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KAÏS AMMARI, ISLAM BOUSSAADA, SILVIU-IULIAN NICULESCU, AND SAMI TLIBA

Abstract. In this paper, we consider the stabilization problem of the wave equation with pointwise delay feedback. We propose a control methodology having the advantage of the assignation of the closed-loop exponential decay. The methodology involves a four-parameter autoregressive control structure for which the design strategy is based on multiplicity manifolds. The proof of the main result is based on spectral analysis, thereby conveying a positive outlook on the control of further classes of partial differential equations.

Contents

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1. Introduction

On the one hand, wave propagation represents a complex dynamical phenomenon ubiquitous in natural and engineered environments. These hyperbolic partial differential equations often exhibit inherent instabilities, posing significant challenges for driving such processes and requiring effective control strategies to ensure stability and appropriate performances. Furthermore, delay effects manifest in various applications and practical problems, with a well-established understanding that even a minute delay can potentially destabilize a system that would otherwise be uniformly asymptotically stable in the absence of delay (as discussed in [19, 18]). The stability analysis of the wave equation with a pure pointwise instantaneous proportional control has been addressed in [7], revealing stability for a specific choice of the control location. Additionally, it has been demonstrated that in the absence of delay, the system remains exponentially stable for any pointwise dissipation. Conversely, the introduction of a delay term likely leads to instability phenomena, as suggested in [20]. In addition, the recent investigation by Ammari and Nicaise [5] is noteworthy, as they comprehensively examined the stabilization of elastic systems via collocated feedback, with or without delay.

On the other hand, conventional control approaches typically rely on instantaneous feedback information, which may be limited in capturing the nuanced dynamics of wave systems. In this paper, we introduce a novel methodology that leverages past data to enhance the stability of wave systems through intelligent control laws. By integrating past information into the control framework, our approach offers a proactive strategy to mitigate instabilities and improve overall system behavior. The main benefit is a new perspective that leverages past data to build the control signal, thereby improving system stability and performance. As a matter of fact, in linear autonomous Functional Differential Equations (FDE), recent works have highlighted a particularly interesting spectral property, called *multiplicity-induced-dominancy (MID)*, which consists in conditions on the system's parameters under which a multiple spectral value corresponds to the spectral abscissa [12, 40]. The first analytic proof of this property has been proposed for first-order FDE in [13], and it relies on an integral representation of the corresponding characteristic function and a contradiction argument. In particular, it appears that a characteristic root of maximal multiplicity (i.e., equal to the degree of the corresponding quasipolynomial) necessarily defines the spectral abscissa of the system. Such a systematic study of the links between roots of large multiplicity and the spectral abscissa was not sufficiently addressed in the literature until the early work [13], even though some hints in this direction were provided in [38] in the case of low-order systems. Since these works, the case of the assignment of a characteristic root with maximal multiplicity, called *generic MID property*, was recently addressed and completely characterized in [35] (retarded case) and in [10] (unifying retarded and neutral cases) for LTI FDEs including a *single delay*.

As discussed in [35, 10], this property opens interesting perspectives in control through the so-called *partial pole placement* method, that is, imposing the multiplicity of a characteristic root of the closed-loop system by an appropriate choice of the controller gains guarantees the exponential stability of the closed-loop system with a prescribed decay rate. Furthermore, the resulting partial poles placement has been widely used in the design of reduced-complexity controllers such as the well-known Proportional-Integral-Derivative (PID) control, see for instance [33].

By this paper, we aim to extend such a control methodology to tackle problems modeled by partial differential equation such as the works [2, 3]. In particular, the middle point stabilization of the wave via autoregressive control is considered. The remainder of the paper is organized as follows. Some prerequisites on stability of continuous-time difference equations are presented and the considered problem is formulated in Section 2. Next, in Section 3, the well posedness of the pointwise control of the wave equation is considered. We provide a study of the asymptotic behavior of solutions in Section 4. Further insights on spectral properties and parametric settings are provided in Sections 5 and 6. The latter is devoted to the estimation of the closed-loop exponential decay. Section 7 is dedicated to the illustration of the main result of the paper. A conclusion ends the paper.

2. Prerequisites and Problem statement

In this section, we start by providing some result from the literature which will be used in our main result. Next, we formulate the considered problem.

2.1. Stability of continuous-time scalar difference equation with interfering delays.

Consider the following scalar dynamical system described by difference equation with interfering delays

$$y(t) + a_1\, y(t-\tau_1) + a_2\, y(t-\tau_2) + a_3\, y(t-\tau_3) = 0, \tag{2.1}$$

where $\tau_k \in \mathbb{R}_+^*$ and $\tau_3 = \tau_1 + \tau_2$. In this case, the characteristic function corresponding to (2.1) reduces to

$$Q(s;\tau_1,\tau_2) := 1 + a_1\, \mathrm{e}^{-\tau_1 s} + a_2\, \mathrm{e}^{-\tau_2 s} + a_3\, \mathrm{e}^{-(\tau_1+\tau_2) s}, \tag{2.2}$$

where s designates the complex Laplace variable. In fact, the Hale-Silkowski criterion completely characterizes the exponential stability of solutions of (2.1), see for instance [28, Chapter 9, Theorem 6.1] and for further refinement and generalization of the above result see [16]. As a matter of fact, since the three involved delays in equation (2.1) (τ_1, τ_2, and $\tau_1+\tau_2$) are rationally dependent, then one can transform (2.1) into an equivalent matrix equation involving only two delays τ_1 and τ_2. By denoting

$$f(t) := \begin{pmatrix} y(t) \\ y(t-\tau_1) \end{pmatrix}, \quad A := \begin{pmatrix} -a_1 & 0 \\ 1 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} -a_2 & -a_3 \\ 0 & 0 \end{pmatrix},$$

equation (2.1) is equivalent to the system:

$$f(t) = A\, f(t-\tau_1) + B\, f(t-\tau_2). \tag{2.3}$$

Furthermore, if the delays τ_1 and τ_2 are rationally independent, then using [27, Chapter 9, Theorem 6.1] one concludes that (2.3) is stable locally in the delays if, and only if,

$$\sup_{\varphi_1,\, \varphi_2 \in [0,\, 2\,\pi]} \mu\left(A\,\mathrm{e}^{i\,\varphi_1} + B\,\mathrm{e}^{i\,\varphi_2}\right) < 1,$$

where $\mu(\cdot)$ designates the spectral radius of a given square matrix.

This entails the following necessary and sufficient stability conditions for the local stability in the delay τ:

$$\begin{cases} 1 - a_1 > \left|a_2 + a_3\right|, \\ 1 + a_1 > \left|a_3 - a_2\right|, \end{cases}$$

However, if delays τ_1 and τ_2 are rationally dependent, that is $\tau_2 = \dfrac{p}{q}\tau_1$ with $p, q \in \mathbb{N}$, then one can further transform (2.3) into an equivalent augmented matrix equation and again using [28, Chapter 9, Theorem 6.1] one can obtain explicit necessary and sufficient conditions.

The importance of the result recalled above is incontestable from a purely qualitative point of view. However, quantitatively, it lacks information on the solution's decay rate. In the following section, we shall recall an interesting stability property generated in the manifold defining a given spectral values' multiplicity to get some insights on the solutions' decay rate.

2.2. **On the MID paradigm.** Notice that the MID property, which consists in conditions on the system's parameters under which a multiple spectral value is dominant, has been fully characterized for single delay retarded as well as neutral equations, see for instance [11]. However, in the multiple delay case, beyond the partial results established in [24] in the scalar retarded differential equations with two delays, and in [3] for the case of continuous-time difference equations, the question remains open. For the sake of self-containment, the latter result will be recalled since it will be exploited in our main result. More precisely, in this section, we shall provide some configurations in which the MID applies; this corresponds to the dominancy of spectral values with a multiplicity which is equal to the degree* of the considered quasipolynomial.

Notice that the degree may vary when some coefficients are set to be zero or when some delays are set to be equal. In particular, the case $\tau_1 = \tau_2$ will be considered separately since it decreases the degree of the quasipolynomial Q.

In this case, the quasipolynomial Q reads as

$$Q(s;\tau_1,\tau_1) := 1 + (a_1 + a_2)\,\mathrm{e}^{-\tau_1 s} + a_3\,\mathrm{e}^{-2\,\tau_1 s} \tag{2.4}$$

which admits a degree equal to two for $a_1 \neq -a_2$ and $a_3 \neq 0$. We recall the following result from [3] which will be used in the sequel.

Theorem 2.1 ([3])**.** *Consider the quasipolynomial $Q(\cdot,\tau_1,\tau_1)$ given by (2.4).*

A given real number s_0 is a double root of (2.4) if, and only if,

$$\begin{cases} a_2 = -a_1 - 2\,\mathrm{e}^{s_0\,\tau_1}, \\ a_3 = \mathrm{e}^{2s_0\,\tau_1}. \end{cases} \tag{2.5}$$

If (2.5) is satisfied then the GMID holds, that is, s_0 corresponds to the spectral abscissa of the quasipolynomial $Q(\cdot;\tau_1,\tau_1)$ given by (2.4). Furthermore, all zeros of (2.4) are double and lie on the vertical axis $\Re\{s\} = s_0$.

Now, let us consider again the quasipolynomial (2.2) where $\tau_1 \neq \tau_2$ and $\sum_{k=1}^{3} a_k^2 \neq 0$, *i.e.*, the case where the quasipolynomial's degree is equal to three.

Theorem 2.2 ([3])**.** *Consider the quasipolynomial $Q(\cdot;\tau_1,\tau_2)$ given by (2.2) and let $\tau_1 \neq \tau_2$.*

A given real number s_0 is a triple root of (2.2) if, and only if,

$$\begin{cases} a_1 = \dfrac{\tau_1+\tau_2}{\tau_1-\tau_2}\,\mathrm{e}^{s_0\,\tau_1}, \\ a_2 = -\dfrac{\tau_1+\tau_2}{\tau_1-\tau_2}\,\mathrm{e}^{s_0\,\tau_2}, \\ a_3 = -\mathrm{e}^{s_0\,(\tau_1+\tau_2)} \end{cases} \tag{2.6}$$

If (2.6) is satisfied and τ_2 is a multiple of τ_1 ($\tau_2 = k\,\tau_1$ with k an integer $k > 1$) then the GMID holds, that is, s_0 corresponds to the spectral abscissa of the quasipolynomial $Q(\cdot;\tau_1,\, k\tau_1)$ given by (2.2).

Notice that The above Theorems have been proven in [3] thanks to a property of self inversive polynomials where a Theorem of A. Cohn [17] and a result of Eneström-Kakeya [31, 23] have been deployed; see also [22, 34, 32, 15] for further insights on polynomials with all zeros on the unit circle.

Remark 2.3. *From a control theory viewpoint, the MID property can be exploited by tuning the control parameters as emphasized above after prescribing a negative number s_0 which corresponds to the closed-loop system solution's decay rate.*

*The degree of a quasipolynomial corresponds to the sum of the degrees of the involved polynomials plus the number of delays.

When the MID property fails, one can prescribe a lower bound for the decay rate as will be discussed in the next section.

2.3. **Estimation of the exponential decay.** In what follows, despite the MID property's failure, we shall provide a lower bound for the solution's exponential decay. By taking the parameters values a_k given in system (2.6) into the expression of Q given in (2.2) and by introducing the variable change

$$z := \frac{\tau_1(s - s_0)}{2}, \tag{2.7}$$

and the new parametrization

$$\tau := \frac{2\,\tau_2}{\tau_1}, \tag{2.8}$$

it comes

$$\tilde{Q}(z;\tau) := 1 - \frac{\tau+2}{\tau-2}\,\mathrm{e}^{-2\,z} + \frac{\tau+2}{\tau-2}\,\mathrm{e}^{-\tau\,z} - \mathrm{e}^{-(\tau+2)\,z}. \tag{2.9}$$

Le us now examine the location of roots of $\tilde{Q}$ with respect to τ. Since the quasipolynomials we are considering are with real coefficients, so that the corresponding zeros are symmetric with respect to the real axis, i.e., the zeros are either real or appear in complex conjugate pairs. The following lemma, which has been shown in [2, 3], underlines an additional symmetry structuring the distribution of zeros of $\tilde{Q}$ with respect to the imaginary axis.

Lemma 2.4 ([3]). *Let $z \in \mathbb{C}$ be a zero of $\tilde{Q}$ defined by expression* (2.9)*. Then, $-z$ is also a zero of $\tilde{Q}$.*

In order to locate the zeros of (2.9), we require the following settings and results from [29], see also [9] for further refinements. Consider the quasipolynomial

$$\Theta(z,\kappa,h) := \sum_{k=0}^{N} \kappa_k\,\mathrm{e}^{-z\,\chi_k.h} \tag{2.10}$$

where $\kappa = (\kappa_1,\ldots,\kappa_N)^T \in \mathbb{R}^N$, $h = (h_1,\ldots,h_M)^T \in \mathbb{R}_+^M$, $\chi_j = (\chi_{j,1},\ldots,\chi_{j,M})$, $\chi_{j,k} \in \mathbb{N}^*$ $(j \in [\![1,N]\!],\ k \in [\![1,M]\!])$ and $\chi_j.h = \sum_{k=1}^{M} \chi_{j,k}h_k$. We also adopt the notations $\kappa_0 = 1$ and $\chi_0 = (0,\ldots,0)$. Define $Z_\Theta(\kappa,h) := \{\Re\{z\} :\ \Theta(z,\kappa,h) = 0\}$ and denote its closure by $\bar{Z}_\Theta(\kappa,h)$.

Lemma 2.5 ([29]). *If the equation $\Theta(x + i\omega,\kappa,h) = 0$ is satisfied for some reals x and ω, then the lengths $\left\{|\kappa_j|\mathrm{e}^{-x\chi_j.h},\ j \in [\![0,N]\!]\right\}$ can form a closed polygon; that is, none of these lengths is larger than the sum of the others: $|\kappa_j|\mathrm{e}^{-x\chi_j.h} \leq \sum_{k\neq j}|\kappa_k|\mathrm{e}^{-x\,\chi_k.h}$ for $j \in [\![0,N]\!]$.*

Also, following Henry [29], let us define $\rho_j = \rho_j(\kappa,\,h)$ $(j \in [\![0,N]\!])$, if they exist, by the relation

$$|\kappa_j|\mathrm{e}^{-\rho_j\,\chi_j.h} = \sum_{k\neq j}|\kappa_k|\mathrm{e}^{-\rho_j\,\chi_k.h} \quad \text{for} \quad j \in [\![0,N]\!]. \tag{2.11}$$

If $\chi_N.h \geq \chi_j.h > 0$ for $j \in [\![1,N-1]\!]$, then ρ_N and ρ_0 are uniquely defined and $\rho_N < \rho_0$ for $N \geq 2$.

Lemma 2.6 ([29]). *If $\chi_N.h \geq \chi_{N-1}.h > \ldots > \chi_1.h > 0$, then*

$$\bar{Z}_\Theta(\kappa,h) \subseteq [\rho_N,\,\rho_0]. \tag{2.12}$$

The following lemma, which has been proved in [3], provides a vertical strip in the complex plane, which is symmetric with respect to the imaginary axis and contains the set of zeros of $\tilde{Q}$:

Lemma 2.7 ([3])**.**

$$\bar{Z}_{\hat{Q}}(\kappa, h) \subseteq [-\rho^*,\, \rho^*], \tag{2.13}$$

where ρ^* is the unique positive zero of

$$\hat{Q}(\rho,\tau) := 1 - \left|\frac{\tau+2}{\tau-2}\right| \mathrm{e}^{-2\rho} - \left|\frac{\tau+2}{\tau-2}\right| \mathrm{e}^{-\tau\rho} - \mathrm{e}^{-(\tau+2)\rho}. \tag{2.14}$$

Now, ρ^* is a root of (2.14) if, and only if,

$$\mathrm{e}^{-(\tau+2)\rho^*} = 1 - \left|\frac{\tau+2}{\tau-2}\right| \mathrm{e}^{-2\rho^*} - \left|\frac{\tau+2}{\tau-2}\right| \mathrm{e}^{-\tau\rho^*} \tag{2.15}$$

Substituting the above expression into $\hat{Q}_\rho$, the first derivative of $\hat{Q}$ with respect to ρ, and evaluating the obtained expression at ρ^* entails:

$$\hat{Q}_\rho(\rho^*,\tau) = -\frac{(\tau+2)\left(\mathrm{e}^{-2\rho^*}\tau - |\tau-2| + 2\,\mathrm{e}^{-\tau\rho^*}\right)}{|\tau-2|}.$$

which never vanish for any $\tau \in \mathbb{R}_+^*\backslash\{2\}$. Indeed, for $\tau < 2$, $\hat{Q}_\rho$ is of constant sign and strictly decreasing with respect to ρ^*. In addition, if we assume that $\hat{Q}_\rho$ vanishes at ρ^* for $\tau > 2$, we get

$$\tau = \frac{-2\,\mathrm{e}^{-\tau\rho^*} - 2}{\mathrm{e}^{-2\rho^*} - 1}. \tag{2.16}$$

However, by eliminating $\mathrm{e}^{-\tau\rho^*}$ from the above expression of $\hat{Q}_\rho$ and substituting it into the expression of $\hat{Q}$, we obtain

$$\hat{Q}(\rho^*,\tau) = \frac{\left(\tau^2-4\right)\mathrm{e}^{-2\rho}}{2|\tau-2|} - \mathrm{e}^{-(\tau+2)\rho} - \frac{\tau}{2} = 0,$$

that is,

$$\tau = \frac{2\,\mathrm{e}^{-(\tau+2)\rho^*} - 2\,\mathrm{e}^{-2\rho^*}}{\mathrm{e}^{-2\rho^*} - 1}, \tag{2.17}$$

which is inconsistent with (2.16) and the fact that $\rho^* > 0$. Consequently, the *Implicit Function Theorem* is then applicable to (2.14) and asserts that $\rho^* = \rho^*(\tau)$ with

$$\rho^{*\prime}(\tau) = \frac{\left(4 + \left(-\tau^2+4\right)\rho^*(\tau)\right)\mathrm{e}^{-2\rho^*(\tau)} + 4\,\mathrm{e}^{-\tau\rho^*(\tau)} + \rho^*(\tau)\,|\tau-2|\,(\tau-2)}{\left(\mathrm{e}^{-2\rho^*(\tau)}\tau - |\tau-2| + 2\,\mathrm{e}^{-\tau\rho^*(\tau)}\right)\left(\tau^2-4\right)}.$$

Lemma 2.8 ([3])**.** *Consider the quasipolynomial $\hat{Q}$ given by (2.14) with $\tau \neq 2$. Then the spectral abscissa σ of $\hat{Q}$ is lower-bounded by $\hat{\rho}(\tau)$ where $\hat{\rho}$ is given by*

$$\hat{\rho}(\tau) := \frac{1}{\min\{\tau, 2\}} \ln\left(1 + 2\,\frac{\tau+2}{|\tau-2|}\right). \tag{2.18}$$

2.4. **Statement of problem** ($\mathcal{P}$)**.** The main goal here is to study the pointwise stabilization of a wave equation with delay-based feedback control. More precisely, we consider the system given by:

$$(\mathcal{P})\qquad \begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0, \quad (0,1)\setminus\{\xi\}\times(0,+\infty) \\ u(\xi^-,t) = u(\xi^+,t), \quad t>0, \\ u_x(\xi^+,t) - u_x(\xi^-,t) + \alpha\,(u_x(\xi^+,t-\tau) - u_x(\xi^-,t-\tau)) = \\ \qquad \beta\, u_t(\xi,t) + \gamma\, u_t(\xi,t-\tau), \quad t>0 \\ u_t(\xi,t-\tau) = 0 = u_x(\xi^+,t-\tau) - u_x(\xi^-,t-\tau), \quad t\in(0,\tau), \\ u(0,t) = 0, \quad u_x(1,t) = 0, \quad t>0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad (0,1), \end{cases}$$

where $\tau > 0, \alpha, \beta, \gamma$ and $\xi \in (0,1)$ are constants.

Delay effects arise in many applications and practical problems and it is well-known that an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay, see e.g. [18, 21, 20, 36]. Nevertheless, recent papers reveal that a particular choice of delays may lead to the exponential stability property, see [25, 26, 41].

We refer also to [1, 6, 36, 5, 37] for stability results for systems with time delay due to the presence of "good" feedbacks compensating the destabilizing delay effect.

Note that the above system is exponentially stable in the absence of time delay, see e.g. $\alpha = \gamma = 0$, $\beta > 0$, ξ admits a coprime factorization $\dfrac{p}{q}$ and p is odd (the best rate is obtained for $\xi = \dfrac{1}{2}$, see e.g. [4]).

3. Formulation via D'Alembert's formula and Well-posedness of problem $(\mathcal{P})$

We look for u solution of $(\mathcal{P})$ in the form:

$$u(x,t) = \psi_-(x+t) - \psi_-(t-x),\ \forall x\in(0,\xi),\ t\geq 0, \tag{3.19}$$

and

$$u(x,t) = \psi_+(x-1+t) + \psi_+(t-x+1),\ \forall x\in(\xi,1),\ t\geq 0, \tag{3.20}$$

where $\psi_- \in H^1_{\text{loc}}(-\xi,+\infty)$ and $\psi_+ \in H^1_{\text{loc}}(-(1-\xi),+\infty)$ have to be determined. From this expression, we directly see that

$$u(0,t) = 0,\ \text{and}\ u_x(1,t) = 0,\ \forall t\geq 0.$$

Hence it remains to impose the initial conditions at $t = 0$ and the transmission conditions at $x = \xi$.

In order to fulfill the initial conditions for $x \leq \xi$, we take

$$\psi_-(x) = -\frac{1}{2}u_0(-x) + \frac{1}{2}\int_0^{-x} u_1(s)\,ds \quad \forall x\in(-\xi,0),$$
$$\psi_-(x) = \frac{1}{2}u_0(x) + \frac{1}{2}\int_0^{x} u_1(s)\,ds \quad \forall x\in[0,\xi).$$

In that way ψ_- is uniquely determined in $(-\xi,\xi)$.

In the same manner to fulfill the initial conditions for $x \geq \xi$, we take

$$\psi_+(y) = \frac{1}{2}u_0(1+y) + \frac{1}{2}\int_0^{1+y} u_1(s)\,ds \quad \forall y \in (-(1-\xi), 0),$$
$$\psi_+(y) = \frac{1}{2}u_0(1-y) - \frac{1}{2}\int_0^{1-y} u_1(s)\,ds \quad \forall y \in [0, 1-\xi).$$

In that way ψ_+ is uniquely determined in $(-(1-\xi), 1-\xi))$.

Note that, by definition, $\psi_- \in H^1(-\xi,\xi)$ and $\psi_+ \in H^1(-(1-\xi), 1-\xi)$. Now, we extend ψ_-, ψ_+ by iteration obtaining functions in $H^1_{\mathrm{loc}}(-\xi,+\infty)$, $H^1_{\mathrm{loc}}(-(1-\xi),+\infty)$ respectively (cfr. [25]). To check $(\mathcal{P})_1$ in $(0,1)\times(0,\tau)$, we need the continuity of u and $(\mathcal{P})_5$ at ξ, that is equivalent to

$$\begin{aligned}
&\psi_-(\xi+t) - \psi_-(t-\xi) = \psi_+(\xi-1+t) + \psi_+(t-\xi+1),\ \forall t \in (0,\tau),\\
&\psi'_+(\xi-1+t) - \psi'_+(t-\xi+1) - \psi'_-(\xi+t) - \psi'_-(t-\xi) =\\
&\qquad\qquad \beta\left(\psi'_+(\xi-1+t) + \psi'_+(t-\xi+1)\right),\ \forall t \in (0,\tau).
\end{aligned}$$

By setting $y = \xi + t$, this is equivalent to

$$\begin{aligned}
&\psi_-(y) - \psi_+(y-2\xi+1) = \psi_-(y-2\xi) + \psi_+(y-1),\ \forall y \in (\xi, \xi+\tau),\\
&\psi'_+(y-1) - \psi'_+(y-2\xi+1) - \psi'_-(y) - \psi'_-(y-2\xi) =\\
&\qquad\qquad \beta\left(\psi'_+(y-1) + \psi'_+(y-2\xi+1)\right),\ \forall y \in (\xi, \xi+\tau).
\end{aligned}$$

Differentiating the first identity in y, taking the sum and the difference of the two identities, we get

$$\psi'_-(y) = \frac{2-\beta}{2}\,\psi'_+(y-1) - \frac{\beta}{2}\,\psi'_+(y-2\xi+1), \quad \forall y \in (\xi, \xi+\tau), \tag{3.21}$$

$$\psi'_+(y+1-2\xi) = -\frac{2}{2+\beta}\,\psi'_-(y-2\xi) - \beta\,\psi'_+(y-1), \quad \forall y \in (\xi, \xi+\tau). \tag{3.22}$$

By iteration this allows to find ψ_- (resp. ψ_+) up to $\tau+\xi$ (resp. $\tau+1-\xi$). Indeed fix $\varepsilon \leq 2\min\{\xi, 1-\xi\}$, then in a first step for $y \in (\xi, \xi+\varepsilon)$, we remark that $y-\ell$ belongs to $(\xi-1, \xi+\varepsilon-1)$ which is included in $(\xi-1, 1-\xi)$ the set where ψ_+ is defined up to now. This allows to obtain $\psi'_-(y)$ for all $y \in (\xi, \xi+\varepsilon)$. In the same manner $\psi'_-(y-2\xi)$ is well-defined and this allows then to obtain $\psi'_+(y+1-2\xi)$ for all $y \in (\xi, \xi+\varepsilon)$. We now iterate this argument, namely for $y \in (\xi+\varepsilon, \xi+2\varepsilon)$, the right-hand sides of (3.21)–(3.22) are meaningful, and consequently we obtain $\psi'_-(y)$ (resp. $\psi'_+(y+1-2\xi)$) for such y. We iterate this procedure up to $y \in (\xi+(k-1)\varepsilon, \xi+k\varepsilon)$, with $k \in \mathbb{N}$ such that

$$\xi + k\varepsilon = \xi + 2.$$

This proves the announced statement.

For $y > \xi + \tau$, we need to take into account $(\mathcal{P})_2$ and $(\mathcal{P})_3$, that take the form

$$\begin{aligned}
&\psi_-(\xi+t) - \psi_-(t-\xi) = \psi_+(\xi-1+t) + \psi_+(t-\xi+1),\ \forall t > \tau,\\
&\psi'_+(\xi-1+t) - \psi'_+(t-\xi+1) - \psi'_-(\xi+t) - \psi'_-(t-\xi) +\\
&\quad \alpha\left(\psi'_+(\xi-1+t-\tau) - \psi'_+(t-\tau-\xi+1) - \psi'_-(\xi+t-\tau) - \psi'_-(t-\tau-\xi)\right) =\\
&\qquad \beta\left(\psi'_+(\xi+t-1) + \psi'_+(t-\xi+1)\right) + \gamma\left(\psi'_+(\xi+t-\tau-1) + \psi'_+(t-\tau-\xi+1)\right),\ \forall t > \tau.
\end{aligned}$$

By setting $y = \xi + t$, this is equivalent to

$$\begin{aligned}
&\psi_-(y) - \psi_-(y-2\xi) = \psi_+(y-1) + \psi_+(y-2\xi+1), \forall y > \xi+\tau, \\
&\psi_+'(y-1) - \psi_+'(y-2\xi+1) - \psi_-'(y) - \psi_-'(y-2\xi) + \\
&\quad \alpha\left(\psi_+'(y-1-\tau) - \psi_+'(y-\tau-2\xi+1) - \psi_-'(y-\tau) - \psi_-'(y-\tau-2\xi)\right) = \\
&\qquad \beta\left(\psi_+'(y-1) + \psi_+'(y-2\xi+1)\right) + \gamma\left(\psi_+'(y-\tau-1) + \psi_+'(y-\tau-2\xi+1)\right), \forall y > \xi+\tau.
\end{aligned}$$

As before differentiating the first equation in y and taking the sum and the difference, we arrive at (compare with (3.21)–(3.22))

$$\begin{aligned}
\psi_-'(y) = {}& \left(1 - \frac{\beta}{2}\right)\psi_+'(y-1) + \frac{\alpha-\gamma}{2}\psi_+'(y-1-\tau) - \frac{\alpha+\gamma}{2}\psi_+'(y-\tau-2\xi+1) \\
& - \alpha\,\psi_-'(y-\tau) - \alpha\,\psi_-'(y-\tau-2\xi) - \frac{\beta}{2}\psi_+'(y-2\xi+1),\ \forall y > \xi+\tau,
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
\psi_+'(y+1-2\xi) = {}& -\frac{2}{2+\beta}\psi_-'(y-2\xi) + \frac{\alpha-\gamma}{2+\beta}\psi_+'(y-1-\tau) \\
& - \frac{\alpha+\gamma}{2+\beta}\psi_+'(y-\tau-2\xi+1) - \frac{\alpha}{2+\beta}\left(\psi_-'(y-\tau) + \psi_-'(y-\tau-2\xi)\right) \\
& - \frac{\beta}{2+\beta}\psi_+'(y-1),\ \forall y > \xi+\tau.
\end{aligned} \tag{3.24}$$

The same iterative argument allows to show that $\psi_-(y)$ (resp. $\psi_+(y)$) is uniquely defined for $y > \tau+\xi$ (resp. $y > \tau+1-\xi$).

This is equivalent to

$$\begin{aligned}
\psi_-'(y) = {}& \frac{2}{2+\beta}\psi_+'(y-1) + \frac{\alpha-\gamma}{2+\beta}\psi_+'(y-1-\tau) \\
& + \frac{(\alpha+\gamma)(2\beta-1)}{2}\psi_+'(y-\tau-2\xi+1) - \alpha\psi_-'(y-\tau) \\
& - \alpha\frac{\beta+4}{2\beta+4}\psi_-'(y-\tau-2\xi) + \frac{\beta}{2+\beta}\psi_+'(y-2\xi),\ \forall y > \xi+\tau,
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
\psi_+'(y+1-2\xi) = {}& -\frac{2}{2+\beta}\psi_-'(y-2\xi) + \frac{\alpha-\gamma}{2+\beta}\psi_+'(y-1-\tau) \\
& - \frac{\alpha+\gamma}{2+\beta}\psi_+'(y-\tau-2\xi+1) - \frac{\alpha}{2+\beta}\left(\psi_-'(y-\tau) + \psi_-'(y-\tau-2\xi)\right) \\
& - \frac{\beta}{2+\beta}\psi_+'(y-1), \forall y > \xi+\tau.
\end{aligned} \tag{3.26}$$

The main point is this last iterative relation between $\psi_-'(y)$, $\psi_+'(y+1-2\xi)$ and previous evaluations.

For $\xi = \frac{1}{2}, \tau = 2$, we can equivalently write (3.25)–(3.26) as the following system

$$\begin{pmatrix} \psi_-'(y) \\ \psi_+'(y) \\ \psi_-'(y-1) \\ \psi_+'(y-1) \\ \psi_-'(y-2) \\ \psi_+'(y-2) \end{pmatrix} = M_{\alpha,\beta,\gamma} \begin{pmatrix} \psi_-'(y-1) \\ \psi_+'(y-1) \\ \psi_-'(y-2) \\ \psi_+'(y-2) \\ \psi_-'(y-3) \\ \psi_+'(y-3) \end{pmatrix}. \tag{3.27}$$

where $M_{\alpha,\beta,\gamma}$ is given by:

$$M_{\alpha,\beta,\gamma} = \begin{pmatrix} \frac{\beta}{2+\beta} & \frac{2}{2+\beta} & -\alpha & \frac{(\alpha+\gamma)(2\beta-1)}{2} & -\alpha\frac{\beta+4}{2\beta+4} & \frac{\alpha-\gamma}{2+\beta} \\ \frac{-2}{2+\beta} & \frac{-\beta}{2+\beta} & \frac{-\alpha}{2+\beta} & -\frac{\alpha+\gamma}{2+\beta} & \frac{-\alpha}{2+\beta} & \frac{\alpha-\gamma}{2+\beta} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \tag{3.28}$$

Let us now take $\xi = \frac{1}{2}, \tau = 2, \alpha = 0, \beta = 0$, then we can equivalently write (3.23)–(3.24) as the following system

$$C(y) = M_\gamma\, C(y-1), \quad \text{where} \quad C(y) := \begin{pmatrix} \psi_-'(y) \\ \psi_+'(y) \\ \psi_+'(y-1) \\ \psi_+'(y-2) \end{pmatrix}, \quad M_\gamma := \begin{pmatrix} 0 & 1 & -\frac{\gamma}{2} & -\frac{\gamma}{2} \\ -1 & 0 & -\frac{\gamma}{2} & -\frac{\gamma}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.29}$$

As in [25, 26] we are reduced to calculate the eigenvalues of the matrix M_γ whose characteristic polynomial is

$$p_\gamma(\lambda) := \lambda^4 + \left(1 + \frac{\gamma}{2}\right)\lambda^2 - \frac{\gamma}{2}.$$

Consequently, the eigenvalues of this matrix are strictly less than 1 in modulus if and only if

$$|\gamma + 2 \pm \sqrt{\gamma^2 + 12\gamma + 4}| < 4. \tag{3.30}$$

In the case $\gamma^2 + 12\gamma + 4 \geq 0$ we see that (3.30) holds if and only if

$$-6 + 4\sqrt{2} < \gamma < 0. \tag{3.31}$$

On the contrary in the case $\gamma^2 + 12\gamma + 4 < 0$ we check that (3.30) holds if and only if

$$-2 < \gamma \leq -6 + 4\sqrt{2}. \tag{3.32}$$

Hence we conclude that (3.30) holds if and only if $\gamma \in (-2, 0)$.

Since

$$p_\gamma'(\lambda) = \lambda(4\lambda^2 + 2 + \gamma),$$

we can conclude that for $\gamma \in (-2, 0)$, all eigenvalues of M_γ are of modulus < 1 and simple. In that case, there exists a matrix V_γ such that

$$M_\gamma = V_\gamma^{-1} D_\gamma V_\gamma,$$

where D_γ is the diagonal matrix made of the eigenvalues of M_γ.

Now coming back to (3.29) and using an inductive argument, we can deduce that for all $j \in \mathbb{N}$, and for all $y \in (\frac{5}{2} + j, \frac{5}{2} + (j+1)]$, we have

$$C(y) = {M_\gamma}^j C(y - j).$$

Therefore using the previous factorization of M_γ, we get

$$C(y) = V_\gamma^{-1} D_\gamma^j V_\gamma C(y - j).$$

Finally, there exists a positive constant C_γ (depending only on γ) such that for all $j \in \mathbb{N}$, and all $y \in (\frac{5}{2} + j, \frac{5}{2} + (j+1)]$, we have

$$\|C(y)\|_2 \leq C_\gamma \rho_\gamma^j \|C(y - j)\|_2, \tag{3.33}$$

where ρ_γ is the spectral radius of D_γ that is < 1 (if $\gamma \in (-2, 0)$).

Now let us consider the total energy of the string given by

$$E(t) := \frac{1}{2}\int_0^1 \left(|u_t(x,t)|^2 + |u_x(x,t)|^2\right) dx. \tag{3.34}$$

By simple calculation, we see that

$$E(t) = \int_{-\frac{1}{2}}^{\frac{1}{2}} (\psi'_-(x+t)^2 + \psi'_+(x+t)^2)\, dx.$$

Now we closely follow the arguments of [25, 26] to conclude the exponential decay of the system. Namely for all $j \in \mathbb{N}$, and for all $t \in (2+j, 2+(j+2)]$, we can apply (3.33) with $y = x+t$ for any $x \in (-\frac{1}{2}, \frac{1}{2})$ and consequently

$$\begin{aligned} E(t) &\leq \int_{-\frac{1}{2}}^{\frac{1}{2}} \|C(x+t)\|_2^2\, dx \\ &\leq C_\gamma^2\, \rho_\gamma^{2j} \int_{-\frac{1}{2}}^{\frac{1}{2}} \|C(x+t-j)\|_2^2\, dx. \end{aligned}$$

Finally as for $t \in (2+j, 2+(j+2)]$ and $x \in (-\frac{1}{2}, \frac{1}{2})$, $x+t-j$ belongs to a compact set, the quantity

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \|C(x+t-j)\|_2^2\, dx$$

is bounded independently of j. This means that we have found a constant K_γ such that for all $j \in \mathbb{N}$, and all $t \in (2+j, 2+(j+2)]$, one has

$$E(t) \leq K_\gamma\, \rho_\gamma^{2j}.$$

This leads to the conclusion because $\rho_\gamma^{2j} = \mathrm{e}^{2j \ln \rho_\gamma} \leq \mathrm{e}^{2t \ln \rho_\gamma}$.

In what follows, we will prove the global existence and the uniqueness of the solution of problem $(\mathcal{P})$. We will first transform the problem $(\mathcal{P})$ to the problem (3.37) by making the change of variables (3.35), and then we use the semigroup approach to prove the existence of the unique solution of problem $(\mathcal{P})$.

To overcome the problem of the boundary delay, we introduce the new variables:

$$z^1(\rho, t) = u_x(\xi^+, t-\tau\rho) - u_x(\xi^-, t-\tau\rho),\ z^2(\rho, t) = u_t(\xi, t-\tau\rho),\ \rho \in (0,1),\ t > 0. \tag{3.35}$$

Then, we have

$$\tau\, z_t^j(\rho, t) + z_\rho^j(\rho, t) = 0, \text{in } (0,1) \times (0, +\infty), j = 1, 2. \tag{3.36}$$

Therefore, problem $(\mathcal{P})$ is equivalent to:

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0, x \in (0,1) \setminus \{\xi\},\ t > 0, \\ u(\xi^+, t) = u(\xi^-, t),\ t > 0, \\ \tau\, z_t^j(\rho, t) + z_\rho^j(\rho, t) = 0,\ \rho \in (0,1),\ t > 0,\ j = 1, 2, \\ u_x(\xi^+, t) - u_x(\xi^-, t) + \alpha\, z^1(1,t) = \beta\, z^2(0,t) + \gamma\, z^2(1,t),\ t > 0, \\ z^1(0,t) = u_x(\xi^+, t) - u_x(\xi^-, t),\ z^2(0,t) = u_t(\xi, t),\ u(0,t) = u_x(1,t) = 0,\ t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in (0,1), \\ z^j(\rho, 0) = 0,\ \rho \in (0,1), j = 1, 2. \end{cases} \tag{3.37}$$

In this section we will give a sufficient condition that guarantees that this problem is well-posed. For this purpose we will use a semigroup formulation of the initial-boundary value problem $(\mathcal{P})$.

If we denote $V := (u, u_t, z_1, z_2)^T$, we define the energy space:

$$\mathscr{H} = H_\ell^1(0,1) \times L^2(0,1) \times L^2(0,1) \times L^2(0,1),$$

where $H_\ell^1(0,1) = \left\{u \in H^1(0,1), u(0) = 0\right\}$.

Clearly, $\mathscr{H}$ is a Hilbert space with respect to the inner product

$$\langle V_1, V_2\rangle_{\mathscr{H}} = \int_0^1 u_{1,x}\, v_{1,x}\, dx + \int_0^1 u_2\, v_2\, dx + \frac{\tau}{\beta}\sum_{j=1}^{2}\int_0^1 z_j\, w_j\, d\rho \tag{3.38}$$

for $V_1 = (u_1, u_2, z_1, z_2)^T$ and $V_2 = (v_1, v_2, w_1, w_2)^T$. Therefore, if $V_0 \in \mathscr{H}$ and $V \in \mathscr{H}$, the problem (3.37) is formally equivalent to the following abstract evolution equation in the Hilbert space $\mathscr{H}$:

$$\begin{cases} V'(t) = \mathscr{A}V(t), \quad t > 0, \\ V(0) = V_0, \end{cases} \tag{3.39}$$

where $'$ denotes the derivative with respect to time t, $V_0 := (u_0, u_1, 0, 0)^T$ and the operator $\mathscr{A}$ is defined by:

$$\mathscr{A}\begin{pmatrix} u \\ v \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} v \\ u_{xx} \\ -\tau^{-1} z_{1,\rho} \\ -\tau^{-1} z_{2,\rho} \end{pmatrix}.$$

The domain of $\mathscr{A}$ is the set of $V = (u, v, z_1, z_2)^T$ such that:

$$(u, v, z_1, z_2)^T \in H^1_\ell(0,1) \times H^1_\ell(0,1) \times H^1(0,1) \times H^1(0,1), \tag{3.40}$$

$$\begin{gathered} u_{|(0,\xi)} \in H^2(0,\xi),\ u_{|(\xi,1)} \in H^2(\xi,1), \\ u_x(1) = 0,\ v(\xi) = z_2(0),\ u_x(\xi^+) - u_x(\xi^-) = z_1(0), \end{gathered} \tag{3.41}$$

$$z_1(0) + \alpha\, z_1(1) = \beta\, z_2(0) + \gamma\, z_2(1). \tag{3.42}$$

Then the well-posedness of problem (3.37) is ensured by:

Theorem 3.1. *Let α, β, γ be such that*

$$\begin{cases} |\alpha| < 1, \\ |\gamma| < \sqrt{1-\alpha^2}, \\ \beta \geq 1, \end{cases} \tag{3.43}$$

Under these conditions, let $V_0 \in \mathscr{H}$, then there exists a unique solution $V \in C(\mathbb{R}_+; \mathscr{H})$ of problem (3.39). Moreover, if $V_0 \in \mathscr{D}(\mathscr{A})$, then

$$V \in C(\mathbb{R}_+; \mathscr{D}(\mathscr{A})) \cap C^1(\mathbb{R}_+; \mathscr{H}).$$

Proof. In order to prove the existence and uniqueness of the solution of problem (3.39) we use the semigroup approach and the Lumer-Phillips' theorem.

Indeed, let $V = (u, v, z_1, z_2)^T \in \mathscr{D}(\mathscr{A})$. By definition of the operator $\mathscr{A}$ and the scalar product of $\mathscr{H}$, we have:

$$\langle \mathscr{A}V, V\rangle_{\mathscr{H}} = \int_0^1 v_x(x) u_x(x)\, dx + \int_0^1 u_{xx}(x) v_x(x)\, dx + \frac{\tau}{\beta}\sum_{j=1}^{2}\int_0^1 \tau^{-1} z_{j\rho}(\rho) z_j(\rho)\, d\rho.$$

We obtain:

$$
\begin{aligned}
\langle \mathscr{A}V, V\rangle_{\mathscr{H}} &= -v(\xi)\, z_1(0) - \frac{1}{2\beta}(z_1^2(1) - z_1^2(0)) - \frac{1}{2\beta}(z_2^2(1) - z_2^2(0)),\\
&= -z_2(0)\,(-\alpha\, z_1(1) + \beta\, z_2(0) + \gamma\, z_2(1)) - \frac{1}{2\beta} z_1^2(1)\\
&\quad + \frac{1}{2\beta}\,(-\alpha\, z_1(1) + \beta\, z_2(0) + \gamma\, z_2(1))^2 - \frac{1}{2\beta}\, z_2^2(1) + \frac{1}{2\beta}\, z_2^2(0),\\
&= z_1(1)\left(\frac{\alpha^2-1}{2\beta}\, z_1(1) - \frac{\alpha\gamma}{2\beta}\, z_2(1)\right) + z_2(0)\,\left(\frac{1}{2\beta} - \frac{\beta}{2}\right)\, z_2(0)\\
&\quad + z_2(1)\,\left(-\frac{\alpha\gamma}{2\beta}\, z_1(1) + \frac{1}{2\beta}\,\left(\gamma^2-1\right)\, z_2(1)\right),\\
&= \begin{pmatrix} z_1(1) & z_2(0) & z_2(1)\end{pmatrix}\, D_{\alpha,\beta,\gamma}\, \begin{pmatrix} z_1(1)\\ z_2(0)\\ z_2(1)\end{pmatrix},
\end{aligned}
\tag{3.44}
$$

where

$$
D_{\alpha,\beta,\gamma} := \begin{pmatrix} \frac{\alpha^2-1}{2\beta} & 0 & -\frac{\alpha\gamma}{2\beta}\\ 0 & \frac{1}{2\beta} - \frac{\beta}{2} & 0\\ -\frac{\alpha\gamma}{2\beta} & 0 & \frac{\gamma^2-1}{2\beta}\end{pmatrix}. \tag{3.45}
$$

One can notice that if (α, β, γ) satisfy the conditions (3.43), then $D_{\alpha,\beta,\gamma}$ given by (3.45) is a definite negative matrix. Indeed, one can explicitly compute its eigenvalues given by $-\dfrac{\beta^2-1}{2\beta}$, $-\dfrac{1}{2\beta}$, $\dfrac{\gamma^2+\alpha^2-1}{2\beta}$. Also, when α is taken to be zero, one recovers the conditions obtained in [1].

According to condition (3.43), we obtain

$$
\begin{aligned}
\langle \mathscr{A}V, V\rangle_{\mathscr{H}} &= -\frac{1-\alpha^2-\gamma^2}{2\beta} z_1^2(1) - \frac{\beta^2-1}{2\beta} z_2^2(0) - \frac{1-\alpha^2-\gamma^2}{2\beta} z_2^2(1)\\
&\quad - \frac{1}{2\beta}\,(\gamma\, z_1(1) + \alpha\, z_2(1))^2\,.
\end{aligned}
\tag{3.46}
$$

Hence $\langle \mathscr{A}V, V\rangle_{\mathscr{H}} \le 0$, $\mathscr{A}$ is thus a dissipative operator.

Now we want to show that $\mathscr{A}$ is invertible.

For $F = (f_1, f_2, f_3, f_4)^T \in \mathscr{H}$, let $V = (u, v, z_1, z_2)^T \in \mathscr{D}(\mathscr{A})$ solution of

$$
\mathscr{A}V = F,
$$

which is:

$$
v = f_1, \tag{3.47}
$$

$$
u_{xx} = f_2, \tag{3.48}
$$

$$
z_{j,\rho} = -\tau\, f_{j+2}, \quad j = 1, 2. \tag{3.49}
$$

Thus, from (3.49), z_j, $j = 1, 2$, are given by:

$$
\begin{aligned}
z_1(\rho) &= z_1(0) - \tau \int_0^\rho f_3(s)\, ds,\ \ \rho \in (0, 1),\\
z_2(\rho) &= f_1(\xi) - \tau \int_0^\rho f_4(s)\, ds,\ \ \rho \in (0, 1).
\end{aligned}
\tag{3.50}
$$

Using the preceding expression and assumption (3.43), we have:

(3.51)

$$u(x)=\begin{cases}\int_0^x\left(\int_0^y f_2(s)\,ds\right)dy\\ \quad+\left\{\int_0^1 f_2(s)\,ds+\frac{\alpha\tau}{1+\alpha}\int_0^1 f_3(s)\,ds-\frac{\gamma\tau}{1+\alpha}\int_0^1 f_4(s)\,ds+\frac{\beta+\gamma}{1+\alpha}f_1(\xi)\right\}x,\ x\in(0,\xi),\\ \int_\xi^x\left(\int_\xi^y f_2(s)\,ds\right)dy-\left(\int_\xi^1 f_2(s)\,ds\right)x+\int_0^\xi\left(\int_0^y f_2(s)\,ds\right)dy\\ \quad+\xi\int_0^\xi f_2(s)\,ds+\frac{\alpha\xi\tau}{1+\alpha}\int_0^1 f_3(s)\,ds-\frac{\gamma\xi\tau}{1+\alpha}\int_0^1 f_4(s)\,ds+\xi\frac{\beta+\gamma}{1+\alpha}f_1(\xi),\ x\in(\xi,1)\end{cases}$$

and

$$z_1(\rho)=\frac{\beta+\gamma}{1+\alpha}f_1(\xi)+\frac{\alpha\tau}{1+\alpha}\int_0^1 f_3(s)ds-\frac{\gamma\tau}{1+\alpha}\int_0^1 f_4(s)ds-\tau\int_0^\rho f_4(s)ds,\ \rho\in(0,1).$$

So, we have found $V=(u,v,z_1,z_2)^T\in\mathscr{D}(\mathscr{A})$, the unique solution of $\mathscr{A}V=F$. Which implies in particular that $0\in\rho(\mathscr{A})$, where $\rho(\mathscr{A})$ is the resolvent set of $\mathscr{A}$.

The operator $\mathscr{A}$ generates a C_0 semigroup of contractions $\mathrm{e}^{t\mathscr{A}}$ on $\mathscr{H}$. Thus from the Lumer-Phillips' theorem, there exists a unique solution $V\in C(\mathbb{R}_+;\mathscr{H})$ of the problem (3.39). This completes the proof of Theorem 3.1. □

Since $\mathscr{D}(\mathscr{A})\subset\left[H^2((0,1)\setminus\{\xi\})\cap H^1_\ell(0,1)\right]\times H^1_\ell(0,1)\times H^1(0,1)\times H^1(0,1)$, the Sobolev's embedding implies that $\mathscr{A}^{-1}$ is a compact operator on $\mathscr{H}$. So, we have the following corollary.

Corollary 3.2. *The spectrum of $\mathscr{A}$, $\sigma(\mathscr{A})$, only consists on eigenvalues of finite multiplicity.*

4. **Asymptotic behavior**

In this section, we show that under condition (3.43), the semigroup $\mathrm{e}^{t\mathscr{A}}$ decays exponentially to the null steady state. To obtain this, our technique is based on a frequency domain method and combines a contradiction argument with the multiplier technique to carry out a special analysis for the resolvent.

Theorem 4.1. *Suppose that condition* (3.43) *holds and $\xi=\frac{1}{2}$. Then there exist constants $C,\omega>0$ such that, for all $V_0\in\mathcal{H}$, the semigroup $\mathrm{e}^{t\mathscr{A}}$ satisfies the following estimate*

$$\left\|\mathrm{e}^{t\mathscr{A}}V_0\right\|_{\mathscr{H}}\le C\,\mathrm{e}^{-\omega t}\ \|V_0\|_{\mathscr{H}},\forall\,t>0. \tag{4.52}$$

Remark 4.2. *For $\alpha=\gamma=0$ and $\beta=1$, the best decay rate, i.e., the best $\omega>0$ such that (4.52) is satisfied for all $V_0\in\mathscr{H}$, is equal to $\frac{\ln(3)}{2}$, according to* [4]. *In other words, in the case where $\alpha=\gamma=0$ and $\beta=1$, the fastest decay rate of the solutions of $(\mathcal{P})$ is obtained if the actuator is located at the middle of the string.*

Proof of Theorem 4.1. We will use the following frequency domain theorem for uniform stability from Huang-Prüss [30, 39] of a C_0 semigroup of contractions on a Hilbert space:

Lemma 4.3. *A C_0 semigroup $\mathrm{e}^{t\mathcal{L}}$ of contractions on a Hilbert space $\mathcal{H}$ satisfies*

$$||\mathrm{e}^{t\mathcal{L}}U_0||_{\mathcal{H}}\le C\,\mathrm{e}^{-\theta t}||U_0||_{\mathcal{H}}$$

for some constant $C>0$ and for $\theta>0$ if and only if

$$\rho(\mathcal{L})\supset\left\{i\delta\ \middle|\ \delta\in\mathbb{R}\right\}\equiv i\mathbb{R}, \tag{4.53}$$

and

$$\limsup_{\delta\in\mathbb{R},|\delta|\to\infty} \|(i\delta I-\mathcal{L})^{-1}\|_{\mathcal{L}(\mathcal{H})}<\infty, \tag{4.54}$$

where $\rho(\mathcal{L})$ denotes the resolvent set of the operator $\mathcal{L}$.

First we look at the point spectrum of $\mathscr{A}$.

Lemma 4.4. *The spectrum of $\mathscr{A}$ contains no point on the imaginary axis if and only if $\xi\in(0,1)$ satisfies the following condition:*

$$\xi\notin\left\{\frac{2k'}{2k+1},\ k',k\in\mathbb{N}\right\}. \tag{4.55}$$

Proof. Since $\mathscr{A}$ has compact resolvent, its spectrum $\sigma(\mathscr{A})$ only consists of eigenvalues of $\mathscr{A}$. We will show that the equation

$$\mathscr{A}Z=i\delta Z \tag{4.56}$$

with $Z=(u,v,z_1,z_2)^T\in\mathscr{D}(\mathscr{A})$ and $\delta\in\mathbb{R}^*$ has only the trivial solution.
Equation (4.56) writes :

$$i\delta u=v, \tag{4.57}$$

$$u_{xx}+\delta^2u=0, \tag{4.58}$$

$$i\delta\tau z_j+z_{j,\rho}=0. \tag{4.59}$$

By taking the inner product of (4.56) with Z and using (3.46), we get:

$$z_1(0)=z_2(1)=0.$$

Thus we have $z_1(\rho)=z_1(0)\mathrm{e}^{-i\delta\tau\rho}=0, z_2(\rho)=z_2(1)\mathrm{e}^{-i\delta\tau(\rho-1)}=0$, we obtain also $u(x)=A\sin(\delta x)$, $x\in(0,1)$ and A is a constant.

So, the only solution of (4.56) is the trivial one if and only if ξ satisfies the condition (4.55). □

According to [8], the Lemma 4.4 shows that

Corollary 4.5. *For all $V_0\in\mathcal{H}$, the semigroup $\mathrm{e}^{t\mathscr{A}}$ satisfies the following strong stability result*

$$\left\|\mathrm{e}^{t\mathscr{A}}V_0\right\|_{\mathscr{H}}\xrightarrow[t\to+\infty]{}0 \tag{4.60}$$

if and only if ξ satisfies (4.55).

The following lemma shows that (4.54) holds with $\mathcal{L}=\mathscr{A}$.

Lemma 4.6. *The resolvent operator of $\mathscr{A}$ satisfies condition* (4.54).

Proof. Suppose that condition (4.54) is false. By the Banach-Steinhaus Theorem (see [14]), there exist a sequence of real numbers $\delta_n\to+\infty$ and a sequence of vectors $Z_n=(u_n,v_n,z_1,z_2)^t\in\mathscr{D}(\mathscr{A})$ with $\|Z_n\|_{\mathscr{H}}=1$ such that

$$||(i\delta_nI-\mathscr{A})Z_n||_{\mathscr{H}}\to0 \quad \text{as} \quad n\to\infty, \tag{4.61}$$

i.e.,

$$i\delta_nu_n-v_n\equiv f_{1,n}\to0 \quad \text{in} \quad H^1_\ell(0,1), \tag{4.62}$$

$$i\delta_nv_n-u_{n,xx}\equiv f_{2,n}\to0 \quad \text{in} \quad L^2(0,1), \tag{4.63}$$

$$i\delta_nz_{j,n}+\frac{1}{\tau}\partial_\rho zj,n\equiv f_{j,n}\to0, j=3,4. \quad \text{in} \quad L^2(0,1), \tag{4.64}$$

Our goal is to derive from (4.61) that $||Z_n||_{\mathscr{H}}$ converges to zero, thus there is a contradiction. The proof is divided in four steps:

First step. We notice that from (4.61) we have

$$||(i\delta_n I - \mathscr{A})Z_n||_{\mathscr{H}} \geq |\Re\{\langle(i\delta_n I - \mathscr{A})Z_n, Z_n\rangle_{\mathscr{H}}\}|. \tag{4.65}$$

Then, by (3.46) and (4.61),

$$z_{j,n}(0), z_{2,n}(1) \to 0,\ j = 1, 2. \tag{4.66}$$

Which implies according (4.62) that $\delta_n u_n(1/2) \longrightarrow 0$. Moreover, since $Z_n \in \mathscr{D}(\mathscr{A})$, we have, by (4.66),

$$z_{1,n}(1), v_n(1/2), u_{n,x}\left(\frac{1}{2}^+\right) - u_{n,x}\left(\frac{1}{2}^-\right) \to 0. \tag{4.67}$$

Thus, we have according to (4.66), that

$$zj,n(\rho) = zj,n(0)\, \mathrm{e}^{-i\delta_n\tau\rho} + \tau\int_0^\rho \mathrm{e}^{-i\delta_n\tau(\rho-s)}\, f_{j+2,n}(s)\, ds$$

$$\Downarrow$$

$$zj,n \longrightarrow 0, L^2(0,1),\ j = 1, 2. \tag{4.68}$$

Second step. We express now v_n in function of u_n from equation (4.62) and substitute it into (4.63) to get

$$-\delta_n^2 u_n - u_{n,xx} = f_{2,n} + i\delta_n f_{1,n} \tag{4.69}$$

Next, we take the inner product of (4.69) with $q(x)u_{n,x}$ in $L^2(0, 1/2)$ where $q(x) \in C^1([0, 1/2])$ and $q(0) = 0$. We obtain that

$$\begin{aligned}\int_0^{\frac{1}{2}} \left(-\delta_n^2\, u_n(x) - u_{nxx}(x)\right) q(x)\, \overline{u}_{n,x}\, dx &= \int_0^{\frac{1}{2}} \left(f_{2,n}(x) + i\delta_n f_{1,n}(x)\right) q(x)\, \overline{u}_{n,x}(x)\, dx,\\ &= \int_0^{\frac{1}{2}} f_{2,n}(x)\, q(x)\, \overline{u}_{n,x}(x)\, dx - i\int_0^{\xi} q(x)\, f_{1,n,x}(x)\, \delta_n \overline{u}_n(x)\, dx\\ &\quad - i\int_0^{\frac{1}{2}} f_{1,n}(x)\, q_x(x)\, \delta_n\overline{u}_n(x)\, dx + i\, f_{1,n}\left(\frac{1}{2}\right)\, q\left(\frac{1}{2}\right)\delta_n\overline{u}_n\left(\frac{1}{2}\right).\end{aligned} \tag{4.70}$$

It is clear that the right-hand side of (4.70) converges to zero since $f_{1,n}, f_{2,n}$ converge to zero in $H_\ell^1(0, 1)$ and $L^2(0, 1)$, respectively.

By a straight-forward calculation,

$$\Re\left\{\int_0^{\frac{1}{2}} -\delta_n^2 u_n(x)\, q(x)\overline{u}_{n,x}\, dx\right\} = -\frac{1}{2}q\left(\frac{1}{2}\right)\left|\delta_n u_n\left(\frac{1}{2}\right)\right|^2 + \frac{1}{2}\int_0^{\frac{1}{2}} q_x(x)|\delta_n u_n(x)|^2\, dx$$

and

$$\Re\left\{\int_0^{\xi} -u_{n,xx}(x)\, q(x)\overline{u}_n\, dx\right\} = -\frac{1}{2}q\left(\frac{1}{2}\right)\left|u_{n,x}\left(\frac{1}{2}\right)\right|^2 + \frac{1}{2}\int_0^{\frac{1}{2}} q_x(x)\left|u_{n,x}(x)\right|^2\, dx.$$

This leads to

$$\int_0^{\frac{1}{2}} q_x(x)\left|\delta_n u_n(x)\right|^2\, dx + \int_0^{\frac{1}{2}} q_x(x)\left|u_{n,x}(x)\right|^2\, dx - q\left(\frac{1}{2}\right)\left|u_{n,x}\left(\frac{1}{2}^-\right)\right|^2 \to 0. \tag{4.71}$$

Similarly, we take the inner product of (4.69) with $q_1(x)u_{n,x}$ in $L^2(\frac{1}{2},1)$ with $q_1 \in C^1([\frac{1}{2},1])$ and $q_1(1)=0$, then repeat the above procedure. This will give us

$$\int_{\frac{1}{2}}^{1} q_{1,x}(x)|\delta_n u_n(x)|^2\,dx + \int_{\frac{1}{2}}^{1} q_{1,x}\left|u_{n,x}(x)\right|^2\,dx + q_1\left(\frac{1}{2}\right)\left|u_{n,x}\left(\frac{1}{2}^+\right)\right|^2 \to 0. \tag{4.72}$$

Third step. Next, we show that both of $\left|u_{n,x}\left(\frac{1}{2}^-\right)\right|$ and $\left|u_{n,x}\left(\frac{1}{2}^+\right)\right|$ converge to zero. To proceed, we have

$$u_n(x) = \begin{cases} A_n\sin(\delta_n x) - \displaystyle\int_0^x \frac{\sin(\delta_n(x-y))}{\delta_n}\left(f_{2,n}(y)+i\delta_n f_{1,n}(y)\right)dy,\ x\in(0,1/2), \\ B_n\cos(\delta_n(1-x)) - \displaystyle\int_x^1 \frac{\sin(\delta_n(x-y))}{\delta_n}\left(f_{2,n}(y)+i\delta_n f_{1,n}(y)\right)dy,\ x\in(1/2,1), \end{cases} \tag{4.73}$$

where A_n, B_n are given by :

$$\begin{aligned} A_n\delta_n = \sin(\delta_n/2)\, & \left[\int_0^{1/2}\sin\left[\delta_n\left(\frac{1}{2}-y\right)\right]\left(f_{2,n}(y)+i\delta_n f_{1,n}(y)\right)dy - \right. \\ & \left. \int_{1/2}^{1}\sin\left[\delta_n\left(\frac{1}{2}-y\right)\right]\left(f_{2,n}(y)+i\delta_n f_{1,n}(y)\right)dy\right] + \\ & \cos(\delta_n/2)\left[\int_0^{1/2}\cos\left[\delta_n\left(\frac{1}{2}-y\right)\right]\left(f_{2,n}(y)+i\delta_n f_{1,n}(y)\right)dy + \right. \\ & \left. \int_{1/2}^{1}\cos\left[\delta_n\left(\frac{1}{2}-y\right)\right]\left(f_{2,n}(y)+i\delta_n f_{1,n}(y)\right)dy\right] - \\ & \cos(\delta_n/2)\left[-\alpha z_{1,n}(1)+\beta z_{2,n}(0)+\gamma z_{2,n}(1)\right], \end{aligned}$$

and

$$\begin{aligned} B_n\delta_n = \sin(\delta_n/2)\, & \left[\int_0^{1/2}\cos\left[\delta_n\left(\frac{1}{2}-y\right)\right]\left(f_{2,n}(y)+i\delta_n f_{1,n}(y)\right)dy + \right. \\ & \left. \int_{1/2}^{1}\cos\left[\delta_n\left(\frac{1}{2}-y\right)\right]\left(f_{2,n}(y)+i\delta_n f_{1,n}(y)\right)dy\right] - \\ & \sin(\delta_n/2)\left[-\alpha z_{1,n}(1)+\beta z_{2,n}(0)+\gamma z_{2,n}(1)\right] - \\ & \cos(\delta_n/2)\left[\int_0^{1/2}\sin\left[\delta_n\left(\frac{1}{2}-y\right)\right]\left(f_{2,n}(y)+i\delta_n f_{1,n}(y)\right)dy - \right. \\ & \left. \int_{1/2}^{1}\sin\left[\delta_n\left(\frac{1}{2}-y\right)\right]\left(f_{2,n}(y)+i\delta_n f_{1,n}(y)\right)dy\right], \end{aligned}$$

which implies that

$$A_n\delta_n, B_n\delta_n \to 0.$$

Thus,

$$u_{n,x}\left(\frac{1}{2}^-\right) = A_n\delta_n\cos(\delta_n/2) - \int_0^{1/2}\cos\left[\delta_n\left(\frac{1}{2}-y\right)\right]\left(f_{2,n}(y)+i\delta_n f_{1,n}(y)\right)dy \to 0. \tag{4.74}$$

and

$$u_{n,x}\left(\frac{1}{2}^+\right) = B_n\delta_n\sin(\delta_n/2) - \int_{1/2}^{1}\cos\left[\delta_n\left(\frac{1}{2}-y\right)\right]\left(f_{2,n}(y)+i\delta_n f_{1,n}(y)\right)dy \to 0. \tag{4.75}$$

Fourth step. Finally, we choose $q(x)$ and $q_1(x)$ so that $\frac{dq}{dx}$ and $\frac{dq_1}{dx}$ are strictly positive. This can be done by taking

$$q(x) = \mathrm{e}^{x} - 1, \qquad q_1(x) = \mathrm{e}^{x-1} - 1. \tag{4.76}$$

Therefore, (4.71) and (4.72) imply

$$\|\delta_n u_n\|_{L^2(0,1)} \to 0, \qquad \|u_{n,x}\|_{L^2(0,1)} \to 0. \tag{4.77}$$

In view of (4.62), we also get

$$\|v_n\|_{L^2(0,1)} \to 0, \tag{4.78}$$

which clearly, with (4.68), contradict $\|Z_n\|_{\mathscr{H}} = 1$.

□

The two hypotheses of Lemma 4.3 are proved by Lemma 4.6. Then (4.52) holds. The proof of Theorem 4.1 is then finished. □

5. **Spectral analysis**

We give here the characterization of the eigenvalues and eigenvectors of $\mathscr{A}$.

Lemma 5.1.. *A complex number $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathscr{A}$ if and only if*

$$\left(1 + \alpha\,\mathrm{e}^{-\tau\,\lambda}\right)\,\cosh\lambda + \left(\beta + \gamma\,\mathrm{e}^{-\tau\,\lambda}\right)\,\cosh\left(\lambda\,\left(\xi - 1\right)\right)\,\sinh\left(\lambda\,\xi\right) = 0, \tag{5.79}$$

or equivalently,

$$\begin{aligned}(2+\beta) + (2-\beta)\,\mathrm{e}^{-2\lambda} - \beta\,\mathrm{e}^{-2\xi\,\lambda} + \beta\,\mathrm{e}^{2(\xi-1)\,\lambda} - \gamma\,\mathrm{e}^{-(\tau+2\xi)\lambda} \\ + \gamma\,\mathrm{e}^{-(\tau-2\xi+2)\lambda} + (2\alpha+\gamma)\,\mathrm{e}^{-\tau\,\lambda} + (2\alpha-\gamma)\,\mathrm{e}^{-(\tau+2)\,\lambda} = 0.\end{aligned} \tag{5.80}$$

Moreover the corresponding eigenfunction F_λ is given by

$$F_\lambda(x,\rho) = \left(u_\lambda(x), \lambda u_\lambda(x), z_\lambda(\rho), w_\lambda(\rho)\right)^T \tag{5.81}$$

where

$$\begin{aligned} u_\lambda(x) &= \begin{cases} \cosh\left(\lambda\,(1-\xi)\right)\,\sinh\left(\lambda\,x\right), & 0 \le x \le \xi, \\ \sinh\left(\lambda\,\xi\right)\,\cosh\left(\lambda\,\left(1-x\right)\right), & \xi \le x \le 1. \end{cases} \\ z_\lambda(\rho) &= -\lambda\,\cosh(\lambda)\,\mathrm{e}^{-\lambda\,\tau\,\rho}, \quad \rho \in (0,1), \\ w_\lambda(\rho) &= \lambda\,\cosh\left(\lambda\,\left(\xi-1\right)\right)\,\sinh\left(\lambda\,\xi\right)\,\mathrm{e}^{-\lambda\,\tau\,\rho}, \quad \rho \in (0,1). \end{aligned} \tag{5.82}$$

Proof. Assume that $(u_\lambda, v_\lambda, z_\lambda, w_\lambda)^T$ is an eigenvector associated to the eigenvalue λ of $\mathscr{A}$. Then $v_\lambda = \lambda\,u_\lambda$ and $(u_\lambda, z_\lambda, w_\lambda)$ satisfy

$$\begin{aligned} \frac{d^2u_\lambda}{dx^2}(x) &= \lambda^2\,u_\lambda(x), \quad x \in (0,\xi) \cup (\xi,1), \\ \frac{dz_\lambda}{d\rho}(\rho) &= -\lambda\tau z_\lambda(\rho), \quad \rho \in (0,1), \\ \frac{dw_\lambda}{d\rho}(\rho) &= -\lambda\tau w_\lambda(\rho), \quad \rho \in (0,1), \end{aligned} \tag{5.83}$$

with boundary conditions

$$u_\lambda(0) = \frac{du_\lambda}{dx}(1) = 0, \tag{5.84}$$

$$u_\lambda(\xi^-) = u_\lambda(\xi^+), \tag{5.85}$$

$$w_\lambda(0) = \lambda u_\lambda(\xi),\; z_\lambda(0) = \frac{du_\lambda}{dx}(\xi^+) - \frac{du_\lambda}{dx}(\xi^-), \tag{5.86}$$

$$\frac{du_\lambda}{dx}(\xi^+) - \frac{du_\lambda}{dx}(\xi^-) = -\alpha z_\lambda(1) + \beta\, w_\lambda(0) + \beta w_\lambda(1). \tag{5.87}$$

Equation (5.83) and (5.84) imply that

$$u_\lambda(x) = \begin{cases} A \sinh(\lambda\, x), & x \in (0,\xi), \\ B \cosh\left(\lambda\,(x-1)\right), & x \in (\xi,1), \end{cases}$$
$$z_\lambda(\rho) = z_\lambda(0)\, \mathrm{e}^{-\lambda\,\tau\,\rho}, \quad \rho \in (0,1),$$
$$w_\lambda(\rho) = w_\lambda(0)\, \mathrm{e}^{-\lambda\,\tau\,\rho}, \quad \rho \in (0,1).$$

Then, (5.87) and (5.85) imply

$$\begin{cases} \sinh\left(\lambda\,\xi\right)\, A - \cosh\left(\lambda\,(\xi-1)\right)\, B = 0, \\ \left(\beta + \gamma\,\mathrm{e}^{-\lambda\,\tau}\right)\sinh\left(\lambda\,\xi\right) + \left(1+\alpha\,\mathrm{e}^{-\lambda\,\tau}\, \cosh\left(\lambda\,\xi\right)\right)\, A - \left(1+\alpha\,\mathrm{e}^{-\lambda\,\tau}\right)\sinh\left(\lambda\,\left(\xi-1\right)\right)\, B = 0. \end{cases}$$

The system (5.83) has a non-trivial solution if and only if the determinant of the coefficients matrix of the above system satisfies:

$$D(\lambda) = \left(1+\alpha\,\mathrm{e}^{-\tau\,\lambda}\right)\, \cosh\lambda + \left(\beta+\gamma\,\mathrm{e}^{-\tau\,\lambda}\right)\, \cosh\left(\lambda\,\left(\xi-1\right)\right)\sinh\left(\lambda\,\xi\right) = 0. \tag{5.88}$$

Then, the solutions of (5.83)-(5.86) have the following form

$$u_\lambda(x) = \begin{cases} \cosh\left(\lambda\,(\xi-1)\right)\sinh\left(\lambda\,x\right), & x \in (0,\xi), \\ \sinh\left(\lambda\,\xi\right)\, \cosh\left(\lambda\,\left(x-1\right)\right), & x \in (\xi,1), \end{cases}$$
$$z_\lambda(\rho) = -\lambda\, \cosh(\lambda)\mathrm{e}^{-\lambda\,\tau\,\rho}, \quad \rho \in (0,1),$$
$$w_\lambda(\rho) = \lambda\, \cosh\left(\lambda\,\left(\xi-1\right)\right)\sinh\left(\lambda\,\xi\right)\, \mathrm{e}^{-\lambda\,\tau\,\rho}, \quad \rho \in (0,1).$$

□

According to Corollary 3.2, $\sigma(\mathscr{A})$ is given by the eigenvalues of $\mathscr{A}$ and under condition (3.43), is localized in $\mathbb{C}_- := \{\lambda \in \mathbb{C} \,|\, \Re\{\lambda\} < 0\}$. More precisely we have the following corollary:

Corollary 5.2. $\sigma(\mathscr{A}) = \{\lambda \in \mathbb{C}_- \,|\, D(\lambda) = 0\}$ *and is symmetrically distributed with respect of real axis.*

6. **A parametric stability analysis via a frequency domain approach**

Let $s \in \mathbb{C}$ be the Laplace variable of the Laplace Transform operator and the delay such that $\tau \in \mathbb{R}_+^*$. For $\xi = \frac{1}{2}$, the characteristic equation in (5.80) which is nothing but $Q(\cdot;2,\tau) = 0$ given in (2.2), with

$$\begin{cases} a_1 = \dfrac{2-\beta}{2+\beta}, \\ a_2 = \dfrac{2\alpha+\gamma}{2+\beta}, \\ a_3 = \dfrac{2\alpha-\gamma}{2+\beta}. \end{cases} \tag{6.89}$$

This allows to the following characteristics function of interest:

$$Q(s;2,\tau) = 1 + \frac{2-\beta}{2+\beta}\,\mathrm{e}^{-2\,s} + \frac{2\alpha+\gamma}{2+\beta}\,\mathrm{e}^{-\tau\,s} + \frac{2\alpha-\gamma}{2+\beta}\mathrm{e}^{-(\tau+2)\,s}. \tag{6.90}$$

The exponential stability of the closed-loop system $(\mathcal{P})$ as well as the one of (2.1) can be achieved if, and only if, the roots of (6.90) have a strictly negative real part. Moreover, the decay-rate σ toward zero, with $\sigma > 0$, for $y(\cdot)$ solution of (2.1) as well as for $u(x,\cdot)$ solution of $(\mathcal{P})$ can be obtained if, and only if, $-\sigma$ is an upper-bound on the real part of any root of (6.90).

For $\beta > 0$ the following set of necessary and sufficient conditions can be derived from [27, Chapter 9, Theorem 6.1] and recalled in Section 2.1:

$$\begin{cases} |\gamma| < 2, \\ |\alpha| < \dfrac{\beta}{2}. \end{cases} \tag{6.91}$$

One can note that such conditions encompass the sufficient conditions obtained in (3.43) for $\beta > 2$.

In the following section, we shall exploit the manifold of spectral values' multiplicities to get some insights on the solutions' decay rates.

6.1. **Validity of the MID property.** Notice that the special case $\tau = 2$ allows to decrease the degree of the quasipolynomial Q. As a matter of fact, the quasipolynomial Q reads

$$Q(s; 2, 2) = 1 + \frac{(2 + 2\alpha - \beta + \gamma)\, \mathrm{e}^{-2s}}{2 + \beta} + \frac{(2\alpha - \gamma)\, \mathrm{e}^{-4s}}{2 + \beta} \tag{6.92}$$

which admits a degree equal to two for $\gamma \neq 2\alpha$ and $\gamma \neq \beta - 2\alpha - 2$. As a direct consequence of (6.92) and Theorem 2.1, we can enunciate the following corollary:

Corollary 6.1. *Consider the quasipolynomial $Q(\cdot; 2, \tau)$ given by* (6.90) *where $\tau = 2$ and α, β, γ are such that $\gamma \neq 2\alpha$ and $\gamma \neq \beta - 2\alpha - 2$.*

A given real number s_0 is a double root of (6.92) *if, and only if,*

$$\begin{cases} \alpha = \dfrac{\left(1 - \mathrm{e}^{2\, s_0}\right)^2}{4} \beta - \mathrm{e}^{2\, s_0} + \dfrac{\mathrm{e}^{4\, s_0}}{2} - \dfrac{1}{2}, \\ \gamma = \left(\dfrac{1}{2} - \mathrm{e}^{2\, s_0} - \dfrac{\mathrm{e}^{4\, s_0}}{2}\right) \beta - \left(1 + \mathrm{e}^{2\, s_0}\right)^2. \end{cases} \tag{6.93}$$

If (6.93) *is satisfied then the GMID holds, that is, s_0 corresponds to the spectral abscissa of the quasipolynomial $Q(\cdot; 2, 2)$ given by* (6.92)*. Furthermore, all zeros of* (6.92) *are double and lie on the vertical axis $\Re\{s\} = s_0$.*

Remark 6.2. *Notice that* (6.93) *allows a parametric freedom since the parameter β can be arbitrarily selected. For a good choice of this a parameter with respect to performance, one can choose β in order for the sufficient condition* (3.43) *of the closed-loop operator dissipativity is guaranteed; that is, $\beta > 1$. This fact shall be illustrated later through the transverse vibration control of a string in Section 7.*

Now, let us consider again the quasipolynomial (6.90) where $\alpha \neq |\frac{\gamma}{2}|$ and $|\beta| \neq 2$, *i.e.*, the case where the quasipolynomial's degree is equal to three. Taking into account (6.90), a direct consequence of Theorem 2.2 gives the following corollary:

Corollary 6.3. *Consider the quasipolynomial $Q(\cdot; 2, 2k)$ given by* (6.90) *and let $k \neq 1$.*

A given real number s_0 is a triple root of (6.90) *if, and only if,*

$$\begin{cases} \alpha = \dfrac{(\tau - 2)\, \mathrm{e}^{2\, s_0} - (\tau + 2)}{(\tau + 2)\, \mathrm{e}^{2\, s_0} - (\tau - 2)}\, \mathrm{e}^{\tau\, s_0}, \\ \beta = -2\, \dfrac{(\tau + 2)\, \mathrm{e}^{2\, s_0} + (\tau - 2)}{(\tau + 2)\, \mathrm{e}^{2\, s_0} - (\tau - 2)}, \\ \gamma = -2\, \dfrac{(\tau - 2)\, \mathrm{e}^{2\, s_0} + (\tau + 2)}{(\tau + 2)\, \mathrm{e}^{2\, s_0} - (\tau - 2)}\, \mathrm{e}^{\tau\, s_0}. \end{cases} \tag{6.94}$$

If (6.94) *is satisfied and* τ *is an even integer strictly greater than 2, then the GMID holds, that is* s_0 *corresponds to the spectral abscissa of the quasipolynomial* $Q(\cdot; 2, 2k)$ *given by* (6.90).

When the MID property fails, one can prescribe a lower bound for the decay rate as will be discussed in the next Subsection.

6.2. **Estimation of the exponential decay.** The following Lemma which is inspired from [3, Section 4, Theorem 5], provides an estimate of the closed-loop system $(\mathcal{P})$ decay rate.

Lemma 6.4. *Consider the quasipolynomial* $\hat{Q}$ *given by* (2.14) *with* $\tau \neq 2$. *Then the spectral abscissa* σ *of* $\hat{Q}$ *is lower-bounded by* $\hat{\rho}(\tau)$ *where* $\hat{\rho}$ *is given by*

$$\hat{\rho}(\tau) := \frac{1}{\min\{\tau, 2\}} \ln\left(1 + 2\,\frac{\tau+2}{|\tau-2|}\right). \tag{6.95}$$

Proof. It is easy to observe that for $\rho > 0$ and $\tau \neq 2$

$$\hat{Q}(\rho; 2, \tau) \geq 1 - \left(1 + 2\frac{\tau+2}{|\tau-2|}\right) \mathrm{e}^{-\tau\rho}.$$

We remark that the right-hand side of this last inequality admits a single root which is upper bounded by $\hat{\rho}$ given by (6.95). In conclusion, for any $\tau \neq 2$, one has $\hat{Q}(\hat{\rho}(\tau), \tau) > 0$, which asserts that $\rho^*(\tau) \leq \hat{\rho}(\tau)$ from the *Intermediate Value Theorem.* □

Thanks to the results of Section 2.1 and the above lemmas, the proof of the following theorem, which gives a certified decay rate's lower-bound for the closed-loop system's solution, is immediate.

Theorem 6.5. *Consider the output feedback stabilization of the problem* $(\mathcal{P})$ *with an arbitrary positive delay* τ*, then the following assertions hold:*

- *If* $\tau = 2$*, then the control parameter tuning prescribed in system* (6.93) *allows to assign the solution's exponential decay rate at an arbitrary* $-s_0$*;*
- *If* $\tau = 2\,k$ *where* k *is an integer greater than one, then the control parameter tuning prescribed in system* (6.94) *allows to assign the solution's exponential decay rate at an arbitrary* $-s_0$*;*
- *If* $\tau \neq 2\,k$*, then the control parameter tuning prescribed in system* (6.94) *allows a closed-loop solution decaying exponentially faster than* $-s_0 - \hat{\rho}(\tau)$*, where* $\hat{\rho}$ *is defined by expression* (6.95).

Proof. The spectral analysis of the operator derived from $(\mathcal{P})$ turns it into the characteristic equation (6.90). Finally, using the normalization (2.7), we end up with expression (2.9). The first assertion is a direct consequence of Corollary 6.1. The second assertion is a direct consequence of Corollary 6.3. The third assertion follows directly from Lemma 6.4. □

Remark 6.6. *Figure 1 shows the locus of* $\hat{\rho}$ *given by* (6.95) *(the proposed upper-bound on the real parts of the zeros of the quasipolynomial* $\hat{Q}$*) as a function of the delay* τ*. Thanks to the linear change of variables* (2.7)*, this enables the selection of an appropriate pair* (s_0, τ) *in the filled gray region, providing an upper-bound on the spectral abscissa of the quasipolynomial* Q *given in* (6.90)*. As asserted in Theorem 6.5, the desired decay rate towards the steady state equilibrium is greater than* $|s_0 + \hat{\rho}(\tau)| = -(s_0 + \hat{\rho}(\tau))$*, since we have proven that any root* $s_i \in \mathbb{C}$ *of equation* (6.90) *(*$i \in \mathbb{N}$*) is such that* $\Re\{s_i\} < s_0 + \rho^* < s_0 + \hat{\rho}(\tau)$*. By prescribing a minimal decay rate* $\sigma > 0$*,* s_0 *is chosen such that* $s_0 + \hat{\rho}(\tau) < -\sigma < 0$ *to ensure the asymptotic stability,* i.e., $s_0 < -\sigma - \hat{\rho}(\tau)$*.*

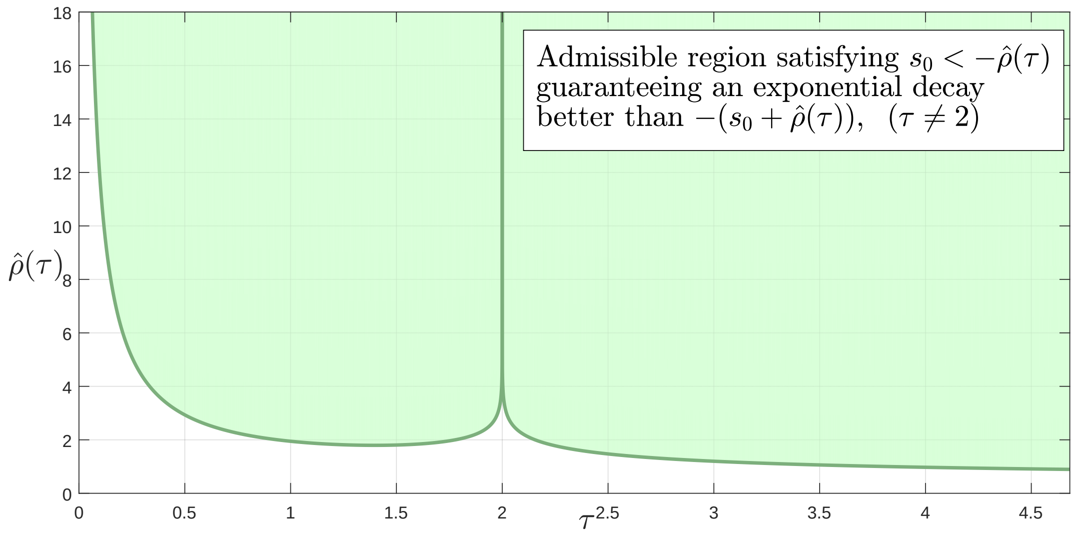

FIGURE 1. Plot of the function $\hat{\rho}(\tau)$ given by expression (6.95).

7. Application to the transverse vibration control of a string

In this section, the transverse vibration of a string is considered for the illustration of the control scheme described in this paper. For a perfectly elastic string of length $l > 0$ with both fixed ends, having constant density and elasticity parameters, when no body forces are considered, the motion equation is described by the following Partial Derivative Equation

$$(\tilde{\mathcal{P}}) \qquad \begin{cases} U_{\mathbf{tt}}(\mathbf{x},\mathbf{t}) - c^2\, U_{\mathbf{xx}}(\mathbf{x},\mathbf{t}) = 0, \quad (\mathbf{x},\mathbf{t}) \in (0,l) \times (0,+\infty) \\ U(0,\mathbf{t}) = 0, \quad U_{\mathbf{x}}(l,\mathbf{t}) = 0, \quad \mathbf{t} > 0, \\ U(\mathbf{x},0) = f(\mathbf{x}), \quad U_{\mathbf{t}}(\mathbf{x},0) = g(\mathbf{x}), \quad \mathbf{x} \in (0,l), \end{cases}$$

where $U(\mathbf{x},\mathbf{t})$ denotes the transversal displacement of a point belonging to the string, located at position $\mathbf{x}$ and given at time $\mathbf{t}$; f and g are smooth univariate functions that are specified further below.

Remark 7.1. *The wave equation problem modeled by equation* $(\mathcal{P})$ *has used some normalized (*i.e.*, dimensionless) time t and position x variables, in such a way that their corresponding relations to the considered true variables are $\mathbf{x} = l\,x$ and $\mathbf{t} = d\,t$ with $d := \dfrac{l}{c}$. In this case, the normalized transversal displacement is related to the original one by the relation*

$$\begin{aligned} u(x,t) &:= U(\mathbf{x},\mathbf{t}), \\ &= U(L\,x, d\,t). \end{aligned} \tag{7.96}$$

The control law is given by the following delay-based autoregressive control law applied in a pointwise manner at the midpoint $x = \xi$ in $(\tilde{\mathcal{P}})$ where now $(\mathbf{x},\mathbf{t}) \in (0,l) \setminus \{\xi := \frac{l}{2}\} \times (0,+\infty)$. It writes

$$(\tilde{\mathcal{C}}) \qquad \begin{cases} U(\xi^-,\mathbf{t}) = U(\xi^+,\mathbf{t}), \quad \xi := \dfrac{l}{2}, \quad \mathbf{t} > 0, \\ U_{\mathbf{x}}(\xi^+,\mathbf{t}) - U_{\mathbf{x}}(\xi^-,\mathbf{t}) + \alpha\,\big(U_{\mathbf{x}}(\xi^+,\mathbf{t}-\boldsymbol{\tau}) - U_{\mathbf{x}}(\xi^-,\mathbf{t}-\boldsymbol{\tau})\big) = \\ \qquad \dfrac{\beta}{c}\,U_{\mathbf{t}}(\xi,\mathbf{t}) + \dfrac{\gamma}{c}\,U_{\mathbf{t}}(\xi,\mathbf{t}-\boldsymbol{\tau}),\ \mathbf{t} > 0, \\ U_{\mathbf{t}}(\xi,\mathbf{t}-\boldsymbol{\tau}) = 0, \quad U_{\mathbf{x}}(\xi^+,\mathbf{t}-\boldsymbol{\tau}) - U_{\mathbf{x}}(\xi^-,\mathbf{t}-\boldsymbol{\tau}) = 0, \quad \mathbf{t} \in (0,\boldsymbol{\tau}), \end{cases}$$

where $\boldsymbol{\tau} := d\,\tau$ is the delay used here as a control parameter.

7.1. **Finite difference scheme.** To check numerically the efficiency of the proposed control scheme, we propose to first perform a finite difference discretization of the above problem by using the following approximations.

Let us define the constant space step $\Delta x > 0$, the constant time step $\Delta t > 0$ and the numerical sequences

$$\mathbf{x}_i := i\,\Delta x\ (i = 0, 1, 2, \ldots, N), \quad \mathbf{t}_j := j\,\Delta x\ (j = 0, 1, 2, \ldots, n) \quad \text{and} \quad U_{i,j} := U(\mathbf{x}_i, \mathbf{t}_j), \tag{7.97}$$

where M, N, $n \in \mathbb{N}$ are such that

$$M\,\Delta x = \frac{l}{2}, \quad N = 2M \quad \text{and} \quad n\,\Delta t = T_f \tag{7.98}$$

where T_f stands for the final time for the simulation. Let us denote $U_{\mathbf{t}^{i,j}} := \left.\dfrac{\partial U(\mathbf{x},\mathbf{t})}{\partial \mathbf{t}}\right|_{\substack{\mathbf{x}=\mathbf{x}_i,\\ \mathbf{t}=\mathbf{t}_j}}$, $U_{\mathbf{x}^{i,j}} := \left.\dfrac{\partial U(\mathbf{x},\mathbf{t})}{\partial \mathbf{x}}\right|_{\substack{\mathbf{x}=\mathbf{x}_i,\\ \mathbf{t}=\mathbf{t}_j}}$, $U_{\mathbf{tt}^{i,j}} := \left.\dfrac{\partial^2 U(\mathbf{x},\mathbf{t})}{\partial \mathbf{t}^2}\right|_{\substack{\mathbf{x}=\mathbf{x}_i,\\ \mathbf{t}=\mathbf{t}_j}}$ and $U_{\mathbf{xx}^{i,j}} := \left.\dfrac{\partial^2 U(\mathbf{x},\mathbf{t})}{\partial \mathbf{x}^2}\right|_{\substack{\mathbf{x}=\mathbf{x}_i,\\ \mathbf{t}=\mathbf{t}_j}}$.

For all $\mathbf{t} > 0$, the boundary conditions in $(\tilde{\mathcal{P}})$ are transcribed as follows with $j \in \mathbb{N}^*$:

$$\begin{aligned} U(0,\mathbf{t}) = 0 \quad &\longrightarrow \quad U_{0,j} = 0,\\ U_{\mathbf{t}}(0,\mathbf{t}) = 0 \quad &\longrightarrow \quad U_{\mathbf{t}^{0,j}} = 0,\\ U_{\mathbf{tt}}(0,\mathbf{t}) = 0 \quad &\longrightarrow \quad U_{\mathbf{tt}^{0,j}} = 0. \end{aligned} \tag{7.99}$$

For all $\forall \mathbf{x} \in (0, l)$, the initial conditions read, for $i = 0, 1, 2, \ldots, N$,

$$\begin{aligned} U(\mathbf{x},0) = f(\mathbf{x}) \quad &\longrightarrow \quad U_{i,0} = f(i\,\Delta x),\\ U_{\mathbf{t}}(\mathbf{x},0) = g(\mathbf{x}) \quad &\longrightarrow \quad U_{\mathbf{t}^{i,0}} = g(i\,\Delta x),\\ U_{\mathbf{x}}(\mathbf{x},0) = f'(\mathbf{x}) \quad &\longrightarrow \quad U_{\mathbf{x}^{i,0}} = f'(i\,\Delta x),\\ U_{\mathbf{xx}}(\mathbf{x},0) = f''(\mathbf{x}) \quad &\longrightarrow \quad U_{\mathbf{xx}^{i,0}} = f''(i\,\Delta x). \end{aligned} \tag{7.100}$$

By continuity of $U(\mathbf{x},\mathbf{t})$ in the variable $\mathbf{t}$, the initial conditions are then

$$\begin{aligned} U(0,0) = 0 \Rightarrow f(0) = 0 \quad &\longrightarrow \quad U_{0,0} = 0,\\ U_{\mathbf{x}}(l,\mathbf{t}) = 0 \Rightarrow U_{\mathbf{x}}(l,0) = 0 \Rightarrow f'(l) = 0 \quad &\longrightarrow \quad U_{\mathbf{x}^{N,j}} = 0,\\ U_{\mathbf{t}}(0,\mathbf{t}) = 0 \Rightarrow U_{\mathbf{t}}(0,0) = 0 \Rightarrow g(0) = 0 \quad &\longrightarrow \quad U_{\mathbf{t}^{0,0}} = 0. \end{aligned} \tag{7.101}$$

Moreover, at $i = 0$ and $j = 0$, it turns out that $f''(0) = 0$. To summarize, at this stage, the functions f and g should satisfy

$$\begin{aligned} &f(0) = 0, \quad f''(0) = 0, \quad f'(l) = 0,\\ &g(0) = 0. \end{aligned} \tag{7.102}$$

Consider the following approximations of the first-order partial derivatives

$$U_{\mathbf{t}^{i,j}} = \frac{1}{2\,\Delta t}\left(3\,U_{i,j} - 4\,U_{i,j-1} + U_{i,j-2}\right), \tag{7.103}$$

$$U_{\mathbf{x}^{i,j}} = \frac{1}{2\Delta x}\left(U_{i+1,j} - U_{i-1,j}\right), \tag{7.104}$$

and the ones for the second-order partial derivatives

$$U_{\mathbf{tt}^{i,j}} = \frac{1}{\Delta t^2}\left(U_{i,j} - 2\,U_{i,j-1} + U_{i,j-2}\right), \tag{7.105}$$

$$U_{\mathbf{xx}^{i,j}} = \frac{1}{\Delta x^2}\left(U_{i+1,j} - 2\,U_{i,j} + U_{i-1,j}\right), \tag{7.106}$$

defined for $i = 1, 2, \dots, N$ and $j = 2, 3, \dots, n$. For $j = 1$, we set

$$\begin{aligned} U_{i,1} &= U_{i,0} + \Delta t\, U_{\mathbf{t}^{i,0}}, \\ &= f(i\,\Delta x) + \Delta t\, g(i\,Dx). \end{aligned} \tag{7.107}$$

The delay-based autoregressive control law $(\tilde{\mathcal{C}})$ is turned into a numerical sequence given by

$$\begin{cases} U_{\mathbf{x}^{M+1,j}} - U_{\mathbf{x}^{M-1,j}} = \dfrac{\beta}{c}\, U_{\mathbf{t}^{M,j}}, \qquad \text{for } j = 0, 1, \dots, \delta - 1, \\ U_{\mathbf{x}^{M+1,j}} - U_{\mathbf{x}^{M-1,j}} = -\alpha\, \left(U_{\mathbf{x}^{M+1,j-\delta}} - U_{\mathbf{x}^{M-1,j-\delta}}\right) + \dfrac{\beta}{c}\, U_{\mathbf{t}^{M,j}} \\ \qquad\qquad + \dfrac{\gamma}{c}\, U_{\mathbf{t}^{M,j-\delta}}, \quad \text{for } j = \delta, \delta + 1, \dots, n \end{cases} \tag{7.108}$$

where $\delta := \lfloor \tau/\Delta t \rfloor$ denotes the integer part of $\tau/\Delta t$ close to $-\infty$, and α, β, γ are real scalar parameters that set the control law as indicated by Theorem 6.5. Here, $U_{M-1,j}$ and $U_{M+1,j}$ stands for $U(\xi^-, \mathbf{t}_j)$ and $U(\xi^+, \mathbf{t}_j)$ respectively. Therefore, it comes

$$U_{M+1,j} = U_{M-1,j} = U_{M,j}. \tag{7.109}$$

for the sake of consistency when $\Delta x \to 0$. The autoregressive control law carries on the gradient jump term $U_{\mathbf{x}}(\xi^+, \mathbf{t}) - U_{\mathbf{x}}(\xi^-, \mathbf{t})$ leading to the following finite difference approximation

$$U_{\mathbf{x}}(\xi^+, \mathbf{t}) - U_{\mathbf{x}}(\xi^-, \mathbf{t}) \longrightarrow U_{\mathbf{x}^{M+1,j}} - U_{\mathbf{x}^{M-1,j}} = \frac{U_{M+2,j} - 2\,U_{M,j} + U_{M-2,j}}{2\,\Delta x},\ \forall j \in \mathbb{N}^*. \tag{7.110}$$

Using (7.109), one should pay attention to the following fact

$$U_{\mathbf{x}}(\xi, \mathbf{t}_j) \longrightarrow U_{\mathbf{x}^{M,j}} = \frac{U_{M+1,j} - U_{M-1,j}}{2\,\Delta x} = 0, \quad \forall j \in \mathbb{N}. \tag{7.111}$$

As a consequence, in addition to (7.102), the following additive constraints should be taken into account for f

$$f'(M\,\Delta x) = 0,\ f''(M\,\Delta x) = 0. \tag{7.112}$$

Now we are ready to write the finite difference scheme for both the uncontrolled wave equation in $(\tilde{\mathcal{P}})$, and the closed-loop system combining $(\tilde{\mathcal{P}})$ and $(\tilde{\mathcal{C}})$.

Let $\mu := \dfrac{c\,\Delta t}{\Delta x}$.

7.2. **Open-loop case.** When there is no output feedback control applied to $(\tilde{\mathcal{P}})$, the finite difference scheme writes

$$\begin{aligned} U_{\mathbf{tt}^{i,j}} &= c^2\, U_{\mathbf{xx}^{i,j}}, \quad i = 0, 1, 2, \dots, N-1, \quad \forall j \in \mathbb{N}\backslash \{0, 1\}, \\ U_{N,j} &= U_{N-1,j}, \quad \forall j \in \mathbb{N}, \end{aligned}$$

leading to the following implicit scheme

$$-\mu^2\, U_{i+1,j} + (1 + 2\,\mu^2)\, U_{i,j} - \mu^2\, U_{i-1,j} = 2\, U_{i,j-1} - U_{i,j-2}. \tag{7.113}$$

7.3. **Closed-loop case.** When the output feedback control given by $(\tilde{\mathcal{C}})$ is applied to $(\tilde{\mathcal{P}})$, the finite difference scheme becomes

$$\begin{aligned} U_{\mathbf{tt}^{i,j}} &= c^2\, U_{\mathbf{xx}^{i,j}}, \quad i = 0, 1, 2, \dots, M-2, M+2, \dots, N-1, \quad \forall j \in \mathbb{N}\backslash \{0, 1\}, \\ U_{N,j} &= U_{N-1,j}, \quad \forall j \in \mathbb{N}, \\ \frac{U_{M+2,j} - 2\,U_{M,j} + U_{M-2,j}}{2\,\Delta x} &= -\alpha\, \frac{U_{M+2,j-\delta} - 2\,U_{M,j-\delta} + U_{M-2,j-\delta}}{2\,\Delta x} \\ &\quad + \frac{\beta}{c}\, \frac{3\,U_{M,j} - 4\,U_{M,j-1} + U_{M,j-2}}{2\,\Delta t} \\ &\quad + \frac{\gamma}{c}\, \frac{3\,U_{M,j-\delta} - 4\,U_{M,j-1-\delta} + U_{M,j-2-\delta}}{2\,\Delta t}, \quad \forall j \in \mathbb{N}\backslash \{0, 1\}, \end{aligned}$$

where, for all $i, j \in \mathbb{N}$, $U_{i,j-k} = 0$ for any $k \in \mathbb{N}$ such that $j - k < 0$. It comes,

$$
(7.114)\quad \begin{cases}
-\mu^2 U_{i+1,j} + (1 + 2\mu^2) U_{i,j} - \mu^2 U_{i-1,j} = 2 U_{i,j-1} - U_{i,j-2}, \quad \forall j \in \mathbb{N} \backslash \{0, 1\}, \\
U_{N,j} = U_{N-1,j}, \quad \forall j \in \mathbb{N}, \\
U_{M+2,j} - \left(2 + 3\beta \dfrac{\Delta x}{\Delta t}\right) U_{M,j} + U_{M-2,j} = -4\beta \dfrac{\Delta x}{\Delta t} U_{M,j-1} + \beta \dfrac{\Delta x}{\Delta t} U_{M,j-2} \\
\quad + \left(2\alpha + 3\gamma \dfrac{\Delta x}{\Delta t}\right) U_{M,j-\delta} - \alpha \left(U_{M+2,j-\delta} - U_{M-2,j-\delta}\right) \\
\quad -4\gamma \dfrac{\Delta x}{\Delta t} U_{M,j-1-\delta} + \gamma \dfrac{\Delta x}{\Delta t} U_{M,j-2-\delta}, \\
\text{where, for all } i, j \in \mathbb{N},\ U_{i,j-k} = 0 \text{ for any } k \in \mathbb{N} \text{ such that } j - k < 0.
\end{cases}
$$

Even if naturally implicit, this numerical scheme can easily be transformed into an explicit numerical scheme and written in a matricial form. The study of stability, consistency, and convergence of both these numerical schemes is abandoned for conciseness.

7.4. **Numerical simulations.** Let us now consider the numerical values of Table 1 for the considered string of problem $(\tilde{\mathcal{P}})$. To ensure stability and convergence of both numerical schemes

Length (m)	l	10
Wave propagation speed (m/s)	c	1.118
Time scaling factor (s)	d	8.9443
Wave magnitude	A	0.5

Table 1. Features of the string.

(7.113) and (7.114), we set $\Delta x = 0.05$, $\Delta t = 0.005$ and $T_f = 100$. The initial conditions are chosen as follow

$$
f(x) = A\left(3\,\frac{\mathbf{x}}{l} - 17\left(\frac{\mathbf{x}}{l}\right)^3 + 27\left(\frac{\mathbf{x}}{l}\right)^4 - 12\left(\frac{\mathbf{x}}{l}\right)^5\right),
$$
$$
g(x) = 0,
$$

to cope with all the requirements described in 7.1.

For illustration purposes, we precise the computation of the Energy function in (3.34) using the previous numerical scheme, for both the open-loop and the closed-loop cases. For $j \geq 2$, this function is computed as

$$
E(j\,\Delta t) = \frac{\Delta x}{2}\left(\left(\frac{u_{1,j} - u_{0,j}}{\Delta x}\right)^2 + \sum_{i=2}^{N-1}\left(\frac{u_{i+1,j} - u_{i-1,j}}{2\,\Delta x}\right)^2 + \left(\frac{u_{N,j} - u_{N-1,j}}{\Delta x}\right)^2 + \sum_{i=0}^{N}\left(\frac{3\,u_{i,j} - 4\,u_{i,j-1} + u_{i,j-2}}{2\,\Delta t}\right)^2\right),
$$

which for $j \geq 2$, given the boundary conditions in $(\tilde{\mathcal{P}})$, reduces to

$$
(7.115)\quad E(j\,\Delta t) = \frac{\Delta x}{2}\left(\frac{{u_{1,j}}^2}{\Delta x^2} + \sum_{i=2}^{N-1}\left(\frac{u_{i+1,j} - u_{i-1,j}}{2\,\Delta x}\right)^2 + \sum_{i=0}^{N}\left(\frac{3\,u_{i,j} - 4\,u_{i,j-1} + u_{i,j-2}}{2\,\Delta t}\right)^2\right).
$$

Moreover, for $j = 0$ and $j = 1$ and by considering (7.107)

$$
(7.116)\quad E(0) = E(1,\Delta t) = \frac{\Delta x}{2}\left(\left(\frac{f(\Delta x)}{\Delta x}\right)^2 + \sum_{i=2}^{N-1}\left(\frac{f((i+1)\,x) - f((i-1)\,x)}{2\,\Delta x}\right)^2\right).
$$

s_0	-2			$-1.005\,\hat{\rho}(\tau)$ $\simeq -14.481$	$-2.01\,\hat{\rho}(\tau)$ $\simeq -28.961$
τ	2	3	4	0.08	
α	0.20463	0.013593	0.0010584	-0.34014	-0.1068
β	3.0	2.1495	1.9969	1.7889	1.7889
γ	0.40758	0.024495	0.0019166	-0.60846	-0.19104
MID	Met	Unmet	Met	Unmet	Unmet
Time plot	Fig. 2d	Fig. 2e	Fig. 2f	Fig. 2b	Fig. 2c

Table 2. Feedback parameters, various cases.

Several simulation cases have been performed, they are summarized in Table 2.

Figure 2a shows the free time response of the wave equation of problem $(\tilde{\mathcal{P}})$, *ie* only subjected to initial conditions and without any control. Figures 2b, 2c, 2d, 2e and 2f show the closed-loop time response to the same initial conditions for the various cases described in Table 2. In Figure 3, the total energy logarithm is plotted for the various closed-loop cases, where the energy is the one defined in (3.34) and computed through (7.115). The blue curve illustrates the case where there is no feedback control. In Figure 4, the time response of the closed-loop control signals for the various cases is plotted. One can notice the periodic occurrence of impulsive peaks on the control signal, whose magnitude seems to be lower as the delay parameter increases. In Figure 3, the exponential decay in closed-loop can be noticed, and especially its controlling by assigning s_0. Moreover, it can also be noticed that, for the same $s_0 = -2$ and for different values of τ, the best decay rate is achieved with the MID case described in Theorem 6.5, say for a triple multiplicity of s_0 with $\tau = 4$. One also notices that this case provides a similar exponential decay as the one with $s_0 \simeq -28.961$ and $\tau = 0.08$, but gives a closed-loop control signal with a lower magnitude in comparison with the other cases.

8. **Conclusion**

This work addressed the control problem of the wave equation via an autoregressive output feedback control law applied at the midpoint. We have first introduced a difference equation with two interfering delays for which we have derived some results about the exponential stability. Then, we proposed an analysis of the operator's dissipativity where the operator was derived from a reformulation of the wave equation and we have exhibited the link between this operator and the previous difference equation. This allowed us to derive some conditions on the parameters of the autoregressive control to ensure the exponential stability of the considered problem with the advantage of prescribing the decay of the solution. Some numerical results have illustrated the efficiency of our approach for the case of a vibrating string satisfying the considered boundary and initial conditions.

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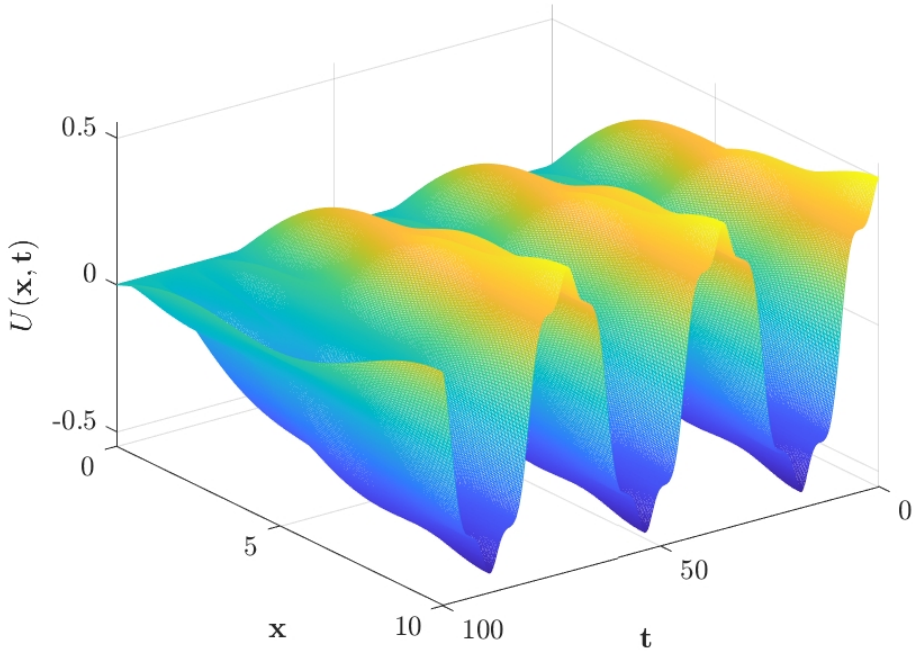
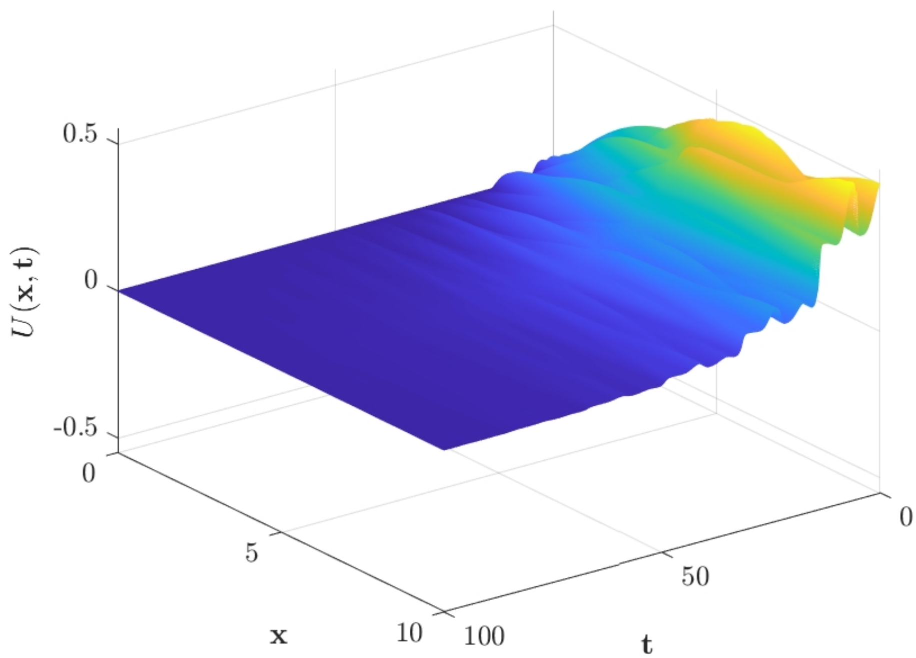
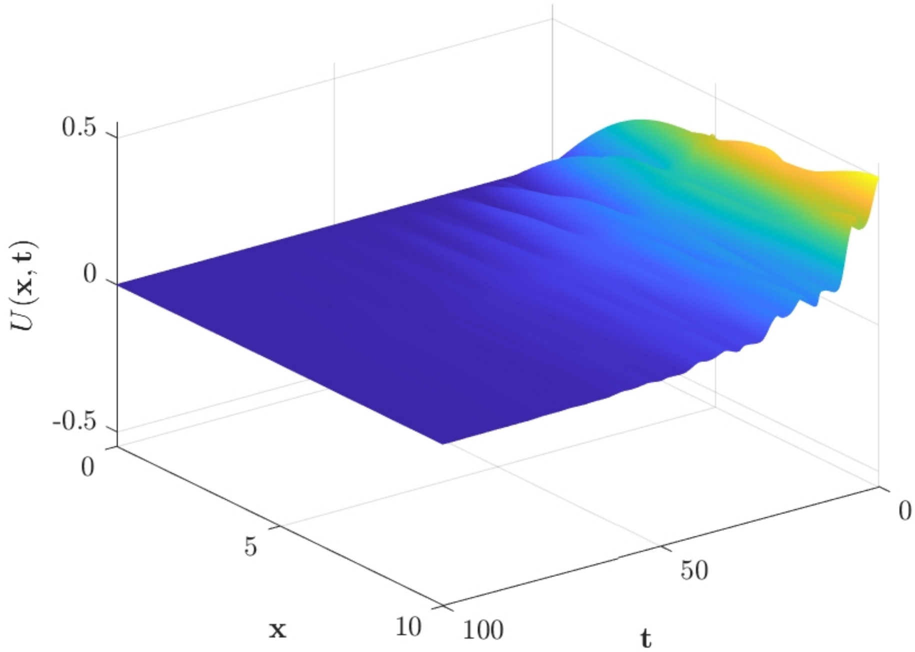


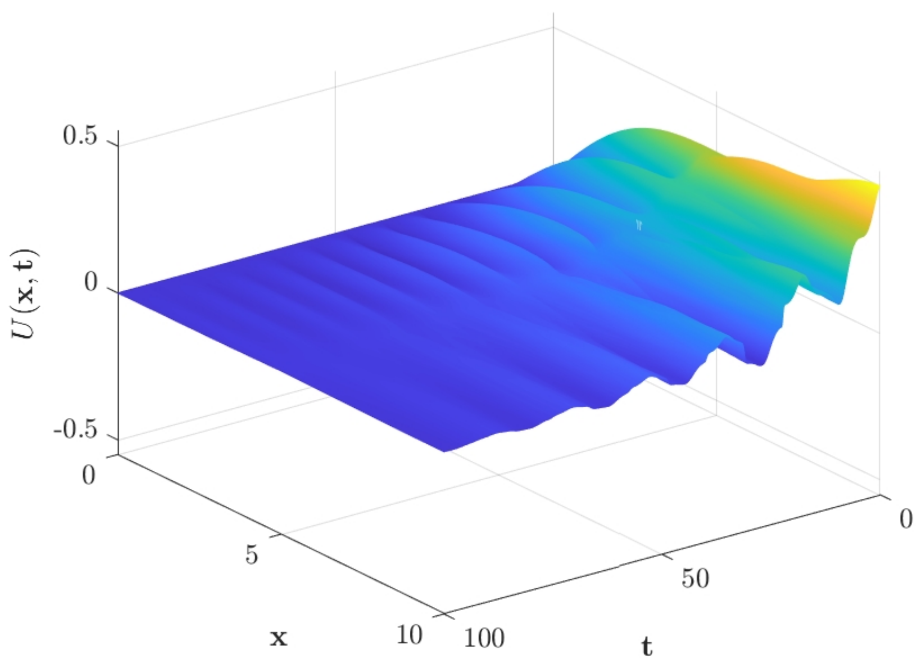
(A) Without control.



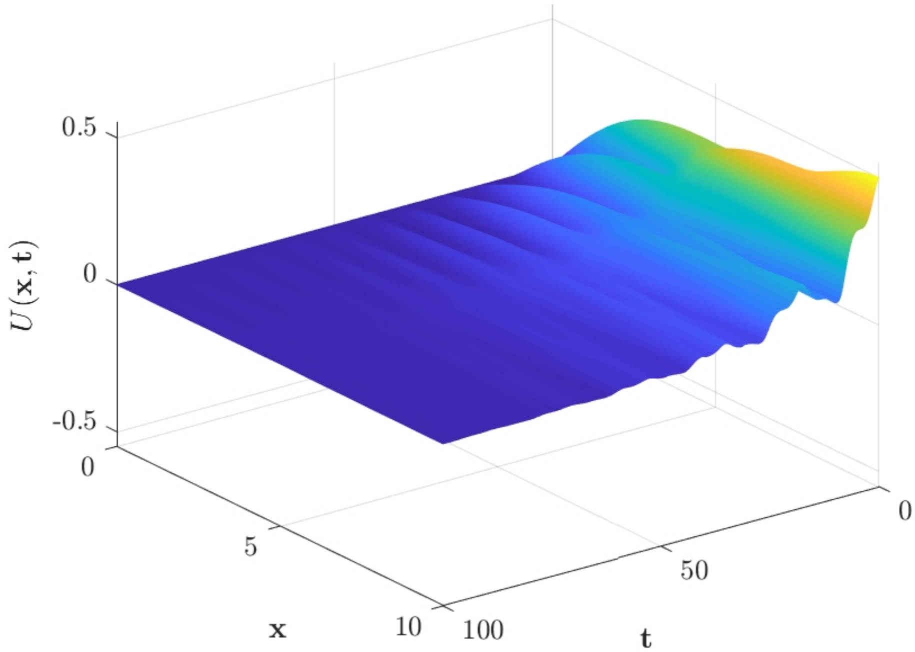
(B) Closed-loop control at the midpoint – case with $\tau = 0.08$ ($d\tau = 0.715$), $\alpha \simeq -0.34014$, $\beta = 1.7889$, $\gamma \simeq -0.60846$ and a spectral abscissa given by $s_0 \simeq -14.481$ triple. This assignment of a triple spectral value has been set thanks to the tuning proposed in Corollary 6.3.



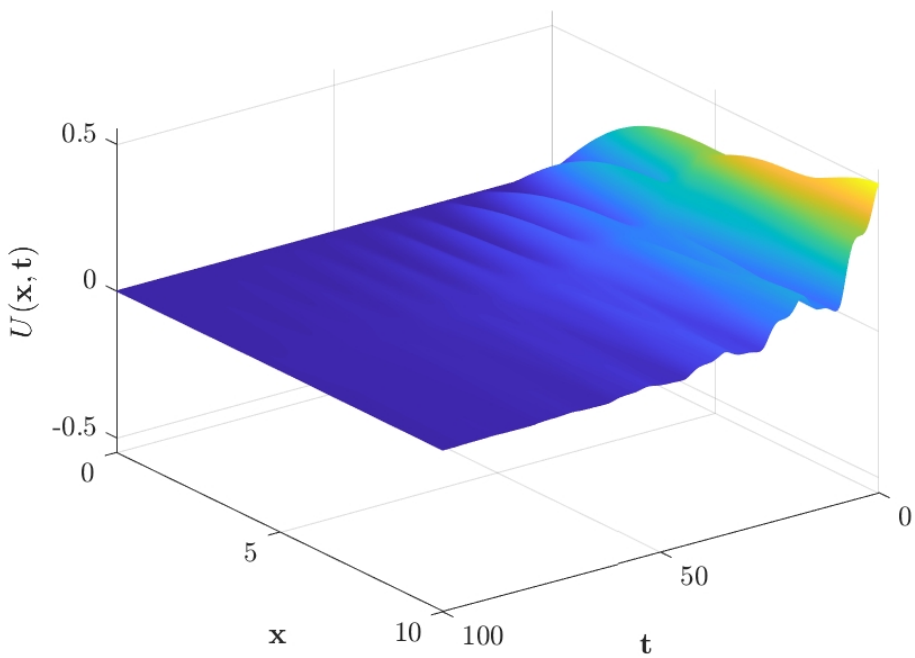
(C) Case $s_0 \simeq -28.961$ triple and $\tau = 0.08$, without MID property.



(D) Case $s_0 = -2$ double and $\tau = 2$, with MID property.



(E) Case $s_0 = -2$ triple and $\tau = 3$, without MID property.



(F) Case $s_0 = -2$ triple and $\tau = 4$, with MID property.

FIGURE 2. Time simulations, various cases.

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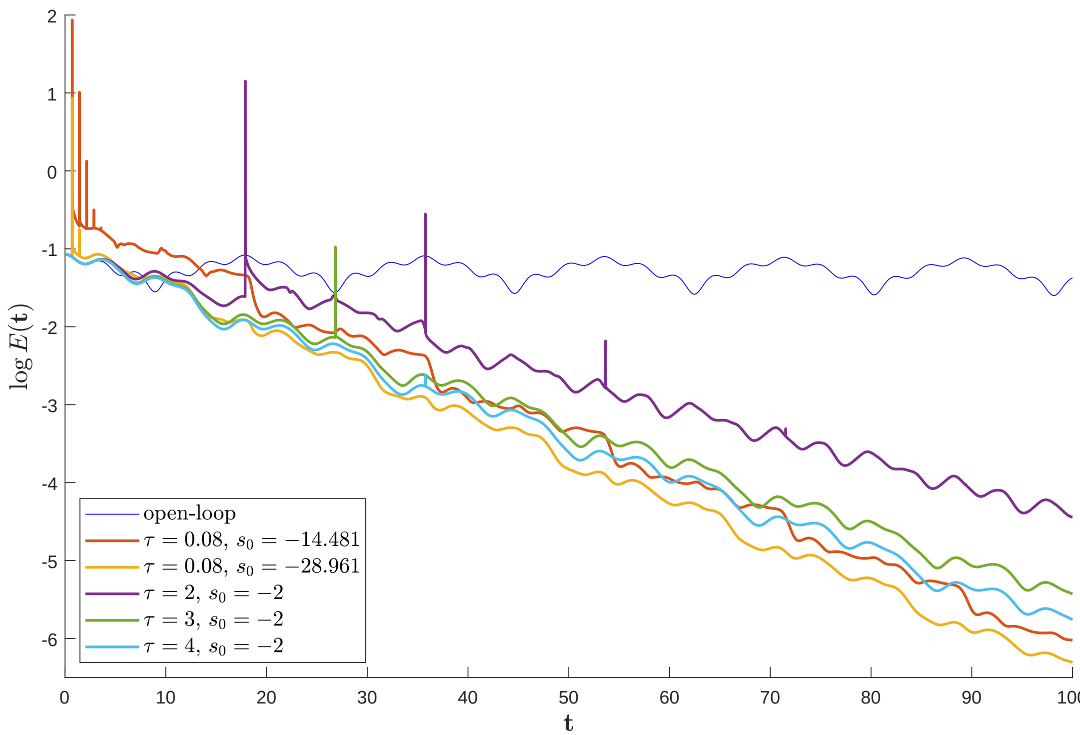

FIGURE 3. Time simulation of the energy logarithm defined in (3.34) for the various cases reported in Tab. 2.

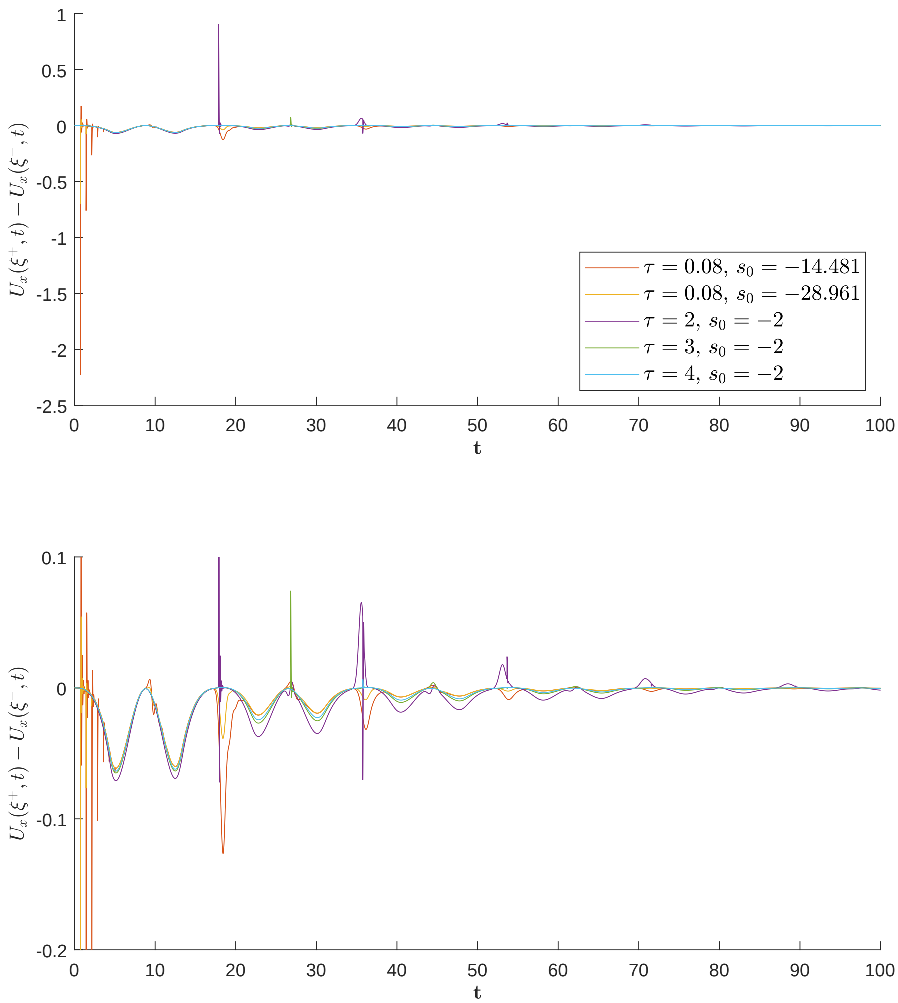

FIGURE 4. (Top) Time simulation of the closed-loop control signal for the various cases reported in Tab. 2. (Bottom) Focus on the control signals around zero.