Further Insights on the Partial Pole Placement in Delay Systems: Simultaneity of the MID and CRRID properties

Fazia Bedouhene, Islam Boussaada, Timothée Schmoderer, Souad Amrane, Silviu-Iulian Niculescu

Further Insights on the Partial Pole Placement in Delay Systems: Simultaneity of the MID and CRRID properties

Fazia Bedouhene[1], Islam Boussaada[2,3], Timothée Schmoderer[4], Souad Amrane[1], and Silviu-Iulian Niculescu[3]

[1]Laboratoire de Mathématiques Pures et appliquées, Mouloud Mammeri University of Tizi-Ouzou, Algeria
[2]Institut polytechnique des sciences avancées, France.
[3]Université Paris-Saclay, CNRS, CentraleSupélec, Inria, Laboratoire des Signaux et Systèmes, France.
[4]PRISME Laboratory, Université d'Orléans & INSA Centre Val de Loire, France

July 29, 2024

Abstract

In recent work, the *multiplicity-induced-dominancy* (MID) property for single delay time-delay systems has been fully characterized in the over-order case, that is when the multiplicity of the spectral abscissa exceeds the system's order. In this note, despite the fact that this assumption is not met, we provide an analytical proof for the MID. The coexistence of real spectral values is the main ingredient. The obtained result is illustrated through the stabilization of an unstable second-order plant by a delayed PD controller.

Keywords Time-delay differential equation, Stability and stabilization, Spectral method, Exponential decay, Partial pole placement, Lower-multiplicity-induced-dominancy property

1 Introduction

Systems with delays are often used in modeling transport, propagation, and communication. Such dynamical systems belong to the class of infinite-dimensional systems and a distinctive feature of such systems is that their rate of evolution can be described by differential equations including information about the *past history* of the system, see for instance [2, 7, 8, 9, 10, 12, 18] and references therein. While pole placement for finite-dimensional systems is a well-established method, for infinite dimensional systems it is more complex and deserves more attention. A series of recent work have highlighted the interest of multiplicity varieties in the characterization of the exponential decay rate for the solution of linear dynamical systems represented by delayed differential equations, see for instance [5, 17] and references therein.

By exploiting the Pólya-Szegő bound (see, e.g., [14]) pertaining to the number of roots of exponential polynomials in horizontal strips, [4] showed that the maximal admissible multiplicity of a characteristic root is given by the degree of the corresponding quasipolynomial.

A recent pole placement analytical paradigm, called *partial-pole-placement* (PPP), has been introduced in [11, 15]. It derives from two properties called respectively *multiplicity-induced-dominancy* (MID) and *coexistent-real-roots -inducing-dominancy* (CRRID), see for instance [1, 16]. These works follow from an observation on the effect of multiple spectral values on the stability of *Delay Differential Equations* (DDE). Indeed, a recent work (see, for instance, [11], [3]) has shown that, for some classes of delay systems, a real root of maximal multiplicity is necessarily the rightmost root, a property we call generic multiplicity-induced-dominancy, or GMID for short. This link between maximal multiplicity and dominance has been suggested in [13] after the study of some simple, low-order cases (scalar and second-order both retarded and neutral), but without any attempt to address the general case.

The objective of the present note is to investigate the MID property when the multiplicity of a given spectral value is not greater than the order of the DDE and, in particular, to explicitly determine conditions when such a root defines the spectral abscissa of the corresponding dynamical system, i.e. the real part of the rightmost (characteristic) root. This latter property is called *lower-multiplicity-induced-dominancy (LMID)*, and, to the best of the authors' knowledge, such a problem is still open.

The remaining paper is organized as follows. Section 2 presents some prerequisites on interpolation problem in quasipolynomial settings. section 3, is dedicated to enunciating and proving the main results of this note.

2 Preliminaries and prerequisites

Consider a controlled dynamical system represented by a general second-order linear differential equation

$$y''(t) + a_1 y'(t) + a_0 y(t) = u(t) \tag{1}$$

with a closed-loop control in the form of a proportional-derivative-delay term given by

$$u(t) = -\alpha_1 y'(t-\tau) - \alpha_0 y(t-\tau). \tag{2}$$

The corresponding closed-loop system is described by

$$y''(t) + a_1 y'(t) + a_0 y(t) + \alpha_1 y'(t-\tau) + \alpha_0 y(t-\tau) = 0, \tag{3}$$

under appropriate initial conditions belonging to the Banach space of continuous functions $\mathcal{C}([-\tau, 0], \mathbb{R})$. To determine the asymptotic behavior of the solutions of equation (3), we investigate the characteristic quasipolynomial function $\Delta : \mathbb{C} \times \mathbb{R}^{+*} \to \mathbb{C}$ defined by:

$$\Delta(s, \tau) = s^2 + a_1 s + a_0 + e^{-\tau s} (\alpha_1 s + \alpha_0), \tag{4}$$

where $(a_1, a_0, \alpha_1, \alpha_0) \in \mathbb{R}^4$. The degree of Δ, defined by the sum of the degrees of the involved polynomials plus the number of delays, is four. Pólya–Szegő result asserts that the maximal number of real roots counted with multiplicity is four. A rightmost root, also called a dominant root, s^* of $\Delta(\cdot, \tau)$ satisfies the condition:

$$\forall\, z \in \mathbb{C}\backslash\{s^*\},\ \Delta(z, \tau) = 0 \ \text{ for some } \tau > 0 \Rightarrow \Re(z) \leq \Re(s^*).$$

Namely, s^* is a root with the largest real part, i.e. it is the spectral abscissa, and, when negative, it determines the exponential decay rate of the solutions of (3). Several configurations of Δ admitting four real roots have been investigated: a single root of multiplicity four (MID) [11], four equidistributed roots (CRRID) [17], and a root of multiplicity three along with another real root (over-order MID) [5]. The definitions of these properties are given hereafter.

Definition 1 (CRRID and MID Properties)**.** *We say that a general quasipolynomial Δ of degree N satisfies the Coexistence-Real-Root-Induced-Dominancy (CRRID) property if it admits N distinct real roots $s_1 > s_2 > \ldots > s_N$, with s_1 being a dominant root of Δ. If, instead, $s_1 = s_2 = \ldots = s_N = s_0$ (hence s_0 is a root of Δ with maximal multiplicity) and is dominant, then this property is referred to as the Multiplicity-Induced-Dominancy (MID) property.*

In all cases, the dominancy of the largest real root have been established. In this work, we explore a new configuration described by the following property:

Definition 2. *We say that the quasipolynomial Δ, given by (4), satisfies the Lower Multiplicity-Induced Dominancy (LMID) property if Δ has a double root at s_0 that is dominant.*

In fact, in single-delay case, the over-order MID property has been fully characterized in [5] thanks to an integral representation introduced in [6]. It appears that such a representation is not valid in lower multiplicities and consequently, there are no results in the literature certifying LMID. To explore the LMID property for the quasipolynomial described in equation (4), we focus on the scenario where the number of real roots of the quasipolynomial is the maximal[1]. This maximization is achieved by assuming that the polynomial has two additional distinct real roots. This corresponds to the concomitance of the two properties MID and CRRID. This combination leverages both properties, particularly demonstrating that the MID property can be preserved even with a minimal multiplicity, i.e., two. It is worth noting that the novelty of this paper lies in the identification of s_0 as a double root of multiplicity 2, inducing dominance.

To state our results, we will use some tools that have been investigated in the CRRID context by [17]; some of their properties are revisited here. Denote $\mathbb{R}^{+*}$ the positive real half line, i.e. $\mathbb{R}^{+*} = (0, +\infty)$. For any distinct real numbers $s_{n+1} < \cdots < s_1$, let $\mathbf{s}_{n+1} \triangleq (s_1, \cdots, s_{n+1})$. Let $\mathcal{T}_n = (t_k)_{k=1}^n$ and, for $\tau > 0$ and $n \geq 0$, we define the function $F_{\tau,n} : \mathbb{R}^{n+1} \to \mathbb{R}^{+*}$ as:

$$F_{\tau,n}(\mathbf{s}_{n+1}) = \underbrace{\int_0^1 \cdots \int_0^1}_{n \text{ times}} l(\mathcal{T}_n) . e^{-\tau h(\mathbf{s}_{n+1}, \mathcal{T}_n)} dt_n \cdots dt_1,$$

[1] The maximal number of real roots of a given quasipolynomial corresponds to the degree of the quasipolynomial.

where $l(\mathcal{T}_n) = \prod_{k=1}^{n-1} (1-t_k)^{n-k}$,

$$h(\mathbf{s}_{n+1}, \mathcal{T}_n) \quad = \quad \left[s_1, \left[s_2, \cdots [s_n, s_{n+1}]_{t_n} \cdots\right]_{t_2}\right]_{t_1},$$

and $[x,y]_t = tx + (1-t)\,y$ for $t \in [0,1]$. The properties of the multivariate functions $F_{\tau,n}$ have been studied in [1]. Let G_n be the multivariate function introduced in [17], namely,

$$G_n(\mathbf{s}_{n+1}, \tau) \triangleq (-\tau)^n F_{\tau,n}(\mathbf{s}_{n+1})\,. \tag{5}$$

Notice that $G_0(s_k, \tau) = e^{-\tau s_k}$. The multivariate functions $G_n(\cdot, \tau)$ exhibit similar properties to those of $F_{\tau,n}$, particularly with respect to shifting:

$$(s_1 - s_{n+1}) G_n(\mathbf{s}_{n+1}, \tau) = G_{n-1}(s_2, \ldots, s_{n+1}, \tau) - G_{n-1}(\mathbf{s}_n, \tau). \tag{6}$$

We can extend the definition of $F_{\tau,n}$ when all the s_i are equal (or partially equal), namely when $[s_0]_{n+1} \triangleq \underbrace{(s_0, s_0, \cdots, s_0)}_{n+1 \text{ times}}$. Indeed, it suffices to identify the convex combination $h\left([s_0]_{n+1}, \mathcal{T}_n\right)$ to the point s_0. The expression of $F_{\tau,n+1}\left([s_0]_{n+1}\right)$, undergoes simplification, resulting in the identity:

$$F_{\tau,n}\left([s_0]_{n+1}\right) = \frac{e^{-\tau s_0}}{n!}.$$

The following proposition encapsulate several fundamental properties validated by $F_{\tau,n}$, and consequently by $G_n(.,\tau)$, in addition to those delineated in [1].

Proposition 1. *The following properties hold:*

(i) If $s_1 \in \mathbb{C}$, we have:

$$|F_{\tau,n}(\mathbf{s}_{n+1})| \leq F_{\tau,n}(\Re(s_1), s_2, \cdots, s_{n+1}) \tag{7}$$

(ii) Let $\mathbf{s}_n \in \mathbb{R}^n$ be fixed. $F_{\tau,n}$ enjoys the decreasing property in the following sense: for any $a > b$,

$$F_{\tau,n}(a, \mathbf{s}_n) < F_{\tau,n}(b, \mathbf{s}_n). \tag{8}$$

(iii) The mapping $G_n(.,\tau)$ is continuous on $\mathbb{R}^{n+1}$, in particular

$$\lim_{s \to z} G_1(s, z, \tau) = G_1([z]_2, \tau) = -\tau G_0(z, \tau). \tag{9}$$

(iv) For every $i = 0 \cdots n$, the mapping $\tau \mapsto G_i(\mathbf{s}_{i+1}, \tau)$ satisfies a first-order linear ordinary differential equation,

$$Y' + s_{i+1} Y + G_{i-1}(\mathbf{s}_i, \tau) = 0, \tag{10}$$

with the initial condition $Y(0) = 1$ if $i = 0$ and $Y(0) = 0$ if $i \geq 1$, where the prime denotes the derivative with respect to τ.

Proof. We proceed only with the proof of (10), as the other points essentially follow immediately from the definition of $F_{\tau,n}$, and the continuity and differentiability of the exponential function. To simplify calculations, we will limit ourselves to the case where $n = 3$, namely

$$G_3'(\mathbf{s}_4, \tau) = -s_4 G_3(\mathbf{s}_4, \tau) - G_2(s_1, s_2, s_3, \tau)$$

However, the proof remains valid for any n. Indeed, if $i = 0$, then $G_0(0, \tau) = e^0 = 1$. For $i \in \{1, \cdots, 4\}$, the expression (5) of the multivariate function G_i implies that $G_i(0, \tau) = 0$. To demonstrate (10), we use the following decomposition:

$$G_3(\mathbf{s}_4, \tau) = \sum_{i=1}^{4} e^{-s_i \tau} \prod_{\substack{j=1 \\ j \neq i}}^{4} d_{ij}^{-1},$$

where $d_{ij} = (s_i - s_j)$, and differentiating with respect to τ, we get

$$G_3'(\mathbf{s}_4, \tau) = -\sum_{i=1}^{4} s_i G_0(s_i, \tau) \prod_{\substack{j=1 \\ j \neq i}}^{4} d_{ij}^{-1}.$$

Using the property $\sum_{i=1}^{4} s_i \prod_{\substack{j=1 \\ j\neq i}}^{4} d_{ij}^{-1} = 0$ and (6), one obtains

$$\begin{aligned} G_3'(\mathbf{s}_4,\tau) &= -\frac{s_1}{d_{12}d_{13}} G_1(s_1,s_4,\tau) - \frac{s_2}{d_{21}d_{23}} G_1(s_2,s_4,\tau) - \frac{s_3}{d_{31}d_{32}} G_2(s_3,s_4,\tau) \\ &= -\frac{s_1}{d_{12}} G_2(s_1,s_3,s_4,\tau) - \frac{s_2}{d_{21}} G_2(s_2,s_3,s_4,\tau) \\ &= -\left(1+\frac{s_2}{d_{12}}\right) G_2(s_1,s_3,s_4,\tau) - \frac{s_2}{d_{12}} G_2(s_2,s_3,s_4,\tau) \\ &= -s_2 G_2(\mathbf{s}_4,\tau) - G_2(s_1,s_3,s_4,\tau). \end{aligned}$$

Thus, G_3 satisfies the differential equation (10). Similarly, it can be demonstrated that G_1 and G_2 also satisfy a corresponding differential equation. The calculations for these cases are more straightforward and are thus omitted. This concludes the proof of the proposition.

□

Remark 1. *Using the shifting property, the following equivalent variants of (10) hold:*

$$\begin{aligned} G_3'(\mathbf{s}_4,\tau) &= -s_3 G_3(\mathbf{s}_4,\tau) - G_2(s_1,s_2,s_4,\tau), \\ &= -s_2 G_3(\mathbf{s}_4,\tau) - G_2(s_1,s_3,s_4,\tau), \\ &= -s_1 G_3(\mathbf{s}_4,\tau) - G_2(s_2,s_3,s_4,\tau). \end{aligned}$$

Also, from (9), by denoting $\mathbf{s} = ([s_0]_2, s_1, s_2)$ with $s_0 > s_1 > s_2$, we have

$$G_3'(\mathbf{s},\tau) = -s_2 G_3(\mathbf{s},\tau) - G_2([s_0]_2, s_1, \tau). \tag{11}$$

The following lemma pertains to the invertibility of a structured functional Vandermonde-type matrix.

Lemma 1. *Let $\mathbf{s} = (s_0, s_1, s_2)$ be distinct real numbers, then the matrix*

$$V(\mathbf{s},\tau) = \begin{pmatrix} 1 & s_0 & e^{-\tau s_0} & s_0 e^{-\tau s_0} \\ 0 & 1 & -\tau e^{-\tau s_0} & e^{-\tau s_0} - s_0 \tau e^{-\tau s_0} \\ 1 & s_1 & e^{-\tau s_1} & s_1 e^{-\tau s_1} \\ 1 & s_2 & e^{-\tau s_2} & s_2 e^{-\tau s_2} \end{pmatrix}$$

is invertible for any $\tau > 0$.

Proof. The proof is based on the rewriting of V in terms of the multivariate functions G_i, as follows:

$$V(\mathbf{s},\tau) = \begin{pmatrix} 1 & s_0 & G_0(s_0,\tau) & -G_0'(s_0,\tau) \\ 0 & 1 & G_1([s_0]_2,\tau) & -G_1'([s_0]_2,\tau) \\ 1 & s_1 & G_0(s_1,\tau) & -G_0'(s_1,\tau) \\ 1 & s_2 & G_0(s_2,\tau) & -G_0'(s_2,\tau) \end{pmatrix}.$$

Using the same procedure as in [17], we obtain the following factorization of the determinant of V:

$$\begin{aligned} v(\mathbf{s},\tau) &\triangleq \det V(\mathbf{s},\tau) = -d_{10}^2 d_{20}^2 Q(\mathbf{s},\tau), \quad \text{where} \\ Q(\mathbf{s},\tau) &= \left(\frac{G_2([s_0]_2, s_2, \tau)}{G_2([s_0]_2, s_1, \tau)}\right)' G_2^2([s_0]_2, s_1, \tau). \end{aligned} \tag{12}$$

The conclusion regarding the invertibility of V is deduced using the same arguments as those in [17, Lemma 1]. The details are omitted. □

3 Main result

In this section, we give our main contributions. First, we show that a quasipolynomial function given by (4) admitting three distinct real roots s_0, s_1, s_2, where s_0 is a double root, is uniquely determined. Second, we show that such a quasipolynomial admits a factorisation by its real roots. Finally, we prove that it satisfies the LMID property, i.e. the largest real root is dominant.

3.1 Assigning real roots of the characteristic function

Proposition 2 (Coexisting real roots). *Given a delay* $\tau > 0$*, the quasipolynomial* (4) *admits* 3 *distinct real spectral values* s_0*,* s_1*, and* s_3*, with* s_0 *being a double root if and only if the real coefficients* a_1*,* a_0*,* α_1*, and* α_0 *are expressed as functions of* τ *and* $\mathbf{s} = ([s_0]_2, s_1, s_2)$ *according to the following relationships:*

$$a_1(\mathbf{s},\tau) = -2s_0 - \alpha_0(\mathbf{s},\tau)G_1([s_0]_2,\tau) + \alpha_1(\mathbf{s},\tau)G_1'([s_0]_2,\tau),$$
$$a_0(\mathbf{s},\tau) = -s_0^2 - a_1(\mathbf{s},\tau)s_0 - \alpha_0(\mathbf{s},\tau)G_0(s_0,\tau) + \alpha_1(\mathbf{s},\tau)G_0'(s_0,\tau),$$

and

$$\alpha_0(\mathbf{s},\tau) = -d_{21}\frac{G_3'(\mathbf{s},\tau)}{Q(\mathbf{s},\tau)} \quad \text{and} \quad \alpha_1(\mathbf{s},\tau) = -d_{21}\frac{G_3(\mathbf{s},\tau)}{Q(\mathbf{s},\tau)}, \tag{13}$$

where $Q(\mathbf{s},\tau)$ *is given by* (12).

Proof. Assume that Δ admits 3 distinct real spectral values s_0, s_1, and s_3, with s_0 being a double root. This means that the coefficients a_0, a_1, α_0 and α_1 satisfy the linear system:

$$\begin{cases} \Delta(s_i,\tau) = 0, \quad i \in \{0,1,2\}, \\ 2s_0 + a_1 + e^{-\tau s_0}\left((1-s_0\tau)\alpha_1 - \tau\alpha_0\right) = 0. \end{cases}$$

Thanks to the invertibility of structured functional Vandermonde type matrix $V(\mathbf{s},\tau)$ as asserted previously, one deals with a Cramer system with respect to the coefficients a_0, a_1, α_0 and α_1. So that, one easily computes these coefficients using the property satisfied by the multivariate G_i allowing to get (13). □

Corollary 1. *Under the conditions of Proposition 2, and in a specific context where* $d_{01} = d_{12} = d > 0$*, the coefficients of the quasipolynomial take the following form, under the notation* $y = e^{\tau d}$*:*

$$\begin{cases} \alpha_0(y,\tau) = -\dfrac{dB(y)s_0 - 2d^2(y-1)^2}{(y-1)(2y\ln y - y^2+1)}e^{\tau s_0} \\ \alpha_1(y,\tau) = \dfrac{dB(y)e^{\tau s_0}}{(y-1)(2y\ln y - y^2+1)} \\ a_1(y,\tau) = -2s_0 - \frac{dA(y)}{(y-1)(2y\ln y - y^2+1)} \\ a_0(y,\tau) = s_0^2 + \dfrac{dA(y)s_0 - 2d^2(y-1)^2}{(y-1)(2y\ln y - y^2+1)} \end{cases} \tag{14}$$

where $A(y) = 3 - y(4-4\ln y) + y^2(1-2\ln y)$*, and* $B(y) = y^2 - 4y + 3 + 2\ln(y)$*. Furthermore, the spectral value* s_0 *is negative if, and only if, there exists* $\tau^* > 0$ *such that*

$$a_1(y^*,\tau^*) + s_0 = 0 \tag{15}$$

with $y^* = e^{\tau^* d}$*. The value of* s_0 *is given by the following expression:*

$$s_0 = d\frac{y^{*2}(2y^*-1) + 4y^*(1-\ln(y^*)) - 3}{(y^*-1)(2y^*\ln(y^*) - y^{*2}+1)}. \tag{16}$$

Proof. Assume that $s_0 < 0$, thanks to the behavior at 0 and ∞ of the function $\tau \mapsto a_1(s_0,\tau) + s_0$:

$$a_1(s_0,\tau) + s_0 \sim \begin{cases} -\frac{4}{\tau d} + O(1) & \text{if} \quad \tau \sim 0 \\ -s_0 & \text{if} \quad \tau \sim \infty \end{cases}$$

we deduce that

$$\begin{aligned} \lim_{\tau\to\infty}\left(a_1(s_0,\tau)+s_0\right) &= -s_0 > 0 \\ \lim_{\tau\to 0}\left(a_1(s_0,\tau)+s_0\right) &= -\infty. \end{aligned}$$

The existence of a number $\tau^* > 0$ satisfying equation (15) is guaranteed by the continuity of a_1 with respect to τ and the Intermediate Value Theorem.

Conversely, if equation (15) is satisfied, then the value of s_0 is given by (16). It remains to examine the sign of s_0, which depends on the respective signs of the two factors $f(y^*) \triangleq 2y^{*2}\ln y^* - 4y^*\ln y^* - y^{*2} + 4y^* - 3$ and $g(y^*) \triangleq 2y^*\ln y^* + y^{*2} - 1$, for $y^* > 1$. To do this, we observe that

$$f(y) = 4\int_1^y (x-1)\ln(x)\,dx$$
$$g(y) = -2y\int_1^y \frac{(x-1)^2}{2x^2}\,dx$$

for all $y > 1$. Hence, we have $f(y) > 0$ and $g(y) < 0$ and we conclude that $s_0 < 0$. Note that s_0 given by (16) is negative regardless of τ, not just at τ^*.

□

3.2 Factorization of the Quasipolynomial Δ

In this subsection, we explore a novel configuration for the factorization of the quasipolynomial Δ, which admits three distinct real roots: s_0, a double root, and two simple roots, s_1 and s_2. This factorization involves the multivariate function G_3, its derivative, and the coefficients given in (13).

Proposition 3. *If the quasipolynomial Δ admits three distinct real roots: s_0, a double root, and two simple roots, s_1 and s_2, then it can be expressed as:*

$$\Delta(s,\tau) = (s-s_0)^2 (s-s_1) R(s,\mathbf{s},\tau) \tag{17}$$

with

$$R(s,\mathbf{s},\tau) = \alpha_0(\mathbf{s},\tau)G_3(s,[s_0]_2,s_1,\tau) - \alpha_1(\mathbf{s},\tau)G_3'(s,[s_0]_2,s_1,\tau).$$

Proof. Let $\delta \triangleq \frac{\Delta(s,\tau)}{(s-s_0^2)(s-s_1)}$, $p_0(s) \triangleq s^2 + a_1 s + a_0$ and $p_1(s) \triangleq (\alpha_1 s + \alpha_0)$. By using the partial fraction decomposition of $\frac{p_0(s)}{(s-s_0^2)(s-s_1)}$ and $\frac{p_1(s)}{(s-s_0^2)(s-s_1)}$ and the equations $\Delta(s_0,\tau) = \Delta(s_1,\tau) = \frac{d}{ds}\Delta(s,\tau)\big|_{s=s_0} = 0$ (and thus implicitly the values of a_1, a_0 and α_0 as functions of α_1 and τ), we have

$$\begin{aligned}\delta = &\frac{-(\alpha_0+\alpha_1 s_0)e^{-s_0\tau}}{(s-s_0)^2(s_0-s_1)} - \frac{(\alpha_0+\alpha_1 s_1)e^{-s_1\tau}}{(s_0-s_1)^2(s-s_1)} + \frac{(\alpha_0+\alpha_1 s_0)e^{-s_0\tau}}{(s_0-s_1)^2(s-s_0)} + \frac{(\tau(\alpha_0+s_0\alpha_1)-\alpha_1)e^{-s_0\tau}}{(s_0-s_1)(s-s_0)}\\ &+\frac{(\alpha_0+\alpha_1 s_0)e^{-s\tau}}{(s-s_0)^2(s_0-s_1)} + \frac{(\alpha_0+\alpha_1 s_1)e^{-s\tau}}{(s_0-s_1)^2(s-s_1)} - \frac{(\alpha_1 s_0+\alpha_0)e^{-s\tau}}{(s_0-s_1)^2(s-s_0)} + \frac{\alpha_1 e^{-s\tau}}{(s_0-s_1)(s-s_0)}.\end{aligned}$$

Now, leveraging the relations:

$$\begin{aligned}(2s_0+a_1) &= \tau(\alpha_0+s_0\alpha_1)e^{-s_0\tau} - \alpha_1 e^{-s_0\tau}\\ &= -(\alpha_0+s_0\alpha_1)G(s_0,s_0,\tau) - \alpha_1 G_0(s_0,\tau)\\ &= -\alpha_0 G_1([s_0]_2,\tau) + \alpha_1 G_1'([s_0]_2,\tau)\end{aligned}$$

and the shifting property, we derive:

$$\begin{aligned}\delta = &\frac{(\alpha_0+\alpha_1 s_0)G_1(s,s_0,\tau)}{(s-s_0)(s_0-s_1)} + \frac{(\alpha_0+\alpha_1 s_1)G_1(s,s_1,\tau)}{(s_0-s_1)^2} - \frac{(\alpha_1 s_0+\alpha_0)G_1(s,s_0,\tau)}{(s_0-s_1)^2}\\ &-\frac{(\alpha_0+s_0\alpha_1)G_1([s_0]_2,\tau)}{(s_0-s_1)(s-s_0)} + \frac{\alpha_1 G_1(s,s_0,\tau)}{(s_0-s_1)}.\end{aligned}$$

To handle the term $\frac{\alpha_1 G_1(s,s_0,\tau)}{(s_0-s_1)}$, we employ the relation

$$\frac{(\alpha_0+\alpha_1 s_1)}{(s_0-s_1)} = -\alpha_1 + \frac{(\alpha_0+s_0\alpha_1)}{(s_0-s_1)}$$

which leads to the following conclusive calculations:

$$\begin{aligned}\delta = &\frac{(\alpha_0+\alpha_1 s_0)G_1(s,s_0,\tau)}{(s-s_0)(s_0-s_1)} + \frac{(\alpha_0+\alpha_1 s_1)G_1(s,s_1,\tau)}{(s_0-s_1)^2} - \frac{(\alpha_1 s_0+\alpha_0)G_1(s,s_0,\tau)}{(s_0-s_1)^2} - \frac{(\alpha_0+s_0\alpha_1)G_1(s_0,s_0,\tau)}{(s_0-s_1)(s-s_0)}\\ &+\frac{(\alpha_0+s_0\alpha_1)G_1(s,s_0,\tau)}{(s_0-s_1)^2} - \frac{(\alpha_0+s_1\alpha_1)G_1(s,s_0,\tau)}{(s_0-s_1)^2}\\ = &\frac{(\alpha_0+\alpha_1 s_0)G_1(s,s_0,\tau)}{(s-s_0)(s_0-s_1)} - \frac{(\alpha_0+s_0\alpha_1)G_1(s_0,s_0,\tau)}{(s_0-s_1)(s-s_0)} + \frac{(\alpha_0+\alpha_1 s_1)G_1(s,s_1,\tau)}{(s_0-s_1)^2} - \frac{(\alpha_0+s_1\alpha_1)G_1(s,s_0,\tau)}{(s_0-s_1)^2}.\end{aligned}$$

By combining pairwise the compatible terms, we obtain the following expression:

$$\delta = \frac{(\alpha_0+\alpha_1 s_1)[G_1(s,s_1,\tau)-G_1(s,s_0,\tau)]}{(s_0-s_1)^2} + \frac{(\alpha_0+s_0\alpha_1)[G_1(s,s_0,\tau)-G_1([s_0]_2,\tau)]}{(s_0-s_1)(s-s_0)}.$$

By applying the shifting property again, we deduce

$$\delta = \frac{-(\alpha_0+\alpha_1 s_1)G_2(s,s_0,s_1,\tau)}{(s_0-s_1)} + \frac{(\alpha_0+s_0\alpha_1)G_2(s,[s_0]_2,\tau)}{(s_0-s_1)}.$$

By exploiting the relationship between $\alpha_0+\alpha_1 s_1$ and $\alpha_0+s_0\alpha_1$, along with the shifting property, we obtain the following expression:

$$\begin{aligned}\delta &= \left(\alpha_1 - \frac{(\alpha_0+s_0\alpha_1)}{(s_0-s_1)}\right)G_2(s,s_0,s_1,\tau) + \frac{(\alpha_0+s_0\alpha_1)G_2(s,[s_0]_2,\tau)}{(s_0-s_1)}\\ &= \alpha_1[G_2(s,s_0,s_1,\tau) + s_0 G_3(s,[s_0]_2,s_1,\tau)] + \alpha_0 G_3(s,[s_0]_2,s_1,\tau).\end{aligned}$$

Finally, in view of (10) and (11), the following expression is derived

$$\Delta(s,\tau) = (s-s_0)^2 (s-s_1) R(s,\mathbf{s},\tau)$$

with

$$R(s,\mathbf{s},\tau) = \alpha_0(\mathbf{s},\tau) G_3(s,[s_0]_2,s_1,\tau) - \alpha_1(\mathbf{s},\tau) G_3'(s,[s_0]_2,s_1,\tau).$$

Thus, the proposition is proven.

□

3.3 LMID property

Theorem 1. *If the quasipolynomial Δ given by (4) has real roots s_0, s_1, and s_2, with s_0 being a double root and $s_2 < s_1 < s_0$, then s_0 is necessarily the corresponding spectral abscissa.*

The proof is based on the factorization of Δ, giving (17), where the expressions of the coefficients α_1 and α_0 are derived from (13), and (10) and (11).

Proof. Suppose there exists $z^* = \eta + i\zeta \in \mathbb{C}$, with $\eta > s_0$, such that $\Delta(z^*, \tau) = 0$, for any $\tau > 0$. This implies:

$$\alpha_0(\mathbf{s},\tau) G_3([s_0]_2, s_1, z^*, \tau) - \alpha_1(\mathbf{s},\tau) G_3'([s_0]_2, s_1, z^*, \tau) = 0, \quad \forall \tau > 0.$$

Using the expression of $\alpha_0(\mathbf{s},\tau)$ and $\alpha_1(\mathbf{s},\tau)$ leads to

$$\det \begin{bmatrix} G_3'([s_0]_2, s_1, s_2, \tau) & G_3'([s_0]_2, s_1, z^*, \tau) \\ G_3([s_0]_2, s_1, s_2, \tau) & G_3([s_0]_2, s_1, z^*, \tau) \end{bmatrix} = 0,$$

for all $\tau > 0$. Using the fact that $F_{\tau,3}([s_0]_2, s_1, s_2) > 0$, after dividing on $F_{\tau,3}^2([s_0]_2, s_1, s_2) > 0$, we deduce that

$$\frac{d}{d\tau}\left(\frac{F_{\tau,3}([s_0]_2, s_1, z^*)}{F_{\tau,3}([s_0]_2, s_1, s_2)}\right) = 0, \quad \forall \tau > 0.$$

Thus,

$$\frac{F_{\tau,3}([s_0]_2, s_1, z^*)}{F_{\tau,3}([s_0]_2, s_1, s_2)} = M, \quad \forall \tau > 0,$$

where M is some constant depending only on (s_i). By continuity, taking $\tau \to 0$ yields that $M = 1$. Thanks to the property (7) and (8), the following estimates hold:

$$\begin{aligned} F_{\tau,3}([s_0]_2, s_1, s_2) &= |F_{\tau,3}([s_0]_2, s_1, s_2)| \\ &= |F_{\tau,3}([s_0]_2, s_1, z^*)| \\ &\leq F_{\tau,3}([s_0]_2, s_1, \Re(z^*)) \\ &< F_{\tau,3}([s_0]_2, s_1, s_0) < F_{\tau,3}([s_0]_2, s_1, s_2). \end{aligned}$$

This leads to a contradiction, proving the dominance of the double root s_0.

□

4 Stabilization of second-order plant

In this section, we consider the application of Theorem 1 in the control of an unstable second-order plant in the form of (1) controlled via a delayed proportional derivative action given by (2). For a comprehensive control design, we exploit the placement of equidistributed real spectral values, i.e. we assign s_0, $s_1 = s_0 - d$ and $s_2 = s_0 - 2d$, with $d > 0$. The parameters a_1 and a_0 of the plant, as well as the delay τ, are defined by the model and the control gains α_1 and α_0 are to be determined as functions of the distance d and the double root s_0. As a matter of fact, consider the case of an unstable second-order plant by taking $a_1 = -2$ and $a_0 = 5$, moreover the delay is fixed to $\tau = 2/3$. Next, using $\Delta(s_0) = \Delta'(s_0) = 0$, we obtain

$$\alpha_1 = -\frac{2}{3} e^{2s_0/3}\left(s_0^2 + s_0 + 2\right) \quad \text{and} \quad \alpha_0 = \frac{2}{3} e^{2s_0/3}\left(2s_0^3 - s_0^2 + 10 s_0 - 15\right).$$

Inserting these values in $\Delta(s_1) = \Delta(s_2) = 0$ we obtain two equations for s_0 and d, which yield $s_0 \approx -0.93$ and $d \approx 1.64$. Hence, we have the following value for the controller gains $\alpha_1 \approx -0.69$ and $\alpha_0 \approx -4.8$.

Figure 1 shows that the roots of the unstable open-loop system (given by $1 \pm 2\mathbf{i}$) are located in the right-half of the complex plane, whereas the infinite number of roots of the closed loop system are located in the left-half plane and are dominated by the double root $s_0 \approx -0.93$, which, therefore, determines the exponential decay rate of the solutions of (3). This numerical example illustrates the partial-pole-placement strategy. Indeed, we observe that by assigning a few number of real roots (four) we are able to guarantee that the rest of the spectrum (a countable infinite set) is located in the left-half plane (actually it is located left to the real part of the largest real root).

Note that the delay τ needs not be fixed by the model, it can also be a control parameter. In that case, we can arbitrarily choose the value of $s_0 < 0$ (namely, we decide the decay rate of the solutions) and we find $d > 0$ and $\tau > 0$ such that (16) is satisfied.

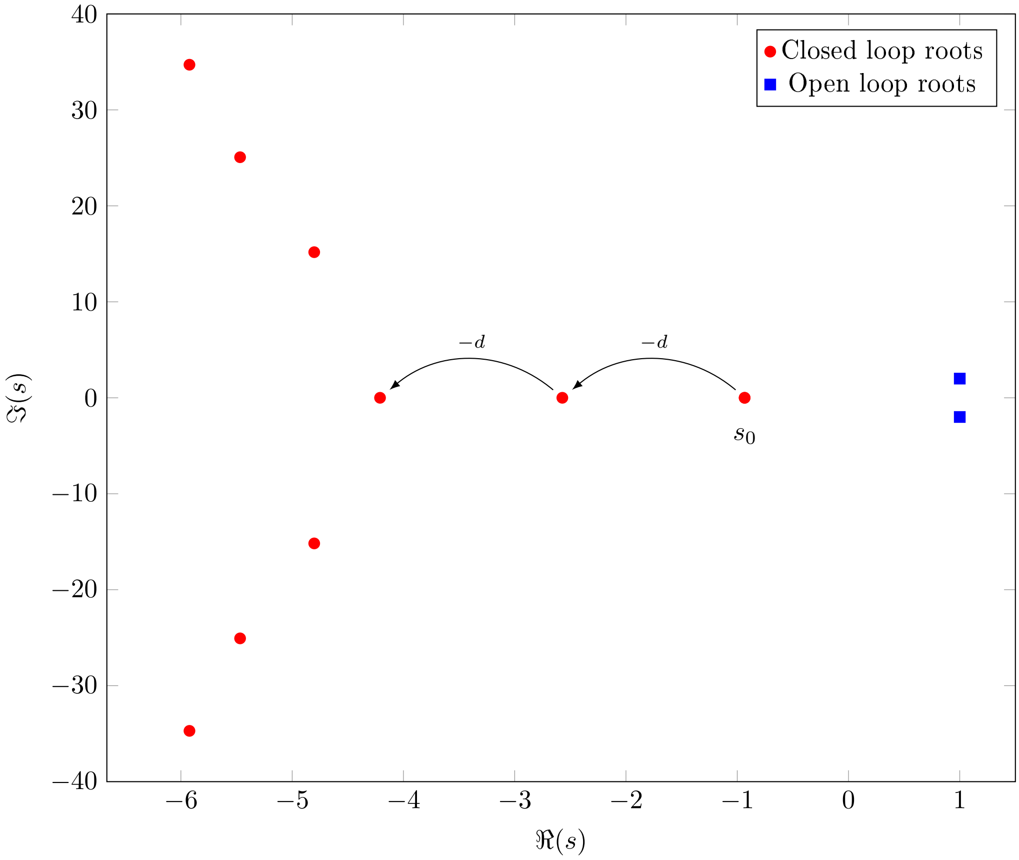

Figure 1: Spectrum distribution of the characteristic function (4) corresponding to the closed-loop system of a second order unstable plant with a delayed proportional-derivative controller.

5 Conclusion

Our study provide a novel analytical proof for the multiplicity-induced-dominancy (MID) property in single-delay time-delay systems. Contrarily to previous research, which primarily focused on the over-order case, the originality of our work is to address scenarios where this assumption is not met. By leveraging the properties of a specific multivariate function, we demonstrate the achievement of MID even in such cases, where real spectral values coexistance is the main ingredient. Our findings emphasize the efficiency of analytical techniques in understanding complex dynamical systems and suggested new prospects of partial poles placement for system stability and control.

References

[1] Fazia Bedouhene, Islam Boussaada, and Silviu-Iulian Niculescu. “Real spectral values coexistence and their effect on the stability of time-delay systems: Vandermonde matrices and exponential decay”. In: *CR. Mathématique* 358.9-10 (2020).

[2] R. Bellman and K.L. Cooke. *Differential-difference equations*. Academic Press: New York, 1963, pp. xvi+462.

[3] I. Boussaada, G/ Mazanti, and S.-I. Niculescu. “The generic multiplicity-induced-dominancy property from retarded to neutral delay-differential equations: When delay-systems characteristics meet the zeros of Kummer functions”. In: *C. R. Math. Acad. Sci. Paris* (2022).

[4] I. Boussaada and S-I. Niculescu. “Characterizing the Codimension of Zero Singularities for Time-Delay Systems”. In: *Acta Appl. Math.* 145.1 (2016), pp. 47–88.

[5] Islam Boussaada, Guilherme Mazanti, and Silviu-Iulian Niculescu. “Over-order multiplicities and their application in controlling delay dynamics. On zeros’ distribution of linear combinations of Kummer hypergeometric functions”. working paper or preprint. 2024.

[6] Islam Boussaada, Hakki Ulaş Ünal, and Silviu-Iulian Niculescu. “Multiplicity and Stable Varieties of Time-Delay Systems: A Missing Link”. In: *Proceedings of the 22nd International Symposium on Mathematical Theory of Networks and Systems (MTNS)*. Minneapolis, MN, USA, 2016, pp. 188–194.

[7] Dimitri Breda. *Controlling Delayed Dynamics: Advances in Theory, Methods and Applications*. Vol. 604. Springer Nature, 2022.

[8] K. Gu, V.L. Kharitonov, and J. Chen. *Stability of time-delay systems*. Control Engineering. Birkhäuser Boston, Inc., Boston, MA, 2003, pp. xviii+353.

[9] Jack K. Hale and Sjoerd M. Verduyn Lunel. *Introduction to functional differential equations*. Vol. 99. Applied Mathematical Sciences. New York: Springer-Verlag, 1993.

[10] V. B. Kolmanovskii and V. R. Nosov. *Stability of functional differential equations*. Academic Press: NY, 1986.

[11] Guilherme Mazanti, Islam Boussaada, and Silviu-Iulian Niculescu. “Multiplicity-induced-dominancy for delay-differential equations of retarded type”. In: *Journal of Differential Equations* 286.15 (2021), pp. 84–118.

[12] Wim Michiels and Silviu-Iulian Niculescu. *Stability, control, and computation for time-delay systems: an eigenvalue-based approach*. Vol. 27. Advances in Design and Control. Philadelphia: SIAM, 2014, pp. I–XXII, 1–435.

[13] Edmund Pinney. *Ordinary difference-differential equations*. University of California Press, 1958, pp. xii+262.

[14] George Pólya and Gabor Szegő. *Problems and theorems in analysis. I*. Classics in Mathematics. Springer-Verlag, 1998, pp. xx+389.

[15] Adrián Ramírez et al. “Design of proportional-integral-retarded (PIR) controllers for second-order LTI systems”. In: *IEEE Trans. Automat. Control* 61.6 (2016), pp. 1688–1693.

[16] Timothée Schmoderer, Islam Boussaada, and Silviu-Iulian Niculescu. “On Boundary Control of the Transport Equation. Assigning Real Spectra & Exponential Decay”. In: *IEEE Control Systems Letters* 7 (2023), pp. 3591–3596.

[17] Timothée Schmoderer et al. “Insights on equidistributed real spectral values in second-order delay systems: Perspectives in partial pole placement”. In: *Systems & Control Letters* 185 (2024), p. 105728.

[18] G. Stépán. *Retarded dynamical systems: stability and characteristic functions*. Vol. 210. Pitman Research Notes Math. Series. Longman Scientific, 1989, pp. viii+151.