ECKMANN-HILTON AND THE HOPF FIBRATION

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The Eckmann-Hilton argument and the Hopf fibration are important constructions in homotopy theory, playing a role in the theory of higher homotopy groups. The Eckmann-Hilton argument constructs an identification $\mathsf{EH}(\alpha,\beta) : \alpha \cdot \beta = \beta \cdot \alpha$ for 2-loops $\alpha,\beta : \Omega^2(X)$, thus proving all higher homotopy groups are commutative. The Hopf fibration is a map $\mathbb{S}^3 \to \mathbb{S}^2$ with fiber $\mathbb{S}^1$. Analyzing the fiber sequence of this map establishes that it is a generator of $\pi_3(\mathbb{S}^2)$. There is an important connection between these two constructions. When applied to surf_2, the generator of $\mathbb{S}^2$, Eckmann-Hilton lends an identification $\mathsf{EH}(\mathsf{surf}_2, \mathsf{surf}_2) : \mathsf{surf}_2 \cdot \mathsf{surf}_2 = \mathsf{surf}_2 \cdot \mathsf{surf}_2$. This is equivalent to an identification of type $\Omega^3(\mathbb{S}^2)$, which we will call eh. Applying the suspension loop space adjunction lends a map $\mathsf{hpf} : \mathbb{S}^3 \to \mathbb{S}^2$. As it so happens, this is the Hopf fibration. This idea has been remarked on before. For example, the HoTT book states "the generating element of $\pi_3(\mathbb{S}^2)$ can be constructed using the interchange law of [the Eckmann-Hilton argument]". Michael Shulman pointed out in a 2011 blog comment that this should be possible to prove in HoTT, with its synthetic notion of ∞-groupoid [2]. But a proof in book HoTT has not previously been given.

At HoTT2023, the current author presented a proof of this claim. We now present an updated and simplified version of the proof, one with connections to work by Kraus and Von Raumer, and recent work by David Wärn. This revised version may be more adaptable to further work in this line, building off of the work of Kristina Sojakova and G. A. Kavvos.

First we introduce a characterization of the fiber of a map in terms of a simple universal property, showing up in similar forms in [3] and [1]. Given a map $h : X \to Y$, we obtain a type family $\mathsf{fib}_h : Y \to U$ over Y. By precomposing with h, we derive the type family $\mathsf{fib}_h \circ h : X \to U$. This induced type family always comes equipped with a section, given by $\delta :\equiv \lambda(x).(x, \mathsf{refl}_{f(x)})$. We can characterize fib_h as being freely generated by this section. That is, fib_h is initial among type families $B : Y \to U$ equipped with sections $(x : X) \to B \circ h(x)$. Another way to state this is that we have an equivalence $((y : Y) \to \mathsf{fib}_h(y) \to B(y)) \simeq (x : X) \to B \circ h(x)$ given by precomposing with δ, for every type family B. This quickly follows from the standard equivalence $\sum_{y:Y} \mathsf{fib}_h(y) \simeq X$. As with any universal property, this characterizes fib_h up to unique equivalence.

When X and Y satisfy nice mapping out properties, and h is defined using the universal property of X, we can further characterize the sections using an adaptation of the ideas of Kraus and Von Raumer in [4]. For example, if $X \equiv \mathsf{unit}$ and $h(\star) = y_0$, then the type of sections $(x : X) \to B \circ h(x)$ is equivalent to $B(y_0)$. This characterizes the fibers of $\mathsf{unit} \to Y$ as a type family generated by a point over y_0, agreeing with the usual universal property of the family of based path types.

Now consider a more exotic case, when $X \equiv \mathbb{S}^3$ and h is defined by choosing a 3-loop $s : \Omega^3(Y, y_0)$. Then sections $(x : X) \to B \circ h(x)$ are equivalent to dependent 3-loops: a point

$u : B(y_0)$ and an identification $\mathsf{tr}^3(s)(u) = \mathsf{refl}_{\mathsf{refl}_u}$, where $\mathsf{tr}^3(s) : \mathsf{Refl\text{-}htpy} \sim \mathsf{Refl\text{-}htpy}$ is the three dimensional transport through the family B. So we can characterize the fiber of the map $h : \mathbb{S}^3 \to Y$ as freely generated by a point and an identification as above.

Then the situation becomes quite interesting when Y too satisfies a mapping out property. In this case, we can adapt methods from [4], characterizing B in term of descent data. Suppose $Y \equiv \mathbb{S}^2$ and consider the map $\mathsf{hpf} : \mathbb{S}^3 \to \mathbb{S}^2$. Then any family B is equivalent to the choice of a type A and a homotopy $\mathsf{id}_A \sim \mathsf{id}_A$. Then, by adapting the theorems of [4] the fiber of hpf will be the type family determined by the inital such descent data, equipped with a point and an identification as before. Explicitly, the descent data of $\mathsf{fib}_{\mathsf{hpf}}$ will be the HIT F freely generated by a homotopy $H : \mathsf{id}_F \sim \mathsf{id}_F$, a point $u : F$ and an identification $\mathsf{tr}^3(\mathsf{eh})(u) = \mathsf{refl}_{\mathsf{refl}_u}$, where we are transporting in the type family induced by the data (F, H).

The tricky part here is calculating $\mathsf{tr}^3(\mathsf{eh})(u) = \mathsf{refl}_{\mathsf{refl}_u}$ in terms of the descent data. We can do this by making a few observations about EH. The EH identification can be realized as the naturality condition of an important homotopy. Consider the family of based path types $\mathsf{Id} : \mathbb{S}^2 \to U$. The 2-loop surf_2 induces a 2-dimensional transport in this family. This can be calculated as a homotopy of type $\mathsf{id}_{\Omega(\mathsf{surf}_2)} \sim \mathsf{id}_{\Omega(\mathsf{surf}_2)}$ given by whiskering $\mathsf{left\text{-}whisk}_{\mathsf{surf}_2}$ (we are omitting a few coherences from the definition of this homotopy). This homotopy has a naturality condition induced by any identification in $\Omega(\mathsf{surf}_2)$. In particular, for the 2-loop $\mathsf{surf}_2 : \Omega^2(X)$, we have a naturality square $\mathsf{nat\text{-}}[\mathsf{left\text{-}whisk}_{\mathsf{surf}_2}](\mathsf{surf}_2) : \mathsf{left\text{-}whisk}_{\mathsf{surf}_2}(\mathsf{refl}) \cdot \mathsf{surf}_2 = \mathsf{surf}_2 \cdot \mathsf{left\text{-}whisk}_{\mathsf{surf}_2}(\mathsf{refl})$. This type is equivalent to $\mathsf{surf}_2 \cdot \mathsf{surf}_2 = \mathsf{surf}_2 \cdot \mathsf{surf}_2$. In fact, under the equivalence, the above naturality square is $\mathsf{EH}(\mathsf{surf}_2, \mathsf{surf}_2)$. This, with a little work, leads to the following lemma: in any type family B, we have an identification $\mathsf{tr}^3(\mathsf{EH}(\mathsf{surf}_2, \mathsf{surf}_2))(b) = \mathsf{nat\text{-}}[\mathsf{tr}^2(\mathsf{surf}_2)](\mathsf{tr}^2(\mathsf{surf}_2)(b))$, for all $b : B$.

Thus, we can characterize the fiber of hpf as a HIT F freely generated by a point $u : F$, a homotopy $H : \mathsf{id}_F \sim \mathsf{id}_F$, and an identification $\mathsf{nat\text{-}}H(H(u)) = \mathsf{refl}_{\mathsf{refl}_u}$. As it turns out, this is exactly $\mathbb{S}^1$. A cubical proof of this has been written by Tom Jack. However, working with recursive HITs is quite difficult in book HoTT. Thus, we will present an alternate proof showing that a type family defined to be $\mathbb{S}^1$ over the base point satisfies the property that it is freely generated by a section.

These latter comments suggest some lines of future work. The presentation of the fiber as HIT that we obtained was a recursive HIT, which can be quite problematic. When the types involved have more complicated mapping out properties than $\mathbb{S}^3$ and $\mathbb{S}^2$, we are faced with an even more problematic recursive HITs. David Wärn recently showed how to unrecursify these types of HIT in the special case of characterizing the fiber of a map $\mathsf{unit} \to P$, where P is a pushout [6]. Wärn's method seems amenable adaptation for characterizing the fiber of a map $X \to P$ for more general X

Another line of work is suggested by Kristina Sojakova and G. A. Kavvos. constructed syllepsis and other higher dimensional analougs of Eckmann-Hilton in book HoTT [5]. We have shown that Eckmann-Hilton gives a generator of $\pi_3(\mathbb{S}^2)$. Freudenthal shows that this is the generator of $\pi_4(\mathbb{S}^3)$. Syllepsis shows that this generator's square is trivial. Thus, all that is missing to provide another calculation of $\pi_4(\mathbb{S}^3)$ is showing that this generator is non-trival. Sojakova and Kavvos also construct a coherence that is believed to give the generator of $\pi_7(\mathbb{S}^4)$. The techniques developed here may be able to be adapted to prove this.

References

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