$\pi_4\mathbb{S}^3 \not\cong 1$ and another Brunerie number in CCHM*

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1 Introduction

Brunerie defined a number n with a proof that $\pi_4\mathbb{S}^3 \cong \mathbb{Z}/n\mathbb{Z}$, and then proved that $n = \pm 2$[2]. Brunerie proposed that the value of n might someday be computed from its definition using an implementation of homotopy type theory, a problem now called "computing Brunerie's number." Recently, Ljungström provided a calculation of a version of Brunerie's number ("a Brunerie number")[9][1] which led to the first successful computation in [10]. Ljungström observed that this calculation provides a standalone proof of $\pi_4\mathbb{S}^3 \leq \mathbb{Z}/2\mathbb{Z}$ using only the Freudenthal suspension theorem and Eckmann-Hilton, and asked in [9] for a direct proof that $\pi_4\mathbb{S}^3$ is nontrivial, which would give a standalone proof of $\pi_4\mathbb{S}^3 \cong \mathbb{Z}/2\mathbb{Z}$.

We report on work in progress inside CCHM[4] cubical type theory using the `cubicaltt`[5] implementation:[2]

- We provide a direct computational proof that $\pi_4\mathbb{S}^3$ is nontrivial, giving one solution to Ljungström's problem. We define a map $\pi_4\mathbb{S}^3 \to \mathsf{bool}$ which computes on a generator in `cubicaltt`, inducing a proof that if $\pi_4\mathbb{S}^3$ were trivial, then $\mathsf{true} = \mathsf{false}$. What is interesting about this proof is what it does not use: no Hopf fibration, no Freudenthal, no Blakers-Massey, no long or short exact sequences, no cohomology.
- Using similar techniques, we define another Brunerie number which computes quickly in `cubicaltt`, using a new isomorphism $\pi_3\mathbb{S}^2 \cong \mathbb{Z}$ and a new definition of the Whitehead product $[i_2, i_2] : \pi_3\mathbb{S}^2$.
- Finally, we give very short proofs of cubical versions of Eckmann-Hilton and syllepsis[14], which almost immediately induce a generator of $\pi_3\mathbb{S}^2$ (and $\pi_4\mathbb{S}^3$) and a proof that the generator of $\pi_4\mathbb{S}^3$ has order 2. Thanks to their alternative cubical statements, these proofs are simpler than Ljungström's formalizations[8] in Cubical Agda, and more immediately related to $\pi_3\mathbb{S}^2$ and $\pi_4\mathbb{S}^3$.

We believe these constructions will lead to another new computational proof of $\pi_4\mathbb{S}^3 \cong \mathbb{Z}/2\mathbb{Z}$, though we have not mechanized this yet. We are also currently attempting to compute $\pi_3\left(\mathbb{S}^2 \vee \mathbb{S}^2\right) \cong \mathbb{Z}^3$ using similar techniques.

We will publish our WIP `cubicaltt` code at https://github.com/pi3js2/pi4s3.

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[1] Computing Brunerie's number *as defined by Brunerie* is still an open problem for implementors, but computations of alternative "Brunerie numbers" are still interesting for the same reasons Brunerie's number was interesting.

[2] We use `cubicaltt` because it is the only implementation we have found where our examples work, so far. In Cubical Agda[16], one of our examples appears to demonstrate a canonicity bug, which we have not been able to diagnose; we hope all of our examples will work in Cubical Agda if the bug is fixed. We have also attempted to do a similar computation in `redtt`[1][15], an implementation of Cartesian cubical type theory, but so far our attempts consume too much memory.

2 $\pi_4\mathbb{S}^3 \not\cong 1$

We define a map $\pi_4\mathbb{S}^3 \to \mathsf{bool}$ which computes in `cubicaltt`, proving by computation that $\pi_4\mathbb{S}^3$ is nontrivial. We define the map as follows:

$$\pi_4\mathbb{S}^3 \xrightarrow{\pi_3 f_1} \pi_3\mathsf{J}\mathbb{S}^2 \xrightarrow{\pi_3 f_2} \pi_3\mathsf{J}_2\mathbb{S}^2 \xrightarrow{\pi_2 f_3} \pi_2\mathsf{T}_2\mathsf{L}\mathsf{J}_2\mathbb{S}^2 \xrightarrow{\pi_2 f_4} \pi_2\mathsf{K}\left(\mathbb{Z}/2\mathbb{Z}, 2\right) \xrightarrow{f_4} \mathsf{bool}$$

We use ad-hoc HIT representations for the James construction $\mathsf{J}\mathbb{S}^2$ and its word-length filtration $\mathsf{J}_2\mathbb{S}^2$. Then, "$\mathsf{T}_2\mathsf{L}\mathsf{J}_2\mathbb{S}^2$" is another ad-hoc HIT, defined like $\mathsf{J}_3\mathbb{S}^1$ but with an extra 3-cell, which we conjecture provides a model of $\|\Omega\mathsf{J}_2\mathbb{S}^2\|_2$. We speculate that this type can be regarded as an instance of the Adams-Hilton construction as described by Carlsson and Milgram[3], closely related to their example 5.2.

The most difficult part of this map is $f_3 : \Omega\mathsf{J}_2\mathbb{S}^2 \to \mathsf{T}_2\mathsf{L}\mathsf{J}_2\mathbb{S}^2$. Ideally, we would define this using a general 4-dimensional corollary of univalence, but we failed to prove this corollary so far. Instead, as suggested by Axel Ljungström, we considered only the specific goal for $\mathsf{T}_2\mathsf{L}\mathsf{J}_2\mathbb{S}^2$. Relying on `cubicaltt`'s normalization, we were able to reduce this to a computation, using the isomorphism $\pi_2\mathsf{J}_3\mathbb{S}^1 \cong \mathbb{Z}$ below and the inclusion $\mathsf{J}_3\mathbb{S}^1 \to \mathsf{T}_2\mathsf{L}\mathsf{J}_2\mathbb{S}^2$. This involves yet another Brunerie number: a certain 10KB term in $\pi_2\mathsf{J}_3\mathbb{S}^1$ which computes to 2, but computes to 0 after making a modification allowed by the extra 3-cell in $\mathsf{T}_2\mathsf{L}\mathsf{J}_2\mathbb{S}^2$. We rely on the fact that the modified term computes to 0 to define the map f_3.

Plugging in a generator of $\pi_4\mathbb{S}^3$ (induced by our cubical Eckmann-Hilton), the map computes as desired, giving a proof that $\pi_4\mathbb{S}^3$ is nontrivial. Our cubical syllepsis then shows that this generator is of order 2.

3 Computing another Brunerie number

We also define an alternative version of Brunerie's number, which computes in the `cubicaltt` implementation. The details are very different from both Brunerie and Ljungström, but at a high level, we follow Brunerie's recipe. First we define a new isomorphism $\pi_3\mathbb{S}^2 \cong \mathbb{Z}$ as follows:

$$\pi_3\mathbb{S}^2 \xrightarrow{\pi_2 f_1} \pi_2\mathsf{J}\mathbb{S}^1 \xrightarrow{\pi_2 f_2} \pi_2\mathsf{J}_3\mathbb{S}^1 \xrightarrow{\pi_2 f_3} \pi_2\mathbb{S}^2 \xrightarrow{f_4[7]} \mathbb{Z}$$

Note that Brunerie's definition of this isomorphism does not go through the James construction at all, but instead involves the total space of the Hopf fibration. Also unlike Brunerie, we use ad-hoc HIT representations for the James construction $\mathsf{J}\mathbb{S}^1$ and its word-length filtration $\mathsf{J}_3\mathbb{S}^1$. After accounting for this difference, the only new part of the map is $f_3 : \mathsf{J}_3\mathbb{S}^1 \to \mathbb{S}^2$. We prove directly that this map induces an equivalence $\|\mathsf{J}_3\mathbb{S}^1\|_2 \simeq \mathbb{S}^1 \times \|\mathbb{S}^2\|_2$, and thus a group isomorphism $\pi_2\mathsf{J}_3\mathbb{S}^1 \cong \pi_2\left(\mathbb{S}^1 \times \|\mathbb{S}^2\|_2\right) \cong \pi_2\mathbb{S}^2$.

We speculate that, under the Pontryagin construction[12][3] relating $\Omega^3\mathbb{S}^2$ to framed links in $\mathbb{R}^3$, the map $\Omega^3\mathbb{S}^2 \to \Omega^2\mathsf{J}\mathbb{S}^1$ computes a *link diagram* from a link. The map $\Omega^2 f_3 : \Omega^2\mathsf{J}_3\mathbb{S}^1 \to \Omega^2\mathbb{S}^2$ then appears to compute the writhe from the link diagram.

The next ingredient for a Brunerie number is the Whitehead product $[i_2, i_2] : \pi_3\mathbb{S}^2$. Brunerie proves that the attaching map of the 4-cell in $\mathsf{J}_2\mathbb{S}^2$ is $[i_2, i_2]$. With our ad-hoc HIT representation of $\mathsf{J}_2\mathbb{S}^2$, we can directly read off the attaching map for the 4-cell, giving a very short direct definition of this element, with only two `hcomp`s.

In the `cubicaltt` implementation, the integer defined by applying the above isomorphism to this element normalizes to 2 in about 0.02s.

[3]We *think* [12] is the original source, but we cannot find it or read Russian. English translations are reportedly available in [6]. For exposition see e.g. [13] or [11].

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