45-th Czech and Slovak Mathematical Olympiad 1996

Category A

1. A sequence $(G_n)_{n=0}^{\infty}$ satisfies $G(0)=0$ and $G(n)=n-G(G(n))$ for each $n\in\mathbb{N}$. Show that

   (a) $G(k)\geq G(k-1)$ for every $k\in\mathbb{N}$;

   (b) there is no integer k for which $G(k-1)=G(k)=G(k+1)$.

2. Let AP,BQ and CR be altirudes of an acute-angled triangle ABC. Show that for any point X inside the triangle PQR there exists a tetrahedron $ABCD$ such that X is the point on the face ABC at the greatest distance from D (measured along the surface of the tetrahedron).

3. Given six three-element subsets of a finite set X, show that it is possible to color the elements of X in two colors so that none of the given subsets is in one color.

4. Points A and B on the rays CX and CY respectively of an acute angle XCY are given so that $CX<CA=CB<CY$. Construct a line meeting the ray CX and the segments AB,BC at K,L,M, respectively, such that $KA\cdot YB=XA\cdot MB=LA\cdot LB\neq 0$.

5. For which integers k does there exist a function $f:\mathbb{N}\to\mathbb{Z}$ such that $f(1995)=1996$ and

$$f(xy)=f(x)+f(y)+kf(\gcd(x,y))\qquad\text{for all } x,y\in\mathbb{N}?$$

6. Let K,L,M be points on sides AB,BC,CA, respectively, of a triangle ABC such that $AK/AB=BL/BC=CM/CA=1/3$. Show that if the circumcircles of the triangles AKM, BLK, CML are equal, then so are the incircles of these triangles.

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