The Steenrod Squares in HoTT Revisited

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In homotopy theory, *Steenrod squares* form a very useful cohomology operation: for any space X, there is a Steenrod square homomorphism $H^m(X, \mathbb{Z}/2\mathbb{Z}) \to H^{m+n}(X, \mathbb{Z}/2\mathbb{Z})$ which can be used both to understand the cohomology of X and to distinguish X from other spaces. In HoTT, cohomology is represented in terms of maps into Eilenberg–MacLane spaces[1] and the Steenrod squares can be directly understood as maps $K(\mathbb{Z}/2\mathbb{Z}, m) \to_\star K(\mathbb{Z}/2\mathbb{Z}, m+n)$ for each m and n. Brunerie [Bru16] defined these maps in HoTT, but left open the verification of the laws governing them.

Here, we present an ongoing development of the Steenrod squares in HoTT. In particular, we present a proof of the *Cartan formula*. The formalisation of these results is in progress and key results such as Theorem 1 have already been formalised in `Cubical Agda`.

In what follows, we let K_n denote the nth Eilenberg–MacLane space over $\mathbb{Z}/2\mathbb{Z}$, i.e. $K_n := K(\mathbb{Z}/2\mathbb{Z}, n)$. We take it to be pointed by its neutral element which we denote by 0. The Steenrod squares can be characterised by the following properties [SE62].

Definition 1 (Axiomatic Steenrod squares)**.** *The Steenrod squares is a set of pointed maps* $\mathsf{Sq}^n : K_m \to_\star K_{m+n}$ *for* $m, n \geq 0$ *satisfying the following identities:*

1. $\mathsf{Sq}^0(x) = x$,
2. $\mathsf{Sq}^n(x) = 0$ *if* $n > m$,
3. $\mathsf{Sq}^n(x) = x \smile x$ *if* $n = m$,
4. $\mathsf{Sq}^n(x \smile y) = \sum_{i+j=n} \mathsf{Sq}^i(x) \smile \mathsf{Sq}^j(y)$ *(the Cartan formula)*

In addition to the axioms above, we expect to prove that Steenrod squares are a *stable* operation, in the sense that the diagram below commutes. Under the $\Sigma \dashv \Omega$ adjunction, this corresponds to the requirement that Steenrod squares respect the suspension isomorphism $H^m(X, \mathbb{Z}/2\mathbb{Z}) \simeq H^{m+1}(\Sigma X, \mathbb{Z}/2\mathbb{Z})$.

$$\begin{array}{ccc} \Omega(K_{m+1}) & \xrightarrow{\Omega(\mathsf{Sq}^n)} & \Omega(K_{(m+1)+n}) \\ \downarrow\wr & & \downarrow\wr \\ K_m & \xrightarrow{\mathsf{Sq}^n} & K_{m+n} \end{array} \tag{1}$$

Furthermore, we would like them to satisfy the *Adem relations* stating that, for $n, k > 0$ with $n < 2k$ we have $\mathsf{Sq}^n \circ \mathsf{Sq}^k = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{k-i-1}{n-2i} \mathsf{Sq}^{n+k-i} \circ \mathsf{Sq}^i$.

There is a very direct way of defining the Steenrod squares in a way that automatically makes axioms Axioms 1–3 in Definition 1, as well as the the suspension axiom (1), hold. This uses the fact that a map $f : K(G, m) \to_\star K(H, m+n)$ has a unique delooping for $m > n$, and for $m = n$ it has a delooping iff it satisfies the homomorphism condition $f(a+b) = f(a) + f(b)$, in which case the delooping is unique. See [Wär23] for a proof of this in HoTT. One can show that the cup product square $K_n \to_\star K_{2n}$ given by $x \mapsto x \smile x$ is a homomorphism, which justifies the following definition.

Definition 2 (First definition of Sq^n)**.** *Given* $m, n : \mathbb{N}$*, we define* $\mathsf{Sq}^n_m : K_m \to_\star K_{n+m}$ *to be constant when* $n > m$, $\mathsf{Sq}^n_n(x) = x \smile x$*, and* $\mathsf{Sq}^n_m = (\Omega^{-1})^{m-n}(\mathsf{Sq}^n_n)$ *when* $n < m$.

While this definition of Sq^n is very direct, it is by no means obvious why it satisfies the Cartan formula. For this reason, we also consider an alternative definition, roughly following that of Brunerie. The construction builds on an analysis of the cohomology of $\mathbb{R}P^\infty$. Here, $\mathbb{R}P^\infty$ denotes the type of 2-element types, i.e. $\mathbb{R}P^\infty := \sum_{X:\mathcal{U}} \| X \simeq 2 \|$. We write $t : \mathbb{R}P^\infty \to_\star K_1$ for the unique pointed equivalence [BR17].

[1] For the original definition of Eilenberg–MacLane spaces in HoTT by Licata and Finster, see [LF14].

One can show using the Gysin sequence that $(\mathbb{R}P^\infty \to K_{m-1}) \simeq (\mathbb{R}P^\infty \to_\star K_m)$ where the forward map is given by $f \mapsto [x \mapsto f(x) \smile t(x)]$. We also have $K_m \times (\mathbb{R}P^\infty \to_\star K_m) \simeq (\mathbb{R}P^\infty \to K_m)$, by $(a, f) \mapsto [x \mapsto a + f(x)]$. Repeatedly applying these equivalences, we get that

$$K_0 \times \cdots \times K_m \simeq (\mathbb{R}P^\infty \to K_m)$$

for all $m \geq 0$, by sending $(a_0, \cdots, a_m)$ to $x \mapsto \sum_{i=0}^m a_i \smile t(x)^{m-i}$. In this way, the Steenrod squares naturally arise from a map $S : K_m \to_\star (\mathbb{R}P^\infty \to K_{2m})$, called the *total square*, such that $\mathsf{Sq}^n(a)$ is the '$(m+n)$th projection' of $S(a, -)$, or equivalently $S(a, X) = \sum_{i=0}^m \mathsf{Sq}^i(a) \smile t(X)^{m-i}$. The total square S should be thought of as an *unordered* generalisation of the cup product square, exponentiating with an arbitrary two-element type. This suggests the notation $S(a, X) = a^X$. Brunerie defines S using unordered smash products, but we use unordered joins instead

Definition 3. *Given a 2-element type $X : \mathbb{R}P^\infty$ and a type family $A : X \to \mathcal{U}$, we define the unordered join $\mathop{\Huge\ast}_{x:X} A(x)$ by the pushout to the right.*

$$\begin{tikzcd} X \times \Pi_{x:X} A(x) \arrow[r, "\mathsf{snd}"] \arrow[d] & \Pi_{x:X} A(x) \arrow[d] \\ \sum_{x:X} A(x) \arrow[r] & \mathop{\Huge\ast}_{x:X} A(x) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Definition 4. *Given a 2-element type $X : \mathbb{R}P^\infty$, a family of pointed types $A : X \to \mathcal{U}_\star$ and a pointed type B, we define the type of unordered bipointed maps (rel. X), written $A \to_\star^X B$, to be the type of pairs (F, p_F) where $F : (\Pi_{x:X} A(x)) \to B$ and $p_F : (a : \Pi_{x:X} A(x)) \to \mathop{\Huge\ast}_{x:X}(a(x) = \star_{A(x)}) \to F(a) = \star_B$.*

We have that $\mathop{\Huge\ast}_{x:\mathbb{2}} A(x) \simeq A_0 * A_1$ and that $(A \to_\star^{\mathbb{2}} B) \simeq (A_0 \wedge A_1 \to_\star B)$, since the smash product is the cofibre of the wedge inclusion which can be defined as a fibrewise join. We always have a trivial unordered bipointed map, where $F(a) = \star_B$ and $p_F(a)(y) = \mathsf{refl}_{\star_B}$.

Let us now define an unordered cup product. Let $X : \mathbb{R}P^\infty$ and consider a function $n : X \to \mathbb{N}$. Since addition on $\mathbb{N}$ is commutative, one can define an unordered sum $\sum n := \sum_{x:X} n(x) : \mathbb{N}$. Now consider the type $K_{n(-)} \to_\star^X K_{\sum n}$. When X is $\mathbb{2}$, this type is equivalent to $K_{n_1} \bigwedge K_{n_2} \to_\star K_{n_1+n_2}$ which is equivalent to $\mathbb{Z}/2$; The unique non-trivial element is the cup product. Since having a unique non-trivial element is a property, $K_{n(-)} \to_\star^X K_{\sum n}$ always has a unique non-trivial element, which we call the unordered cup product: for $a : (x : X) \to K_{n(x)}$, we write it as $\smile_{x:X} a(x) : K_{\sum n}$. The total square $S(a, X)$ is simply the diagonal of this unordered bipointed map, i.e. $S(a, X) = \smile_{x:X} a$, where $n : X \to \mathbb{N}$ is constant.

Axioms 2 and 3 in Definition 1 follow directly for this construction. It turns out that the Cartan formula and Adem relation both have the same conceptual explanation; they follow from a Fubini-like interchange rule.

Theorem 1. *For $X, Y : \mathbb{R}P^\infty$, $n : X \to Y \to \mathbb{N}$ and $a : (x : X)(y : Y) \to K_{n(x,y)}$, we have*

$$\smile_{x:X} \smile_{y:Y} a(x, y) = \smile_{y:Y} \smile_{x:X} a(x, y),$$

We obtain the Cartan formula by taking $Y = \mathbb{2}$ and having $a(x, y)$ depend only on y. We obtain the Adem relations by taking $a(x, y)$ to be constant [BM82]. The stability of Steenrod squares can be deduced from the Cartan formula together with the fact that $\mathsf{Sq}^0 x = x$ for $x : K_1$, which we expect to be able to verify explicitly [Hat02]. Stability in turn directly implies $\mathsf{Sq}^0 x = x$ for $x : K_n$ for all $n : \mathbb{N}$. Thus it remains only to explain the proof Theorem 1. It can be reduced to the analogous interchange law for joins of spaces; in fact it follows already from the following.

Theorem 2. *For $X, Y : \mathbb{R}P^\infty$ and $A : X \to Y \to \mathcal{U}$, we have a map*

$$\mathop{\Huge\ast}_{x:X} \mathop{\Huge\ast}_{y:Y} A(x, y) \to \mathop{\Huge\ast}_{y:Y} \mathop{\Huge\ast}_{x:X} A(x, y).$$

As always with Fubini-like theorems, this interchange law would follow from an appropriate definition of the join of an unordered 4-tuple of spaces. However defining such an operation, let alone reasoning about it, seems to require an inordinate amount of higher path algebra. It is at this point that the arguments in HoTT seem to get more complicated than their classical counterparts. We have however formalised a proof of Theorem 2 in `Cubical Agda`.

Many important questions remain open. For example, is it possible to define and reason about Steenrod powers for odd primes in HoTT? Is it possible give a better treatment of unordered joins in HoTT, or perhaps to define the join of an unordered n-tuple for each $n : \mathbb{N}$?

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