frac{\partial \Phi}{\partial z}(a, b, z) - a\Phi(a, b, z) = 0. \tag{3}$$

As discussed in [13, 17, 23], for every $a, b, z \in \mathbb{C}$ such that $\Re(b) > \Re(a) > 0$, Kummer functions also admit the integral representation

$$\Phi(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b - a)} \int_0^1 \mathrm{e}^{zt} t^{a-1} (1 - t)^{b-a-1} \, \mathrm{d}t, \tag{4}$$

where Γ denotes the Gamma function. This integral representation has been exploited in [21] to characterize the spectrum of some functional differential equations of retarded type.

Notice that Kummer functions satisfy some recurrence relations, often called *contiguous relations*, see for instance [23]. We collect, in the next lemma, some relations of interest for our paper.

Lemma 2.1 ([23, p. 325])**.** *Let a, b, $z \in \mathbb{C}$ be such that $-b \notin \mathbb{N}$. Then the following relations hold true.*

$$(b - a)\, \Phi(a - 1, b, z) + (2a - b + z)\, \Phi(a, b, z) - a\, \Phi(a + 1, b, z) = 0, \tag{5a}$$

$$b(b - 1)\, \Phi(a, b - 1, z) + b(1 - b - z)\, \Phi(a, b, z) + z(b - a)\, \Phi(a, b + 1, z) = 0, \tag{5b}$$

$$(a-b+1)\ \Phi(a,b,z) - a\ \Phi(a+1,b,z) + (b-1)\ \Phi(a,b-1,z) = 0, \tag{5c}$$
$$b\ \Phi(a,b,z) - b\ \Phi(a-1,b,z) - z\ \Phi(a,b+1,z) = 0, \tag{5d}$$
$$b(a+z)\ \Phi(a,b,z) + z(a-b)\ \Phi(a,b+1,z) - ab\ \Phi(a+1,b,z) = 0, \tag{5e}$$
$$(a-1+z)\ \Phi(a,b,z) + (b-a)\ \Phi(a-1,b,z) + (1-b)\ \Phi(a,b-1,z) = 0, \tag{5f}$$
$$(a+1)z\ \Phi(a+2,b+2,z) + (b+1)(b-z)\ \Phi(a+1,b+1,z) - b(b+1)\ \Phi(a,b,z) = 0. \tag{5g}$$

The following result from [8] gives the location of zeros of Kummer functions and will be used in the main results.

Lemma 2.2 ([8, Corollary 3.8])**.** *Let a, $b \in \mathbb{R}$ be such that $b \geq 2$.*

1. *If $b = 2a$, then all nontrivial roots z of $\Phi(a,b,\cdot)$ are purely imaginary.*
2. *If $b > 2a$, then all nontrivial roots z of $\Phi(a,b,\cdot)$ satisfy $\Re(z) > 0$.*
3. *If $b < 2a$, then all nontrivial roots z of $\Phi(a,b,\cdot)$ satisfy $\Re(z) < 0$.*
4. *If $b \neq 2a$, then all nontrivial roots z of $\Phi(a,b,\cdot)$ satisfy $(b-2a)^2\Im(z)^2 - (4a(b-a)-2b)\,\Re(z)^2 > 0$.*

2.2 Whittaker functions

Kummer confluent hypergeometric functions have close links with *Whittaker functions*. More precisely, for $k, l \in \mathbb{C}$ with $-2l \notin \mathbb{N}^*$, the *Whittaker function* $\mathcal{M}_{k,l}$ is the function defined for $z \in \mathbb{C}$ by

$$\mathcal{M}_{k,l}(z) = \mathrm{e}^{-\frac{z}{2}} z^{\frac{1}{2}+l} \Phi(\tfrac{1}{2}+l-k, 1+2l, z) \tag{6}$$

(see, e.g., [23]). Note that, if $\frac{1}{2}+l$ is not an integer, the function $\mathcal{M}_{k,l}$ is a multi-valued complex function with branch point at $z=0$. The nontrivial roots of $\mathcal{M}_{k,l}$ coincide with those of $\Phi(\frac{1}{2}+l-k, 1+2l, \cdot)$ and $\mathcal{M}_{k,l}$ satisfies the (second-order) *Whittaker differential equation*

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}z^2}(z) = \left(\frac{1}{4} - \frac{k}{z} + \frac{l^2-\frac{1}{4}}{z^2}\right)\varphi(z). \tag{7}$$

Since $\mathcal{M}_{k,l}$ is a nontrivial solution of the second-order linear differential equation (7), any nontrivial root of $\mathcal{M}_{k,l}$ is necessarily simple.

2.3 Modified Bessel functions

Given $\nu \in \mathbb{C}$, the modified Bessel function I_ν is defined by

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^k}{k!\Gamma(\nu+k+1)}, \qquad z \in \mathbb{C},$$

where Γ is the Gamma function [23, Chapter 10]. Note that I_ν is a multi-valued function whenever ν is not an integer and it satisfies the (second-order) differential equation

$$z^2 \frac{\partial^2 I_\nu}{\partial z^2}(z) + z\frac{\partial I_\nu}{\partial z}(z) - \left(z^2+\nu^2\right) I_\nu(z) = 0. \tag{8}$$

Kummer functions can be related to modified Bessel functions by (see, e.g., [23, Section 13.6])

$$I_\nu(z) = \frac{e^{-z}\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)} \Phi(\nu+\tfrac{1}{2}, 2\nu+1, 2z), \qquad z \in \mathbb{C}. \tag{9}$$

3 Iterated integrals of an exponential kernel

As indicated in the Introduction, the aim of this paper is to provide links between confluent hypergeometric functions and iterated integrals of an exponential kernel. More precisely, we consider the iterated integrals provided in the next definition.

Definition 3.1. Let m, n be nonnegative integers with $m \geq 2$. We define the entire function $\mathcal{I}(m, n, \cdot) \colon \mathbb{C} \to \mathbb{C}$ by setting, for $z \in \mathbb{C}$,

$$\mathcal{I}(m,n,z) = \underbrace{\int_0^1 \int_0^{t_{n+m-1}} \cdots \int_0^{t_{n+1}}}_{m \text{ integrals}} \underbrace{\int_{t_n}^{t_{n+1}} \cdots \int_{t_2}^{t_3} \int_{t_1}^{t_2}}_{n \text{ integrals}} \mathrm{e}^{z t_0} \mathrm{d}t_0 \mathrm{d}t_1 \cdots \mathrm{d}t_{n-1} \mathrm{d}t_n \cdots \mathrm{d}t_{n+m-2} \mathrm{d}t_{n+m-1}. \tag{10}$$

In order to study the iterated integrals $\mathcal{I}(m, n, z)$ of the exponential kernel $e^{z t_0}$, we find it useful to consider some auxiliary iterated integrals of bivariate monomials, as given in the next definition.

Definition 3.2. Let k, j, m, n be nonnegative integers with $m \geq 2$. We define the real number $I(k, j, m, n)$ by

$$I(k,j,m,n) = \underbrace{\int_0^1 \int_0^{t_{n+m-1}} \cdots \int_0^{t_{n+1}}}_{m \text{ integrals}} \underbrace{\int_{t_n}^{t_{n+1}} \cdots \int_{t_2}^{t_3} \int_{t_1}^{t_2}}_{n \text{ integrals}} t_0^k t_1^j \mathrm{d}t_0 \mathrm{d}t_1 \cdots \mathrm{d}t_{n-1} \mathrm{d}t_n \cdots \mathrm{d}t_{n+m-2} \mathrm{d}t_{n+m-1}. \tag{11}$$

A few remarks concerning $\mathcal{I}(m, n, z)$ and $I(k, j, m, n)$ are now in order.

Remark 3.3. The condition $m \geq 2$ is important for both $I(k, j, m, n)$ and $\mathcal{I}(m, n, z)$ to be well-defined. Indeed, in both (11) and (10), the innermost n integrals define a function depending on t_n and t_{n+1}. For $I(k, j, m, n)$ and $\mathcal{I}(m, n, z)$ to be independent of the variables of integration $t_0, \ldots, t_{n+m-1}$, one should thus integrate the innermost n integrals at least twice more, one with respect to t_n and another with respect to t_{n+1}, justifying the requirement that $m \geq 2$.

The only exceptions are the cases $n = 0$ in (10) and $n = j = 0$ in (11), in which case $m = 1$ makes sense, and we then write

$$\mathcal{I}(1,0,z) = \int_0^1 e^{z t_0} \mathrm{d}t_0, \qquad I(k,0,1,0) = \int_0^1 t_0^k \mathrm{d}t_0 = \frac{1}{k+1}. \tag{12}$$

Note that, using (4), it is immediate to verify that $\mathcal{I}(1, 0, z) = \Phi(1, 2, z)$.

Remark 3.4. The m outer integrals in (10) and (11) correspond to an integration over the domain

$$\{(t_n, \ldots, t_{n+m-1}) \in \mathbb{R}^m \mid 0 \leq t_n \leq t_{n+1} \leq \cdots \leq t_{n+m-1} \leq 1\}.$$

As for the n inner integrals, note that their domains of integration are not necessarily oriented according to the standard order[2], i.e., we may have $t_j > t_{j+1}$ for some $j \in [\![1, n]\!]$. Indeed, the outermost of the inner n integrals is an integral on t_{n-1} between t_n and t_{n+1} and, since $t_n \leq t_{n+1}$, we have

$$t_n \leq t_{n-1} \leq t_{n+1},$$

and, in particular, this integral is in the standard order. The second outermost of the inner n integrals is an integral on t_{n-2} between t_{n-1} and t_n and, since $t_n \leq t_{n-1}$, we have

$$t_n \leq t_{n-2} \leq t_{n-1},$$

[2] Here, we say that (the domain of integration of) an integral $\int_a^b$ is in the *standard order* if $a \leq b$

and, in particular, this integral is not in the standard order. Proceeding in a similar fashion, we obtain that the domain of integration of the inner n integrals is

$$\left\{(t_0,\dots,t_{n-1})\in\mathbb{R}^n \;\middle|\; \begin{matrix} t_{n-j+1}\le t_{n-j}\le t_{n-j+2} & \text{if } j\in[\![1,n]\!] \text{ is odd} \\ t_{n-j+2}\le t_{n-j}\le t_{n-j+1} & \text{if } j\in[\![1,n]\!] \text{ is even}\end{matrix}\right\},$$

which can be rewritten as

$$\{(t_0,\dots,t_{n-1})\in\mathbb{R}^n \mid t_n\le t_{n-2}\le t_{n-4}\le\cdots\le t_{n-3}\le t_{n-1}\le t_{n+1}\}.$$

In particular, in the n inner integrals in (10) and (11), the domains of integration alternate between being in the standard order or not, and the total number of domains of integration in these integrals not in standard order is $\lfloor \frac{n}{2}\rfloor$, where the symbol $\lfloor x\rfloor$ designates the integer part of x. If $z\in\mathbb{R}$, then $\mathrm{e}^{zt_0}\ge 0$ for all $t_0\in\mathbb{R}$, and thus the signs of $\mathcal{I}(m,n,z)$ and $I(k,j,m,n)$ coincide for all nonnegative integers k,j and $z\in\mathbb{R}$, and they are equal to $(-1)^{\lfloor \frac{n}{2}\rfloor}$.

4 Main results

After the preliminaries on confluent hypergeometric functions from Section 2 and the definition of the iterated integrals of interest in Section 3, we are now in position to state and prove our main results.

4.1 Iterated integrals of an exponential kernel in terms of Kummer functions

The main result of our paper is the following, expressing the iterated integrals (10) in terms of Kummer functions.

Theorem 4.1. *Let m,n be nonnegative integers with $m\ge 2$. Then, for all $z\in\mathbb{C}$, we have*

$$\mathcal{I}(m,n,z)=\begin{cases}\dfrac{(-1)^{\frac{n}{2}}}{(m+n)!}\Phi(\frac{n+2}{2},m+n+1,z) & \text{if } n \text{ is even,}\\ \dfrac{(-1)^{\frac{n-1}{2}}}{(m+n)!}\Phi(\frac{n+3}{2},m+n+1,z) & \text{if } n \text{ is odd.}\end{cases} \tag{13}$$

As a preliminary step to prove Theorem 4.1, we first characterize the iterated integrals of bivariate monomials from (11).

Lemma 4.2. *Let k,j,m,n be nonnegative integers with $m\ge 2$. Then*

$$I(k,j,m,n)=\begin{cases}\dfrac{(-1)^{\frac{n}{2}}(k+j+\frac{n+2}{2})!}{(k+\frac{n+2}{2})(k+j+m+n)!(\frac{n}{2})!} & \text{if } n \text{ is even,}\\ \dfrac{(-1)^{\frac{n-1}{2}}(k+j+\frac{n+1}{2})!}{(j+\frac{n+1}{2})(k+j+m+n)!(\frac{n-1}{2})!} & \text{if } n \text{ is odd.}\end{cases} \tag{14}$$

Remark 4.3. Note that, in the cases of (12) in which $\mathcal{I}(m,n,z)$ and $I(k,j,m,n)$ are defined for $m=1$, the formulas from (13) and (14) are trivially satisfied.

Proof of Lemma 4.2. Notice first that, for every $k,j\in\mathbb{N}$ and $t_1,t_2\in\mathbb{R}$, we have

$$\int_{t_1}^{t_2} t_0^k t_1^j \mathrm{d}t_0=\frac{t_1^j t_2^{k+1}-t_1^{k+j+1}}{k+1}.$$

For $m,n \in \mathbb{N}$ with $m \geq 2$ and $n \geq 1$, inserting the above identity in (11), we obtain at once that, for every $k,j,m,n \in \mathbb{N}$ with $m \geq 2$ and $n \geq 1$, we have

$$I(k,j,m,n) = \frac{1}{k+1}\left[I(j,k+1,m,n-1) - I(k+j+1,0,m,n-1)\right]. \tag{15}$$

We will thus prove (14) by induction on n, using (15) for the inductive step.

Let us first consider the case $n = 0$. We have

$$I(k,j,m,0) = \underbrace{\int_0^1 \int_0^{t_{m-1}} \cdots \int_0^{t_2} \int_0^{t_1}}_{m \text{ integrals}} t_0^k t_1^j \mathrm{d}t_0 \mathrm{d}t_1 \cdots \mathrm{d}t_{m-2} \mathrm{d}t_{m-1}, \tag{16}$$

and we compute the value of this integral by induction on m. In the case $m = 2$, we have

$$I(k,j,2,0) = \int_0^1 \int_0^{t_1} t_0^k t_1^j \mathrm{d}t_0 \mathrm{d}t_1 = \int_0^1 \frac{t_1^{k+j+1}}{k+1} \mathrm{d}t_1 = \frac{1}{(k+1)(k+j+2)}.$$

On the other hand, using that

$$\int_0^{t_1} t_0^k t_1^j \mathrm{d}t_0 = \frac{t_1^{k+j+1}}{k+1},$$

we obtain from (16) the recursive formula

$$I(k,j,m,0) = \frac{1}{k+1} I(k+j+1,0,m-1,0).$$

Thus, by an immediate inductive argument using the above recursive formula, we obtain that, for every $k,j,m \in \mathbb{N}$ with $m \geq 2$, we have

$$\begin{aligned} I(k,j,m,0) &= \frac{(k+j+1)!}{(k+1)(k+j+m-2)!} I(k+j+m-2,0,2,0) \\ &= \frac{(k+j+1)!}{(k+1)(k+j+m)!}, \end{aligned}$$

and hence (14) holds with $n = 0$ for every $k,j,m \in \mathbb{N}$ with $m \geq 2$.

Let now $n \in \mathbb{N}$ be such that (14) holds for that value of n and for every $k,j,m \in \mathbb{N}$ with $m \geq 2$. Fix $k,j,m \in \mathbb{N}$ with $m \geq 2$. We split the sequel of the proof according to whether n is odd or even.

If n is odd, we compute, using (15) and (14), that

$$\begin{aligned} I(k,j,m,n+1) &= \frac{1}{k+1}\left[I(j,k+1,m,n) - I(k+j+1,0,m,n)\right] \\ &= \frac{(-1)^{\frac{n-1}{2}}(k+j+\frac{n+3}{2})!}{(k+1)(k+j+m+\frac{n+1}{2})!(\frac{n-1}{2})!}\left[\frac{1}{k+\frac{n-3}{2}} - \frac{1}{\frac{n+1}{2}}\right] \\ &= \frac{(-1)^{\frac{n+1}{2}}(k+j+\frac{n+3}{2})!}{(k+\frac{n+3}{2})(k+j+m+n+1)!(\frac{n+1}{2})!}, \end{aligned}$$

which agrees with (14) since $n+1$ is even.

If n is even, using once again (15) and (14), we compute

$$\begin{aligned} I(k,j,m,n+1) &= \frac{1}{k+1}\left[I(j,k+1,m,n) - I(k+j+1,0,m,n)\right] \\ &= \frac{(-1)^{\frac{n}{2}}(k+j+\frac{n}{2}+2)!}{(k+1)(k+j+m+n+1)!(\frac{n}{2})!}\left[\frac{1}{j+\frac{n}{2}+1} - \frac{1}{k+j+\frac{n}{2}+2}\right] \\ &= \frac{(-1)^{\frac{n}{2}}(k+j+\frac{n}{2}+1)!}{(j+\frac{n}{2}+1)(k+j+m+n+1)!(\frac{n}{2})!}, \end{aligned}$$

which once again agrees with (14) since now $n+1$ is odd. □

As an immediate consequence of Lemma 4.2, we obtain the formula for iterated integrals of the monomial t_0^k provided in the next result.

Corollary 4.4. *Let k, m, n be nonnegative integers with $m \geq 2$. Then*

$$I(k,0,m,n) = \begin{cases} \dfrac{(-1)^{\frac{n}{2}}(\frac{n}{2}+1)_k}{(m+n)!(m+n+1)_k} & \text{if } n \text{ is even,} \\ \dfrac{(-1)^{\frac{n-1}{2}}(\frac{n+3}{2})_k}{(m+n)!(m+n+1)_k} & \text{if } n \text{ is odd.} \end{cases}$$

We are now in position to provide the proof of Theorem 4.1.

Proof of Theorem 4.1. Using the Taylor–Maclaurin series expansion of the exponential function, we obtain at once that

$$\underbrace{\int_0^1 \int_0^{t_{n+m-1}} \cdots \int_0^{t_{n+1}}}_{m \text{ integrals}} \underbrace{\int_{t_n}^{t_{n+1}} \cdots \int_{t_2}^{t_3} \int_{t_1}^{t_2}}_{n \text{ integrals}} \mathrm{e}^{zt_0} \mathrm{d}t_0 \mathrm{d}t_1 \cdots \mathrm{d}t_{n-1} \mathrm{d}t_n \cdots \mathrm{d}t_{n+m-2} \mathrm{d}t_{n+m-1}$$
$$= \sum_{k=0}^{\infty} \frac{z^k}{k!} I(k,0,m,n).$$

By Corollary 4.4, if n is even, then

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} I(k,0,m,n) = \frac{(-1)^{\frac{n}{2}}}{(m+n)!} \sum_{k=0}^{\infty} \frac{(\frac{n+2}{2})_k}{(m+n+1)_k} \frac{z^k}{k!} = \frac{(-1)^{\frac{n}{2}}}{(m+n)!} \Phi(\tfrac{n}{2}+1, m+n+1, z),$$

while, if n is odd, we have

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} I(k,0,m,n) = \frac{(-1)^{\frac{n-1}{2}}}{(m+n)!} \sum_{k=0}^{\infty} \frac{(\frac{n+3}{2})_k}{(m+n+1)_k} \frac{z^k}{k!} = \frac{(-1)^{\frac{n-1}{2}}}{(m+n)!} \Phi(\tfrac{n+3}{2}, m+n+1, z),$$

yielding the result. ☐

4.2 Confluent hypergeometric functions as iterated integrals

As a first consequence of Theorem 4.1, we obtain immediately two representations of Kummer functions with integer parameters as iterated integrals, stated in the next result.

Corollary 4.5. *Let $a, b \in \mathbb{N}^*$.*

1. *If $b \geq 2a+1$, then*

$$\Phi(a,b,z) = (-1)^{a-1}(b-1)!\, \mathcal{I}(b-2a+1, 2a-2, z), \qquad \text{for every } z \in \mathbb{C}.$$

2. *If $a \geq 2$ and $b \geq 2a$, then*

$$\Phi(a,b,z) = (-1)^{a}(b-1)!\, \mathcal{I}(b-2a+2, 2a-3, z), \qquad \text{for every } z \in \mathbb{C}.$$

Similarly, combining (6) and Corollary 4.5, we obtain the following representation of Whittaker functions with integer or half-integer parameters in terms of iterated integrals.

Corollary 4.6. *Let k, l be such that $2k, 2l \in \mathbb{N}$.*

1. *If $k \geq \frac{1}{2}$, $l \geq k + \frac{1}{2}$, and $2(l-k)$ is odd, then*

$$\mathcal{M}_{k,l}(z) = (-1)^{l-k-\frac{1}{2}}(2l)!\, e^{-\frac{z}{2}} z^{\frac{1}{2}+l}\, \mathcal{I}(2k+1, 2l-2k-1, z), \qquad \text{for every } z \in \mathbb{C}.$$

2. *If $k \geq 0$, $l \geq k+1$, and $2(l-k)$ is even, then*

$$\mathcal{M}_{k,l}(z) = (-1)^{l-k+\frac{1}{2}}(2l)!\, e^{-\frac{z}{2}} z^{\frac{1}{2}+l}\, \mathcal{I}(2k+2, 2l-2k-2, z), \qquad \text{for every } z \in \mathbb{C}.$$

Finally, (9) and Corollary 4.5 also allow one to express the modified Bessel function I_ν in terms of iterated integrals when ν is a half-integer greater than or equal to $\frac{3}{2}$.

Corollary 4.7. *Let ν be such that 2ν is an odd positive integer and $\nu \geq \frac{3}{2}$. Then*

$$I_\nu(z) = \frac{(-1)^{\nu-\frac{3}{2}}(2\nu)!}{\Gamma(\nu+1)}\, \mathrm{e}^{-z} \left(\frac{z}{2}\right)^\nu \mathcal{I}(2, 2\nu-2, 2z).$$

4.3 Further properties of iterated integrals

Using the properties of Kummer functions presented in Section 2.1, Theorem 4.1, and Corollary 4.5, we can provide nontrivial properties of iterated integrals of the form (10). Inspired by the contiguous relations of Kummer functions listed in [23] and recalled in Lemma 2.1, one is able to prove the following contiguous relations for iterated integrals, which seem to be new in the open literature.

Corollary 4.8. *Let $m, n \in \mathbb{N}$.*

1. *Assume that $m \geq 4$ and $n \geq 2$.*

   (a) *If n is even, then, for every $z \in \mathbb{C}$,*

$$\left(m + \frac{n}{2}\right) \mathcal{I}(m+2, n-2, z) - (1+z-m)\, \mathcal{I}(m,n,z) - \frac{n+2}{2}\, \mathcal{I}(m-2, n+2, z) = 0. \tag{17a}$$

   (b) *If n is odd, then, for every $z \in \mathbb{C}$,*

$$\left(m + \frac{n-1}{2}\right) \mathcal{I}(m+2, n-2, z) - (2+z-m)\, \mathcal{I}(m,n,z) - \frac{n+3}{2}\, \mathcal{I}(m-2, n+2, z) = 0. \tag{17b}$$

2. *Assume that $m \geq 3$. Then, for every $z \in \mathbb{C}$,*

$$\mathcal{I}(m-1, n, z) - (z+m+n)\, \mathcal{I}(m,n,z) + z \left(m + \left\lfloor \frac{n}{2} \right\rfloor\right) \mathcal{I}(m+1, n, z) = 0. \tag{17c}$$

3. *Assume that $m \geq 4$.*

   (a) *If n is even, then, for every $z \in \mathbb{C}$,*

$$\left(\frac{2-n}{2} - m\right) \mathcal{I}(m,n,z) + \frac{n+2}{2}\, \mathcal{I}(m-2, n+2, z) + \mathcal{I}(m-1, n, z) = 0. \tag{17d}$$

   (b) *If n is odd, then, for every $z \in \mathbb{C}$,*

$$\left(\frac{3-n}{2} - m\right) \mathcal{I}(m,n,z) + \frac{n+3}{2}\, \mathcal{I}(m-2, n+2, z) + \mathcal{I}(m-1, n, z) = 0. \tag{17e}$$

4. *Assume that $m \geq 2$ and $n \geq 2$. Then, for every $z \in \mathbb{C}$,*

$$\mathcal{I}(m,n,z) + \mathcal{I}(m+2, n-2, z) - z\, \mathcal{I}(m+1, n, z) = 0. \tag{17f}$$

5. *Assume that* $m \geq 4$.

   (a) *If* n *is even, then, for every* $z \in \mathbb{C}$,

$$\left(z + \frac{n+2}{2}\right) \mathcal{I}(m,n,z) - z\left(m + \frac{n}{2}\right) \mathcal{I}(m+1,n,z) + \frac{n+2}{2} \mathcal{I}(m-2,n+2,z) = 0. \tag{17g}$$

   (b) *If* n *is odd, then, for every* $z \in \mathbb{C}$,

$$\left(z + \frac{n+3}{2}\right) \mathcal{I}(m,n,z) - z\left(m + \frac{n-1}{2}\right) \mathcal{I}(m+1,n,z) + \frac{n+3}{2} \mathcal{I}(m-2,n+2,z) = 0. \tag{17h}$$

6. *Assume that* $m \geq 3$ *and* $n \geq 2$.

   (a) *If* n *is even, then, for every* $z \in \mathbb{C}$,

$$\left(z + \frac{n}{2}\right) \mathcal{I}(m,n,z) - \left(m + \frac{n}{2}\right) \mathcal{I}(m+2,n-2,z) - \mathcal{I}(m-1,n,z) = 0. \tag{17i}$$

   (b) *If* n *is odd, then, for every* $z \in \mathbb{C}$,

$$\left(z + \frac{n+1}{2}\right) \mathcal{I}(m,n,z) - \left(m + \frac{n-1}{2}\right) \mathcal{I}(m+2,n-2,z) - \mathcal{I}(m-1,n,z) = 0. \tag{17j}$$

7. *Assume that* $m \geq 4$.

   (a) *If* n *is even, then, for every* $z \in \mathbb{C}$,

$$\frac{n+4}{2} z\, \mathcal{I}(m-2,n+4,z) + (z-1-m-n)\, \mathcal{I}(m-1,n+2,z) - \mathcal{I}(m,n,z) = 0. \tag{17k}$$

   (b) *If* n *is odd, then, for every* $z \in \mathbb{C}$,

$$\frac{n+5}{2} z\, \mathcal{I}(m-2,n+4,z) + (z-1-m-n)\, \mathcal{I}(m-1,n+2,z) - \mathcal{I}(m,n,z) = 0. \tag{17l}$$

Proof. Notice that relations (17a)–(17b) are derived from (5a), relation (17c) is derived from (5b), relations (17d)–(17e) are derived from (5c), relation (17f) is derived from (5d), relations (17g)–(17h) are derived from (5e), relations (17i)–(17j) are derived from (5f), and relations (17k)–(17l) are derived from (5g). We only provide an argument for (17a) and (17b), the other relations being proved in a similar way.

Let $m, n \in \mathbb{N}^*$ be such that $m \geq 4$ and $n \geq 2$ is even. Set $a = \frac{n}{2} + 1$ and $b = m + n + 1$ and note that $a, b \in \mathbb{N}^*$ with $a \geq 2$ and $b \geq 2a + 3$. Combining (5a) and Corollary 4.5.1, we deduce that, for every $z \in \mathbb{C}$,

$$(b-a)\, \mathcal{I}(b-2a+3, 2a-4, z) - (2a-b+z)\, \mathcal{I}(b-2a+1, 2a-2, z) - a\, \mathcal{I}(b-2a-1, 2a, z) = 0,$$

which yields (17a).

Let us now assume that $m, n \in \mathbb{N}^*$ are such that $m \geq 4$ and $n \geq 2$ is odd. Set $a = \frac{n+3}{2}$ and $b = m + n + 1$ and note that $a, b \in \mathbb{N}^*$ with $a \geq 3$ and $b \geq 2a + 2$. We now combine (5a) with Corollary 4.5.2 to obtain that, for every $z \in \mathbb{C}$,

$$(b-a)\, \mathcal{I}(b-2a+4, 2a-5, z) - (2a-b+z)\, \mathcal{I}(b-2a+2, 2a-3, z) - a\, \mathcal{I}(b-2a, 2a-1, z) = 0,$$

which proves (17b). □

As a consequence of the relations (2) and (5d), we also obtain immediately from Corollary 4.5 the following expression for the derivative of the iterated integrals (10).

Corollary 4.9. *Let $m, n \in \mathbb{N}$ with $m \geq 2$. Then, for all $z \in \mathbb{C}$, we have*

$$\frac{\partial \mathcal{I}}{\partial z}(m,n,z) = \begin{cases} -\dfrac{n+2}{2}\,\big[(z+m+n+2)\,\mathcal{I}(m,n+2,z) \\ \qquad -z\,(m+\frac{n}{2}+1)\,\mathcal{I}(m+1,n+2,z)\big] & \text{if } n \text{ is even,} \\ -\dfrac{n+3}{2}\,\big[(z+m+n+2)\,\mathcal{I}(m,n+2,z) \\ \qquad -z\,(m+\frac{n+1}{2})\,\mathcal{I}(m+1,n+2,z)\big] & \text{if } n \text{ is odd.} \end{cases}$$

Remark 4.10. If $m \geq 3$, one obtains from (2) and Corollary 4.5 the more compact formula

$$\frac{\partial \mathcal{I}}{\partial z}(m,n,z) = \begin{cases} -\dfrac{n+2}{2}\,\mathcal{I}(m-1,n+2,z) & \text{if } n \text{ is even,} \\ -\dfrac{n+3}{2}\,\mathcal{I}(m-1,n+2,z) & \text{if } n \text{ is odd.} \end{cases}$$

Since iterated integrals of the form (10) can be expressed in terms of confluent hypergeometric functions thanks to Theorem 4.1, and the latter satisfy nonautonomous second-order differential equations of Kummer type (3), one deduces that iterated integrals from (10) also satisfy nonautonomous second-order differential equations. More precisely, we derive the following result immediately from Corollaries 4.8 and 4.9.

Corollary 4.11. *Let m, n be nonnegative integers with $m \geq 2$ and $z \in \mathbb{C}$.*

1. *If n is even, then $\mathcal{I}(m,n,z)$ satisfies the second-order differential equation*

$$z\frac{\partial^2 \mathcal{I}}{\partial z^2}(m,n,z) + (m+n+1-z)\frac{\partial \mathcal{I}}{\partial z}(m,n,z) - \left(\frac{n}{2}+1\right)\mathcal{I}(m,n,z) = 0.$$

2. *If n is odd, then $\mathcal{I}(m,n,z)$ satisfies the second-order differential equation*

$$z\frac{\partial^2 \mathcal{I}}{\partial z^2}(m,n,z) + (m+n+1-z)\frac{\partial \mathcal{I}}{\partial z}(m,n,z) - \frac{n+3}{2}\,\mathcal{I}(m,n,z) = 0.$$

Finally, as a consequence of Theorem 4.1, we also obtain immediately from Lemma 2.2 the location of zeros of the iterated integrals of the exponential kernel (10).

Corollary 4.12. *Let m, n be nonnegative integers with $m \geq 2$. The following properties hold:*

1. *If $m = 2$ and n is odd (respectively, even), then all nontrivial roots z of $\mathcal{I}(m,n,\cdot)$ are purely imaginary (respectively, satisfy $\Re(z) > 0$).*

2. *If $m > 2$, then all nontrivial roots z of $\mathcal{I}(m,n,\cdot)$ satisfy $\Re(z) > 0$.*

3. *If $m > 2$ and n is odd, then all nontrivial roots z of $\mathcal{I}(m,n,\cdot)$ satisfy*

$$(m-2)^2\Im(z)^2 - \big(2mn + n^2 + 4m - 5\big)\,\Re(z)^2 > 0.$$

4. *If n is even, then all nontrivial roots z of $\mathcal{I}(m,n,\cdot)$ satisfy*

$$(m-1)^2\Im(z)^2 - \big(2mn + n^2 + 2m - 2\big)\,\Re(z)^2 > 0.$$

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