Edge Guarding Plane Graphs

Master Thesis of

Paul Jungeblut

At the Department of Informatics
Institute of Theoretical Informatics

Reviewers:	Prof. Dr. Dorothea Wagner
	Prof. Dr. Peter Sanders
Advisor:	Dr. Torsten Ueckerdt

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Statement of Authorship

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Paul Jungeblut, Karlsruhe, 25.10.2019

Abstract

Let $G = (V, E)$ be a plane graph. We say that a face f of G is *guarded* by an edge $vw \in E$ if at least one vertex from $\{v, w\}$ is on the boundary of f. For a planar graph class $\mathcal{G}$ the function $\Gamma_{\mathcal{G}} : \mathbb{N} \to \mathbb{N}$ maps n to the minimal number of edges needed to guard all faces of any n-vertex graph in $\mathcal{G}$.

This thesis contributes new bounds on $\Gamma_{\mathcal{G}}$ for several graph classes, in particular on $\Gamma_{\triangle,\text{stacked}}$ for stacked triangulations, on $\Gamma_{\square}$ for quadrangulations and on Γ_{sp} for series parallel graphs. Specifically we show that

- $\lfloor (2n-4)/7 \rfloor \leqslant \Gamma_{\triangle,\text{stacked}}(n) \leqslant \lfloor 2n/7 \rfloor$,
- $\lfloor (n-2)/4 \rfloor \leqslant \Gamma_{\square}(n) \leqslant \lfloor n/3 \rfloor$ and
- $\lfloor (n-2)/3 \rfloor \leqslant \Gamma_{sp}(n) \leqslant \lfloor n/3 \rfloor$.

Note that the bounds for stacked triangulations and series parallel graphs are tight (up to a small constant). For quadrangulations we identify the non-trivial subclass of 2-degenerate quadrangulations for which we further prove $\Gamma_{\square,\text{2-deg}}(n) \leqslant \lfloor n/4 \rfloor$ matching the lower bound.

Deutsche Zusammenfassung

Eine Facette f eines eingebetteten planaren Graphen $G = (V, E)$ wird von einer Kante $vw \in E$ *überwacht*, wenn mindestens einer der Knoten aus $\{v, w\}$ auf dem Rand von f liegt. Für eine Unterklasse $\mathcal{G}$ der planaren Graphen ordnet die Funktion $\Gamma_{\mathcal{G}} : \mathbb{N} \to \mathbb{N}$ jeder natürliche Zahl n die minimale Anzahl an Kanten zu, mit der alle Facetten jedes Graphen mit n Knoten aus $\mathcal{G}$ überwacht werden können.

Diese Masterarbeit liefert neue Schranken für $\Gamma_{\mathcal{G}}$ für mehrere Unterklassen planarer Graphen, insbesondere für Stacked-Triangulierungen ($\Gamma_{\triangle,\text{stacked}}$), für Quadrangulierungen ($\Gamma_{\square}$) und für Serien-Parallele Graphen (Γ_{sp}). Wir zeigen, dass

- $\lfloor (2n-4)/7 \rfloor \leqslant \Gamma_{\triangle,\text{stacked}}(n) \leqslant \lfloor 2n/7 \rfloor$,
- $\lfloor (n-2)/4 \rfloor \leqslant \Gamma_{\square}(n) \leqslant \lfloor n/3 \rfloor$ und
- $\lfloor (n-2)/3 \rfloor \leqslant \Gamma_{sp}(n) \leqslant \lfloor n/3 \rfloor$.

Die Schranken für Stacked-Triangulierungen und Serien-Parallele Graphen sind scharf (bis auf kleine Konstanten). Für Quadrangulierungen betrachten wir weiter die nicht-triviale Unterklasse der 2-degenerierten Quadrangulierungen und zeigen für diese, dass $\Gamma_{\square,\text{2-deg}}(n) \leqslant \lfloor n/4 \rfloor$. Diese Schranke entspricht der unteren Schranke, ist also scharf.

Contents

1. Introduction

1.1. Motivation and History

In 1975 Chvátal [7] laid the foundation for the widely studied field of *art gallery problems* by answering a question that was posed by Victor Klee in 1973 [20]. We rephrase this question using one of its geometric interpretations:

> What is the smallest number $f(n)$ of guards necessary, such that in any simple, n-sided polygon P each interior point is visible by at least one guard? The guards must be positioned in the corners of P and a point p is visible by a guard g, if and only if the line segment connecting g and p lies completely inside the polygon P.

We can imagine the polygon P to be the floor plan of an art gallery containing exhibits both on its walls and in its interior. For example, consider the gallery shown in Figure 1.1. It is bounded by a 20-sided polygon P and can be guarded by four guards. For each guard we shaded the area of P that is in its field of vision. In fact, this gallery needs at least four guards but this is neither obvious nor is it easy to compute a minimal set of guard positions. Lee and Lin [16] showed that the decision problem, whether an n sided art gallery can be guarded by k guards is NP-complete.

Chvátal showed that $\lfloor n/3 \rfloor$ guards are occasionally necessary and always sufficient. He presented an infinite family of art galleries bounded by polygons P_k with $3k + 2$ vertices

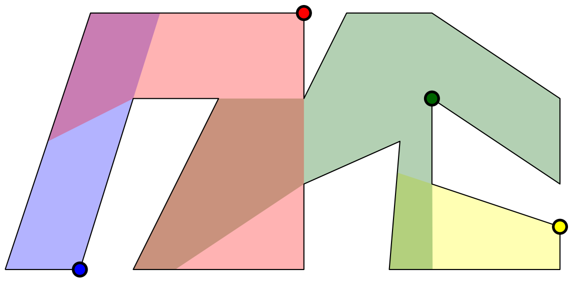

Figure 1.1.: An art gallery bounded by a polygon P with $n = 20$ vertices. As we can see, four guards are sufficient to guard any interior point. To illustrate this, each guard is assigned a color and its field of vision in the interior of P is shaded in this color.

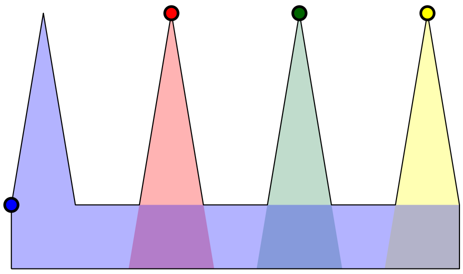

Figure 1.2.: An art gallery bounded by a polygon P with $n = 14$ sides. It has four spikes, each of them needing its own guard. This example can easily be extended to contain more spikes so that more guards are needed.

that need k guards (for $k = 4$ this is shown in the Figure 1.2). Polygon P_k contains k spikes and each of them needs its own guard. To prove sufficiency he already employed graph theoretic methods, but his original proof was rather complicated. Only a few years later, Fisk [13] gave a very short – six lines (!) – and elegant proof: Triangulate the interior of P and find a proper 3-coloring of the vertices of the obtained graph G (such a 3-coloring exists by [26]). Then every face of G is triangular and incident to vertices of all three colors. Since each face is convex, it can be guarded from any of its boundary vertices. The smallest color class contains at most $\lfloor n/3 \rfloor$ vertices and can therefore be used as a guard set. Figure 1.3 shows a possible triangulation of the art gallery from Figure 1.1 and a proper 3-coloring of the vertices. There are seven red vertices, seven green vertices and six blue vertices, so the smallest color class is the blue one and its vertices guard the whole art gallery. However, this strategy does not necessarily provide a minimal number of guards: We know from above that four guards are sufficient in this case.

Until now, all guards were so called *vertex guards*. Starting in 1983, O'Rourke [21] considered more powerful types of guards, which he called *mobile guards*. These guards are not fixed at a vertex but are allowed to move in a restricted area R and their field of vision is defined as all points visible from at least one point in R. If they are wisely placed, fewer guards might be necessary in this setting. Countless variants and specializations of art gallery problems have been developed, a survey of results is given by Shermer [29] and O'Rourke [20] published a whole book about it.

In another attempt to generalize the original question, we can free ourselves from the geometric notion of visibility and ask what the smallest number of vertex guards for any n-vertex plane graph $G = (V, E)$ is, such that each face of G is guarded. Here a vertex guard $g \in V$ guards all faces that have g on their boundary. Note that f does not need to be convex in this generalization. Bose et al. [5] answered this question in 1997 showing that $\lfloor n/2 \rfloor$ vertex guards are sometimes necessary and always sufficient. Mobile guards are also considered in this setting: Out of these the so called *edge guards* received most

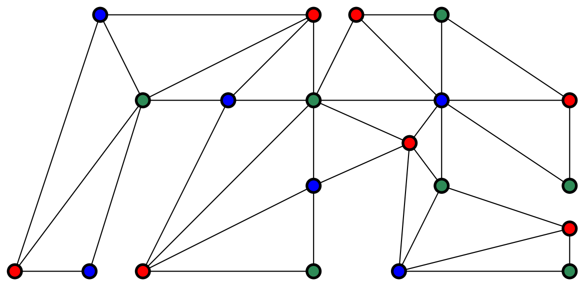

Figure 1.3.: The plane graph obtained from triangulating the art gallery from Figure 1.1 and a proper 3-coloring of the vertices. Each color class is a possible guard set.

attention in the literature and are what this thesis is about. In our graph theoretic setting, an edge guard is just an edge vw of the plane graph and it guards all faces that have at least one vertex from $\{v, w\}$ on their boundary. The ultimate goal is to answer the following open question giving a conjecture about the minimal number of edge guards needed for all n-vertex plane graphs:

> Can any n-vertex plane graph be guarded by $\lfloor n/3 \rfloor$ edge guards?

It is easy to see that $\lfloor n/3 \rfloor$ edge guards are sometimes necessary: Assume that $n = 3k$ for some $k \in \mathbb{N}$ and let $G = \left(\bigcup_{i=1}^{k}\{v_i^1, v_i^2, v_i^3\}, \bigcup_{i=1}^{k}\{v_i^1v_i^2,\ v_i^2v_i^3,\ v_i^3v_i^1\}\right)$ be the disconnected graph consisting of pairwise unconnected triangles. Each of them needs its own edge guard. Even though upper bounds were considered in several publications [2, 5, 12] and gradually improved, there is still a theoretic gap between the best currently known lower and upper bounds. This led to the study of several subclasses of plane graphs, like triangulations [5, 12] and (maximal) outerplanar graphs [5, 7, 21].

1.2. Variants and Related Work

This thesis is about edge guard sets, i.e. edges guarding all faces of a plane graph $G = (V, E)$. In the previous section we saw how this question evolved since the original art gallery problem from 1972. Now we alter the rules to discover variants related to this question: A plane graph consists of three types of combinatorial structures, namely the vertices, the edges and the faces. Each of these three can be the guarding entity and the object to be guarded giving rise to nine different combinations. Some of these combinations are well known and widely studied under their own names while others have not yet received further attention. In the following we shortly explore each of these combinations.

- Let us say that a vertex v guards itself and all vertices in its neighborhood. The problem to find a subset of the vertices guarding all vertices in V is exactly the dominating set problem. A *dominating set* $D \subseteq V$ is a subset of the vertices, such that each $v \notin D$ has a neighbor in D. Mathworld[1] shows bounds on the domination number for several (not necessarily planar) graph classes. Matheson and Tarjan [18] conjectured in 1996 that $\lfloor n/4 \rfloor$ vertices are always enough to dominate any n-vertex plane triangulation. Their conjecture remains unsettled. Good bounds were only shown for extremely restricted graph classes as in [15, 25]. The best known upper bound is $\lfloor 17n/53 \rfloor$ given by Špacapan [30].
- Another classical graph theoretic problem appears when we assume that vertices guard their incident edges. A *vertex cover* is a subset $X \subseteq V$ of the vertices, such that each edge $e \in E$ has at least one endpoint in X. Again, the vertex cover problem has been studied for many different graph classes[2]. For n-vertex planar graphs we can easily construct a vertex cover X of size $\lfloor 3n/4 \rfloor$ by applying the 4-Color Theorem and selecting the three smallest color classes. All remaining vertices have the same color and therefore form an independent set, so X is indeed a vertex cover.
  Recently even stronger vertex guards have been considered, where a vertex v guards all edges incident to v or one of its neighbors [22].
- The setting in which vertices guard faces was already mentioned in the previous section. Recall that Bose et al. [5] showed that $\lfloor n/2 \rfloor$ vertex guards are sometimes necessary and always sufficient.

[1] `http://mathworld.wolfram.com/DominationNumber.html`
[2] `http://mathworld.wolfram.com/VertexCover.html`

- Now we consider all variants where faces guard some object of the graph. There is no established name for any of the three problems, because they are just the duals of the ones above: For a planar graph G consider its dual G^*. Then the faces of G are the vertices of G^*, so face guards can be treated as vertex guards in the dual graph.

- This thesis is about edges guarding the faces of a plane graph. Here we say that an edge vw guards a face f if at least one vertex from $\{v, w\}$ is on the boundary of f. Bose et al. [4] called this *weak edge coverage* and defined *strong edge coverage* to require that both v and w are on the boundary of f. Under this circumstances, each edge can guard at most two faces. Any maximal plane n-vertex graph contains $2n - 4$ faces, so $n - 2$ strong edge guards are necessary. The dual of a maximal plane graph G is cubic and 3-connected, so by Petersen's Theorem [23] it contains a perfect matching. This matching can be used to group the faces of G into pairs that can share a strong edge guard. Therefore $n - 2$ strong edge guards are also sufficient for maximal plane graphs. For a general plane graph G consider a maximal plane graph $\tilde{G}$ on the same vertex set that contains G as a subgraph. Any strong edge guard set $\tilde{\Gamma}$ for $\tilde{G}$ can be augmented to a strong edge guard set for G of the same size: Every $e \in \tilde{\Gamma}$ with $e \notin E(G)$ is a chord of a unique face f in G, so we can choose any edge incident to f instead.

- Assuming that edges guard the vertices of a graph $G = (V, E)$ leads us to another well known problem. A subset $C \subseteq E$ of the edges is known as an *edge cover*, if each vertex $v \in V$ is the endpoint of at least one edge in C. The problem to find edge covers also arises if we look at the dual graph in the setting of strong edge coverage from the previous item. If G contains a perfect matching M^*, this is obviously also a minimal edge cover as each edge in M^* guards two vertices that are not guarded by any other edge. If on the other hand no perfect matching exists, a minimal edge cover is found by computing a maximum matching M and taking one additional edge per vertex that is not guarded by M. Nishizeki [19] presented bounds on the size of a maximum matching in planar graphs depending on its minimum degree. These then give upper bounds for the size of edge covers.

- For the last remaining setting we assume that edges guard other edges of $G = (V, E)$. This is again a classical problem known as the edge dominating set problem. An *edge dominating set* $D \subseteq E$ is a subset of the edges such that each $e \notin D$ is adjacent to an edge in D. When removing all edges and their endpoints in D from G, the remaining vertices are all independent, because otherwise any remaining edge would not be dominated. Using this observation we see that we can use inclusion maximal matchings as edge dominating sets. The smallest cardinality of an inclusion maximal matching of a graph G is also known as its *saturation number* $s(G)$. Bounds for the saturation number were found for example for fullerene graphs [9].

1.3. Outline of this Thesis

This thesis explores three more graph classes, namely the stacked triangulations, the quadrangulations and the series parallel graphs. In each case we present new lower and upper bounds on the number of edge guards that are sometimes necessary and always sufficient. The document is organized as follows:

Chapter 2 introduces the needed mathematical notation and concepts. We repeat some classic graph theoretic results that are used throughout this thesis (without proving them).

In **Chapter 3** we give an outline of the current best known results for different graph classes. In some cases we restrict ourselves to citing the theorems from the corresponding publications while in other cases we also present their proofs. This can be either because they are particularly elegant or because we use a similar approach later on for one of our own results.

The next three chapters describe our own contribution, starting with **Chapter 4** giving matching lower and upper bounds for stacked triangulations. We see that they have a hierarchical structure and use it to describe a finite list of subgraphs, at least one of which must appear in the graph. By locally changing these subgraphs we obtain a smaller stacked triangulation, thus allowing us to use induction on the number of vertices. To show that this bound is tight, we borrow a technique that is also used to prove the best known lower bound for general triangulations.

In **Chapter 5** we study quadrangulations. Quadrangular faces have proven to be problematic in previous strategies to find small edge guard sets. We improve the current best known upper bound for the number of edge guards, even though we conjecture that our given bound is not yet tight. As a first step towards a better bound, we further look at the subclass of 2-degenerate quadrangulations. For those we are able to present a better upper bound that also matches the lower bound.

The third considered graph class are the series parallel graphs in **Chapter 6**. Again we present a matching lower and upper bound. It is noteworthy that the upper bound is achieved using only vertex guards, so for this graph class edge guards provide no benefit over vertex guards. An important subclass are the maximal series parallel graphs, which are exactly the 2-trees. We show that they need the same number of guards.

The conclusion in **Chapter 7** looks back on this thesis. We repeat our and all previous results and put them into context. Gaps between lower and upper bounds directly lead to a list of open questions.

The **Appendix** is split into two parts. We conducted experiments to find edge guard sets for small instances of different graph classes. The results are presented in Appendix A. These gave first hints towards some of the bounds proven in the main part of this thesis. The second part in Appendix B contains several proofs for 2-degenerate quadrangulations that were skipped in Chapter 5 for space reasons and improved readability.

2. Preliminaries

This chapter introduces the mathematical and graph theoretic terms and definitions used throughout this thesis. We adopt most notation from the book "Graph Theory" by Reinhard Diestel [8].

2.1. General Definitions

By $\mathbb{N}$ we denote the set of natural numbers where $0 \notin \mathbb{N}$, so zero is not a natural number. To include the zero we use $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For a non-negative real number $x \in \mathbb{R}$ we define the *floor*-function as $\lfloor x \rfloor := \max\{n \in \mathbb{N}_0 \mid n \leqslant x\}$.

A *graph* $G = (V, E)$ is a pair of disjoint sets. Here V contains the *vertices* and $E \subseteq V \times V$ contains the *edges*. Often we also use the functions $V(G) := V$ and $E(G) := E$ to get the vertex and edge set of a graph. The number of vertices $|G| := |V(G)|$ is called the *order* of a graph and the number of edges $\|G\| := |E(G)|$ is its *size*. Typically we use the variables $n := |G|$ and $m := \|G\|$ for this. An edge $e = \{v, w\}$ is a 2-element subset of V and expresses a relation between the vertices v and w. All graphs considered here are undirected, so edge $e = \{v, w\} = \{w, v\}$. As a shorthand notation we write $e = vw$ in the following. For a vertex $v \in V$ the set $N(v) := \{w \in V \mid vw \in E\}$ is its *neighborhood*. Further the *degree* of v is the size $|N(v)|$ of its neighborhood. A graph H is a *subgraph* of G, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $E(H) = \{vw \in E \mid v, w \in V(H)\}$ then H is an *induced subgraph*. For the induced subgraph obtained by removing a subset X of the vertices we write $G - X$. A *matching* $M \subseteq E(G)$ is a subset of the edges, such that no two edges in M share an endpoint. Matching M is *maximal (by inclusion)*, if there is no bigger matching $\tilde{M}$ with $M \subset \tilde{M}$ (i.e. M is a proper subset of $\tilde{M}$).

There are countless subclasses of graphs, the "Information System on Graph Classes and their Inclusions" [27] lists about 1600 of them. A few important ones appear in this thesis: In a *complete graph* $K_n := \big(\{v_1, \dots, v_n\}, \{v_i v_j \mid 1 \leqslant i < j \leqslant n\}\big)$ every possible edge exists. A *path* $P_n := \big(\{v_1, \dots, v_n\}, \{v_1 v_2, \dots, v_{n-1} v_n\}\big)$ is a sequence of vertices joined by edges between consecutive vertices. The *length* of a path is its number of edges. By adding another edge $v_n v_1$ we get a cycle $C_n := \big(V(P_n), E(P_n) \cup \{v_n v_1\}\big)$. We say that a graph is *connected*, if there is a path between any pair of vertices, otherwise it is *disconnected*. A connected graph without cycles is called a *tree*. This can be generalized to k-*trees*: For $k \in \mathbb{N}$ a graph G is a k-tree, if and only if it is the complete graph K_{k+1} or a vertex v exists, such that $N(v)$ induces a copy of K_k and $G - \{v\}$ is a k-tree. Trees are exactly the 1-trees. A graph G where every subgraph H has a vertex of degree at most k is called k-*degenerate*.

A graph G can be drawn in the plane with its vertices mapped to points and its edges mapped to curves connecting the corresponding endpoints. Graphs that are already drawn such that the curves for no two edges cross (except in common endpoints) are called *plane graphs* and such drawings are *plane drawings*. Graphs for which a plane drawing exists are *planar graphs*. In a plane drawing the edges subdivide the plane into disjoint regions, called *faces*. The only unbounded face is the *outer face*, all others are *inner faces*. Let $G = (V, E)$ be a plane graph and let f be an arbitrary face. By $\partial f := \{v \in V \mid v \text{ is incident to } f\}$ we denote the set of *boundary vertices*. The *degree* of f is its number $|\partial f|$ of boundary vertices. If $\partial f = \{v_1, \dots, v_k\}$ appear in counterclockwise order along the boundary of f, we also write $f = (v_1, \dots, v_k)$. Two faces f, g are h*-hop apart*, if the shortest path connecting a vertex from ∂f with a vertex from ∂g has length h.

This thesis focuses on subclasses of plane graphs. The *maximal plane graphs* are plane graphs that contain the maximum possible number of edges. Any additional edge cannot be drawn without crossing one of the existing ones. An *outerplane graph* is a plane graph where every vertex is incident to the outer face. Similarly the *maximal outerplane graphs* are outerplane graphs that contain the maximum number of edges. The corresponding subclasses of planar graphs are defined analogously.

Given a plane graph $G = (V, E)$ with face set F we define its dual graph $G^* = (V^*, E^*)$. Here $V^* := F$, so each dual vertex corresponds to one of the faces in F. For each edge $e \in E$ let $f, g \in F$ be the two faces on the two sides of e. Then there is an associated dual edge $e^* \in E^*$ connecting the dual vertices f^* and g^* originating from f and g. Note that G^* can be a *multigraph*: If there is an edge touching the same face on both of its sides, then G^* contains a *loop* (an edge where both endpoints coincide). If there are two adjacent faces sharing at least two boundary edges, G^* contains a *multiedge* (several simple edges between the same pair of vertices). The dual graph G^* has an inherited plane drawing: For each face $f \in F$ place the dual vertex f^* inside f and draw a the dual edge e^* such that is crosses the corresponding edge $e \in E$ exactly once.

Now that we introduced all graph theoretic basics we finish this section defining the central concept of this thesis: Guard sets. A vertex $v \in V$ of a plane graph $G = (V, E)$ *guards* all faces f where $\partial f \cap \{v\} \neq \emptyset$. Now a set of vertices $\Lambda \subseteq V$ is a *vertex guard set* if any face of G is guarded by at least one vertex in Λ. Similarly, an edge $e = vw \in E$ guards all faces f where $\partial f \cap \{v, w\} \neq \emptyset$, so all faces that are guarded by v and/or w. Subset $\Gamma \subseteq E$ is an *edge guard set* if any face of G is guarded by at least one edge in Γ. Even though Γ is a set of edges, we write $V(\Gamma)$ to denote the set of vertices that are endpoints of at least one edge in Γ (as if Γ were a graph).

For a graph class $\mathcal{G}$ we use the short notation $\Gamma_{\mathcal{G}}(n)$ to denote the smallest number $k \in \mathbb{N}$, such that every n-vertex $G \in \mathcal{G}$ can be guarded by an edge guard set of size k. For the following graph classes $\Gamma_{\mathcal{G}}(n)$ appears in this thesis:

- $\Gamma_{\text{plane}}(n)$ for general plane graphs
- $\Gamma_{\triangle}(n)$ for triangulations
- $\Gamma_{\triangle,\text{stacked}}(n)$ for stacked triangulations
- $\Gamma_{\triangle,\text{4-con}}(n)$ for 4-connected triangulations
- $\Gamma_{\square}(n)$ for quadrangulations
- $\Gamma_{\square,\text{2-deg}}(n)$ for 2-degenerate quadrangulations
- $\Gamma_{\text{sp}}(n)$ for series parallel graphs
- $\Gamma_{\text{2-tree}}(n)$ for 2-trees

To avoid writing constant terms too often, we introduce an asymptotic notation for $\Gamma_{\mathcal{G}}(n)$: We write $\Gamma_{\mathcal{G}}(n) \sim cn$, if the quotient between $\Gamma_{\mathcal{G}}(n)$ and cn approaches 1 as n goes to infinity, so $\lim_{n\to\infty}(\Gamma_{\mathcal{G}}(n)/cn) = 1$.

2.2. Classical Theorems

The previous section laid the graph theoretic foundation. In this section we continue with some additional concepts amended with classical results from the respective area.

Let $G = (V, E)$ be a graph and C be a set of colors. A function $\chi : V \to C$ is called a *coloring* of G. If for any edge $e = vw \in E$ the condition $\chi(v) \neq \chi(w)$ holds, we say that χ is a *proper coloring*. Colorings play a central role in several chapters of this thesis. One of the most important breakthroughs regarding colorings of planar graphs is the following 4-Color Theorem, that was applied to construct guard sets in the literature.

Theorem 2.1 (Appel, Haken 1976 [1]) *For every planar graph there is a proper* 4*-coloring.*

The theorem was proven in 1976 by a computer assisted case distinction considering 1936 different cases [1]. This was later reduced by Robertson et al. [28] to 633 cases, but there is still no non-assisted, purely combinatorial proof. Given a coloring χ of a plane graph G we call edges and faces *monochromatic* if all incident vertices are assigned the same color by χ. In this case we say that the edge/face is colored in the same color as its incident vertices.

We proceed with a plane graph $G = (V, E)$ and faces F. Euler observed and proved that:

Theorem 2.2 (Euler 1758 [11]) *For every connected plane graph we get* $|V| - |E| + |F| = 2$.

An important consequence is that the number of edges and faces in triangulations and quadrangulations (graphs where every face has degree four) only depends on their number of vertices. The proofs only differ in the numbers plugged in, so we only give one of them.

Corollary 2.3 *An* n*-vertex triangulation has* $3n - 6$ *edges and* $2n - 4$ *faces.*

Proof A triangulation $G = (V, E)$ is a connected plane graph. By Euler's Formular we know that $|V| - |E| + |F| = 2$. Any face is incident to three edges and any edge touches exactly two faces, therefore we get $3|F| = 2|E|$. We rewrite this as $|E| = 3|F|/2$ and plugging into Euler's Formular gives $|V| - 3|F|/2 + |F| = 2$ which solves to $|F| = 2|V| - 4$. By plugging in $|F| = 2|E|/3$ instead, we get $|E| = 3|V| - 6$. ∎

Corollary 2.4 *An* n*-vertex quadrangulation has* $2n - 4$ *edges and* $n - 2$ *faces.*

To conclude this section we again consider general graphs. A graph is called k-regular if every vertex has degree k. Further, a d-factor F of a graph G is a spanning subgraph such that F is d-regular. Here *spanning* means that $V(G) = V(F)$. Petersen gave a sufficient condition for a graph to contain a 2-factor in his 2-Factor Theorem.

Theorem 2.5 (Petersen 1891 [23]) *Every* k*-regular graph with even* k *has a 2-factor.*

3. Previous Work

Several papers regarding edge guard sets in plane graphs have been published over the last 40 years. Here we present some of this work, displaying the currently best known lower and upper bounds. The authors mainly distinguish between general plane graphs and triangulations (the maximal plane graphs). While we limit ourselves to stating the best known results for general plane graphs, we quote the work of Bose et al. [5] and Biniaz et al. [2] for triangulations in more detail. On the one hand because we correct a small error in a constant term of the published lower bound, on the other hand because the upper bound can be generalized to general graphs. We use that same technique in Chapter 5 to prove an upper bound for quadrangulations.

3.1. Triangulations

We start by looking at $\Gamma_{\triangle}(n)$, the minimal number of edge guards needed to guard all faces of any n-vertex triangulation. For a lower bound we repeat a construction by Bose et al. [5]. Their idea is to create a plane triangulation G that contains several copies of a special subgraph H, all of them at least 2-hop apart from all others. Graph H was chosen such that it needs a big number of edge guards relative to its small number of vertices.

Theorem 3.1 (Theorem 3.6 in [5]) *It is* $\lfloor (4n-8)/13 \rfloor \leqslant \Gamma_{\triangle}(n)$.

Proof Consider the 6-vertex triangulation shown in Figure 3.1 that is known as the octahedron graph. A single edge guard is not enough to guard all of its faces, an edge guard set of size two is shown in the figure. We now show how to create a graph that contains many copies of the octahedron graph, any two of them at least 2-hop apart. Let S

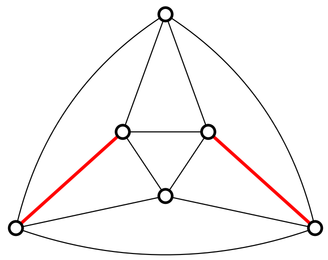

Figure 3.1.: The octahedron graph with two edge guards (shown in red). No single edge guards all of its eight faces.

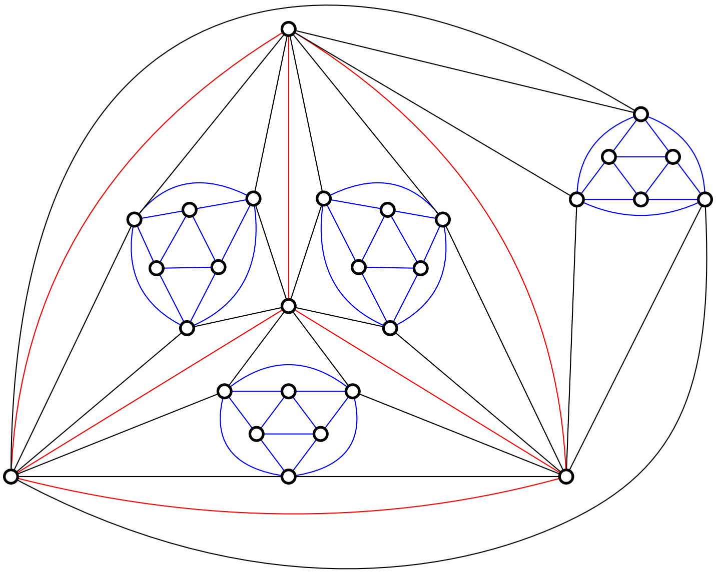

Figure 3.2.: A triangulation with 28 vertices needing at least eight edge guards. The red subgraph is the skeleton graph S, the copies of the octahedron graph are blue and the black edges were added to obtain a triangulation.

be a plane triangulation with ν vertices and call this the skeleton triangulation. It has $2\nu - 4$ faces. Put a copy of the octahedron graph into each face of S and triangulate the resulting plane graph in an arbitrary way to obtain a triangulation G. The number of vertices of G is $n = \nu + 6 \cdot (2\nu - 4) = 13\nu - 24$ which leads to $\nu = (n + 24)/13$. The shortest path between any two copies of the octahedron graph must contain a vertex of the skeleton graph S, therefore they are at least 2-hop apart. In total there are $2\nu - 4$ such copies and any edge guard set Γ contains two edge guards for each of them. Plugging in we get

$$\Gamma_\triangle(n) \geqslant |\Gamma| \geqslant (2\nu - 4) \cdot 2 = \left(2\frac{n + 24}{13} - 4\right) \cdot 2 = \frac{4n - 8}{13}.$$

A complete example of a triangulation obtained by this construction is shown in Figure 3.2. ■

Remark 3.2 *Bose et al. [5] give a lower bound of* $\lfloor (4n - 4)/13 \rfloor \leqslant \Gamma_\triangle(n)$ *in their publication. They obtain this bound the same way as in the proof above, but they put three copies of the octahedron graph into the outer face of the skeleton triangulation* S*. However, the resulting graph is impossible to triangulate such that all copies of the octahedron are 2-hop apart, therefore their argument that each copy exclusively needs two edge guards does not work here.*

Remark 3.3 *To further improve the lower bound, Bose et al. posed the question whether there is a* 9*-vertex triangulation needing three edge guards. If so, this could be used instead of the octahedron graph to obtain a better lower bound. From our experiments presented in Appendix A we now know that such a graph does not exist (and neither do* 12/15/18*-vertex triangulations needing* 4/5/6 *edge guards).*

Let us now consider the upper bound for triangulations that was originally proved by Bose et al. [4]. They start by defining a specific non-proper 2-coloring of the vertex set that can be used to construct a guard set. Then they show the existence of such a coloring for triangulations.

Definition 3.4 (Guard Coloring) *A* guard coloring *of a plane graph* G *is a 2-coloring of the vertices such that no face is monochromatic but each face is incident to a monochromatic edge.*

Lemma 3.5 *Let* G *be a plane graph with* n *vertices. If* G *has a guard coloring, then it can be guarded by* $\lfloor n/3 \rfloor$ *edge guards.*

Proof Without loss of generality let 1 and 2 be the two colors of the guard coloring and let G_1 and G_2 be the two subgraphs of G induced by the vertices colored 1 and 2, respectively. Further let M_1 and M_2 be maximal matchings in G_1 and G_2. We claim that M_i (for $i \in \{1,2\}$) guards any face that has a monochromatic edge $e = vw$ of color i. Otherwise we would have $v, w \notin V(M_i)$ and $M_i \cup \{vw\}$ would be a bigger matching and thus M_i would not be maximal. Therefore $\Gamma_{12} := M_1 \cup M_2$ is an edge guard set of size $|\Gamma_{12}| = |M_1| + |M_2|$.

We define two more edge guard sets: For $i \in \{1,2\}$ define $\Gamma_i := M_i \cup E_i$ where E_i is a set of edges containing one edge incident to each vertex in $V(G_i) \setminus V(M_i)$. Because no face is monochromatic, Γ_i touches every face. It is an edge guard set and has size $|\Gamma_i| = |M_i| + |V(G_i)| - 2|M_i| = |V(G_i)| - |M_i|$.

The total size of all three edge guard sets is

$$\begin{aligned} |\Gamma_{12}| + |\Gamma_1| + |\Gamma_2| &= |M_1| + |M_2| + |V(G_1)| - |M_1| + |V(G_2)| - |M_2| \\ &= |V(G_1)| + |V(G_2)| \\ &= |V(G)| \\ &= n \end{aligned}$$

and therefore the smallest one of these three must be of size at most $\lfloor n/3 \rfloor$. ∎

Theorem 3.6 *Any triangulation* G *has a guard coloring, therefore* $\Gamma_\triangle(n) \leqslant \lfloor n/3 \rfloor$.

Proof Apply the 4-Color Theorem to obtain a proper 4-coloring of G with colors from $\{1,2,3,4\}$. Now group colors 1 and 2 into a new color A and colors 3 and 4 into a new color B. The resulting 2-coloring (with colors from $\{A, B\}$) is a guard coloring for G. This is true, because each face of G is a triangle and thus has exactly three different colors from $\{1,2,3,4\}$ among its boundary vertices. Therefore each face has a boundary vertex of both colors from $\{A, B\}$, so there are no monochromatic faces. Further each non-monochromatic 2-coloring of a triangle contains a monochromatic edge. ∎

3.2. General Plane Graphs

It is conjectured that $\Gamma_{\text{planar}}(n) \stackrel{?}{=} \lfloor n/3 \rfloor$ edge guards are sometimes necessary and always sufficient for an n-vertex plane graph. A lower bound is trivially given by a set of copies of triangles (a disconnected graph), each of them needing exactly one edge guard (see Figure 3.3). Stating an upper bound is much more difficult and there are two different strategies used throughout the previous work.

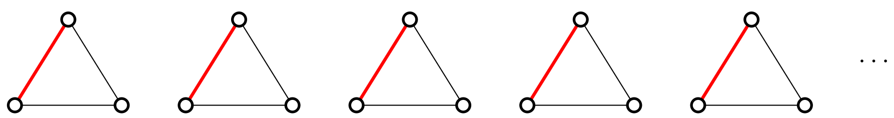

Figure 3.3.: A graph consisting of pairwise unconnected triangles. Each triangle needs its own edge guard.

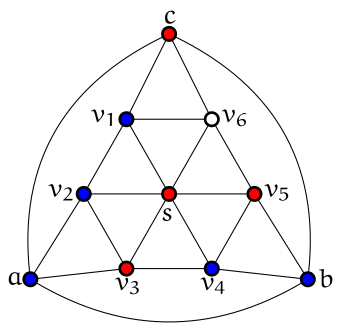

Figure 3.4.: A plane graph that does not admit a guard coloring. Assume that we want to find a guard coloring in red and blue. Face (a, b, c) must not be monochromatic, so without loss of generality we choose a and b to be blue while c is red. Then either v_1 or v_6 needs to be blue. The graph is symmetric, so we choose v_1 to be blue. Then v_2 must be blue as well to have a monochromatic edge incident to face (a, v_2, v_1, c). In the next step, we color v_3 and s red, because otherwise we would get monochromatic faces. For the same reason v_4 then becomes blue and v_5 becomes red. This leaves only v_6 uncolored. It cannot be red, as triangle (s, v_5, v_6) would become monochromatic, and it cannot be blue, as triangle (c, v_6, v_5, b) would have no monochromatic edge.

3.2.1. Iterative Guarding

The following was shown by Biniaz et al. [2].

Theorem 3.7 (Theorem 3 in [2]) $\Gamma_{\text{planar}}(n) \leqslant \lfloor 3n/8 \rfloor$ *for plane graphs with* $n \geqslant 3$ *vertices.*

To prove this, the authors use a technique they call *iterative guarding* that can be described as follows: For a graph $G_0 = (V_0, E_0)$ they start with an empty partial edge guard set $\Gamma_0 := \emptyset$ and iteratively extend it until it guards every face. In step $i \geqslant 1$ they choose two sets $V_i' \subseteq V_{i-1}$ and $E_i' \subseteq E_{i-1}$, such that E_i' guards all faces of G_{i-1} incident to vertices in V_i'. The edges in E_i' are added to the partial guard set, so $\Gamma_i := \Gamma_{i-1} \cup E_i'$. Then the vertices in V_i' as well as all incident edges are removed from the graph: $G_i := (V_i, E_i)$ with $V_i := V_{i-1} \setminus V_i'$ and $E_i := E_{i-1} \setminus \{vw \mid v \in V_i' \text{ or } w \in V_i'\}$. If the sets V_i' and E_i' are carefully chosen – in each step such that $|E_i'| \leqslant c|V_i'|$ for some constant c – this leads to a complete guard set Γ of size at most cn.

Biniaz et al. [2] use a theorem by Borodin [3] about local structures in planar graphs. It guarantees them at least one of eight configurations to appear, which they use to find sets V_i' and E_i', such that $|E_i'| \leqslant (3/8) \cdot |V_i'|$ leading to the desired bound.

3.2.2. Guarding by Coloring

We used guard colorings to show that $\lfloor n/3 \rfloor$ edge guards are always sufficient for triangulations but this method is much more powerful. In fact, Bose et al. [4] show that any plane graph without quadrilateral faces has a guard coloring and they further give an algorithm that can compute one in linear time without applying the 4-Color Theorem (for which no simple and constructive proof is known). However there are graphs that do not allow a guard coloring, one of them is shown in Figure 3.4 that was presented originally by Biniaz et al. [2].

Theorem 3.8 (Theorem 5 in [2]) *For every plane graph with* $n \geqslant 3$ *vertices and* α *quadrilateral faces* $\lfloor n/3 + \alpha/9 \rfloor$ *edge guards are sufficient.*

The proof of Theorem 3.8 is very similar to the proof of Theorem 3.6. The authors describe how to triangulate a plane graph G in such a way, that each non-quadrilateral face has vertices from at least three of the four color classes on its boundary. When grouping two of the four colors each into new colors A and B, we get a 2-coloring that fulfills the requirements of a guard coloring for all non-quadrilateral faces. There are three pairwise non-symmetric ways to define A and B ($A_1 := \{1, 2\}$, $A_2 := \{1, 3\}$, $A_3 := \{1, 4\}$ and $B_i := \{1, 2, 3, 4\} \setminus A_i$ for $i \in \{1, 2, 3\}$). Using the technique form Lemma 3.5 this leads to nine guard sets with a total size of $3n$ (that guard all non-quadrilateral faces). Now their key observation is that any quadrilateral face is always guarded by at least eight of the nine guard sets. So for each such face only one edge needs to be added to only one guard set. We get nine guard sets guarding all faces each and they haves a total size of $3n + \alpha$. The smallest one must be of size at most $\lfloor n/3 + \alpha/9 \rfloor$.

4. Stacked Triangulations

In the previous section we saw that n-vertex triangulations can be guarded by $\lfloor n/3 \rfloor$ edge guards. However, no family of triangulations is known for which this number of edge guards is actually necessary. This chapter considers a subclass of triangulations, the so called stacked triangulations and develops bounds for $\Gamma_{\triangle,\text{stacked}}(n)$, the minimal number of edge guards needed to guard any n-vertex stacked triangulation. They have a hierarchical structure which allows us to prove that $\Gamma_{\triangle,\text{stacked}}(n) \sim 2n/7$ by giving a lower and an upper bound.

First we define what stacked triangulations are and explore their hierarchical structure. Then we present a construction for stacked triangulations needing many edge guards for the lower bound in Section 4.1. This chapter finishes with an inductive proof for the upper bound in Section 4.2.

Definition 4.1 (Stacked Triangulations) *A plane triangulation is a* stacked triangulation *if it can be formed by the following recursive process:*

- *A triangle is a stacked triangulation.*
- *Let* $G = (V, E)$ *be a stacked triangulation and* f *be an inner face with* $\partial f = \{x, y, z\}$*. Then the graph* $G' = (V \cup \{v\}, E \cup \{xv, yv, zv\})$ *formed by adding a new vertex* v *into* f *subdividing it into three smaller triangles is also a stacked triangulation.*

The stacked triangulations are also known as *Apollonian Networks*. Our definition immediately shows that they are exactly the planar 3-trees. Further it can be shown that the class is equivalent to the maximal planar chordal graphs [17] and to the uniquely 4-colorable planar graphs [14].

Figure 4.1 shows two graphs: Subfigure (a) shows a stacked triangulation and the vertices are numbered in the order they could be added according to Definition 4.1. Subfigure (b) shows the octahedron graph, which is not a stacked triangulation. We can give two different reasons for this: Firstly, none of the three inner vertices is connected with all three outer vertices. Secondly, the last added vertex must be of degree three, but the octahedron graph does not contain such a vertex.

Defining a stacked triangulation $G = (V, E)$ as a sequence of face subdivisions as in Definition 4.1 gives it a hierarchical structure. Number the vertices $v_1, v_2, \ldots, v_{|V|}$ in the order in which they were added. Note that v_1, v_2 and v_3 form the outer face.

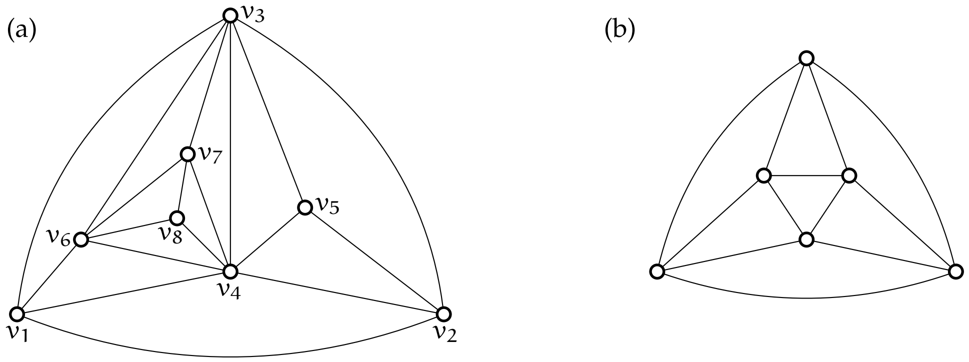

Figure 4.1.: (a) A stacked triangulation. The vertices are numbered in the order they were added. (b) The octahedron graph is not a stacked triangulation, because it does not have any vertex of degree 3.

Definition 4.2 (Level, Parent) *The function* level : $V \to \mathbb{N}$ *assigns each vertex an integer level giving its depth in the subdivision hierarchy:*

$$\text{level}(v_i) \mapsto \begin{cases} 0 & \textit{for } 1 \leqslant i \leqslant 3 \\ \max\limits_{\substack{j<i \\ v_j v_i \in E}} \text{level}(v_j) + 1 & \textit{otherwise} \end{cases}$$

Further the function parent : $V \to V$ *assigns each vertex to its unique neighbor of the previous level (exceptions for the first four vertices):*

$$\text{parent}(v_i) \mapsto \begin{cases} v_1 & \textit{for } 1 \leqslant i \leqslant 4 \\ v_j & \textit{such that } v_j v_i \in E \textit{ and } \text{level}(v_j) + 1 = \text{level}(v_i) \end{cases}$$

The parent function defines an implicit tree T on the vertices. By $S(v)$ we denote the subtree of $v \in V$ in T including v itself. If we do not want v to be included, we define $S^\circ(v) := S(v) \setminus \{v\}$. A vertex that has $k \in \{0, 1, 2, 3\}$ children in T is called a k-*vertex*.

Definition 4.3 (3-Wheel) *Let* $G = (V, E)$ *be a stacked triangulation. A* 3-wheel *is a triangle* (x, y, z) *containing exactly one vertex* v *inside it, subdividing it into three triangular faces.*

4.1. Lower Bound

To construct a stacked triangulation needing many edge guards, we apply the same technique as we did for general triangulations in Theorem 3.1. We start with a skeleton graph S – itself a stacked triangulation – and further subdivide each face. Then we analyze how many edge guards are necessary for the resulting graph.

Theorem 4.4 *For stacked triangulations we have* $\lfloor (2n-4)/7 \rfloor \leqslant \Gamma_{\triangle,\text{stacked}}(n)$.

Proof Start with a stacked triangulation S with v vertices. By Euler's Theorem it has $2v-4$ triangular faces. Into each face f with $\partial f = \{x, y, z\}$ we add as in Figure 4.2:

1. Vertex a_f and edges $a_f x$, $a_f y$ and $a_f z$ into face f.

2. Vertex b_f and edges $b_f a_f$, $b_f y$ and $b_f z$ into face (a_f, y, z).

3. Vertex c_f and edges $c_f a_f$, $c_f b_f$ and $c_f z$ into face (a_f, b_f, z).

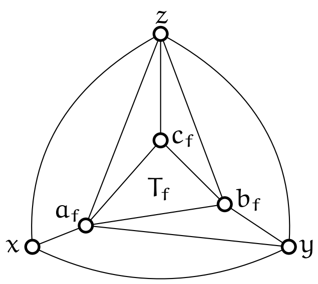

Figure 4.2.: Face f of S with boundary vertices $\partial f = \{x, y, z\}$ gets subdivided by the three vertices a_f, b_f and c_f, which form a face in G.

Call the resulting graph G. It is still a stacked triangulation, which is easy to see by the order in which we added a_f, b_f and c_f. Vertices (a_f, b_f, c_f) form a triangle T_f and no edge from any edge guard set Γ of G guarding T_f can also guard triangle $T_g = (a_g, b_g, c_g)$ in another face g of S, because they are at least 2-hop apart.

The number n of vertices of G is given by $n = (2v - 4) \cdot 3 + v = 7v - 12$. Isolating v we get $v = (n + 12)/7$. For any edge guard set Γ of G we now have:

$$\begin{aligned}
\Gamma_{\triangle,\text{stacked}}(n) &\geqslant |\Gamma| \\
&\geqslant 2v - 4 && \text{(at least one edge per face of S)} \\
&= \frac{2n - 4}{7} && \text{(substituting } v\text{)}
\end{aligned}$$

■

4.2. Upper Bound

For a stacked triangulation G we want to use induction on the number of vertices, so we need to create a smaller stacked triangulation G′. We can apply the induction hypothesis on G′ to find an edge guard set Γ′ and this edge guard set can then be turned into an edge guard set Γ for G. We guarantee that $(|\Gamma| - |\Gamma'|)/(|G| - |G'|) \leqslant 2/7$ which allows us to find an edge guard set containing about $2n/7$ edge guards.

Before describing how to create G′ in which situation, we state several lemmas that can be grouped into two kinds.

1. We start with two *forcing lemmas*. They describe how to extend a stacked triangulation such that some of its vertices are forced to appear in any minimum cardinality edge guard set.

2. Next we construct edge guard sets for several small stacked triangulations that appear as induced subgraphs in G′. Applying these lemmas allows us to augment Γ′ to an edge guard set Γ with only one or two additional edges.

Lemma 4.5 (Weak Forcing Lemma) *Let* f *be a face of a stacked triangulation* G *with boundary vertices* $\partial f = \{x, y, z\}$. *By subdividing* f *with two vertices* a *and* b *and edges* $\{xa, xb, ya, yb, za, ab\}$ *we get another stacked triangulation* H. *Figure 4.3 (a) shows how* f *is subdivided. Then an edge guard set* Γ *of* H *exists that has minimum cardinality among all edge guard sets for* H *and for which* $x, y \in V(\Gamma)$.

Proof Let Γ be any edge guard set of minimum cardinality for H. If both $x, y \in V(\Gamma)$, we are done. So from now on, at most one of them is in $V(\Gamma)$. Consider the triangular face (x, y, b). At least one of its three boundary vertices must be in $V(\Gamma)$.

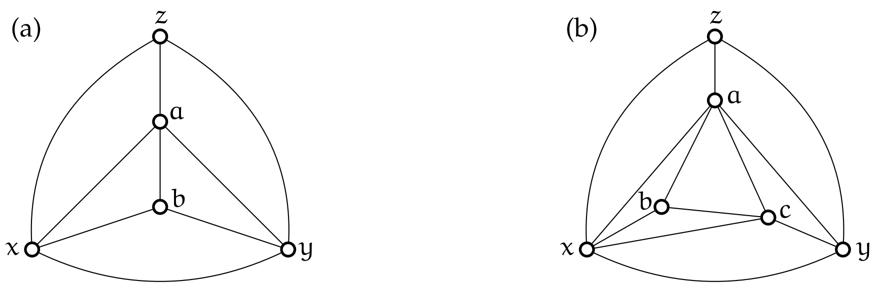

Figure 4.3.: (a) The face $f = (x, y, z)$ subdivided by vertices a and b for the Weak Forcing Lemma. (b) The face $f = (x, y, z)$ subdivided by vertices a, b and c for the Strong Forcing Lemma.

Case 1: $b \in V(\Gamma)$:
Then there must be an edge $vb \in \Gamma$ for some $v \in V(H)$. We can set $\tilde{\Gamma} := (\Gamma \setminus \{vb\}) \cup \{xy\}$ to get an edge guard set of the same size with $x, y \in V(\tilde{\Gamma})$. This is possible, because for every neighbor v of b, edge xy guards a superset of what vb guards.

Case 2: $b \notin V(\Gamma)$:
Without loss of generality assume that $x \in V(\Gamma)$ (and neither b nor y). But then face (a, b, y) can only be guarded through a and vertex a must be part of an edge wa for some $w \in V(H)$. But every neighbor of a is also a neighbor of y, so we can set $\tilde{\Gamma} := (\Gamma \setminus \{wa\}) \cup \{wy\}$ to get an edge guard set of the same size with $x, y \in V(\tilde{\Gamma})$. ∎

The Weak Forcing Lemma allows us (at the cost of adding two vertices) to assume that an edge guard set Γ' given by applying the induction hypothesis contains a given edge e (or at least that $V(e) \subseteq V(\Gamma')$). This is important, because not any Γ' might be augmentable to an edge guard set Γ for G not needing too many edges. Next we see how to enforce an even stronger property of Γ' by adding three vertices into a face.

Lemma 4.6 (Strong Forcing Lemma) *Let f be a face of a stacked triangulation G with boundary $\partial f = \{x, y, z\}$. Create a stacked triangulation H by subdividing f with three vertices $\{a, b, c\}$ and edges $\{xa, xb, xc, ya, yc, za, ab, ac, bc\}$. This is shown in Figure 4.3 (b). Then an edge guard set Γ of H exists that has minimum cardinality among all edge guard sets of H and for which $x \in V(\Gamma)$ and $|\{xa, xb, xc, ya, yc, za\} \cap \Gamma| \geqslant 1$.*

Proof Let Γ be any edge guard set of minimum cardinality of H. Assume that Γ does not yet fulfill the requirements.

Case 1: $\exists uv \in \Gamma$ with $u, v \in \{a, b, c\}$:
We define an edge guard set $\tilde{\Gamma} := (\Gamma \setminus \{uv\}) \cup \{xa\}$ which fulfills the requirements. This is possible, because edge xa guards a superset of the faces that uv guards.

Case 2: otherwise:
We can deduce several things from the fact that Γ does not fulfill all requirements and that Case 1 does not apply: Firstly, there must be an edge uv guarding face (a, b, c) with $u \in \{y, z\}$ and $v \in \{a, b, c\}$. Secondly, we follow that $x \notin V(\Gamma)$ which further leads to $b \notin V(\Gamma)$. This leaves vertex c to be the only possible vertex to guard face (x, c, b), and vertex a to be the only possible vertex to guard face (x, b, a). Thus there are two edges $av_1, cv_2 \in \Gamma$ with $v_1, v_2 \in \{y, z\}$. If $v_1 = v_2 = y$, we define $\tilde{\Gamma} := (\Gamma \setminus \{ay\}) \cup \{ax\}$. Otherwise we have $v_1 = z$ and $v_2 = y$ and we define $\tilde{\Gamma} := (\Gamma \setminus \{cy\}) \cup \{xy\}$.

So in both cases either Γ already fulfilled the requirements or we could locally change it to an edge guard set $\tilde{\Gamma}$ of the same size that does so. ∎

Lemma 4.7 (3-Wheel Lemma) *Let $G = (V, E)$ be a stacked triangulation and let (x, y, z) be a triangle with exactly one vertex v inside it, subdividing it into three triangular faces (so these four vertices form a 3-wheel). For any edge guard set Γ of G, we have $|\{x, y, z, v\} \cap V(\Gamma)| \geqslant 2$.*

Proof Let Γ be an edge guard set for G. Vertex v is the only vertex guarding all three faces (x, y, v), (z, y, v) and (x, v, z), but $V(\Gamma) \cap \{x, y, z, v\} = \{v\}$ is impossible, because v must be part of an edge vw with $w \in \{x, y, z\}$. Now assume without loss of generality that $x \in V(\Gamma)$. Then face (v, y, z) is not guarded, so it must be $V(\Gamma) \cap \{v, y, z\} \neq \emptyset$. ■

Let H be a 3-wheel with x, y, z and v as described above in a graph G with an edge guard set Γ. We can use the 3-Wheel Lemma and one additional edge e to make sure that $\{x, y, z, v\} \subseteq V(\Gamma \cup \{e\})$. This then guards the three faces (x, y, v), (z, y, v) and (x, v, z) each at all three boundary vertices, which is a property that we further explore in Lemma 4.8.

Lemma 4.8 (Doubly/Triply Guarded Faces) *Let G be a stacked triangulation, $f := (x, y, z)$ be one of its faces and Γ be an edge guard set.*

1. *If $|\partial f \cap V(\Gamma)| = 2$, we call f* doubly guarded. *Then Γ is also an edge guard set for the stacked triangulation H obtained from G by subdividing f with a single vertex v into three smaller triangular faces.*
2. *If $|\partial f \cap V(\Gamma)| = 3$, we say that f is* triply guarded. *Then Γ is also an edge guard set for every H that is obtained by subdividing f with a single vertex v into three smaller triangular faces and optionally subdividing each of them one more time into even smaller triangular faces.*

Proof For part 1 assume that $x, y \in V(\Gamma)$ (all other cases are symmetric). Each of the triangular faces (x, y, v), (z, y, v) and (x, v, z) has at least one boundary vertex in $\{x, y\}$, so they are all guarded by Γ. For part 2 note that after subdividing with vertex v, each of the three new triangular faces (x, y, v), (z, y, v) and (x, v, z) is doubly guarded by Γ. Therefore part 1 can be applied to show that even another subdivision of each of them would still be guarded by Γ. ■

Now we have enough knowledge about stacked triangulations that we can construct edge guard sets for stacked triangulations with $n \in \{6, 7, 8, 9, 10\}$ vertices. To find an edge guard set for a bigger stacked triangulation G, we find copies of these small instances in it and use one of the following lemmas to obtain an edge guard set for G.

Lemma 4.9 *Let G be a stacked triangulation with $n = 6$ vertices. Then:*

1. *G can be guarded by a single (and unique) edge guard.*
2. *If an outer vertex x is given as a vertex guard, then the remaining faces of G can be guarded with one additional edge guard $e = vw$, where $w \neq x$ is another outer vertex.*

Proof There is only a single stacked triangulation G with $n = 6$ vertices. To prove Statement 1, Figure 4.4 shows its three different planar embeddings[1]. In each case, only the red edge uv guards all faces, where we always name the vertices such that u is an outer vertex.

For Statement 2 first assume that $x \neq u$. Then edge uv can be used together with x and u is a different outer vertex. If on the other hand $x = u$, edge vw guards the remaining faces of G, where $w \neq x$ can be any other outer vertex. In Figure 4.4 we see, that v is adjacent to all three outer vertices (or is one of them and adjacent to the other two), so this is always possible. ■

[1] Considering different planar embeddings is not necessary to prove this lemma. However, all three can appear as subgraphs inside bigger stacked triangulations and the endpoints u and v of the unique edge guard have different levels depending on the embedding.

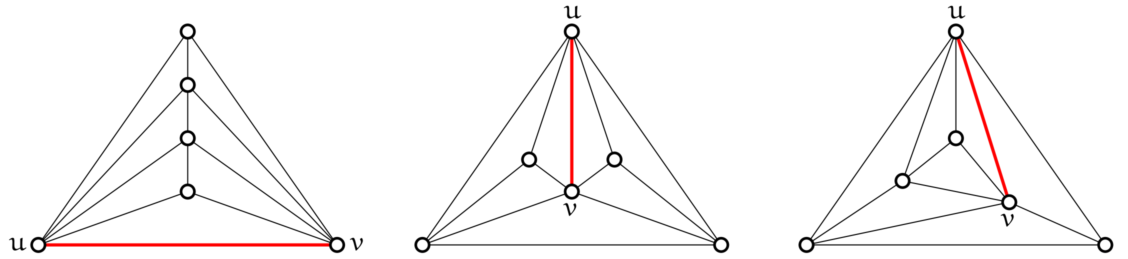

Figure 4.4.: The unique stacked triangulation with six vertices. There are three possible planar embeddings.

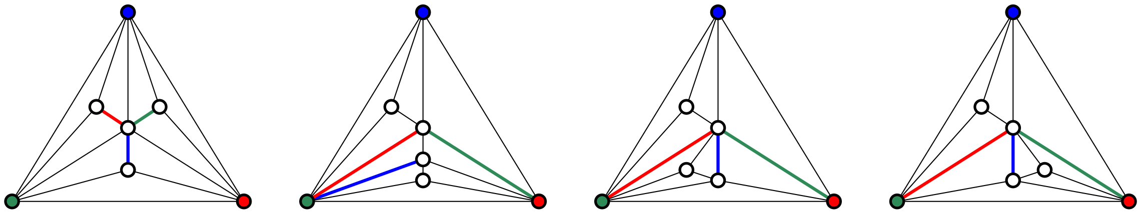

Figure 4.5.: The four possible stacked triangulations with seven vertices that are not formed from a stacked triangulation with six vertices and an additional outer vertex. The colors encode how to form an edge guard set using one additional edge: If for example the green outer vertex is a vertex guard, the thick green edge can be used to guard the remaining faces.

Lemma 4.10 *Let* G *be a stacked triangulation with* $n = 7$ *vertices and outer vertices* x, y *and* z. *Further let* v *be the unique other vertex that is adjacent to all three outer vertices. If any of the three outer vertices is given as a vertex guard, the rest of* G *can be guarded with one additional edge guard.*

Proof We start by considering the case that one of the three outer vertices (without loss of generality z) can be removed to get a stacked triangulation $\triangle_6$ with six vertices. If z was given as a vertex guard, we could just guard $\triangle_6$ with one additional edge by Lemma 4.9. Otherwise, if $w \in \{x, y\}$ was given as a vertex guard, we can form a guard set Γ for $\triangle_6$ with an edge containing a vertex from $\{x, y, v\} \setminus \{w\}$ by Lemma 4.9. Then both $\{x, v, z\} \cap V(\Gamma) \neq \emptyset$ and $\{y, z, v\} \cap V(\Gamma) \neq \emptyset$, so all faces not belonging to $\triangle_6$ are also guarded.

Now assume that it is impossible to remove one outer vertex to get a stacked triangulation. There are only four such stacked triangulations, all shown in Figure 4.5. For each of them we give three guard sets in different colors: They all consist of one vertex guard at an outer vertex and one edge guard. ■

The two above lemmas can be seen as base cases for six and seven vertices. Now we state three lemmas that handle stacked triangulations that are particularly hard cases from the proof of Theorem 4.14. Treating them in isolation here, allows us to formulate a cleaner proof later on.

Lemma 4.11 *Let* $G = (V, E)$ *be a stacked triangulation with* $n = 8$ *vertices and outer face* (x, y, z) *such that the following configuration from Figure 4.6 (a) applies:*

- *Vertex* $v \in V$ *is the unique vertex adjacent to all* x, y *and* z.
- *Triangle* (x, y, v) *and all vertices inside it form a stacked triangulation* $\triangle_6$ *with six vertices.*
- *Triangle* (x, v, z) *and all vertices inside it form a stacked triangulation* $\triangle_4$ *with four vertices.*
- *Triangle* (v, y, z) *is a face* $\triangle_3$.

Then any edge guard set Γ' for the subgraph G' induced by $\{x, y, z, v\} \subseteq V$ can be extended by one edge e to an edge guard set for G.

Proof By the 3-Wheel Lemma 4.7 we have $|\{x, y, z, v\} \cap V(\Gamma')| \geqslant 2$. Face $\triangle_3$ is then already guarded. If we even have $|\{x, z, v\} \cap V(\Gamma')| \geqslant 2$, then triangle (x, v, z) is doubly guarded and therefore $\triangle_4$ is guarded by Lemma 4.8. In this case set e to be the unique edge guarding $\triangle_6$, which exists by Lemma 4.9.

If otherwise $|\{x, z, v\} \cap V(\Gamma')| = 1$, then we have $y \in V(\Gamma')$, thus $\triangle_3$ is guarded. By Lemma 4.9 an edge $e = ab$ exists, such that $\triangle_6$ is completely guarded by e and y and further $a \in \{x, v\}$. If $a \notin V(\Gamma')$, then it is a second guarded outer vertex of $\triangle_4$, such that it is guarded by Lemma 4.8. Otherwise if $a \in V(\Gamma')$, then edge $e' = a'b$ can be used instead, where $a' \neq a$ is the other vertex from $\{x, v\}$. ∎

Lemma 4.12 *Let $G = (V, E)$ be a stacked triangulation with $n = 9$ vertices and outer face (x, y, z) such that the following configuration from Figure 4.6 (b) applies:*

- *Vertex $v \in V$ is the unique vertex adjacent to all x, y and z.*
- *Triangle (x, y, v) and all vertices inside it form a stacked triangulation $\triangle_6$ with six vertices.*
- *Triangle (x, v, z) and all vertices inside it form a stacked triangulation $\triangle_5$ with five vertices.*
- *Triangle (v, y, z) is a face $\triangle_3$.*

Then we can create a new stacked triangulation G' with $n' = 5$ vertices, such that any edge guard set Γ' for G' can be augmented to an edge guard set Γ for G with $|\Gamma| = |\Gamma'| + 1$. Further Γ is constructed to doubly guard $\triangle_3$.

Proof We create G' by removing all inner vertices and adding two vertices a and b as in the Weak Forcing Lemma 4.5. This allows us to force two adjacent vertices of $\{x, y, z\}$ to be in $V(\Gamma')$. Note that a stacked triangulation with five vertices can always be guarded with a single outer edge. Depending on which edge from $\{xz, xv, vz\}$ guards $\triangle_5$ we force different outer vertices. Let $e = uw$ be the unique edge guarding $\triangle_6$ such that $u \in \{v, x, y\}$ is an outer vertex of $\triangle_6$.

Case 1: xz guards $\triangle_5$:

Place a and b such that $x, z \in V(\Gamma')$. Then $\triangle_5$ and face $\triangle_3$ are guarded by the forced vertices. If edge e contains a vertex from $\{v, y\}$, we can use it additionally to guard the faces of $\triangle_6$ and to get another outer vertex of $\triangle_3$. Otherwise Lemma 4.9 allows us to change e to an edge e', such that it guards $\triangle_6$ and contains a second outer vertex of $\triangle_3$.

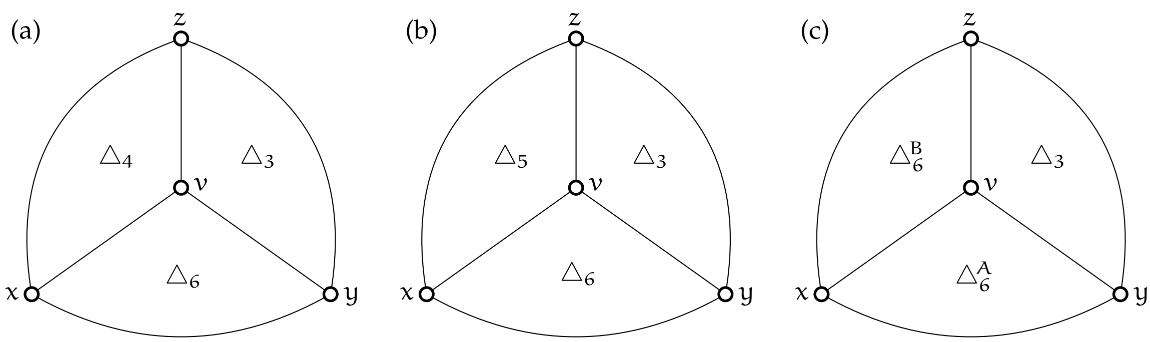

Figure 4.6.: (a) Stacked triangulation as in Lemma 4.11. (b) Stacked triangulation as in Lemma 4.12. (c) Stacked triangulation as in Lemma 4.13.

Case 2: xv guards $\triangle_5$:
Place a and b such that $x, y \in V(\Gamma')$. Then face $\triangle_3$ is already guarded. If $u = v$, set $\Gamma := \Gamma' \cup \{e\}$ to guard $\triangle_5$ and $\triangle_6$ and to doubly guard $\triangle_3$. Otherwise we can change e to $e' := vw$ by Lemma 4.9 and use $\Gamma := \Gamma' \cup \{e'\}$ instead.

Case 3: vz guards $\triangle_5$:
In any of the following possibilities we get $v, z \in V(\Gamma)$, so $\triangle_3$ is doubly guarded. If $u = v$, we can place a and b to force $y, z \in V(\Gamma')$. Then $\Gamma := \Gamma' \cup \{e\}$ guards $\triangle_6$ and $\triangle_5$. Otherwise, place a and b so that $u, z \in V(\Gamma')$ and set $e' := vw$ by Lemma 4.9. Edge guard set $\Gamma := \Gamma' \cup \{e'\}$ fulfills the requirements. ■

Lemma 4.13 *Let* $G = (V, E)$ *be a stacked triangulation with* $n = 10$ *vertices and outer face* (x, y, z) *such that the following configuration from Figure 4.6 (c) applies:*

- *Vertex* $v \in V$ *is the unique vertex adjacent to all* x, y *and* z.
- *Triangle* (x, y, v) *and all vertices inside it form a stacked triangulation* $\triangle_6^A$ *with six vertices.*
- *Triangle* (x, v, z) *and all vertices inside it form a stacked triangulation* $\triangle_6^B$ *with six vertices.*
- *Triangle* (v, y, z) *is a face* $\triangle_3$.

Then we can create a new stacked triangulation G' *with* $n' = 6$ *vertices, such that any edge guard set* Γ' *for* G' *can be used to construct an edge guard set* Γ *for* G *with* $|\Gamma| = |\Gamma'| + 1$.

Proof We create G' by removing all inner vertices and adding three vertices a, b and c as in the Strong Forcing Lemma 4.6. This allows us to force one vertex of $\{x, y, z\}$ to be in $V(\Gamma')$. Further we have an edge $e = uw \in \Gamma'$, such that $u \in \{x, y, z\}$ and $w \in \{a, b, c\}$ which we can change to any $uw' \in E$.

First we name two special edges: Edge $u_A w_A$ is the unique edge guarding $\triangle_6^A$ and $u_B w_B$ is the unique edge guarding $\triangle_6^B$ by Lemma 4.9. Let the naming be so that u_A and u_B are incident to the outer face of $\triangle_6^A$ and $\triangle_6^B$, respectively. Depending on which vertices of G the vertices u_A and u_B correspond to, we apply the Strong Forcing Lemma differently. In any case we then show which edge uw' to use instead of edge uw and which other edge to add. To avoid further case distinction we might end up with edges where both endpoints coincide. We can then either interpret those as vertex guards or use an arbitrary neighbor in G to form a real edge.

Case 1: $x = u_A = u_B$:
Force vertex x. If $u = x$, use edge $uw' = xw_A$ and additionale edge $u_B w_B$. If otherwise $u \neq x$, it must be either $u = y$ or $u = z$. Without loss of generality we assume $u = y$ (the other case $u = z$ works symmetrically). Then use $uw' = yw_A$ and additional edge $u_B w_B$.

Case 2: $x = u_A \neq u_B$: ($x = u_B \neq u_A$ can be handled symmetrically)
Force vertex x. Now if $u = x$, use edge $uw' = xw_A$ and additional edge $u_B w_B$. If $u = y$, use $uw' = yw_A$ and additional edge $u_B w_B$. If otherwise $u = z$, use $uw' = zw_B$ and additional edge $w_A v$.

Case 3: $v = u_A = u_B$:
Force vertex x. Then if $u = x$ or $u = y$, use $uw' = uw_A$ and additional edge vw_B. If otherwise $u = z$, use $uw' = zw_B$ and additional edge vw_A.

Case 4: $y = u_A$: ($z = u_B$ can be handled symmetrically)
Force vertex y. If $u = y$ or $u = x$, use $uw' = uw_A$ and additional edge $u_B w_B$. If otherwise $u = z$, use $uw' = zw_B$ and additional edge $w_A v$. This always works, because $u_B \in \{v, z\}$ since otherwise Case 1 applies.

In all cases $\{u_A, w_A, u_B, w_B\} \subseteq V(\Gamma)$, so all faces of $\triangle_6^A$ and $\triangle_6^B$ are guarded. If face $\triangle_3$ is also guarded by Γ, we are done, so assume it is not. Then vertex x must appear twice inside $\{u_A, w_A, u_B, w_B\}$, because both $u_A w_A$ and $u_B w_B$ contain an outer vertex of $\triangle_6^A$ and $\triangle_6^B$, respectively. Without loss of generality, assume that $u_A = u_B = x$. Then we can change the edge containing u_B to end at v instead. ∎

We now have everything we need to prove the main theorem of this section giving an upper bound for the number of edge guards that are always sufficient for stacked triangulations.

Theorem 4.14 *For stacked triangulations we have* $\Gamma_{\triangle,\text{stacked}}(n) \leqslant \lfloor 2n/7 \rfloor$.

Proof Let $G = (V, E)$ be a stacked triangulation with n vertices. We give an inductive proof. As the induction base we first note that if G has $n \leqslant 6$ vertices, it can be guarded with a single edge by Lemma 4.9. So from now on assume $n > 6$. The induction hypothesis is that every plane stacked triangulation with $n' < n$ vertices can be guarded by $\lfloor 2n'/7 \rfloor$ edge guards.

We describe how to construct a new graph G' with $n - k$ vertices for wich we use the induction hypothesis to find an edge guard set Γ' of size $\lfloor 2(n-k)/7 \rfloor$. Here $k := |V^-| - |V^+|$ where $V^- \subseteq V$ is a subset of vertices removed from G and V^+ is a set of new vertices added to it. We then show, that we can augment Γ' to an edge guard set Γ for G with $|\Gamma| = |\Gamma'| + l$. By guaranteeing that $l/k \leqslant 2/7$ we then get $|\Gamma| \leqslant \lfloor 2n/7 \rfloor$.

Let T be the tree that is defined by G's parent function. We start with any vertex $w \in V$ that is a leaf in T of maximum level and define $x := \text{parent}(w)$, $y := \text{parent}(x)$ and $z := \text{parent}(y)$.

Case 1: x is a 3-vertex:
Set $V^- := S(x)$, that is x and its three children in T. They all lie inside the same face f of G' and at least one boundary vertex in ∂f is in $V(\Gamma')$. The vertices in $V^- \cup \partial f$ induce a seven-vertex stacked triangulation, so by Lemma 4.10 we then need one additional edge to guard G.

Case 2: x is a 2-vertex:
Then $|S(x)|$ contains only three vertices, so we need to step up to its parent y. Note that y cannot be incident to the outer face of G, because otherwise G would have only six vertices.

Case 2.1: y is a 1-vertex:
Set $V^- := S(y)$ to be all four vertices in y's subtree. As above they lie in the same face of G' which is guarded by Γ'. By Lemma 4.10 one additional edge suffices.

Case 2.2: y is a 2- or 3-vertex:
Now $|S^\circ(y)| \geqslant 4$, so we can set $V^- := S^\circ(y)$. Then G' contains a 3-wheel with y as its center. By the 3-Wheel Lemma 4.7 we can use one additional edge e to have y and its three neighbors in G' all in $V(\Gamma' \cup \{e\})$. Then each face incident to y in G' is triply guarded. Because w was a vertex of maximum level, by Lemma 4.8 this then guards all faces of G.

Case 3: x is a 1-vertex:
In this case w is the only child of x in T. We step up to its parent y, which again cannot be incident to the outer face, because otherwise G would have only five vertices.

Case 3.1: y is a 3-vertex:
With the same reasoning as in Case 2.2, we can set $V^- := S^\circ(y)$ and use the 3-Wheel Lemma.

Case 3.2: y is a 2-vertex:
If $|S(y)| = 4$, we set $V^- := S(y)$. Then V^- lies inside a single face f of G' and $V^- \cup \partial f$ induces a stacked triangulation with seven vertices, so we can use Lemma 4.10 to only need one additional edge. If otherwise $|S(y)| \geqslant 5$, set $V^- := S^\circ(y)$ and use the 3-Wheel Lemma as in Case 2.2.

Case 3.3: y is a 1-vertex:
We have $|S(y)| = 3$, so we need to step up to y's parent z. The first thing to note is, that for any child y' of z, we have $|S(y')| \leqslant 3$, otherwise one of the earlier cases applies for that subtree. Therefore $|S(z)| \leqslant 10$. Additionally note that z cannot be incident to the outer face, because otherwise G would contain only six vertices.

Case 3.3.1: z is a 1-vertex:
We can set $V^- := \{w, x, y, z\}$ and they all lie in the same face f of G'. Again, $V^- \cup \partial f$ induces a stacked triangulation with seven vertices, so we can use Lemma 4.10 to use just one additional edge.

Case 3.3.2: z is a 2-vertex:
We need to distinguish how big $|S(z)|$ is. For this let $y' \neq y$ be the other child of z in T .

If y' is a leaf in T, we can set $V^- := \{w, x, y, y'\}$. Lemma 4.11 describes how the 3-Wheel Lemma can be applied then so that one additional edge suffices.

If $|S(y')| = 2$, then y' is a 1-vertex and has exactly one vertex x' below it in T. We can set $V^- := \{w, x, y, z, x', y'\}$ and $V^+ := \{a, b\}$, where a and b are added to use the Weak Forcing Lemma. Lemma 4.12 describes how a and b need to be inserted into G, so that one additional edge suffices.

If $|S(y')| = 3$, we set $V^- := S(z)$ and $V^+ := \{a, b, c\}$, where a, b and c are added to use the Strong Forcing Lemma. Lemma 4.13 describes in detail, how one additional edge suffices.

Case 3.3.3: z is a 3-vertex:
Again we need to distinguish how big $|S(z)|$ is. For this let $y', y'' \neq y$ be the two other children of z in T.

If without loss of generality $|S(y')| = 3$, remove $S(y')$ from G. We can then solve the remaining graph as in Case 3.3.2. Using one additional edge we can then guard all still unguarded faces incident to a vertex from $S(y')$. This is possible, because in Case 3.3.2 we removed at least four vertices and needed only one additional edge. Combined with $S(y')$ we made G' at least $k = 7$ vertices smaller than G and needed $l = 2$ additional edges, so we end up with a ratio of $l/k \leqslant 2/7$.

If $|S(y')| = 2$ and $|S(y'')| = 2$, set $V^- := S^\circ(z)$. This way we removed seven vertices and it remains a 3-wheel with z at its center. By the 3-Wheel Lemma 4.7 we can use one additional edge, to guard the whole 3-wheel and a second additional edge to guard all faces incident to vertices in $S(y)$. Since the whole 3-wheel is guarded, all faces of it are triply guarded and therefore by Lemma 4.8 all faces incident to vertices in $S(y')$ and $S(y'')$ are also guarded.

If otherwise without loss of generality $|S(y')| = 2$ and $|S(y'')| = 1$, remove $S(y'')$ from G. The remaining graph can then again be solved using Case 3.3.2. This needs Lemma 4.12, which is strong enough to doubly guard the triangle containing $S(y'')$. Since $|S(y'')| = 1$, all faces incident to the unique vertex in it are then guarded by Lemma 4.8.

In all remaining cases both y' and y'' are leaves in T. Let (v_1, v_2, v_3) be the triangle in G that contains $S(z)$ in it such that $\triangle := (v_1, v_2, z)$ is the triangle containing $S(y)$. Define $V^- := \{w, x, y, z, y', y''\}$ and $V^+ := \{a, b\}$. We use the Weak Forcing Lemma to place a and b, such that $v_1, v_2 \in V(\Gamma')$. Triangle $\triangle$ together with $S(y)$ induce a six-vertex stacked triangulation $\triangle_6$, so by Lemma 4.9 there is an edge e with $v \in V(e)$ that guards the remaining faces of $\triangle_6$. Then all faces inside $\triangle_6$ are guarded. Further, both triangles (v_1, v, v_3) and (v_2, v_3, v) are doubly guarded, so by Lemma 4.8 all faces incident to their only subdivision vertices y' and y'' are also guarded. ■

5. Quadrangulations

In this chapter we consider *quadrangulations* i.e. plane graphs where every face f is quadrangular. We develop bounds for $\Gamma_{\square}(n)$ which is the smallest number of edge guards needed for any n-vertex quadrangulation. These quadrangulations are interesting for different reasons: Firstly, they are exactly the maximal planar bipartite graphs. Secondly, faces of degree four seem to be problematic for edge guarding strategies: Biniaz et al. [2] showed that $\lfloor n/3 + \alpha/9 \rfloor$ edge guards suffice for general plane graphs on n vertices where α is the number of their quadrangular faces. In a quadrangulation with n vertices we have $\alpha = n - 2$, so that until now the best known upper bound for large quadrangulations is $\lfloor 3n/8 \rfloor$ also shown in [2]. This bound is valid for all plane graphs, therefore not using any of the structure a quadrangulation has.

We start this chapter with a construction in Theorem 5.1 showing that asymptotically $n/4$ edge guards can be necessary. Then we prove an upper bound of $\lfloor n/3 \rfloor$ edge guards in Theorem 5.2 using guard colorings that were introduced by Bose et al. [4] and also described here in Definition 3.4. The two bounds are not yet matching and how to further improve them stands as an open question. As a first step towards an answer, we conclude this chapter by looking at 2-degenerate quadrangulations. These form a subclass that has a similar structure to the stacked triangulations considered in Chapter 4 allowing us to prove an upper bound of $\lfloor n/4 \rfloor$. This is best possible, because the quadrangulations constructed in Theorem 5.1 are 2-degenerate.

5.1. Lower Bound

Via a simple construction we prove a lower bound for the number of edge guards that are sometimes necessary for quadrangulations. We construct such quadrangulations in a way that they can easily be extended by four more vertices so that they need one additional edge guard.

Theorem 5.1 *For any $k \in \mathbb{N}$ there is a quadrangulations Q_k with $n = 4k + 2$ vertices that needs k edge guards. Therefore $\lfloor (n-2)/4 \rfloor \leqslant \Gamma_{\square}(n)$.*

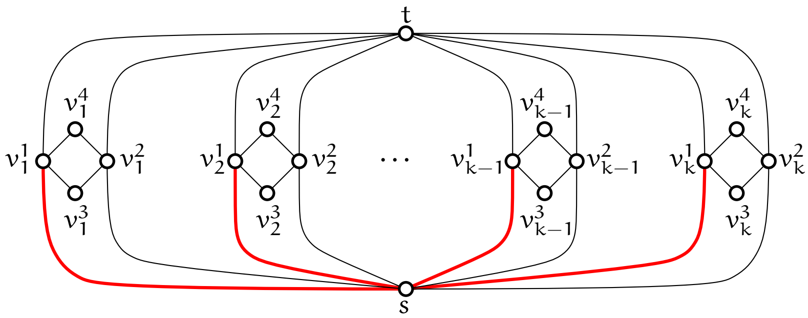

Figure 5.1.: A quadrangulation with $4k + 2$ vertices needing k edge guards. The thick, red edges show an edge guard set of minimum cardinality.

Proof For $k \in \mathbb{N}$ we construct a quadrangulation $Q_k = (V, E)$ with $n = 4k + 2$ vertices that needs k edge guards. Solving for k yields $k = (n - 2)/4$ as needed. Define

$$V := \{s, t\} \cup \bigcup_{i=1}^{k} \{v_i^1, v_i^2, v_i^3, v_i^4\} \quad \text{and}$$

$$E := \bigcup_{i=1}^{k} \{sv_i^1,\ sv_i^2,\ tv_i^1,\ tv_i^2,\ v_i^1 v_i^3,\ v_i^1 v_i^4,\ v_i^2 v_i^3,\ v_i^2 v_i^4\}\,.$$

Figure 5.1 shows this and a planar embedding. Now for any $1 \leqslant i, j \leqslant k$ with $i \neq j$ the two quadrilateral faces $(v_i^1, v_i^2, v_i^3, v_i^4)$ and $(v_j^1, v_j^2, v_j^3, v_j^4)$ are 2-hop apart. Therefore, no two of them can share an edge guard and we need at least k edge guards for Q_k. On the other hand it is easy to see that $\{sv_1^1, \ldots, sv_k^1\}$ is an edge guard set of size k, so Q_k needs exactly k edge guards. ∎

5.2. Upper Bound

For the upper bound we use guard colorings, a technique introduced by Bose et al. [4]. In earlier publications [2, 4, 5] this and similar coloring approaches were used to prove upper bounds of $\lfloor n/3 \rfloor$ edge guards for all n-vertex plane graphs not containing any quadrangular faces. They all however fail to construct guard colorings for graphs with such quadrangular faces. In this section we show how to find a guard coloring for plane graphs containing only quadrangular faces using a classical theorem by Petersen about 2-factors. We are explicitly not using the 4-Color Theorem here and our construction can easily be turned into an efficient algorithm.

Theorem 5.2 *It is* $\Gamma_\square(n) \leqslant \lfloor n/3 \rfloor$.

Proof Let $G = (V, E)$ be a quadrangulation with n vertices. We show that G has a guard coloring. By Lemma 3.5 it can then be guarded by $\lfloor n/3 \rfloor$ edge guards. Remember that a guard coloring is a 2-coloring of the vertex set, where no face is monochromatic but each face is incident to a monochromatic edge.

Consider the dual graph $G^* = (V^*, E^*)$ of G with its inherited plane embedding, so each vertex $f^* \in V(H)$ is placed inside the face f of G that it corresponds to. Since every face of G is of degree four, its dual graph G^* is 4-regular. Using Petersen's 2-Factor Theorem [23][1] we get that G^* contains a 2-factor H. Any vertex of H is of degree 2, so H is a set of

[1] Diestel [8, Corollary 2.1.5] gives a very short and elegant proof of this theorem in his book. He only considers simple graphs there, but all steps in the proof (including the given proof of Hall's Theorem [8, 24, Theorem 2.1.2]) also work for multigraphs like G^* that have at most two edges between any pair of vertices.

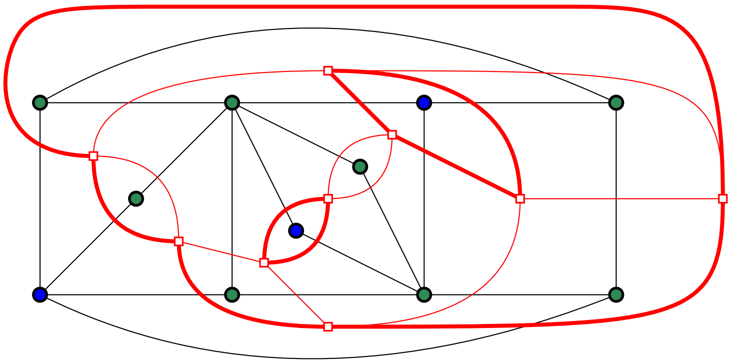

Figure 5.2.: A graph G (with black edges) and its dual graph G^* (red vertices and edges) with its inherited embedding. The thick red edges form a 2-factor H of G^* and subdivide the plane in which the vertices of G lie. Coloring the vertices in these regions in green and blue where touching regions get different colors yields a guard coloring.

vertex-disjoint cycles. These cycles subdivide the plane into a set of closed regions R that are nested inside each other. For each region $r \in R$ we define $\mathrm{depth}(r)$ to be the number of cycles that r lies in. The only unbounded region r_0 has $\mathrm{depth}(r_0) = 0$ and for each level of nesting the depth increases by one.

Now define a 2-coloring C for the vertices of G with the colors from $\{0, 1\}$: Every vertex $v \in V$ of G lies inside exactly one region $r \in R$. The color of v is determined by the parity of $\mathrm{depth}(r)$ as

$$C(v) := \mathrm{depth}(r) \mod 2.$$

We claim that this yields a guard coloring of G: Any edge $e = ab \in E$ has a corresponding dual edge e^*. If $e^* \in E(H)$, vertices a and b lie in different but touching regions $r_a, r_b \in R$, so $\mathrm{depth}(r_a) = \mathrm{depth}(r_b) \pm 1$ and therefore $C(a) \neq C(b)$. Otherwise $e \notin E(H)$, so its two endpoints are in the same region $r_{a,b} \in R$, therefore $C(a) = C(b)$ and e is monochromatic. Because H is a 2-factor, each face has exactly two monochromatic edges. ■

Figure 5.2 shows an example graph together with its dual and a 2-factor of it. These are then used to get the shown guard coloring. The upper bound of $\lfloor n/3 \rfloor$ presented here does not match the lower bound of $\lfloor (n-2)/4 \rfloor$ presented in the previous section. To bridge this gap we look at a subclass of quadrangulations in the next section for which we can show a stronger upper bound.

5.3. 2-Degenerate Quadrangulations

A 2-*degenerate quadrangulation* G is a quadrangulation where any subgraph has a vertex of degree 2. In this section we look at $\Gamma_{\square,\text{2-deg}}(n)$, the minimal number of edge guards needed to guard any face of an n-vertex 2-degenerate quadrangulation. The construction from Theorem 5.1 yields 2-degenerate quadrangulations, so we know that $\lfloor (n-2)/4 \rfloor \leqslant \Gamma_{\square,\text{2-deg}}(n)$. In the following we present a matching upper bound.

2-degenerate quadrangulations are very similar to the stacked triangulations considered in Chapter 4 and we might also call them *stacked quadrangulations*, because they can be constructed iteratively:

- A 4-cycle C_4 is a stacked quadrangulation.

- Let G be a stacked quadrangulation and f be an inner face of it with two non-adjacent boundary vertices x and y. Then the graph obtained by adding a new vertex v inside f and edges vx and vy gives another stacked quadrangulation.

As the similarity of the definitions might suggest, the strategy used to prove an upper bound is the same as for stacked triangulations. Before proving the theorem, we introduce some notation and state a few lemmas covering special cases. The conditions for these special cases might seem a bit technical at first but they allow us to formulate a short proof for Theorem 4.14.

Defining a 2-degenerate quadrangulation $G = (V, E)$ as a sequence of face subdivisions as described above gives it a hierarchical structure. Number the vertices $v_1, v_2, \ldots, v_{|V|}$ in the order in which they were added. The outer face is (v_1, v_2, v_3, v_4).

Definition 5.3 (Level) *The function* level : $V \to \mathbb{N}$ *assigns each vertex an integer level that is its depth in the subdivision hierarchy:*

$$\text{level}(v_i) \mapsto \begin{cases} 0 & \text{for } 1 \leqslant i \leqslant 4 \\ \max\limits_{\substack{j<i \\ v_i v_j \in E}} \text{level}(v_j) + 1 & \text{otherwise} \end{cases}$$

When a vertex v is added into a face $f := (w, x, y, z)$ it subdivides f into two new faces, a left one $f_L := (w, v, y, z)$ and a right one $f_R := (w, x, y, v)$. By $S^\circ_L(v)$ we denote the set of vertices that further subdivide f_L and symmetrically the vertices placed inside f_R are in $S^\circ_R(v)$. Combining them we define $S^\circ(v) := S^\circ_L(v) \cup S^\circ_R(v)$ and $S(v) := S^\circ(v) \cup \{v\}$.

Definition 5.4 (Height) *For a vertex v of a 2-degenerate quadrangulation, we further define the function* height : $\{S(v) \mid v \in V\} \to \mathbb{N}$ *as*

$$\text{height}(S(v)) \mapsto \left(\max_{w \in S(v)} \text{level}(w) \right) - \text{level}(v).$$

We now state three lemmas covering most of the cases that need to be considered for an upper bound. The proof of Theorem 5.9 then just combines them. Verifying the lemmas themselves is a lot of case work, so we skip their proofs here and give them in detail in Appendix B.

Definition 5.5 *Let $G = (V, E)$ be a 2-degenerate quadrangulation. We define the following two properties (that G can but does not need to satisfy):*

(P1) $\forall v \in V : \text{height}(S(v)) = 1 \implies |S(v)| \leqslant 3$

(P2) $\forall v \in V : \text{height}(S(v)) = 2 \implies |S(v)| = 3$

Lemma 5.6 *Let $G = (V, E)$ be a 2-degenerate quadrangulation satisfying (P1) and $v \in V$ be a vertex with $4 \leqslant |S(v)| \leqslant 7$ and $|S^\circ_L(v)|, |S^\circ_R(v)| \leqslant 3$. Then we can construct a 2-degenerate quadrangulation G' with $|G'| \leqslant |G| - 4$ and an edge guard set Γ' for it that can be augmented to an edge guard set Γ of G with $|\Gamma| = |\Gamma'| + 1$.*

Lemma 5.7 *Let $G = (V, E)$ be a 2-degenerate quadrangulation satisfying (P1) and $v \in V$ be a vertex with* $\text{height}(S(v)) = 2$ *and* $\max\{|S^\circ_L(v)|, |S^\circ_R(v)|\} \geqslant 4$. *Then we can construct a 2-degenerate quadrangulation G' with $|G'| \leqslant |G| - 4$ and an edge guard set Γ' for it that can be augmented to an edge guard set Γ of G with $|\Gamma| = |\Gamma'| + 1$.*

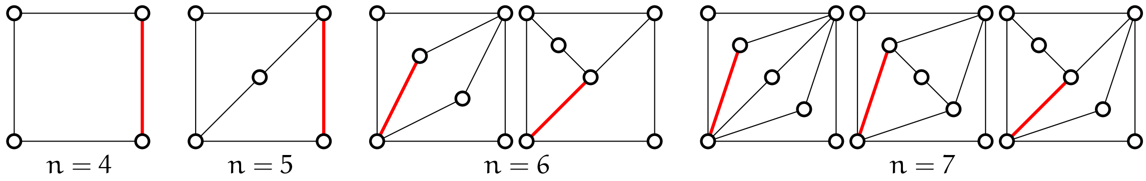

Figure 5.3.: All 2-degenerate quadrangulations with $4 \leqslant n \leqslant 7$ vertices. Each of them can be guarded by a single edge, drawn in red.

Lemma 5.8 *Let $G = (V, E)$ be a 2-degenerate quadrangulation satisfying (P1) and (P2) and $v \in V$ be a vertex with $\text{height}(S(v)) = 3$ and $\max\{|S_L^\circ(v)|, |S_R^\circ(v)|\} \geqslant 4$. Then we can construct a 2-degenerate quadrangulation G' with $|G'| \leqslant |G| - 4$ and an edge guard set Γ' for it that can be augmented to an edge guard set Γ of G with $|\Gamma| = |\Gamma'| + 1$.*

Theorem 5.9 *For 2-degenerate quadrangulations we have $\Gamma_{\square,\text{2-deg}}(n) \leqslant \lfloor n/4 \rfloor$.*

Proof Let $G = (V, E)$ be a 2-degenerate quadrangulation with n vertices. We give an inductive proof. For $4 \leqslant n \leqslant 7$ all possible 2-degenerate quadrangulations are shown in Figure 5.3. As shown, they can be guarded by a single edge guard. Now assume that $n \geqslant 8$. Our induction hypothesis is that for any 2-degenerate G' with $|G'| < n$ an edge guard set Γ' of size $|\Gamma'| = \lfloor |G'|/4 \rfloor$ exists. In the inductive step, we describe how to create G' with $|G'| \leqslant |G| - 4$, such that we can augment Γ' to an edge guard set Γ for G with $|\Gamma| = |\Gamma'| + 1 \leqslant \lfloor (n-4)/4 \rfloor + 1 = \lfloor n/4 \rfloor$.

Case 1: $\exists v \in V : \text{height}(S(v)) = 1$ and $|S(v)| \geqslant 4$:
Let G' be the induced subgraph $G' := G - S(v)$. Any edge guard set Γ' for G' can be extended to a guard set for G by a vertex guard at v, therefore we can set $\Gamma := \Gamma' \cup \{vw\}$ for an arbitrary neighbor w of v.

From now on we can assume that any vertex $v \in V$ with $\text{height}(S(v)) = 1$ has $|S(v)| \leqslant 3$, because otherwise we could use Case 1. This condition is exactly property (P1) that we stated above.

Case 2: $\exists v \in V : 4 \leqslant |S(v)| \leqslant 7$ and $|S_L^\circ(v)|, |S_R^\circ(v)| \leqslant 3$:
When this case applies, its conditions and property (P1) satisfy the requirements of Lemma 5.6. It describes how to find G' with $|G'| \leqslant |G| - 4$ and the induction hypothesis gives an edge guard set Γ' of size $\lfloor |G'|/4 \rfloor$. Using the lemma again, we can turn this into an edge guard set Γ for G of size $\lfloor n/4 \rfloor$.

Case 3: $\exists v \in V : \text{height}(S(v)) = 2$ and $\max\{|S_L^\circ(v)|, |S_R^\circ(v)|\} \geqslant 4$:
Together with property (P1) these conditions allow us to use Lemma 5.7. The same way as above we can construct an edge guard set for G.

If neither of the Cases 1, 2 nor 3 applied, then we can follow that any vertex $v \in V$ with $\text{height}(S(v)) = 2$ has $|S(v)| = 3$, which is the definition of property (P2). Therefore all graphs considered in the last case satisfy both properties (P1) and (P2).

Case 4: $\exists v \in V : \text{height}(S(v)) = 3$ and $\max\{|S_L^\circ(v)|, |S_R^\circ(v)|\} \geqslant 4$:
Graph G satisfies properties (P1) and (P2), so we can use Lemma 5.8 in this case. Combined with the induction hypothesis we again get a valid edge guard set for G. ■

6. Series Parallel Graphs

This chapter is about series parallel graphs. They can be motivated as a model for series and parallel compositions in electrical circuits but also have applications in graph drawing, for example to display flow charts. The main results of this chapter are a lower and an upper bounds for series parallel graphs with n vertices showing that $\Gamma_{sp}(n) \sim n/3$. We start with some necessary definitions before showing the lower bound in Theorem 6.3 by a construction for a family of series parallel graphs needing many edge guards. Then we use the fact that all series parallel graphs are 3-colorable to construct a small guard set in Theorem 6.6. We finish this chapter by taking a look at the maximal series parallel graphs, showing that the same number of edge guards is required for them.

Definition 6.1 (Series Parallel Graphs) *A connected graph* G *with two distinguished vertices called* terminals *is a* series parallel graph *if it can be recursively built up by one of the following operations:*

Base Case: *The complete graph* $K_2 = (V{=}\{s,t\}, E{=}\{st\})$ *is a series parallel graph. The two vertices* s *and* t *are its terminals.*

S**-composition:** *Let* G_1 *(with terminals* s_1 *and* t_1*) and* G_2 *(with terminals* s_2 *and* t_2*) be two series parallel graphs. A* series composition *identifies the terminal* t_1 *of* G_1 *with the terminal* s_2 *of* G_2*, such that the resulting graph has terminals* s_1 *and* t_2*.*

P**-composition:** *Let* G_1 *and* G_2 *be as above. A* parallel composition *identifies* s_1 *with* s_2 *into a new terminal vertex* s *and symmetrically identifies terminal* t_1 *with* t_2 *into a new terminal vertex* t*. This is only allowed, if the resulting graph remains a simple graph.*

Figure 6.1 shows both types of compositions for some series parallel graphs G_1 *and* G_2*.*

This recursive definition allows us to construct a series parallel graph G from smaller ones via a sequence of S- and P-compositions. Let G_1 and G_2 be the two series parallel graphs that were composed into G. Obviously the order of compositions forming G_1 is independent of the order of compositions forming G_2 but all of them need to be done strictly before G_1 and G_2 are composed into G. This induces a partial order on the compositions and motivates the following definition:

Definition 6.2 (Decomposition Tree) *Let* G *be a plane series parallel graph. A* decomposition tree *for* G *is a binary tree* T *with vertices of three types:*

- *If* G *is a single edge,* T *consists only of a single vertex of type* Q*.*

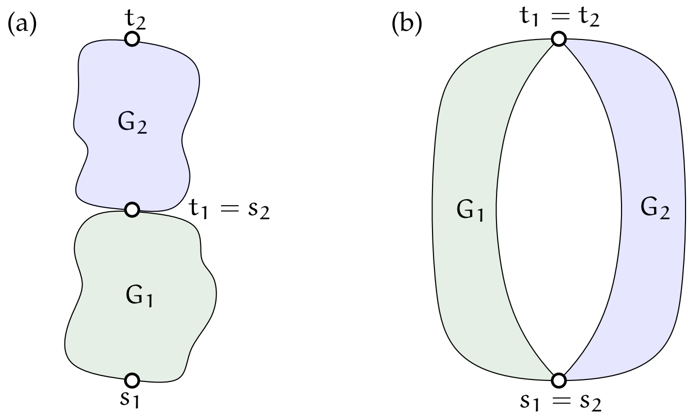

Figure 6.1.: S- and P-compositions for two series parallel graphs G_1 (with terminals s_1 and t_1) and G_2 (with terminals s_2 and t_2). (a) An S-composition: Terminals t_1 of G_1 and s_2 of G_2 have been identified. (b) A P-composition: Terminal s_1 was identified with terminal s_2 and t_1 was identified with t_2.

- *If G is the series composition of two series parallel graphs G_1 and G_2, let the root of T be an S-type vertex with two subtrees: The decomposition trees of G_1 and G_2.*
- *If G is the parallel composition of two series parallel graphs G_1 and G_2, let the root of T be a P-type vertex with two subtrees: The decomposition trees of G_1 and G_2.*

Figure 6.2 shows an example series parallel graph and one of its decomposition trees.

Note that the decomposition tree for a series parallel graph G may not be unique. Consider again Figure 6.2 as an example. The root node is of type P and decomposes the graph into two subgraphs with four and six vertices. A decomposition into three and seven vertices would have also been possible, separating the left path between the two terminals from the rest of the graph.

6.1. Lower Bound

We show the lower bound by constructing a family of n-vertex series parallel graphs that all need about $n/3$ edge guards. Our strategy here is to iteratively add three vertices such that a new triangular face arises that is 2-hop apart from all previous inner faces of the graph, thus needing a new edge guard.

Theorem 6.3 *For any $k \in \mathbb{N}$ there is series parallel graph G_k with $n = 3k + 2$ vertices needing k edge guards. Therefore we have $\lfloor (n-2)/3 \rfloor \leqslant \Gamma_{sp}(n)$.*

Proof Graph G_k is the parallel composition of k smaller series parallel graphs $G_k^1, \ldots, G_k^k$. For $i \in \{1, \ldots, k\}$ define

$$V\left(G_k^i\right) := \left\{s_i,\, v_i^1,\, v_i^2,\, v_i^3,\, t_i\right\} \quad \text{and}$$
$$E\left(G_k^i\right) := \left\{s_i v_i^1,\, v_i^1 v_i^2,\, v_i^1 v_i^3,\, v_i^3 v_i^2,\, v_i^2 t_i\right\}$$

and choose a planar embedding as in Figure 6.3 (a). Subfigure (b) shows a decomposition tree of G_k^i, proving that it is indeed series parallel. The parallel composition of $G_k^1, \ldots, G_k^k$, where vertices $s_1, \ldots, s_k$ and $t_1, \ldots, t_k$ have been identified to the new vertices s and t is then shown in subfigure (c). The edges $\left\{sv_1^1, \ldots, sv_k^1\right\}$ form an edge guard set of size k. On the other hand, for any two different $i, j \in \{1, \ldots, k\}$ the two triangles $\left(v_i^1, v_i^2, v_i^3\right)$ and $\left(v_j^1, v_j^2, v_j^3\right)$ are 2-hop apart, so they cannot be guarded by the same edge. Therefore G_k needs exactly k edge guards. ∎

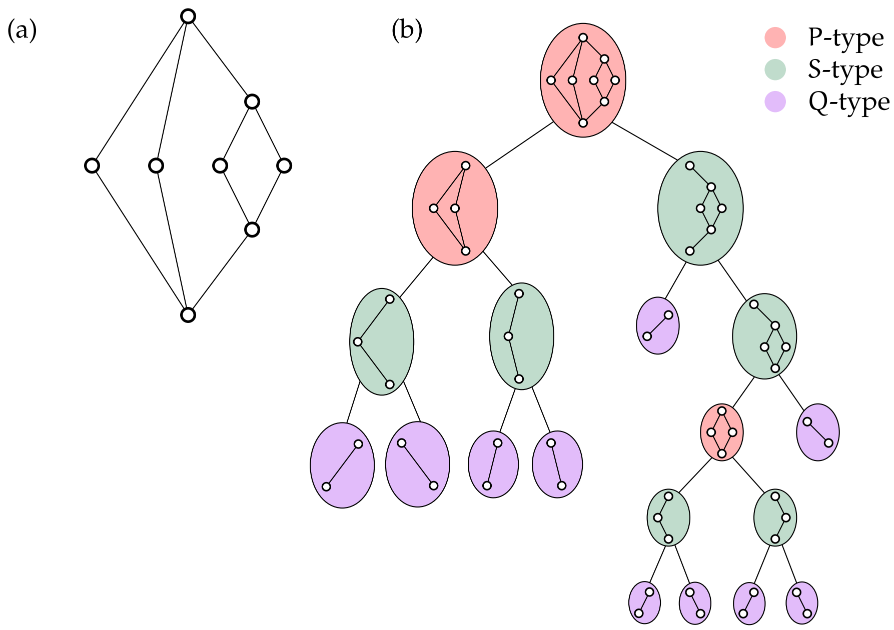

Figure 6.2.: (a) A series parallel graph G. (b) One of the decomposition trees of G. The colors of the vertices show their type.

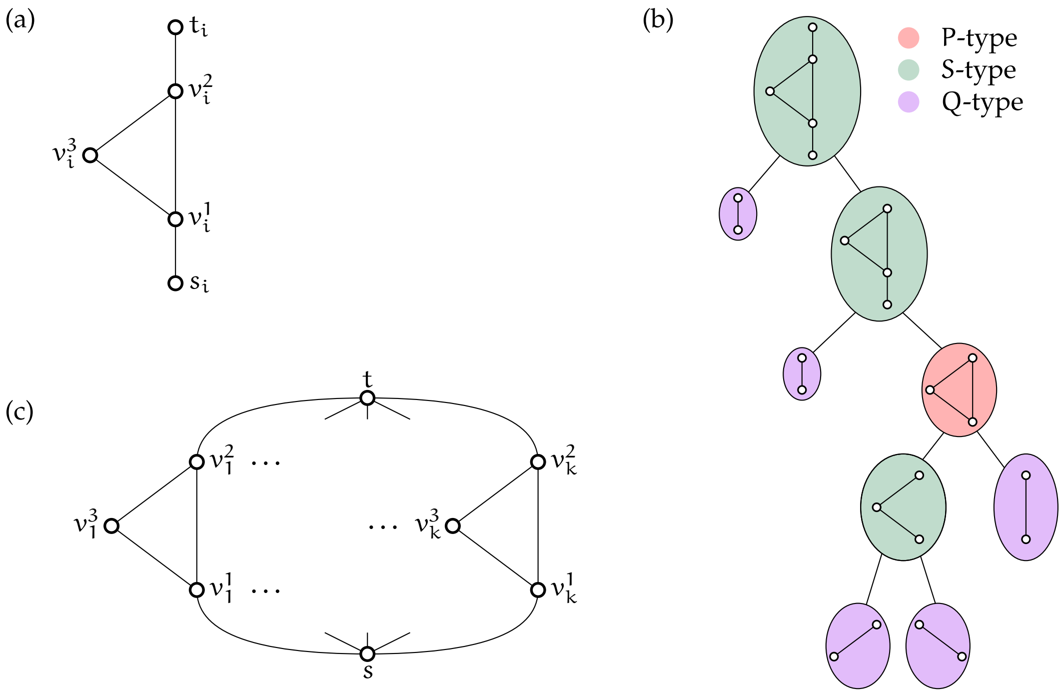

Figure 6.3.: (a) Series parallel graph G_k^i, a building block of G_k. (b) The decomposition tree of G_k^i to show that it is series parallel. (c) The parallel composition G_k of $G_k^1, \ldots, G_k^k$.

Remark 6.4 *We can show the same lower bound indirectly via the relation between series parallel and outerplanar graphs. Duffin [10, Corollary 1] shows that outerplanar graphs are a subclass of series parallel graphs. From Bose [5] we know that an n-vertex outerplanar graph might need $\lfloor n/3 \rfloor$ edge guards. This is therefore also a lower bound for series parallel graphs.*

6.2. Upper Bound

The upper bound is based on a proper 3-coloring of a series parallel graph G. We carefully add edges to G, that keep it series parallel but allow us to use color classes as guard sets. The smallest color class can then be a used as a vertex guard set.

Lemma 6.5 *Series parallel graphs are 3-colorable.*

Proof We show a slightly stronger claim, namely that series parallel graphs can be 3-colored such that the two terminal vertices have a different color. The proof is by structural induction using the decomposition tree. Let G be a series parallel graph, T be its decomposition tree and r be the root of T. The base case is when r is of type Q and G is a single copy of K_2: Its endpoints are the terminals and they are of different color in all proper colorings.

Now assume that G has at least three vertices and terminals s and t. Graph G was constructed from two of its series parallel subgraphs G_1 (with terminals s_1 and t_1) and G_2 (with terminals s_2 and t_2) via a series or parallel composition. By the induction hypothesis both G_1 and G_2 have a proper 3-coloring such that their terminals have different colors. We need to distinguish, whether r is of type S or of type P.

First assume that r is of type S, so the last composition was a series composition. Permute the three colors of G_2, such that the color of t_1 equals the one of s_2 and that the color of s_1 is different to the one of t_2. This is always possible, because we assumed that in both G_1 and G_2 the colors of the two terminals were different. Then the graph G obtained by identifying t_1 with s_2 is properly 3-colored and its terminals s and t have different colors.

Now assume that r is of type P, so the last composition was a parallel composition. We permute the colors of G_2, this time such that the colors of s_1 and s_2 are equal and that the colors of t_1 and t_2 are equal. Again, this is always possible, because both G_1 and G_2 had their terminals colored differently. Then the graph G obtained by identifying s_1 with s_2 and t_1 with t_2 is properly 3-colored and its terminals s and t have different colors. ∎

Knowing that series parallel graphs are 3-colorable already implies that they are so even when we require the terminals s and t to be of a different color. In any series parallel graph G we can add the edge st (if it is not already there) by a parallel composition with another copy of K_2. Then any proper coloring of G has different colors for s and t.

Theorem 6.6 *Any series parallel graph with n vertices can be guarded by $\lfloor n/3 \rfloor$ vertex guards, so $\Gamma_{sp}(n) \leqslant \lfloor n/3 \rfloor$.*

Proof Definition 6.1 tells us that a series parallel graph can be built from a set of copies of K_2 by a sequence of S- and P-compositions. We now define a third type of composition. Let G_1 and G_2 be two series parallel graphs with terminals s_1, t_1 and s_2, t_2, respectively. The S'-composition is a combination of an S- and a P-composition as follows:

1. Do an S-composition of the two series parallel graphs G_1 and G_2 to get $G_{1,2}$.
2. Do a P-composition of $G_{1,2}$ with a new copy of K_2. If this leads to a multigraph, ignore this step.

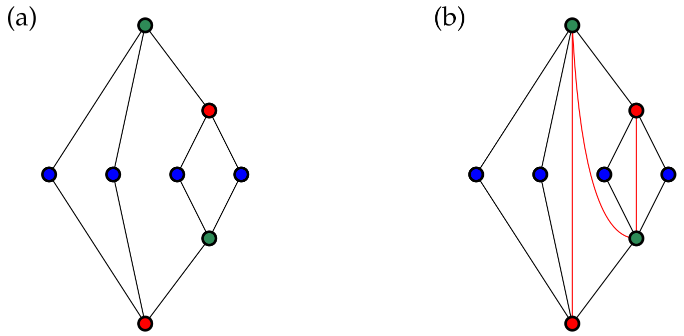

Figure 6.4.: (a) A series parallel graph G. (b) Series parallel graph G' obtained by replacing all of the S-compositions in the creation process of G by S'-compositions. The red edges were added in this process. They guarantee that any face created by a P-composition is incident to three vertices of pairwise different color.

An S'-composition guarantees that the terminal vertices s_1 and t_2 of $G_{1,2}$ are connected and therefore have different colors in all proper colorings.

Let G be a series parallel graph with n vertices and m edges. Further let $C_1, \ldots, C_{m-1}$ be all S- and P-compositions that create G from a set of m copies of K_2. Replace any S-composition in $C_1, \ldots, C_{m-1}$ by an S'-composition and call the resulting series parallel graph G'.

Now use Lemma 6.5 to find a 3-coloring χ of G'. Graphs G and G' have the same vertex set and $E(G) \subseteq E(G')$, so we can use coloring χ for both of them. We claim that any face of G is incident to vertices of all three color classes and therefore the smallest color class is a vertex guard set of size $\lfloor n/3 \rfloor$.

The key observation here is that exactly one new face arises under a P-composition, while the face set is invariant under an S-composition. So for any face f of G consider the P-composition P^* that formed it by combining two series parallel graphs G_1 and G_2 into a series parallel graph $G_{1,2}$. Let s, t be the two terminal vertices of $G_{1,2}$. Face f is bounded by two st-paths P_1 and P_2, subgraphs of G_1 and G_2, respectively. Since we only consider simple graphs we know that $|\partial f| \geqslant 3$ and therefore without loss of generality, path P_1 consists of at least three vertices. Let S^* be the last S-composition that was done in G_1 before P^* and $v \in V(P_1)$ be the vertex where the two subgraphs of G_1 were glued together by S^*. There is an S'-composition S'^* corresponding to S^* that was used to create G'. It guarantees that in any 3-coloring of G' the vertices s, t and v have three different colors. Since we used the coloring of G' for G, these three boundary vertices of f are of different colors in G, too. ∎

Figure 6.4 (a) shows a series parallel graph G and subfigure (b) shows a corresponding graph G' on the same vertex set that was created by replacing the S-compositions with S'-compositions. In both graphs the same proper vertex coloring is used and we can see that any face of G has indeed vertices of all three color classes on its boundary.

In Theorem 6.6 we showed that $\lfloor n/3 \rfloor$ vertex guards are enough for any series parallel graph with n vertices. Of course we only need at most the same number of edge guards. Since this asymptotically matches the lower bound from Theorem 6.3, we cannot get a better upper bound by considering edge guards.

6.3. Maximal Series Parallel Graphs

To end this chapter, we consider the maximal series parallel graphs. These are series parallel graphs that contain the maximum possible number of edges: Any additional edge

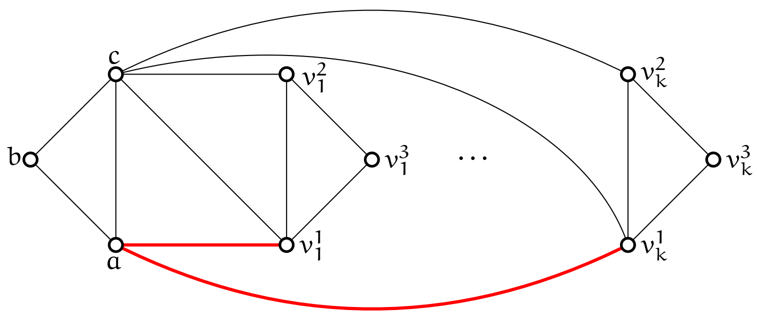

Figure 6.5.: A 2-tree that needs k edge guards. The thick red edges form an edge guard set.

would lead to a graph that is not series parallel any more. The class of maximal series parallel graphs is equal to the class of 2-trees, so this also relates to Chapter 4 about stacked triangulations. Remember that the stacked triangulations are exactly the planar 3-trees (we do not consider k-trees for $k \geqslant 4$ in this thesis, because they are not planar).

Considering the maximal graphs of a graph class is motivated by the fact, that sometimes fewer edge guards are necessary than for the non-maximal graphs of the same class. For example we know that outerplanar graphs on n vertices sometimes need $n/3$ edge guards [7] while $\lfloor n/4 \rfloor$ edge guards are always sufficient for maximal outerplanar graphs [21]. Additionally we know that general plane graphs sometimes need $n/3$ edge guards[1], but we conjecture that triangulations (which are the maximal plane graphs) can be guarded by strictly less than that, even though this is not yet proven.

We show here that the maximal series parallel graphs do not follow this trend by presenting how to construct 2-trees that asymptotically need $n/3$ edge guards. The upper bound is then given by Theorem 6.6 above which is valid for all series parallel graphs.

Theorem 6.7 *For any $k \in \mathbb{N}$ there is a maximal series parallel graph G_k with $n = 3 + 3k$ vertices that needs k edge guards. Therefore $\lfloor (n-3)/3 \rfloor \leqslant \Gamma_{\text{2-tree}}(n)$.*

Proof For $k \in \mathbb{N}$ we construct a 2-tree $G_k = (V, E)$ that has $n = 3 + 3k$ vertices and needs exactly $k = (n-3)/3$ edge guards. The vertex and edge sets are given by

$$V := \{a, b, c\} \cup \bigcup_{i=1}^{k} \{v_i^1, v_i^2, v_i^3\} \quad \text{and}$$

$$E := \{ab, bc, ac\} \cup \bigcup_{i=1}^{k} \{av_i^1,\, cv_i^1,\, v_i^1v_i^2,\, cv_i^2,\, v_i^1v_i^3,\, v_i^2v_i^3\}\,.$$

To see that G_k is indeed a 2-tree, we describe how to construct it step by step from an initial triangle formed by the vertices $\{a, b, c\}$ by always adding one new vertex and connecting it with exactly two adjacent ones of the previous vertices. Repeat the following steps k times. In step i ($1 \leqslant i \leqslant k$):

1. Add vertex v_i^1 and connect it with neighbors a and c.
2. Add vertex v_i^2 and connect it with neighbors v_i^1 and c.
3. Add vertex v_i^3 and connect it with neighbors v_i^1 and v_i^2.

A complete example can be seen in Figure 6.5. For any two different $i, j \in \{1, \dots, k\}$ the two triangles (v_i^1, v_i^2, v_i^3) and (v_j^1, v_j^2, v_j^3) are 2-hop apart. Therefore they cannot share an edge guard and we need at least k edge guards. Further $\{av_1^1, \dots, av_i^k\}$ is an edge guard set of size k, so that G_k needs exactly k edge guards. ■

[1] As discussed in Chapter 3, general plane graphs on n vertices might need even more than $n/3$ edge guards. This is still an open question.

7. Conclusion and Open Questions

This thesis broadened the understanding of edge guard sets for plane graphs by providing new bounds for stacked triangulations, quadrangulations and series parallel graphs. All previous and new results are summarized in Table 7.1, where we either cite the original source or link to the corresponding theorem in this thesis.

We started with matching lower and upper bounds for stacked triangulations and showed that $\Gamma_{\triangle,\text{stacked}}(n) \sim 2n/7$. The lower bound was achieved by reusing a technique that was already applied to obtain the best know lower bound for general triangulations. However, the best known upper bound for triangulations ($\lfloor n/3 \rfloor$ edge guards) relies on 4-colorings of the vertex set and chances are that this bound is not yet tight. We developed a new approach to locally reduce the order of a stacked triangulation, allowing us to use induction on the number of vertices and to get an upper bound of strictly less than $\lfloor n/3 \rfloor$ edge guards for a non-trivial subclass of triangulations.

Next up we considered quadrangulations, motivated by the fact that the previous coloring approaches stumbled over quadrangular faces. We were able to improve the best known upper bound from $\lfloor 3n/8 \rfloor$ edge guards to $\lfloor n/3 \rfloor$ by describing how to construct guard colorings for quadrangulations. These colorings can be turned into small edge guard sets as described in Lemma 3.5. Although this is an improvement, this result does not yet match the lower bound. We have $\lfloor (n-2)/4 \rfloor \leqslant \Gamma_{\square}(n) \leqslant \lfloor n/3 \rfloor$ raising the following open question:

> What is the minimal number of edge guards necessary to guard any n-vertex quadrangulation?

The results from the experiments in Appendix A together with our lower bound lead to the conjecture that $\lfloor n/4 \rfloor$ edge guards are always sufficient. With the 2-degenerate quadrangulations we presented a subclass for which we could improve the upper bound to match the lower bound, so $\Gamma_{\square,\text{2-deg}}(n) \sim n/4$. For the proof we applied the same technique that we originally developed for stacked triangulations.

Our third contribution are bounds for series parallel graphs, we showed that $\Gamma_{\text{sp}}(n) \sim n/3$. We even showed the stronger claim that $\lfloor n/3 \rfloor$ vertex guards are sufficient, but the selected vertices always form an independent set, so we cannot directly decrease this number by switching to edge guards. In fact, we constructed an infinite family of series parallel graphs needing at least $n/3$ edge guards, showing that edge guards provide no benefit over vertex guards for this graph class.

Table 7.1.: Overview of previous and new bounds for n-vertex planar graphs of different classes. Parameter α denotes the number of quadrilateral faces.

Graph Class	Lower Bound	Upper Bound
Outerplanar	$\lfloor n/3 \rfloor$ [5]	$\lfloor n/3 \rfloor$ [7]
Maximal Outerplanar	$\lfloor n/4 \rfloor$ [21]	$\lfloor n/4 \rfloor$ [21]
Planar	$\lfloor n/3 \rfloor$ [5]	$\lfloor \min\{n/3 + \alpha/9, 3n/8\} \rfloor$ [2]
Triangulation	$\lfloor (4n-8)/13 \rfloor$ [5]	$\lfloor n/3 \rfloor$ [12]
Stacked Triangulation (= Planar 3-Tree)	$\lfloor (2n-4)/7 \rfloor$ [Thm. 4.4]	$\lfloor 2n/7 \rfloor$ [Thm. 4.14]
Quadrangulation	$\lfloor (n-2)/4 \rfloor$ [Thm. 5.1]	$\lfloor n/3 \rfloor$ [Thm. 5.2]
2-Degenerate Quadrangulation	$\lfloor (n-2)/4 \rfloor$ [Thm. 5.1]	$\lfloor n/4 \rfloor$ [Thm. 5.9]
Series Parallel	$\lfloor (n-2)/3 \rfloor$ [Thm. 6.3]	$\lfloor n/3 \rfloor$ [Thm. 6.6]
2-Tree (= Maximal Series Parallel)	$\lfloor (n-3)/3 \rfloor$ [Thm. 6.7]	$\lfloor n/3 \rfloor$ [Thm. 6.6]

On a closing note we would like to repeat two open questions concerning general plane graphs and triangulations that already received attention in the literature, but as we can see in Table 7.1, their known lower and upper bounds do not match, calling for further investigation.

What is the minimal number of edge guards for any n-vertex plane graph?

What is the minimal number of edge guards for any n-vertex triangulation?

While the conjectured answer for general plane graphs is $\lfloor n/3 \rfloor$, this most probably needs new ideas to be proven. The currently known bounds either fail on quadrilateral faces or seem to be limited by their local operations only considering a small subgraph in each step. For triangulations we cannot conjecture a fixed value yet, but our result on stacked triangulations and our experiments on 4-connected triangulations lead us to the assumption that it is strictly less than $n/3$.

Appendix

A. Experiments

In order to get an intuition for the number of edge guards needed for different graph classes, we computed minimum cardinality edge guard sets for small graphs within each class. For each graph class $\mathcal{G}$, we used the program `plantri` provided by Brinkmann and McKay [6] to exhaustively generate all (up to isomorphism) planar embeddings of all contained graphs. For each graph $G \in \mathcal{G}$ with $|V(G)| = n$, we then computed a minimum cardinality edge guard set Γ to find the value

$$\Gamma_{\mathcal{G}}(n) := \max_{\substack{G \in \mathcal{G} \\ |G| = n}} \left\{ |\Gamma| \mid \Gamma \text{ is a minimum cardinality edge guard set of } G \right\}.$$

To find a minimum cardinality edge guard set for an embedded graph $G = (V, E) \in \mathcal{G}$ with face set F, we first build a bipartite auxiliary graph H with vertex set $V(H) := E \dot{\cup} F$ and edge set $E(H) := \{ef \mid \text{edge } e \in E \text{ is incident to face } f \in F \text{ in } G\}$. With a highly optimized backtracking algorithm we then computed a minimum cardinality hitting set $S \subseteq E$, that is a minimum cardinality subset of E, such that each face $f \in F$ has at least one neighbor $e \in S$ in H. Since these computations are independent for any two different graphs in $\mathcal{G}$, the program can be naively parallelized.

A.1. General Triangulations

Table A.1 shows the experimental results for triangulations with up to $n = 18$ vertices. Sequence `A000109`[1] from the On-Line Encyclopedia of Integer Sequences (OEIS) gives the number of triangulations with n vertices.

From Bose et al. [5] we know that $\lfloor (4n - 8)/13 \rfloor$ edge guards are sometimes necessary. In their paper they note that this can be improved, if a 9-vertex triangulation existed that needs three edge guards. They could not find one and raised the question whether such a triangulation exists. With the results from this experiment we can negatively answer this question. This suggests that their lower bound might actually be the best possible but as seen in Chapter 3 the best known upper bound is still $\lfloor n/3 \rfloor$.

[1] `https://oeis.org/A000109`

Table A.1.: Experimental results for triangulations. Each line shows how many triangulations there are with n vertices and what the maximal size among all minimum cardinality edge guard sets is.

n	A000109	$\Gamma_{\triangle}(n)$
3	1	1
4	1	1
5	1	1
6	2	2
7	5	2
8	14	2
9	50	2
10	233	3

n	A000109	$\Gamma_{\triangle}(n)$
11	1249	3
12	7595	3
13	49 566	3
14	339 722	4
15	2 406 841	4
16	17 490 241	4
17	129 664 753	4
18	977 526 957	5

Table A.2.: Experimental results for stacked triangulations. Each line shows how many stacked triangulations there are with n vertices and what the maximal size among all minimum cardinality edge guard sets is.

n	A027610	$\Gamma_{\triangle,\text{stacked}}(n)$
3	1	1
4	1	1
5	1	1
6	1	1
7	3	2
8	7	2
9	24	2
10	93	2
11	434	3

n	A027610	$\Gamma_{\triangle,\text{stacked}}(n)$
12	2110	3
13	11 002	3
14	58 713	3
15	321 776	4
16	1 792 133	4
17	10 131 027	4
18	57 949 430	4
19	334 970 205	5
20	1 953 890 318	5

A.2. Stacked Triangulations

Table A.2 shows the experimental results for stacked triangulations with up to $n = 20$ vertices. Sequence A027610[2] from the OEIS gives the number of stacked triangulations with n vertices.

By looking at the values, we would expect the number of edge guards that are always sufficient to be $\lfloor (n-2)/4 \rfloor$. However, in Chapter 4 we showed that $\Gamma_{\triangle,\text{stacked}}(n) \sim 2n/7$ (Theorems 4.4 and 4.14). The smallest stacked triangulation known to us needing strictly more than $\lfloor (n-2)/4 \rfloor$ edge guards contains $n = 30$ vertices and can be obtained through the construction presented in Theorem 4.4 [3].

[2] https://oeis.org/A027610

[3] Using the notation from the proof of Theorem 4.4: Choose a stacked triangulation S with $\nu = 6$ vertices and subdivide it as described in the proof. The resulting stacked triangulation G with $n = 30$ vertices needs at least 8 edge guards.

Table A.3.: Experimental results for 4-connected triangulations. Each line shows how many 4-connected triangulations there are with n vertices and what the maximal size among all minimum cardinality edge guard sets is.

n	A007021	$\Gamma_{\triangle,\text{4-con}}(n)$	n	A007021	$\Gamma_{\triangle,\text{4-con}}(n)$
4	1	1	14	1357	3
5	1	1	15	6244	4
6	1	2	16	30 926	4
7	1	2	17	158 428	4
8	2	2	18	836 749	4
9	4	2	19	4 504 607	5
10	10	2	20	24 649 284	5
11	25	3	21	136 610 879	5
12	87	3	22	765 598 927	5
13	313	3	23	4 332 047 595	6

A.3. 4-connected Triangulations

Table A.3 shows the experimental results for 4-connected triangulations with up to $n = 23$ vertices. Sequence A007021[4] from the OEIS gives the number of 4-connected triangulations with n vertices.

The values suggest that $\Gamma_{\triangle,\text{4-con}}(n) \sim n/4$. Note that the unique 4-connected triangulation with six vertices needs two edge guards and therefore contradicts this bound. This triangulation is the octahedron graph, wich was already considered in Section 3.1. However, no 4-connected triangulation G with more than six vertices can contain the octahedron graph as a subgraph, because otherwise G would contain a separating triangle[5]. We state a lower bound on $\Gamma_{\triangle,\text{4-con}}(n)$ to substantiate our conjectured bound. How to prove a matching upper bound remains an open problem.

Theorem A.1 *We have* $\lfloor (n-2)/4 \rfloor \leqslant \Gamma_{\triangle,\text{4-con}}(n)$.

Proof We use the same strategy as for general triangulations in Theorem 3.1 and for stacked triangulations in Theorem 4.4. Let S be a quadrangulation with ν vertices. By Euler's Theorem it has $\nu - 2$ faces and into each face f we put three vertices forming a triangular face t_f. Triangulate the obtained graph in an arbitrary way. The resulting graph G is 4-connected, because it does not contain any separating triangles. Further we have $n := |V(G)| = \nu + (\nu - 2) \cdot 3 = 4\nu - 6$ and isolating ν we get $\nu = (n+6)/4$. For any edge guard set Γ of G we now have:

$$\begin{aligned} \Gamma_{\triangle,\text{4-con}}(n) &\geqslant |\Gamma| \\ &\geqslant \nu - 2 && \text{(at least one edge per face of S)} \\ &= \frac{n-2}{4} && \text{(substituting } \nu\text{)} \end{aligned}$$

■

[4] https://oeis.org/A007021

[5] A separating triangle is a triangle $(x, y, z) \subseteq V(G)$, such that there are vertices inside and outside of it in any planar embedding of G. It is a well known fact, that a graph is 4-connected, if and only if it does not contain any separating triangle.

Table A.4.: Experimental results for quadrangulations. Each line shows how many quadrangulations there are with n vertices and what the maximal size among all minimum cardinality edge guard sets is.

n	A113201	$\Gamma_\square(n)$
4	1	1
5	1	1
6	2	1
7	3	1
8	9	2
9	18	2
10	62	2
11	198	2
12	803	2
13	3378	3
14	15 882	3
15	77 185	3
16	393 075	3
17	2 049 974	4
18	10 938 182	4
19	59 312 272	4
20	326 258 544	4
21	1 815 910 231	4
22	10 213 424 233	5
23	57 974 895 671	5

A.4. Quadrangulations

Table A.4 shows the experimental results for quadrangulations with up to $n = 23$ vertices. Sequence A113201[6] from the OEIS gives the number of quadrangulations with n vertices.

The values suggest that $\lfloor(2n+2)/9\rfloor$ edge guards would be always sufficient but in Chapter 5 we showed that this is not enough. In fact, Theorems 5.1 and 5.2 show that $\lfloor(n-2)/4\rfloor \leqslant \Gamma_\square(n) \leqslant \lfloor n/3\rfloor$. Our presented upper bound of $\lfloor n/3\rfloor$ edge guards from Theorem 5.2 is most probably not tight. For the subclass of 2-degenerate quadrangulations we could show a better bound of $\Gamma_{\square,\text{2-deg}}(n) \leqslant \lfloor n/4\rfloor$ that matches the lower bound.

[6] https://oeis.org/A113201

B. Skipped Proofs for 2-Degenerate Quadrangulations

This section contains the proofs for Lemmas 5.6, 5.7 and 5.8 that were skipped in Chapter 5 for improved clarity. The three proofs all make use of the following two observations which we state in the form of *forcing lemmas* as we already did similarly in Chapter 4 for stacked triangulations. They allow us to assume certain properties of the edge guard sets that are given by the induction hypothesis in the proof of Theorem 5.9.

Lemma B.1 (Weak Forcing Lemma) *Let* f *be a face of a quadrangulation* G *with boundary vertices* $\partial f = \{w, x, y, z\}$*. By subdividing* f *with two vertices* a *and* b *and edges* $\{xa, xb, za, zb\}$ *we get another quadrangulation* H *shown in Figure B.1 (a). Then an edge guard set* Γ *of* H *exists that has minimum cardinality among all edge guard sets for* H *and for which* $|\{x, z\} \cap V(\Gamma)| \geqslant 1$.

Proof Face $f = (x, b, z, a)$ of H must be guarded by Γ, so we have $|\{x, b, z, a\} \cap V(\Gamma)| \geqslant 1$. Assume that $\{x, y\} \cap V(\Gamma) = \emptyset$, so f is guarded by a and/or b. Since Γ consists only of edge guards (and not vertex guards), at least one of their neighbors must also be in $V(\Gamma)$, but $N(a) \cup N(b) = \{x, z\}$, so we get $|\{x, z\} \cap V(\Gamma)| \geqslant 1$, a contradiction to our assumption. ∎

Lemma B.2 (Strong Forcing Lemma) *Let* f *be a face of a quadrangulation* G *with boundary vertices* $\partial f = \{w, x, y, z\}$*. By subdividing* f *with three vertices* $\{a, b, c\}$ *and edges* $\{xa, xb, za, zb, ac, bc\}$ *we get another quadrangulation* H *shown in Figure B.1 (b). Then an edge guard set* Γ *of* H *exists that has minimum cardinality among all edge guard sets for* H *and for which one of the following two properties holds:*

1. $\{x, z\} \subseteq V(\Gamma)$ *or*
2. $\exists vw \in \Gamma$ *with* $v \in \{x, z\}$ *and* $w \in \{a, b\}$

Proof We assume that property 1 does not hold, because otherwise we are done. Then we must have an edge $vw \in \Gamma$ with $\{v, w\} \subseteq \{x, z, a, b, c\}$, to guard the two faces (x, b, c, a) and (z, a, c, b). Either property 2 holds for this edge vw or we have $\{v, w\} \subseteq \{a, b, c\}$. In the latter case, we can set $\tilde{\Gamma} = (\Gamma \setminus \{vw\}) \cup \{xa\}$ to get a guard set of equal cardinality fulfilling the requirements. ∎

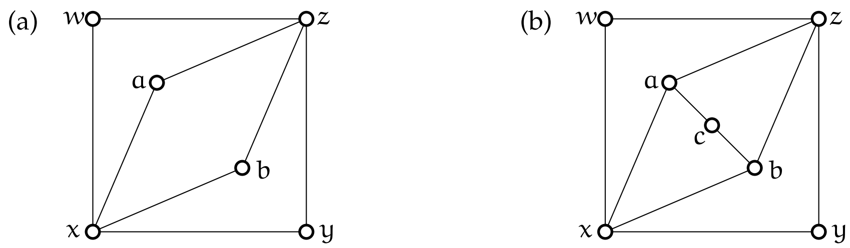

Figure B.1.: (a) The face f subdivided by vertices a and b. Any edge guard set contains at least one vertex from $\{x, z\}$.
(b) The face f subdivided by vertices a, b and c. Any edge guard set contains either both $\{x, z\}$ or an edge with both endpoints in $\{a, b, c, x, z\}$.

B.1. Proof of Lemma 5.6

Proof Let $G = (V, E)$ be a 2-degenerate quadrangulation. The conditions of the lemma require that G satisfies property (P1) and has a vertex $v \in V$ with $4 \leqslant |S(v)| \leqslant 7$ and $|S^{\circ}_{L}(v)|, |S^{\circ}_{R}(v)| \leqslant 3$. When vertex v was added, it subdivided a quadrangular face (w, x, y, z) into two: $f_L = (w, x, y, v)$ and $f_R = (w, v, y, z)$ (this is visualized in Figure B.2). There are only a few ways how a single face incident to v can be subdivided by at most three

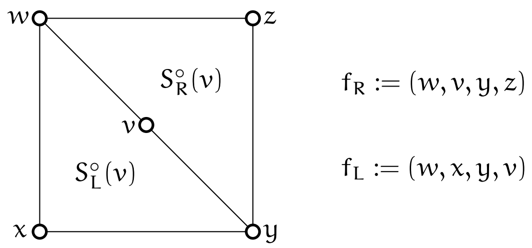

Figure B.2.: Vertex v lies inside quadrangle (w, x, y, z) and subdivides it into two faces $f_L := (w, x, y, v)$ and $f_R := (w, v, y, z)$. Both of them can be further subdivided by vertices in $S^\circ(v)$ where $S^\circ_L(v)$ is inside f_L and $S^\circ_R(v)$ is inside f_R.

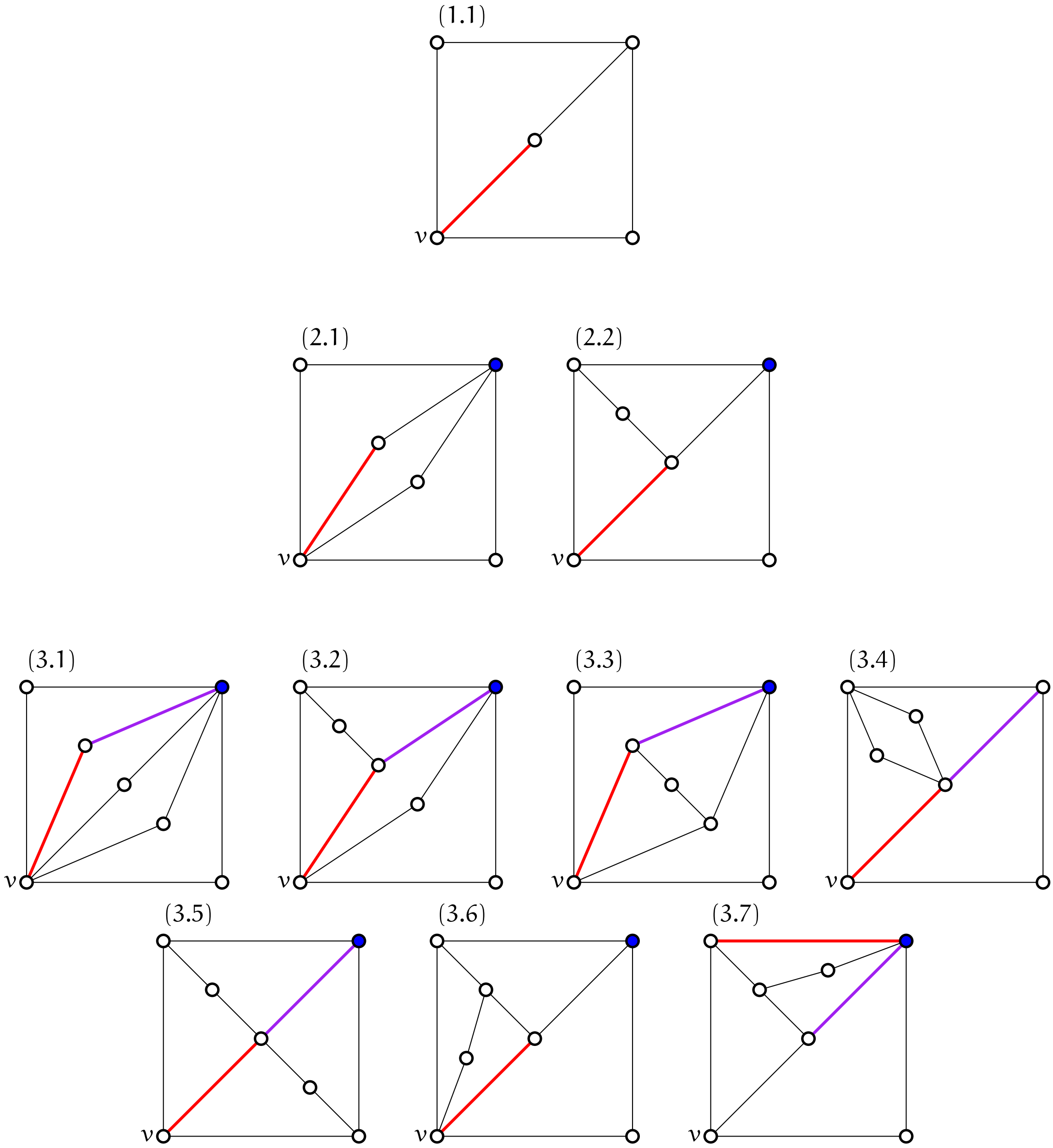

Figure B.3.: All possible ways to subdivide a quadrangular face with up to three vertices. The colors show (partial) guard sets. Their exact meaning is described in the proof of Lemma 5.6.
(1.1): The only way to subdivide with a single vertex.
(2.1)–(2.2): The two possible ways to subdivide with two vertices.
(3.1)–(3.7): All seven ways to subdivide with three vertices.

additional vertices such that the vertices are in $S(v)$: All of them are shown in Figure B.3. The vertices in $S^\circ_L(v)$ and $S^\circ_R(v)$ each resemble one of these ways such that their combination satisfies the lemma's requirements.

In any of these combinations we remove $S(v)$ from V and sometimes add a set V^+ of new vertices to it (together with some new edges). We call the resulting graph G' and guarantee that $|G'| \leqslant |G| - 4$. Now let Γ' be an edge guard set of G'. We use it to construct a new edge guard set Γ for G of size $|\Gamma| = |\Gamma'| + 1$. How this works is dependent on $|S(v)|$. To avoid looking at symmetric cases, we assume $|S^\circ_L(v)| \leqslant |S^\circ_R(v)|$ in the following.

Case 1: $|S(v)| = 4$:

The three vertices in $S^\circ(v)$ can be distributed to the left and the right side in two ways. If $|S^\circ_L(v)| = 0$ and $|S^\circ_R(v)| = 3$, we can extend Γ' by the red edge shown for the corresponding 3-vertex subdivision of $S^\circ_R(v)$ in Figure B.3. It guards all faces incident to vertices in $S^\circ_R(v)$ and also the remaining face left of v.

If otherwise $|S^\circ_L(v)| = 1$ and $|S^\circ_R(v)| = 2$, we can extend Γ' by the red edge shown in subdivisions (2.1) or (2.2). It guards all faces incident to vertices in $S^\circ_R(v)$ and contains vertex v, which alone guards the two faces incident to the unique vertex in $S^\circ_L(v)$.

Case 2: $|S(v)| = 5$:

Assume first that $|S^\circ_L(v)| = 1$ and $|S^\circ_R(v)| = 3$. If the vertices in $S^\circ_R(v)$ follow one of the combinations (3.1)–(3.6), we can extend Γ' by the red edge shown for the respective subdivision. This edge has v as an endpoint, so that it also guards the faces incident to vertices in $S^\circ_L(v)$. It remains the case that the vertices in $S^\circ_R(v)$ are as in (3.7). Figure B.4 shows how to guard this case.

Now we consider the case $|S^\circ_L(v)| = 2$ and $|S^\circ_R(v)| = 2$. If at least one side is subdivided as in (2.1), we use the red edge from the combination of the other side. It contains v and therefore guards all needed faces incident to vertices in $S(v)$. Otherwise both sides are as in (2.2) and again we reference to Figure B.4 showing how to guard these cases (there are two, because (2.2) can be reflected).

Case 3: $|S(v)| = 6$:

In this case we have $|S^\circ_L(v)| = 2$ and $|S^\circ_R(v)| = 3$. Let us first consider the cases where the vertices in $S^\circ_R(v)$ are as in (3.1), (3.2), (3.3), (3.5) or (3.6). They can all be guarded by three vertex guards at v, x and z (x and z correspond to the blue vertices in the figures). Since $|S(v)| = 6$, we can afford to add two new vertices to G after removing $S(v)$. So we use the Weak Forcing Lemma B.1 to get $|\{x, z\} \cap V(\Gamma')| \geqslant 1$. If $x \in V(\Gamma')$, we can use the red edge of the subdivision for the vertices in $S^\circ_R(v)$. Otherwise, if $z \in V(\Gamma')$, we can use the red edge of the subdivision for the vertices in $S^\circ_L(v)$. In both cases, we have $\{v, x, z\} \subseteq V(\Gamma)$, so all faces are guarded.

The remaining cases are when $S^\circ_R(v)$ is subdivided as in (3.4) or (3.7). Figure B.4 shows how to guard these cases.

Case 4: $|S(v)| = 7$:

Now we have both $|S^\circ_L(v)| = |S^\circ_R(v)| = 3$, so $|S(v)| = 7$. In this case $|S(v)| = 7$ is big enough to use the Strong Forcing Lemma B.2 for the vertices x and z. Then one of the two cases applies:

- We have $\{x, z\} \subseteq V(\Gamma')$. Any combination where neither side is (3.4) is guarded by the three vertex guards $\{x, z, v\}$, so we just need to add another edge containing v as an endpoint to Γ'. If exactly one side is (3.4), we can use the red edge of (3.4) to extend Γ'. The case that both sides are subdivides as in (3.4) is again shown in Figure B.4 (now using the Weak Forcing Lemma).

- We have an edge $xa \in \Gamma'$ with $a \in S^\circ_L(v)$ (or symmetrically an edge $zb \in \Gamma'$ with $b \in S^\circ_R(v)$). All subdivisions for $S^\circ_L(v)$ except for (3.6) can be guarded by a single edge containing x. This edge is shown in purple in drawings of Figure B.3 and we use it instead of xa in Γ'. The other side can then be guarded by the red edge for the subdivision of $S^\circ_R(v)$.

  If however $S^\circ_L(v)$ is subdivided as in (3.6) but $S^\circ_R(v)$ is not (3.7), we can again use the purple edge for $S^\circ_L(v)$ and the red edge for $S^\circ_R(v)$, because the latter one contains v and $\{x, v\}$ is a vertex guard set for (3.6). The only remaining case is a combination of (3.6) and (3.7) Guardings for them are again shown with the other special cases in Figure B.4. ■

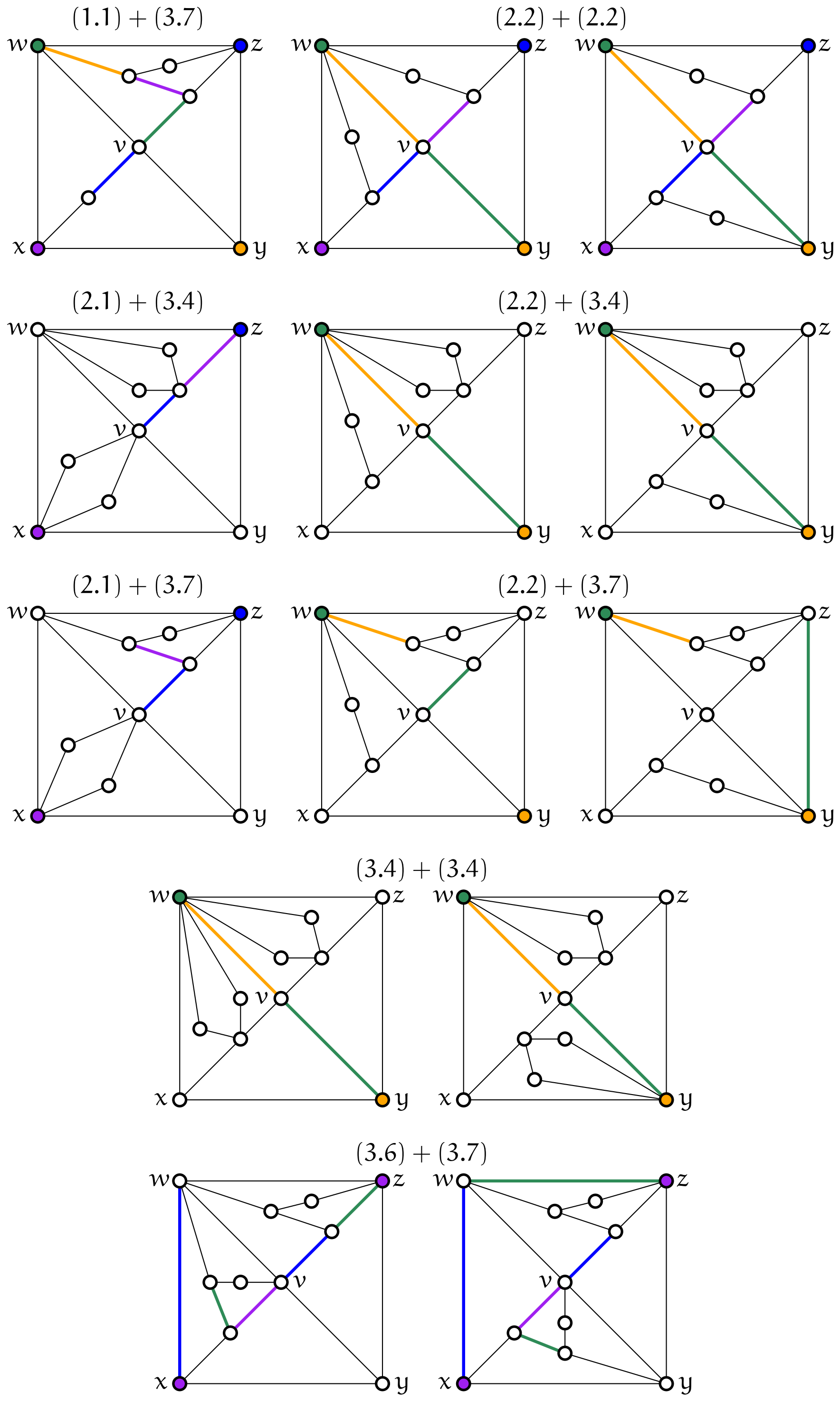

Figure B.4.: Special cases from the proof of Lemma 5.6 that need to be considered one by one. Any edge guard set Γ' contains an edge with an endpoint among $\{w, x, y, z\}$. We color code these four possibilities: If for example the green vertex $w \in V(\Gamma')$, then the green edge works to set $\Gamma := \Gamma' \cup \{\text{green edge}\}$.
When we only give edge guard sets for two opposing corners ($\{x, z\}$ or $\{w, y\}$), this is because we have $|S(v)| \geqslant 6$ and we can use the Weak Forcing Lemma B.1 to guarantee that Γ' contains at least one of them.
The two combinations for $(3.6) + (3.7)$ use the Strong Forcing Lemma B.2. Here a purple edge completes an edge guard set Γ', if $x, z \in V(\Gamma')$. Blue and green edges show how to find Γ, if Γ' contained an edge from x (blue) or z (green).

B.2. Proof of Lemma 5.7

Proof Let $G = (V, E)$ be a 2-degenerate quadrangulation satisfying property (P1). We assume that a vertex v exists with

1. $\text{height}(S(v)) = 2$ and
2. $\max\{|S_L^\circ(v)|, |S_R^\circ(v)|\} \geqslant 4$. Without loss of generality, we assume that $|S_R^\circ(v)| \geqslant 4$.

We show how to find at least four vertices from $S_R^\circ(v)$ that we can remove to get G'. Further we describe how any edge guard set Γ' for G' can be turned into an edge guard set Γ for G of size $|\Gamma| = |\Gamma'| + 1$.

Let (v, x, y, z) be the 4-cycle containing $S_R^\circ(v)$ and $w_1, w_2, \ldots, w_k \in N(v) \cap S_R^\circ(v)$ be the common neighbors of v and y inside $S_R^\circ(v)$ in clockwise order (see Figure B.5). Each w_i (for $1 \leqslant i \leqslant k$) can have further neighbors, but we have $|S(w_i)| \leqslant 3$ (since property (P1) applies) and all vertices in $S^\circ(w_i)$ must be on the same level (because $\text{height}(S(v)) = 2$ and therefore $\text{height}(S(w_i)) \leqslant 1$). Note that $k \geqslant 2$, because $|S_R^\circ(v)| \geqslant 4$ and with $k \in \{0, 1\}$ we would get $|S_R^\circ(v)| \leqslant 3$.

To start, assume there is an $i \in \{1, \ldots, k-1\}$, such that w_i and w_{i+1} share two neighbors $\{a, b\}$ of deeper level as shown in Figure B.6. Then by above observation these must be their only neighbors except for v and y, so we have $S(w_i) \cup S(w_{i+1}) = \{w_i, w_{i+1}, a, b\}$ and we can remove these four vertices from G to get G'. Any edge guard set Γ' for G' can then be extended by edge vw_i to be an edge guard set Γ for G. In all following cases, we assume that no such i exists.

Note that $S_R^\circ(v)$ can contain arbitrarily many elements. We can consider only a finite number of cases, so we do not remove the complete set $S_R^\circ(v)$. Instead we identify induced 4-cycles $C_{i,j} := (v, w_i, y, w_j)$ ($0 \leqslant i < j \leqslant k+1$ with $w_0 := x$ and $w_{k+1} := z$) that contain at least four vertices inside of them. We show that we can always find one, that allows us to remove all its interior vertices.

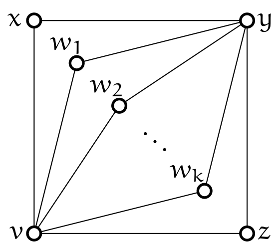

Figure B.5.: Face (v, x, y, z) was further subdivided by vertices $w_1, w_2, \ldots, w_k$. The resulting faces might be further subdivided, so $S_R^\circ(v) = \bigcup_{i=1}^{k} S(w_i)$.

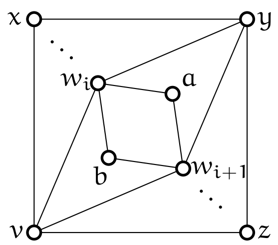

Figure B.6.: Vertices w_i and w_{i+1} share two neighbors a and b of a deeper level. Then $S(w_i) \cup S(w_{i+1}) = \{w_i, w_{i+1}, a, b\}$ and all four of them can be removed. Edge vw_i extends any edge guard set of the obtained smaller graph to an edge guard set for the original graph.

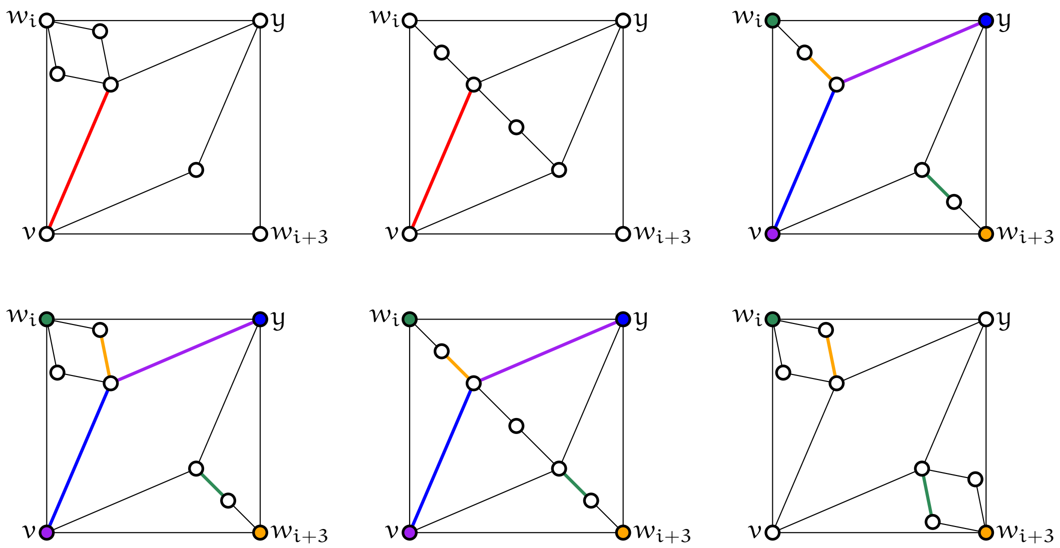

Figure B.7.: All six ways how the interior of $C_{i,i+3} = (v, w_i, y, w_{i+3})$ can be subdivided, such that it contains at least four vertices. Configurations that are symmetric when the order of w_i and w_{i+3} is changed, are not displayed. The color coding is as in Figure B.4.
The first, fourth and sixth configuration can only appear if $w_i = x$, because otherwise we would have a configuration as in Figure B.6. For the same reason the sixth configuration is further only possible, if $w_{i+3} = z$.

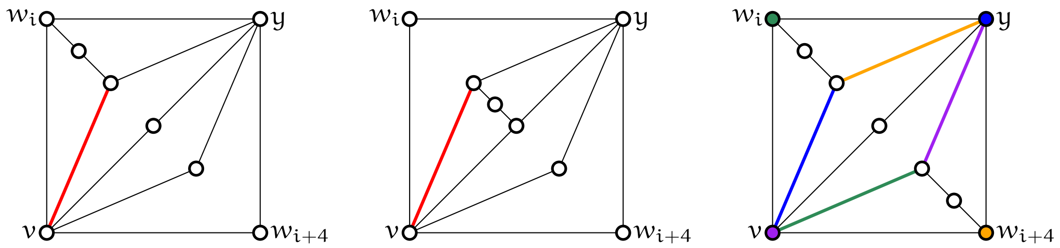

Figure B.8.: All three ways how the interior of $C_{i,i+4} = (v, w_i, y, w_{i+4})$ can be subdivided, such that it contains at least four vertices and none of the configurations from Figure B.7. Configurations that are symmetric when the order of w_i and w_{i+4} is changed, are not displayed. The color coding is as in Figure B.4.

Assume there is an i ($0 \leqslant i \leqslant k-2$), such that $C_{i,i+3} = (v, w_i, y, w_{i+3})$ has at least four vertices inside it. Figure B.7 shows all six possible ways how its inside can look like. In each case, we remove all inner vertices to obtain G'[7]. The color coding then shows how to extend and edge guard set Γ' to an edge guard set Γ for G by a single edge.

If above case does not apply, we instead look for an i ($0 \leqslant i \leqslant k-3$), such that the 4-cycle $C_{i,i+4} = (v, w_i, y, w_{i+4})$ has at least four vertices inside it. Since neither $C_{i,i+3}$ nor $C_{i+1,i+4}$ may have four vertices inside them, there are only three ways how the interior of $C_{i,i+4}$ can be subdivided, all shown in Figure B.8. Again, we remove all inner vertices and the color coding shows how to handle the three cases.

It is possible that none of the above cases applies. Then we can always find an i ($0 \leqslant i \leqslant k-4$), such that $C_{i,i+5} = (v, w_i, y, w_{i+5})$ contains at least four vertices, which must be exactly the common neighbors of v and y. This is shown in Figure B.9. There

[7] In one case we need to apply the Weak Forcing Lemma B.1, so after deleting the inner vertices we add two new ones. Since the inside of $C_{i,i+3}$ contained six vertices in this case, we still get $|G'| = |G| - 4$.

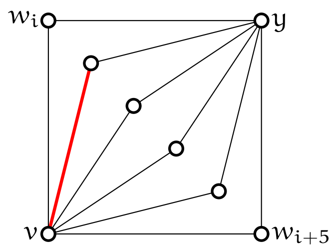

Figure B.9.: The unique way how the interior of $C_{i,i+5} = (v, w_i, y, w_{i+5})$ must be subdivided, such that it contains at least four vertices and none of the configurations from Figures B.7 and B.8. The color coding is as in Figure B.4.

are no further subdivisions inside $C_{i,i+5}$, because otherwise an earlier case would apply. Edge vw_{i+1} suffices to extend any edge guard set of G' to become an edge guard set of G. ∎

B.3. Proof of Lemma 5.8

Proof Let $G = (V, E)$ be a 2-degenerate quadrangulation satisfying properties (P1) and (P2). We assume that a vertex v exists with

1. $\text{height}(S(v)) = 3$ and
2. $\max\{|S^\circ_L(v)|, |S^\circ_R(v)|\} \geqslant 4$. Without loss of generality, we assume that $|S^\circ_R(v)| \geqslant 4$.

We show how to find at least four vertices from $S^\circ_R(v)$ that we can remove to get G'. Further we describe how any edge guard set Γ' for G' can be turned into an edge guard set Γ for G of size $|\Gamma| = |\Gamma'| + 1$.

The proof is very similar to the one of Lemma 5.7 which is given above. Again we let (v, x, y, z) be the 4-cycle containing $S^\circ_R(v)$ and $w_1, w_2, \dots, w_k \in N(v) \cap S^\circ_R(v)$ be the common neighbors of v and y inside $S^\circ_R(v)$ in clockwise order (see Figure B.5 from the previous proof). Further remember the definition of the 4-cycles $C_{i,j} := (v, w_i, y, w_j)$ ($0 \leqslant i < j \leqslant k+1$ with $w_0 := x$ and $w_{k+1} := z$).

First, assume that there is an i ($1 \leqslant i \leqslant k-1$), such that there are two vertices a, b with $\{a, b\} \subseteq S^\circ(w_i) \cap S^\circ(w_{i+1})$. The three possible ways for this are shown in Figure B.10. Property (P1) then gives us that $S(w_i) \cup S(w_{i+1}) = \{w_i, w_{i+1}, a, b\}$ and we can remove these four vertices to get G'. Any edge guard set Γ' for G' can be extended by a single edge (which is drawn red in the subfigures). In the following we assume that no such i exists.

For all remaining cases we look at 4-cycles $C_{i,j}$ ($0 \leqslant i < j \leqslant k+1$) that have at least four vertices inside and where $j - i$ is minimal. If the vertices inside $C_{i,j}$ are of at most two different levels, one of the cases from the proof of Lemma 5.7 applies and we saw there

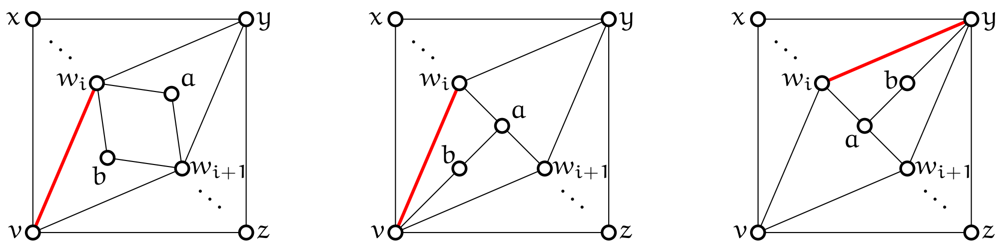

Figure B.10.: We have $S(w_i) \cup S(w_{i+1}) = \{w_i, w_{i+1}, a, b\}$ and all four of them can be removed. Each subfigure contains a thick, red edge that extends any edge guard set of the obtained smaller graph to an edge guard set for the original graph.

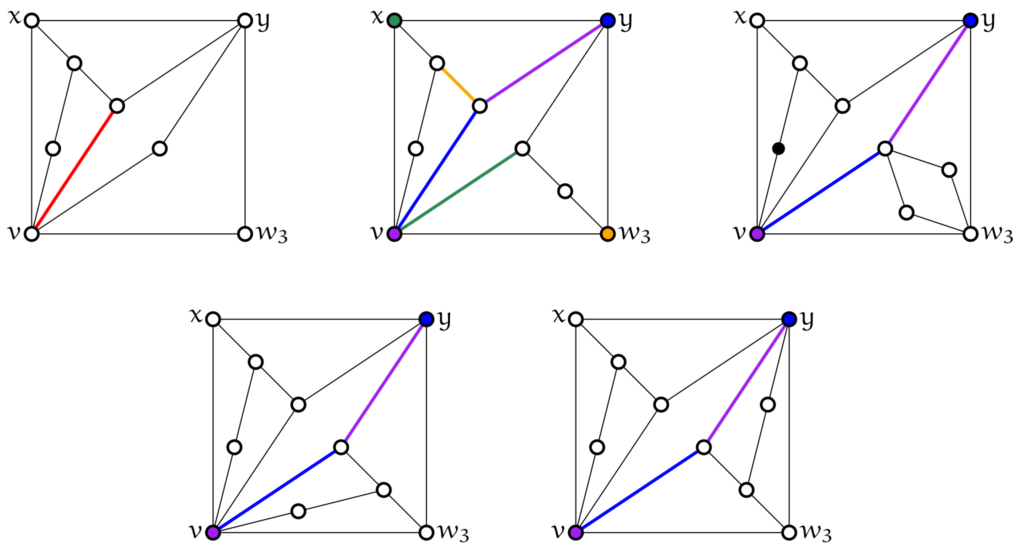

Figure B.11.: All ways that $C_{0,3}$ can be subdivided by at least four vertices. Here we ignore configurations that are obtained by changing the role of v and y. The color coding is as in Figure B.4.

how to handle them. Thus we only need to consider such $C_{i,j}$ that contain vertices of three different levels inside of them. The only way this is possible such that $C_{i,j}$ does not contain a configuration from Figure B.10 is that we have $i = 0$ and/or $j = z$. Without loss of generality we restrict ourselves here to the case that $i = 0$ (the other case works symmetrically). Because of property (P1), the 4-cycles $C_{0,1}$ and $C_{0,2}$ can contain at most three vertices, so we look at all possible ways, how $C_{0,3}$ can be subdivided: Figure B.11 shows all of those possibilities. In all cases we can remove the inner vertices to get G'. Any edge guard set Γ' for G' can be extended by a single edge to an edge guard set Γ for G as shown by the given color coding. ∎

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