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H_∞-STABILITY ANALYSIS OF FRACTIONAL DELAY SYSTEMS OF NEUTRAL TYPE*

LE HA VY NGUYEN†, CATHERINE BONNET†, AND ANDRÉ RICARDO FIORAVANTI‡

Abstract. In this paper we consider linear fractional systems of commensurate orders and with commensurate delays, whose characteristic equation is a polynomial in the two variables s^α $(0 < \alpha < 1)$ and $e^{-s\tau}$ $(\tau > 0)$. These systems may have single or multiple chains of poles asymptotic to the imaginary axis. Location of poles of large modulus belonging to these chains are determined by approximation and simple necessary and sufficient H_∞-stability conditions are derived.

Key words. Fractional systems, neutral systems, delay effects, H_∞-stability

AMS subject classifications. 93C23, 93D25, 93C05

1. Introduction. In the time domain, fractional models involve derivatives and/or integrals of orders which are not necessarily integers. Similarly, they involve in the frequency domain non (necessarily) integer powers of the Laplace variable s. These models have found applications in many fields, for instance electromagnetics [11, 19], mechanics [6, 12], and biology [8], since they have been showed to better model certain phenomena than integer-order models.

In control engineering, a lot of results are available on fractional controllers and their implementation. See for example [16, 17, 14], and the references therein.

With the spreading of fractional systems including both plants and controllers, it is natural to think about fractional systems with delays since delays are commonly encountered in real systems. However, not all fractional systems with delays are obtained in this manner. One example is an acoustic system subjected to viscothermal perturbations, which was first modeled by generalized partial differential equations [13] and then was modeled in some ranges of low frequencies by an approximated transfer function of fractional type with a delay [10].

Recently, an increasing interest is paid towards fractional systems with delays, both for stability aspects and also for stabilization problems. For stability aspects, most of the results are obtained in the frequency domain, i.e. by considering the spectrum of the characteristic equation. The characteristics of fractional systems with delays are similar to those of classical systems with delays, i.e. there are infinitely many poles in chains which can be classified in retarded, advanced and neutral types [9, 5].

To the authors' knowledge, the stability of fractional systems with delays is first examined in [9] where some sufficient as well as necessary conditions for BIBO and H_∞ stabilities of the simple model of the aforementioned acoustic system are derived. In [4], the most general class of systems is considered. For systems of retarded type, it is proved that the classical stability condition "no poles in the closed right half-plane" is a necessary and sufficient condition for BIBO-stability. However, this is only a necessary condition for neutral systems since, in the critical case where poles approach the imaginary axis, the system may be unstable even though all poles are

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in the open left half-plane. This interesting phenomenon is also present in classical delay systems.

In such a delicate situation, H_∞-stability of fractional systems with one delay is studied in [5] where simple necessary and sufficient conditions are derived. Within the same framework, [3] studies H_∞-stability of some classes of classical systems with commensurate delays and with single chains of poles asymptotic to the imaginary axis.

In this paper, we examine the H_∞-stability of fractional systems of commensurate orders and with commensurate delays whose characteristic equation is a polynomial in two variables s^α $(0 < \alpha < 1)$ and $e^{-s\tau}$ $(\tau > 0)$. Specially, these systems may have multiple chains of poles asymptotic to the imaginary axis. We are interested in the effects of poles of large modulus on the stability and will not pay attention to poles of small modulus, which can be computed using numerical methods such as QPmR [18] and YALTA [1].

This paper extends the works presented in [7, 15] and makes some corrections for the latter which will be announced right after the related result.

The rest of the paper is organized as follows. Section 2 presents the fractional delay system of interest and some preliminaries. In §3 we approximate the characteristic equation around its poles of large modulus. The obtained expression is repeatedly used in the next two sections to determine pole location with respect to asymptotic axes and estimate the magnitude of the characteristic equation on the imaginary axis, which allows one to conclude about H_∞-stability of the system. Section 4 examines single neutral chains of poles while §5 is dedicated to multiple chains. Illustrative examples are given in §6. We conclude the paper with §7.

Here, we denote $\mathbb{N}_N = \{1, \ldots, N\}$, which is a set of natural numbers, $\mathbb{Z}_+ = \{x \in \mathbb{Z} \mid x \geq 0\}$, $\mathbb{C}_+ = \{s \in \mathbb{C} \mid \Re s > 0\}$, $\ln(\cdot)$ denotes the real logarithm function, $\mathrm{Arg}(x)$ the argument of $x \in \mathbb{C}$ satisfying $-\pi < \mathrm{Arg}(x) < \pi$, and $[x]$ the integer part of $x \in \mathbb{R}$.

2. A class of fractional neutral time-delay systems. We consider fractional neutral time-delay systems with transfer function of the form

$$G(s) = \frac{t(s^\mu)}{p(s^\mu) + \sum\limits_{k=1}^{N} q_k(s^\mu) e^{-ks\tau}}, \tag{2.1}$$

where $\tau > 0$ is the delay, t, p, and q_k for all $k \in \mathbb{N}_N$ are real polynomials in s^μ, $0 < \mu < 1$, $-\pi < \arg(s) < \pi$ in order to have a single-valued transfer function, $\deg p \geq \deg t$, $\deg p \geq \deg q_k$ for all $k \in \mathbb{N}_N$, and $\deg p = \deg q_k$ at least for one $k \in \mathbb{N}_N$ in order to deal with proper neutral systems. Note that degrees of the polynomials in this paper stand for the degrees in s^μ.

The function s^μ where μ is a real number, $0 < \mu < 1$, has a branch point at $s = 0$. To study this function we make a cut in the complex plane at $\mathbb{R}_-$ and consider the domain $\mathbb{C} \setminus \mathbb{R}_-$.

In this paper, we consider stability of systems in a H_∞-sense. We recall that a system is H_∞-stable if and only if for input signals that are bounded in the norm L_2 it provides output signals that are also bounded in the same norm. Recall that a linear time-invariant system is H_∞-stable if and only if its transfer function belongs to $H_\infty := \{f : \mathbb{C}_+ \mapsto \mathbb{C} \mid f \text{ is analytic in } \mathbb{C}_+ \text{ and } \sup_{s\in\mathbb{C}_+} |f(s)| < \infty\}$.

Since $\deg p \geq \deg q_k$ for all $k \in \mathbb{N}_N$, we can obtain the following expression for each k

$$\frac{q_k(s^\mu)}{p(s^\mu)} = \alpha_k + \frac{\beta_k}{s^\mu} + \frac{\gamma_k}{s^{2\mu}} + \frac{\delta_k}{s^{3\mu}} + \frac{\epsilon_k}{s^{4\mu}} + \mathcal{O}(s^{-5\mu}) \text{ as } |s| \to \infty. \tag{2.2}$$

The coefficient of the highest degree term of the denominator of the transfer function (2.1) can be written as a multiple of the following polynomial in z (which can be called formal polynomial as in the case of standard delay systems)

$$\widetilde{c}_d(z) = 1 + \sum_{k=1}^{N} \alpha_k z^k, \tag{2.3}$$

where $z = e^{-s\tau}$.

Let us denote s_n a root of the denominator of the transfer function. According to Rouché's theorem, s_n is asymptotic to a root r of $\widetilde{c}_d(e^{-s\tau})$ as $|s_n| \to \infty$ and thus is approximated by

$$s_n\tau = \lambda_n + o(1), \tag{2.4}$$

where

$$\lambda_n = -\ln|r| - \jmath\mathrm{Arg}(r) + \jmath 2\pi n, \quad n \in \mathbb{Z}, n \to \infty. \tag{2.5}$$

We can say that to each root r of $\widetilde{c}_d(z)$ is associated a chain of poles and this chain is said to be of neutral type.

As a consequence, when $n \to \infty$ the neutral chain of poles asymptotically approaches the vertical line

$$\Re(s) = -\frac{\ln|r|}{\tau}. \tag{2.6}$$

If the vertical line is on the right or on the left of the imaginary axis, which happens when $|r| < 1$ or $|r| > 1$, then poles asymptotic to this vertical line are respectively on the right or on the left of the imaginary axis, and then their effects on H_∞-stability are easily concluded. Therefore, in these cases, we do not need to further analyze in order to determine on which side of the asymptotic axis the neutral chains of poles lie.

However, when a vertical line coincides with the imaginary axis, a more precise analysis is required in order to determine in which half-plane the neutral chains of poles is located.

REMARK 1. *Note that the basic results recalled in this section are also valid for the classical case, i.e.* $\mu = 1$.

3. Approximation of the characteristic equation. We already have the first approximation of neutral poles corresponding to a root r of $\widetilde{c}_d(z)$ in (2.4). Our objective is to find the next *non-zero* approximation term of these poles, which are denoted by s_n. Let us write

$$s_n\tau = \lambda_n + \nu_{n,1} + o(n^{-y_1}),$$

where

$$\nu_{n,1} = \frac{\nu_1}{n^{y_1}}, \quad \nu_1 \neq 0,\ y_1 > 0,\ n \in \mathbb{Z}, n \to \infty,$$

and we recall that n^{y_1} is defined on the branch where $-\pi < \arg(n) \leq \pi$.

REMARK 2. *We will see later that in general y_1 is not an integer multiple of μ. For example, $y_1 = \mu$ in certain cases of single chains, but $y_1 = \mu/m$ in certain cases of multiple chains, where m is the multiplicity of r.*

We have $\Re(s_n) = [\Re(\lambda_n) + \Re(\nu_{n,1}) + o(n^{-y_1})]/\tau$. Therefore, the sign of $\Re(\nu_{n,1})$ indicates the location of poles of the neutral chain with respect to the asymptotic axis.

REMARK 3. *Note that for neutral chains of poles relative to a root r*

$$\Re(\nu_{n,1})_{n>0} = \frac{\Re(\nu_1)}{n^{y_1}},$$
$$\Re(\nu_{n,1})_{n<0} = \frac{\Re(\nu_1)\cos(y_1\pi) + \Im(\nu_1)\sin(y_1\pi)}{|n|^{y_1}}.$$

Since the signs of $\Re(\nu_{n,1})_{n>0}$ and $\Re(\nu_{n,1})_{n<0}$, which are determined by the signs of $\Re(\nu_1)$ and $(\Re(\nu_1)\cos(y_1\pi) + \Im(\nu_1)\sin(y_1\pi))$ respectively, may be different, so are the locations around the asymptotic axis of poles of large modulus in the upper and lower half-planes.

To determine the approximation term $\nu_{n,1}$ of neutral poles of the system, we will need the series expansion of the characteristic equation around s_n as $n \to \infty$.

Let us denote $d(s)$ the characteristic equation of $G(s)$, that is

$$d(s) := p(s^\mu) + \sum_{k=1}^{N} q_k(s^\mu)e^{-ks\tau}.$$

Since s_n is a pole of $G(s)$, we have $d(s_n) = 0$. We will now compute the series expansion as $n \to \infty$ of $d(s_n)/p(s_n^\mu)$ which is

$$1 + \sum_{k=1}^{N} \frac{q_k(s_n^\mu)}{p(s_n^\mu)} e^{-ks_n\tau} = 0 \tag{3.1}$$

assuming that s_n has the form

$$s_n\tau = \lambda_n + \nu_{n,1} + \nu_{n,2} + \ldots + \nu_{n,M} + o(n^{-4\mu})$$

with $\nu_{n,i} = \nu_i n^{-y_i}$, $i = 1, \ldots, M$ where $\nu_i \neq 0$ and $0 < y_1 < \ldots < y_M \leq 4\mu$.

We choose indeed a development of order 4μ which will allow us to analyze in this paper several cases of interest.

In the next sections, we will show that by collecting terms of the same orders from the series expansion of $d(s_n)/p(s_n^\mu)$ and by matching the two sides of (3.1), we will be able to determine $\nu_{n,1}$.

In the series expansion (see Appendix A for detailed computations), the following sums appear

$$A_r^L := \sum_{k=1}^{N} \alpha_k r^k k^L, \qquad B_r^L := \sum_{k=1}^{N} \beta_k r^k k^L, \qquad C_r^L := \sum_{k=1}^{N} \gamma_k r^k k^L,$$
$$D_r^L := \sum_{k=1}^{N} \delta_k r^k k^L, \qquad E_r^L := \sum_{k=1}^{N} \epsilon_k r^k k^L,$$

with $L \in \mathbb{Z}_+$. These sums will be shown to play an important role in the sequel.

In order to facilitate the collection of the terms of highest order of the development, we can decompose the series expansion of $d(s_n)/p(s_n^\mu)$ into three components as follows

$$\frac{d(s_n)}{p(s_n^\mu)} = g_1 + g_2 + g_3 + o(n^{-4\mu}) = 0 \tag{3.2}$$

where

$$g_1 = 1 + A_r^0 + \frac{\tau^\mu B_r^0}{(\jmath 2\pi)^\mu} n^{-\mu}\left(1 + O(n^{-1})\right) + \frac{\tau^{2\mu} C_r^0}{(\jmath 2\pi)^{2\mu}} n^{-2\mu}\left(1 + O(n^{-1})\right) + \frac{\tau^{3\mu} D_r^0}{(\jmath 2\pi)^{3\mu}} n^{-3\mu} + \frac{\tau^{4\mu} E_r^0}{(\jmath 2\pi)^{4\mu}} n^{-4\mu}, \tag{3.3}$$

$$g_2 = \sum_{(l_1,\ldots,l_M)\in\mathcal{L}(4\mu)} \frac{(-1)^{\sum_{i=1}^M l_i}\left(\prod_{i=1}^M \nu_i^{l_i}\right) A_r^{\sum_{i=1}^M l_i}}{\left(\prod_{i=1}^M l_i!\right)} n^{-\sum_{i=1}^M l_i y_i}, \tag{3.4}$$

and

$$\begin{aligned} g_3 = {} & \frac{\tau^\mu}{(\jmath 2\pi)^\mu} n^{-\mu}\left(1 + O(n^{-1})\right) \\ & \times \sum_{(l_1,\ldots,l_M)\in\mathcal{L}(3\mu)} \frac{(-1)^{\sum_{i=1}^M l_i}\left(\prod_{i=1}^M \nu_i^{l_i}\right) B_r^{\sum_{i=1}^M l_i}}{\left(\prod_{i=1}^M l_i!\right)} n^{-\sum_{i=1}^M l_i y_i} \\ & + \frac{\tau^{2\mu}}{(\jmath 2\pi)^{2\mu}} n^{-2\mu}\left(1 + O(n^{-1})\right) \\ & \times \sum_{(l_1,\ldots,l_M)\in\mathcal{L}(2\mu)} \frac{(-1)^{\sum_{i=1}^M l_i}\left(\prod_{i=1}^M \nu_i^{l_i}\right) C_r^{\sum_{i=1}^M l_i}}{\left(\prod_{i=1}^M l_i!\right)} n^{-\sum_{i=1}^M l_i y_i} \\ & + \frac{\tau^{3\mu}}{(\jmath 2\pi)^{3\mu}} n^{-3\mu} \sum_{(l_1,\ldots,l_M)\in\mathcal{L}(\mu)} \frac{(-1)^{\sum_{i=1}^M l_i}\left(\prod_{i=1}^M \nu_i^{l_i}\right) D_r^{\sum_{i=1}^M l_i}}{\left(\prod_{i=1}^M l_i!\right)} n^{-\sum_{i=1}^M l_i y_i}, \end{aligned} \tag{3.5}$$

where

$$\mathcal{L}(x) := \left\{(l_1,\ldots,l_M) \mid l_i \in \mathbb{Z}_+, \sum_{i=1}^M l_i \geq 1 \text{ and } \sum_{i=1}^M l_i y_i \leq x\right\}. \tag{3.6}$$

Note that for example in g_1, the highest order in n is $-\mu$ if $B_r^0 \neq 0$. It is -2μ if $B_r^0 = 0$ and $C_r^0 \neq 0$. To find the highest order in n for g_2, note that $A_r^{\sum_{i=1}^M l_i} = 0$ if $\sum_{i=1}^M l_i < m$ and it is zero if $\sum_{i=1}^M l_i = m$ (see Appendix B). In this case we have $\sum_{i=1}^M l_i y_i \geq m y_1$ and the highest order in n is $-my_1$. Hence, the highest order of the sum $(g_1 + g_2 + g_3)$ may be a function of y_1.

As $g_1 + g_2 + g_3 + o(n^{-4\mu}) = 0$, the term of highest order of the left expression is thus zero and so is the term of highest order of the sum $(g_1 + g_2 + g_3)$. As we will see in the sequel, this allows us to derive y_1 and ν_1.

We have already seen that an important role is played by the coefficients A_r^0, B_r^0, ... In the following sections, we will derive y_1 and ν_1 for classes of systems which may have some of these coefficients vanishing.

We start with the case of single chains, i.e. $m = 1$, for which the analysis for systems with vanishing or non vanishing coefficients does not differ too much. More precisely, we will consider the case $B_r^0 \neq 0$ and show that the results can then be expanded to other possible cases by similar arguments.

On the other hand, the analysis in the case of multiple chains, i.e. $m \geq 2$, needs in each case (vanishing or non vanishing coefficients) an appropriate development to get y_1 and ν_1 from the highest order of development in the terms g_1, g_2, and g_3. Three cases will be investigated:

- $B_r^0 \neq 0$,
- $B_r^0 = 0$, $B_r^1 \neq 0$, and $C_r^0 \neq 0$,
- $B_r^0 = 0$, $B_r^1 = 0$, $B_r^2 \neq 0$, and $C_r^0 \neq 0$.

4. Single chains. To avoid complexity, we begin by studying thoroughly the simplest case, whose results can be exactly adapted to other cases.

4.1. The case where $B_r^0 \neq 0$.

First, we give a more precise approximation of roots of large modulus of the characteristic equation than that given in (2.4).

THEOREM 4.1. *Let $G(s)$ be a fractional neutral delay system defined by (2.1) and suppose that at least one root of the formal polynomial $\widetilde{c}_d(z)$ defined in (2.3) has multiplicity one. If such a root, denoted by r, satisfies*

$$B_r^0 \neq 0, \tag{4.1}$$

then for large enough $n \in \mathbb{Z}$ poles of the neutral chain relative to r are approximated by

$$s_n \tau = \lambda_n + \nu_{n,1} + o(n^{-\mu})$$

with λ_n given by (2.5) and

$$\nu_{n,1} = \frac{\tau^\mu K_r}{(\jmath 2\pi)^\mu} n^{-\mu}, \tag{4.2}$$

where

$$K_r = \frac{B_r^0}{A_r^1}. \tag{4.3}$$

Proof. Under the condition (4.1), the highest order in n of g_1 is $-\mu$. Obviously, the highest order of g_3 is smaller than that of g_2, which is $-y_1$. Therefore, $y_1 = \mu$ and (3.2) can be rewritten as

$$\frac{\tau^\mu B_r^0}{(\jmath 2\pi)^\mu} n^{-\mu} - \nu_1 A_r^1 n^{-y_1} + o(n^{-\mu}) = 0,$$

which completes the proof. ∎

Given a more precise approximation of neutral poles of large modulus as above, now our interest is mainly on which side of vertical line the poles are, in other words, to find out the sign of $\Re(\nu_{n,1})$ for n sufficiently large. This question is particularly important when the asymptotic axis is the imaginary axis.

Recall from Remark 3 that for a chain relative to a root of $\widetilde{c}_d(s)$, its poles in the upper and lower half-planes may lie on different sides of the asymptotic axis. Fortunately, for the case considered in this subsection, these two parts of the chain may be on the same side. This behavior is characterized in the next two corollaries.

COROLLARY 4.2. *Let $0 < \mu < 1$, $\nu_{n,1}$ be given by (4.2). Then $\Re(\nu_{n,1}) < 0$ for all $n \in \mathbb{Z}$ if and only if*

$$\Re(K_r) < -\tan\left(\frac{\mu\pi}{2}\right)|\Im(K_r)| \tag{4.4}$$

with K_r defined in (4.3).

Proof. Besides K_r, the only term of interest is $J = (\jmath n)^{-\mu}$, as $\mathrm{sgn}(\Re(\nu_{n,1})) = \mathrm{sgn}(\Re(JK_r))$. Since n can be both positive or negative, this term is given by

$$J = \frac{1}{|n|^\mu}\left(\cos\left(\frac{\mu\pi}{2}\right) \mp \jmath \sin\left(\frac{\mu\pi}{2}\right)\right).$$

Multiplying J by K_r and getting its real part leads to

$$\Re(JK_r) = \frac{1}{|n|^\mu}\left(\cos\left(\frac{\mu\pi}{2}\right)\Re(K_r) \pm \sin\left(\frac{\mu\pi}{2}\right)\Im(K_r)\right)$$

from which (4.4) follows from the fact that $0 < \mu < 1$. ☐

Some remarks can be drawn from this corollary. First, the condition (4.4) does not depend on the delay τ. This means that for all $\tau > 0$ the chain of poles does not change side with respect to the vertical line in question when the delay varies. Second, the condition (4.4) still holds if r is replaced by its complex conjugate $\bar{r}$, which is also a root of the formal polynomial $\widetilde{c}_d(z)$. Therefore, the chain relative to $\bar{r}$ lies on the same side as the one relative to r.

As K_r is independent of μ, we can reformulate the previous corollary to give the critical value of μ as follows

COROLLARY 4.3. *Let $0 < \mu < 1$, $\nu_{n,1}$ be given by (4.2) and its associated K_r by (4.3). Then, if $\Re(K_r) < 0$, all poles of the respective chain asymptotic to the vertical line $\Re(s) = -\ln|r|/\tau$ will be on the left of this line if*

$$\mu < \frac{2}{\pi}\arctan\left(-\frac{\Re(K_r)}{|\Im(K_r)|}\right). \tag{4.5}$$

Proof. This follows directly from Corollary 4.2. ☐

In the case where $\Re(\nu_{n,1}) = 0$, further analysis is needed to determine the location of poles. However, the procedure is similar to the one given in [3] and therefore will be omitted.

Now, we are interested in answering the question of stability of $G(s)$ in the H_∞-sense.

For systems without chains of poles asymptotic to the left of the imaginary axis, the stability can be concluded if there is no poles in the closed right half-plane.

On the other hand, if there exist neutral chains of poles approaching the imaginary axis from the left, we may have to consider the magnitude of the transfer function on the axis in order to answer the question of H_∞-stability. This is the objective of the proposition below.

We will refer to poles in the closed right half-plane $\bar{\mathbb{C}}_+$ as *unstable poles.*

Proposition 4.4. *Let $G(s)$ be a transfer function given as in (2.1) and suppose that the formal polynomial $\widetilde{c}_d(z)$ defined in (2.3) has at least one simple root of modulus one, the other roots being of modulus strictly greater than one. We also suppose that every root of modulus one which is denoted by r satisfies (4.1).*

1. *Suppose that $\Re(\nu_{n,1}) < 0$ for all r and that G has no unstable pole of small modulus (which could exist only in a finite number), then G is H_∞-stable if and only if* $\deg p \geq \deg t + 1$.
2. *If there exists a root r for which $\Re(\nu_{n,1}) = 0$, then the condition* $\deg p \geq \deg t + 1$ *is necessary for H_∞-stability.*

Proof. Let $s = s_n + \eta \in \jmath\mathbb{R}$, we have

$$\begin{aligned}
&\left| p(s^\mu) + \sum_{k=1}^{N} q_k(s^\mu) e^{-ks\tau} \right| \\
&\approx |\eta| \quad \left| p'(s_n^\mu)\mu s^{\mu-1} + \sum_{k=1}^{N} \left(q_k'(s_n^\mu)\mu s^{\mu-1} - k\tau q_k(s_n^\mu) \right) e^{-ks_n\tau} \right| \\
&\approx |\eta| \; |p(s_n^\mu)| \; \left| \frac{p'(s_n^\mu)}{p(s_n^\mu)}\mu s^{\mu-1} + \sum_{k=1}^{N} \left(\frac{q_k'(s_n^\mu)}{p(s_n^\mu)}\mu s^{\mu-1} - k\tau \frac{q_k(s_n^\mu)}{p(s_n^\mu)} \right) e^{-ks_n\tau} \right| \\
&\approx |\eta| \; |p(s_n^\mu)| \; \left| \sum_{k=1}^{N} k\tau \frac{q_k(s_n^\mu)}{p(s_n^\mu)} e^{-ks_n\tau} \right| \\
&\approx \tau \; |\eta| \; |p(s_n^\mu)| \; \left| A_r^1 \right|
\end{aligned}$$

as $n \to \infty$, $n \in \mathbb{Z}$.

Recall that A_r^1 is non zero by assumption.

If $\Re(\nu_{n,1}) \neq 0$, then η is at least of order $n^{-\mu}$ and a necessary and sufficient condition of H_∞-stability is that $\deg p \geq \deg t + 1$. If $\Re(\nu_{n,1}) = 0$ the condition is still necessary. ◻

The results of this subsection are illustrated later by Example 1 in §6.

In the next section, the same stability analysis will be realized for other cases of systems with single chains of poles, thus completing the analysis for this class of systems.

4.2. Other cases.

Returning to the approximation of the characteristic equation around poles of a single chain, we see that the terms of highest order are only constituted from those of g_1 and g_2 given in (3.3) and (3.4) respectively since the highest order of g_3 (3.5) is smaller than that of g_2. While that term of g_2 remains the same, i.e. $(-\nu_{n,1}A_r^1)$, for all cases of single chains, that of g_1 is decided by its non-zero terms. Some examples are cases where

- $B_r^0 = 0$ and $C_r^0 \neq 0$ (the term of highest order of g_1 is $\tau^{2\mu} C_r^0/(\jmath 2\pi n)^{2\mu}$),
- $B_r^0 = 0$, $C_r^0 = 0$, $D_r^0 \neq 0$ (the term of highest order of g_1 is $\tau^{3\mu} D_r^0/(\jmath 2\pi n)^{3\mu}$),
- and so on.

Similarly, we obtain easily $\nu_{n,1}$ for the cases above by noting that in the development of the characteristic equation around a pole the coefficient of the highest order is zero. In general,

$$\nu_{n,1} = \left(\frac{\tau}{\jmath 2\pi n} \right)^{x_r \mu} \tilde{K}_r \tag{4.6}$$

with $\tilde{K}_r$ a function in r and the coefficients $\alpha_k, \beta_k, \dots$ in (2.2), and $x_r \in \mathbb{N}$. Note that we get a value of x_r for each root r of multiplicity one of $\widetilde{c}_d(z)$ (2.3), where comes the subscript. For example, if r satisfies the first case above, i.e. $B_r^0 = 0$ and $C_r^0 \neq 0$, then $x_r = 2$ and $\tilde{K}_r = C_r^0/(A_r^1)$.

Now, as in the previous subsection, we can realize a similar analysis about the location of the chain of poles relative to r. Here, $\mathrm{sgn}(\Re(\nu_{n,1})) = \mathrm{sgn}(\Re(\jmath^{-x_r\mu}\tilde{K}_r))$. Therefore, $\Re(\nu_{n,1}) < 0$ for all $n \in \mathbb{Z}$, i.e. the chain is on the left of the asymptotic axis, if and only if

$$\cos\left(\frac{x_r\mu\pi}{2}\right)\Re(\tilde{K}_r) < -\left|\sin\left(\frac{x_r\mu\pi}{2}\right)\Im(\tilde{K}_r)\right|. \tag{4.7}$$

However, (4.7) does not hold for $x_r\mu = 2k+1$ for $k \in \mathbb{N}$, $0 < \mu < 1$. In that case, we will be in the same situation as the case $\mu = 1$ [3], where either $\Re(\nu_{n,1}) = 0$ (when $\Im(\tilde{K}_r) = 0$) or $\Re(\nu_1) = \pm c$ (when $\Im(\tilde{K}_r) \neq 0$), meaning that respectively more approximation terms are needed or we conclude to have both left and right chains.

From all the above analyzes about the location of poles of large modulus, the H_∞-stability condition for systems with single chains asymptotic to the imaginary axis can be restated as follows.

PROPOSITION 4.5. *Let $G(s)$ be a transfer function given as (2.1) and suppose that the formal polynomial $\widetilde{c}_d(z)$ defined in (2.3) has at least one simple root of modulus one, denoted r, the other roots being of modulus strictly greater than one.*

1. *Suppose that $\Re(\nu_{n,1}) < 0$ for all r and that G has no unstable pole of small modulus, then G is H_∞-stable if and only if $\deg p \geq \deg t + \max_r\{x_r\}$, where, for each r, $x_r = y_1/\mu$ and $-y_1$ is the order in n of $\nu_{n,1}$.*
2. *If $\Re(\nu_{n,1}) = 0$ for any r, then the condition $\deg p \geq \deg t + \max_r\{x_r\}$ is necessary for H_∞-stability.*

Proof. As in the proof of Proposition 4.4, let us consider the numerator of $G(s)$ at s on the imaginary axis near poles of the neutral chain relative a root r. Let $s = s_n + \eta \in \jmath\mathbb{R}$, we also have

$$\left|p(s^\mu) + \sum_{k=1}^{N} q_k(s^\mu)e^{-ks\tau}\right| \approx \tau\ |\eta|\ |p(s_n^\mu)|\ \left|A_r^1\right|$$

as $n \to \infty$, $n \in \mathbb{Z}$.

Here, if $\Re(\nu_{n,1}) \neq 0$, then η is at least of order $n^{-x_r\mu}$. □

REMARK 4. *Because the mentioned cases are defined by algebraic relations involving some equality ones, systems which belong to one of these cases may easily change category due to small changes in their coefficients. For example, in the case where $B_r^0 = 0$ and $C_r^0 \neq 0$, if the coefficients β_k, $k = 1, \dots, n$ change a little, it is likely that the equality $B_r^0 = 0$ is no longer satisfied and the system now belongs to the case studied in the previous subsection. Hence, the stability of the system may as well change.*

This is the same situation for other cases that we will consider in the sequel since they will also be defined by the same type of conditions.

5. Multiple chains. While the stability analysis of single chains under different conditions results in similar conclusions, the stability of multiple chains differs significantly from case to case.

In this section, we do not aim for a complete analysis of general cases but for a large class of systems. This analysis reveals interesting different behaviors.

5.1. The case where $m \geq 2$ and $B_r^0 \neq 0$.

Under the same condition, an exhaustive H_∞-stability analysis for neutral chains relative to roots of multiplicity one of (2.3) has been conducted in Subsection 4.1. In this section, multiple chains will be studied. The first step is also to approximate the pole location. However, the analysis based on this approximation ends shortly.

THEOREM 5.1. *Let $G(s)$ be a fractional neutral delay system defined by (2.1), and suppose that at least one root of the formal polynomial $\widetilde{c}_d(z)$ defined by (2.3) has multiplicity $m > 1$. If for such a root, denoted by r, the condition (4.1) is satisfied, then for large enough $n \in \mathbb{Z}$, poles of neutral chains relative to those m identical roots are approximated by*

$$s_n\tau = \lambda_n + \nu_{n,1} + o(n^{-\mu/m}),$$

with λ_n given by (2.5) and

$$\nu_{n,1} = \nu_1 n^{-\mu/m}, \tag{5.1}$$

where

$$\nu_1^m = (-1)^{m+1}\frac{m!\tau^\mu B_r^0}{(\jmath 2\pi)^\mu A_r^m}, \tag{5.2}$$

which gives rise to m different neutral chains, sharing the same λ_n, the same $y_1 = \mu/m$, but different values of ν_1.

Proof. Because of the condition (4.1), the highest order in n of g_1 is $-\mu$, which is obviously higher than that of g_3. Therefore, in order to vanish the highest order of $d(s_n)/p(s_n^\mu)$, those of g_1 and g_2 must be equal. Recall that the highest order of g_2 is $-my_1$. Then $y_1 = \mu/m$ and from (3.2), we obtain

$$\frac{\tau^\mu B_r^0}{(\jmath 2\pi)^\mu}n^{-\mu} + \frac{(-1)^m A_r^m}{m!}\nu_1^m n^{-my_1} + o(n^{-\mu}) = 0,$$

then (5.2) holds, which completes the proof. ∎

It is interesting to see that, in this case, the order in n of $\nu_{n,1}$ is no longer a multiple of μ as in the cases of single chains but $-\mu/m$.

Also, note that (5.1) and (5.2) are identical to (4.2) for $m = 1$.

Although approximations of poles of single and multiple chains seem to share a similar form, we will show in the next corollary that they have a different position relative to their asymptotic axis.

COROLLARY 5.2. *Let $G(s)$ be a fractional neutral delay system defined by (2.1). If a root r of multiplicity $m > 1$ of the formal polynomial $\widetilde{c}_d$ defined in (2.3) satisfies (4.1), then there exist neutral chains of poles on both sides of the corresponding asymptotic axis $\Re(s) = -\ln|r|/\tau$.*

Proof. Under the assumptions, $\nu_{n,1}$ is given by (5.1) and (5.2) for neutral chains relative to r.

Recall from Remark 3 that the location of poles of large modulus around the asymptotic axis is decided by the sign of $\Re(\nu_1)$ in the upper half-plane, i.e. $n > 0$, and by the sign of $(\Re(\nu_1)\cos(\mu\pi/m) + \Im(\nu_1)\sin(\mu\pi/m))$ in the lower half-plane, i.e. $n < 0$.

First, we consider $\Re(\nu_1)$. Note that the equation of ν_1^m (5.2) has m distinct roots that are equally distributed on a circle centered at the origin in the complex plane.

If $m \geq 3$, it is obvious that there exist both roots with positive and negative real part.

If $m = 2$, the two roots are symmetric with respect to the origin. Hence, there is always one root with positive real part and the other root with negative real part except for the case of two purely imaginary roots.

In that case, $\Re(\nu_1) = 0$ and $\Im(\nu_1) = \pm c \neq 0$. Hence, $\Re(\nu_1)\cos(\mu\pi/m) + \Im(\nu_1)\sin(\mu\pi/m) = \pm c' \neq 0$ and thus in the lower half-plane there are one chain on the left and one chain on the right of the asymptotic axis. ∎

In conclusion, if any multiple root of modulus one of (2.3) satisfies the condition (4.1), then the system is unstable. Clearly, this condition does not depend on τ and μ, with $0 < \mu < 1$.

In the next subsections, we progress in the analysis of the remaining cases and we start in Subsection 5.2 with the case of $B_r^0 = 0$.

5.2. The case where $m \geq 2$, $B_r^0 = 0$, $B_r^1 \neq 0$, $C_r^0 \neq 0$.

In the previous case, all neutral chains relative to the same root r of (2.3) approach the asymptotic axis at the same rate since the corresponding approximation terms have the same order in n. This may no longer occur for the current case as well as for other cases that we will study later.

THEOREM 5.3. *Let $G(s)$ be a neutral delay system defined by (2.1), and suppose that one of the roots of the formal polynomial $\widetilde{c}_d(z)$ defined in (2.3) has multiplicity $m > 1$. If this root, denoted by r, satisfies*

$$B_r^0 = 0, \tag{5.3}$$

$$B_r^1 \neq 0, \tag{5.4}$$

$$C_r^0 \neq 0, \tag{5.5}$$

then, for large enough $n \in \mathbb{Z}$, poles of neutral chains relative to those m identical roots are approximated by

$$s_n\tau = \lambda_n + \nu_{n,1} + o(n^{-y_1}),$$

with λ_n given by (2.5) and

$$\nu_{n,1} = \nu_1 n^{-y_1},$$

where for $m = 2$, $y_1 = \mu$ and ν_1 satisfies the equation

$$\frac{A_r^2}{2}\nu_1^2 - \frac{\tau^\mu B_r^1}{(\jmath 2\pi)^\mu}\nu_1 + \frac{\tau^{2\mu}}{(\jmath 2\pi)^{2\mu}}C_r^0 = 0, \tag{5.6}$$

and for $m \geq 3$, (y_1, ν_1) takes m different pair of values below

$$y_1 = \mu, \qquad \nu_1 = \frac{\tau^\mu C_r^0}{(\jmath 2\pi)^\mu B_r^1}, \tag{5.7}$$

$$y_1 = \frac{\mu}{m-1}, \quad \nu_1^{m-1} = \frac{(-1)^m m! \tau^\mu B_r^1}{(\jmath 2\pi)^\mu A_r^m}, \tag{5.8}$$

the latter gives $m-1$ different values of ν_1 corresponding to the same value of $y_1 = \mu/(m-1)$.

Proof. From the conditions (5.3)-(5.5), we deduce that the highest orders in n of g_1, g_2, and g_3, which are given by (3.3), (3.4), and (3.5), are -2μ, $-my_1$, and $-\mu - y_1$ respectively.

The following cases may occur in order to eliminate the terms of highest order of the denominator at s_n

$$2\mu = my_1 < \mu + y_1 \tag{5.9}$$

$$2\mu = \mu + y_1 < my_1 \tag{5.10}$$

$$my_1 = \mu + y_1 < 2\mu \tag{5.11}$$

$$my_1 = \mu + y_1 = 2\mu \tag{5.12}$$

The case (5.9) is eliminated as it cannot be satisfied for $m \geq 2$.

The case (5.12) is equivalent to $y_1 = \mu$, $m = 2$ and, from (3.2), we have

$$\frac{\tau^{2\mu} C_r^0}{(\jmath 2\pi)^{2\mu}} n^{-2\mu} + \frac{A_r^2}{2} \nu_1^2 n^{-2\mu} - \frac{\tau^\mu B_r^1}{(\jmath 2\pi)^\mu} \nu_1 n^{-2\mu} + o(n^{-2\mu}) = 0$$

and then (5.6) follows immediately.

When $m > 2$, it is easy to see that both (5.10) and (5.11) are satisfied.

From (5.10), we deduce that $y_1 = \mu$ and thus (3.2) can be rewritten as

$$\frac{\tau^{2\mu} C_r^0}{(\jmath 2\pi)^{2\mu}} n^{-2\mu} - \frac{\tau^\mu B_r^1}{(\jmath 2\pi)^\mu} \nu_1 n^{-2\mu} + o(n^{-2\mu}) = 0,$$

giving one value ν_1 in (5.7).

Other values of ν_1 are derived from the case (5.11), where $y_1 = \frac{\mu}{m-1}$. In turn, (3.2) becomes

$$\frac{(-1)^m A_r^m}{m!} \nu_1^m n^{-my_1} - \frac{\tau^\mu B_r^1}{(\jmath 2\pi)^\mu} \nu_1 n^{-(\mu+y_1)} + o(n^{-(\mu+y_1)}) = 0,$$

giving $m-1$ non-zero values of ν_1 in (5.8). □

REMARK 5. *A previous version of the above theorem was stated in [15]. However, the result about $\nu_{n,1}$ for the case $m \geq 3$ was incomplete. Indeed, only the value (5.7) of $\nu_{n,1}$ was given and the others values with different order in n were missing.*

As we have seen from (5.7) and (5.8) in the above theorem, due to different order of $\nu_{n,1}$, the chains of poles relative to a multiple root r with $m \geq 3$ approach the asymptotic axis with different rates. An example of such a system is given in §6, Example 3.

We recognize that ν_1^{m-1}, $m \geq 3$ in (5.8) has the same pattern as ν_1^m, $m \geq 2$ in (5.2), leading to the same conclusion on stability.

COROLLARY 5.4. *Let $G(s)$ be a neutral delay system defined by (2.1), and suppose that at least one root r of the formal polynomial $\widetilde{c}_d(z)$ defined in (2.3) has multiplicity $m \geq 3$, satisfies (5.3) and (5.4). Then there exist neutral chains of poles on both sides of the asymptotic axis $\Re(s) = -\ln|r|/\tau$.*

Proof. The proof is similar to the one of Corollary 5.2. □

REMARK 6. *The condition (5.5) is omitted in the above corollary. Indeed, whether or not the condition holds does not affect ν_1^{m-1} in (5.8), and thus the existence of a neutral chain on the right of the asymptotic axis. Furthermore, the result in the corollary does not depend on τ and μ, with $0 < \mu < 1$.*

Under the conditions in Theorem 5.3, two chains relative to r of multiplicity two may both lie on the left of the asymptotic axis. We will see such a system later in §6, Example 2. Therefore, the complementary condition to ensure H_∞-stability of the system in that situation is the objective of the following proposition.

PROPOSITION 5.5. *Let $G(s)$ be a neutral delay system defined by (2.1), and suppose that the formal polynomial $\widetilde{c}_d(z)$ defined in (2.3) has at least one root of modulus one of multiplicity two, the other roots being of modulus strictly greater than one. We also suppose that each root of modulus one of $\widetilde{c}_d(z)$ satisfies (5.3)-(5.5). If G has no unstable poles of small modulus and $\Re(\nu_{n,1}) < 0$ then G is H_∞-stable if and only if* $\deg p \geq \deg t + 2$.

Proof. Under the assumptions, all the poles of $G(s)$ are in the open left half-plane. Now, $G(s)$ is H_∞-stable if and only if $G(s)$ is bounded on the imaginary axis. Therefore, let us consider the magnitude of $G(s)$ on the imaginary axis by first examining its denominator $d(s)$.

Let $s = s_n + \eta_n \in j\mathbb{R}$, where s_n is one of poles of the neutral chain relative to a root r of modulus one and of multiplicity two of $\widetilde{c}_d(z)$. Recall that $s_n = (\lambda_n + \nu_1 n^{-\mu})/\tau + o(n^{-\mu})$ and note that $\Re(\lambda_n) = 0$. Since $\Re(\nu_{n,1}) \neq 0$, then η_n is at least of order $n^{-\mu}$. In this case, we can write $\eta_n = \eta n^{-\mu} + o(n^{-\mu})$, and thus $s = [\lambda_n + (\nu_1 + \eta\tau)n^{-\mu}]/\tau + o(n^{-\mu})$, which is of the same form as s_n recalled earlier if we replace $\nu_1' = \nu_1 + \eta\tau$.

Therefore, the developments of the denominator of G around s and s_n are the same. Note that the development of $d(s_n)$ as $|s_n| \to \infty$ is obtained from (3.2) by collecting terms of highest order of g_1, g_2, g_3 as follows

$$d(s_n) = p(s_n^\mu)\left(\frac{f(\nu_1)}{n^{2\mu}} + o(n^{-2\mu})\right)$$

where $f(\nu_1)$ is the left expression of (5.6). Similarly, $d(s)$ as $|s| \to \infty$, $s \in j\mathbb{R}$ near s_n is given by

$$d(s) = p(s^\mu)\left(\frac{f(\nu_1 + \eta\tau)}{n^{2\mu}} + o(n^{-2\mu})\right)$$

Now, we will prove that $f(\nu_1 + \eta\tau) \neq 0$. Let us denote $\nu_1^{(1)}$ and $\nu_1^{(2)}$ two roots of $f(\nu_1)$ and first consider $f(\nu_1^{(1)} + \eta\tau)$. We see that $f(\nu_1^{(1)} + \eta\tau) = 0$ if and only if $\nu_1^{(1)} + \eta\tau = \nu_1^{(2)}$, which is in turn equivalent to

$$\eta = \frac{\nu_1^{(2)} - \nu_1^{(1)}}{\tau}. \tag{5.13}$$

However, this condition cannot be satisfied because

$$\Re(\eta) \neq \frac{\Re(\nu_1^{(2)}) - \Re(\nu_1^{(1)})}{\tau}. \tag{5.14}$$

Indeed, $\Re(\eta) = -\Re(\nu_1^{(1)})/\tau$ since $s \in j\mathbb{R}$ and $\Re(\nu_2^{(2)}) \neq 0$ under the assumption $\Re(\nu_{n,1}) < 0$. Therefore, $f(\nu_1^{(1)} + \eta\tau) \neq 0$. Similarly, we can prove that $f(\nu_1^{(2)} + \eta\tau) \neq 0$. Hence, the order in n^μ of the denominator of $G(s)$ is $\deg p - 2$. □

While systems considered in the previous subsection were all unstable, we have been able to find in this subsection systems with multiple chains asymptotic to the imaginary axis which are H_∞-stable. We will then continue our analysis for another case in order to see in which situation there may exist H_∞-stable systems.

5.3. The case where $m \geq 2$, $B_r^0 = 0$, $B_r^1 = 0$, $B_r^2 \neq 0$, and $C_r^0 \neq 0$.

As for the previous cases, pole location is considered first.

THEOREM 5.6. *Let $G(s)$ be a neutral delay system defined by (2.1), and suppose that one of the roots of the formal polynomial $\widetilde{c}_d(z)$ defined in (2.3) has multiplicity $m > 1$. If this root, denoted by r, satisfies*

$$B_r^0 = 0,$$

$$B_r^1 = 0,$$

$$B_r^2 \neq 0, \tag{5.15}$$

$$C_r^0 \neq 0, \tag{5.16}$$

then, for large enough $n \in \mathbb{Z}$, poles of neutral chains relative to those m identical roots are approximated by

$$s_n\tau = \lambda_n + \nu_{n,1} + o(n^{-y_1}),$$

with λ_n given by (2.5) and

$$\nu_{n,1} = \nu_1 n^{-y_1},$$

where for $2 \leq m \leq 3$, $y_1 = 2\mu/m$ and

$$\nu_1^m = \frac{(-1)^{m+1} m! \tau^{2\mu} C_r^0}{(\jmath 2\pi)^{2\mu} A_r^m}, \tag{5.17}$$

for $m = 4$, $y_1 = \mu/2$ and ν_1 satisfies

$$\frac{\nu_1^4}{4!} A_r^4 + \frac{\nu_1^2 \tau^\mu}{2(\jmath 2\pi)^\mu} B_r^2 + \frac{\tau^{2\mu}}{(\jmath 2\pi)^{2\mu}} C_r^0 = 0, \tag{5.18}$$

and for $m \geq 5$, (y_1, ν_1) takes one of m different pairs of values

$$y_1 = \frac{\mu}{2}, \qquad \nu_1^2 = -\frac{2\tau^\mu C_r^0}{(\jmath 2\pi)^\mu B_r^2}, \tag{5.19}$$

$$y_1 = \frac{\mu}{m-2}, \qquad \nu_1^{m-2} = \frac{(-1)^{m+1} m! \tau^\mu B_r^2}{2(\jmath 2\pi)^\mu A_r^m}. \tag{5.20}$$

REMARK 7. *Here again it is clear that (5.17) gives rise to m different complex values of ν_1, (5.18) gives rise to four, (5.19) to two, and (5.20) to $m-2$.*

Proof. Under the assumptions, the terms of highest order in n of g_1, g_2, and g_3, which are given by (3.3), (3.4), and (3.5), are -2μ, $-my_1$, and $\max\{-\mu - 2y_1, -2\mu - y_1\}$ respectively. Obviously, we just need to compare the first three orders as $-2\mu > -2\mu - y_1$. For further steps to determine y_1 and ν_1, which are the same as in the proof of Theorem 5.3, see Appendix C. ∎

REMARK 8. *Note that in the case $2 \leq m \leq 3$, condition (5.15) is not necessary and in the case $m \geq 5$ condition (5.16) is not necessary (and we may conclude as well on the presence of chains of poles in the right half-plane).*

Some quick observation leads to the following conclusions on the stability of the system in the current case.

COROLLARY 5.7. *Let $G(s)$ be a neutral delay system defined by (2.1), and suppose that at least one root of the formal polynomial $\widetilde{c}_d(z)$ defined in (2.3) has multiplicity $m \geq 2$, satisfies $B_r^0 = 0$, $B_r^1 = 0$ and*

- *for* $2 \le m \le 3$, $C_r^0 \neq 0$,
- *for* $m = 4$, $B_r^2 \neq 0$ *and* $C_r^0 \neq 0$
- *for* $m \ge 5$, $B_r^2 \neq 0$.

Then there exist neutral chains of poles on both sides of the asymptotic axis $\Re(s) = -\ln|r|/\tau$.

Proof. For $2 \le m \le 3$ and $m \ge 5$, the proof is similar to that of Corollary 5.2. Now, we consider the case of $m = 4$. By replacing $\nu_1^2 = x$ in (5.18), we obtain

$$\frac{x^2}{4!}A_r^4 + \frac{x\tau^\mu}{2(\jmath 2\pi)^\mu}B_r^2 + \frac{\tau^{2\mu}}{(\jmath 2\pi)^{2\mu}}C_r^0 = 0$$

Let us denote x_1 and x_2 the two roots of the above equation. Equation (5.18) has at least one value of ν_1 with positive real part except the case where both roots x_1, x_2 are negative. However, we will demonstrate that this case does not exist.

The two roots of the equation satisfy

$$x_1 + x_2 = -\frac{12\tau^\mu B_r^2}{(\jmath 2\pi)^\mu A_r^4} = -\frac{12\tau^\mu}{(\jmath 2\pi)^\mu}\hat{K}_r$$

where $\hat{K}_r = B_r^2/A_r^4$.

We consider $r \in \mathbb{R}$ and $r \in \mathbb{C}\backslash\mathbb{R}$.

If r is real, then $x_1 + x_2$ is not real. Therefore, x_1 and x_2 cannot be both real.

If r is not real, then $\bar{r}$ is also a root of (2.3). Denote x_1' and x_2' roots corresponding to $\bar{r}$. Hence, they satisfy

$$x_1' + x_2' = -\frac{12\tau^\mu}{(\jmath 2\pi)^\mu}\overline{\hat{K}_r}.$$

Therefore

$$x_1 + x_2 + x_1' + x_2' = -\frac{24\tau^\mu}{(\jmath 2\pi)^\mu}\Re(\hat{K}_r),$$

indicating that x_1, x_2, x_1', and x_2' cannot be all real. ∎

Corollary 5.7 shows that if (2.3) possesses a multiple root of modulus one satisfying the conditions in Theorem 5.6 then the system is unstable.

6. Examples. The numerical examples in this section illustrate several results obtained in the preceding sections. They are however proposed for illustrative purpose only and do not come from models found in the literature.

Example 1. First, we consider the system with the transfer function given by

$$G_1(s) = \frac{s^\mu + 1}{s^{2\mu} + (-1.9s^{2\mu} + s^\mu)e^{-s} + (s^{2\mu} - s^\mu + 0.3)e^{-2s}}.$$

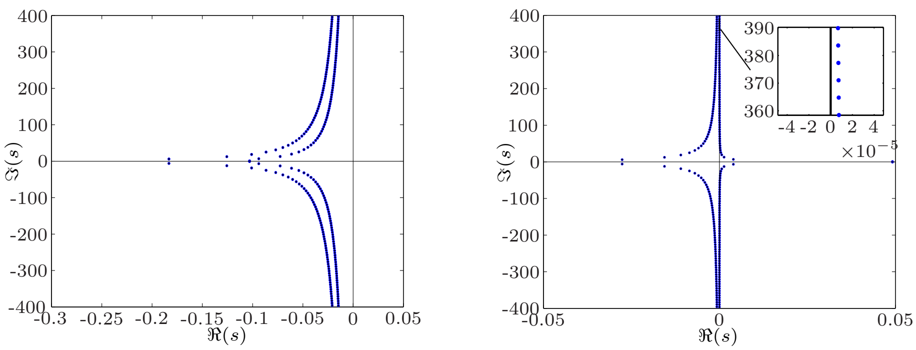
For this system, the fractional order is μ and the delay is $\tau = 1$. It is easy to see that the coefficients of the development $q_k(s^\mu)/p(s^\mu)$ are $\alpha_1 = -1.9$, $\beta_1 = 1$, $\alpha_2 = 1$, $\beta_2 = -1$, and thus the formal polynomial is $\widetilde{c}_d(z) = 1 - 1.9z + z^2$, which has two complex conjugate roots $r = (19 \pm \jmath\sqrt{39})/20$ of multiplicity $m = 1$. Since $|r| = 1$ for each r, then the asymptotic axis defined by $\Re(s) = -\ln|r|/\tau$ is the imaginary axis.

As $\sum_{k=1}^{2} \beta_k r^k \neq 0$ for both r, Theorem 4.1 is applied.

If $\mu = 0.5$, from Theorem 4.1 we obtain $\nu_{n,1} = (-0.1636 + \jmath 0.1185)/n^{0.5}$ for $r = (19 + \jmath\sqrt{39})/20$ and $\nu_{n,1} = (-0.1185 + \jmath 0.1636)/n^{0.5}$ for $r = (19 - \jmath\sqrt{39})/20$.

Therefore, in the upper half-plane, i.e. $n > 0$, the two neutral chains of poles are on the left of the imaginary axis. So are the chains in the lower half-plane since poles of $G(s)$ are symmetric about the real axis, which is due to the fact that the denominator of $G(s)$ is a quasi-polynomial with real coefficients.

The same conclusion about the location of neutral poles can be drawn using Corollary 4.3. The critical value of μ is $\mu_c = (2/\pi)\arctan(-\Re(K_r)/|\Im(K_r)|) = 0.8989$ with $K_r = \sum_{k=1}^{2}\beta_k r^k / \sum_{k=1}^{2} k\alpha_k r^k$. Recall that μ_c is the same for both r. Since $\mu = 0.5 < \mu_c$, then the two neutral chains of poles relative to r are on the left of the imaginary axis as we can see in Figure 1a.



(a) $\mu = 0.5$ (b) $\mu = 0.9$

Fig. 1: Neutral chains of poles for $G_1(s)$

In addition, all poles of small modulus of the system are in the open left half-plane. Then Proposition 4.4 shows that G is H_∞-stable since $\deg t = \deg p - 1$. Indeed, $G(s)$ is bounded on the imaginary axis, which can be seen in Figure 2.

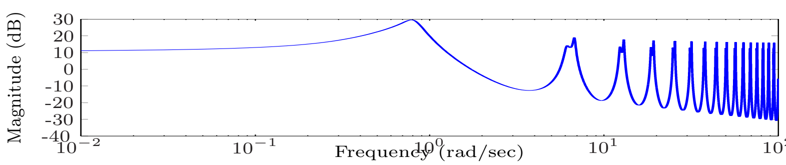


Fig. 2: Bode diagram for $G_1(s)$

If $\mu = 0.9 > \mu_c$, then chains of poles go to the right of the imaginary axis (see Figure 1b).

Example 2. The system is given by

$$G_2(s) = (s + (-2s + s^{0.5} + 0.25)e^{-s} + (s - s^{0.5})e^{-2s})^{-1}.$$

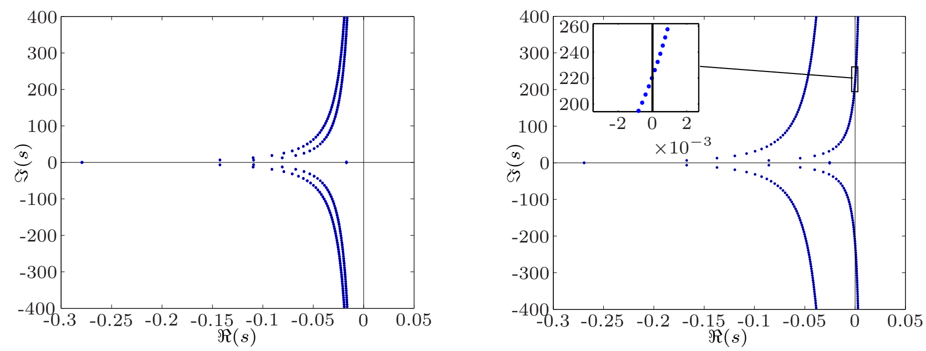
We see that the delay is $\tau = 1$ and the fractional order is $\mu = 0.5$. The polynomial $\widetilde{c}_d(z)$ given in (2.3) has root $r = 1$ of multiplicity two, then the system has two chains of poles asymptotic to the imaginary axis. The system satisfies $\sum_{k=1}^{2}\beta_k r^k = 0$, $\sum_{k=1}^{2} k\beta_k r^k \neq 0$, and $\sum_{k=1}^{2}\gamma_k r^k \neq 0$, then Theorem 5.3 is applied. Equation (5.6) has a double root, which gives $\nu_{n,1} = (-0.1410 + 0.1410\jmath)/n^{0.5}$ for $n \to +\infty$. Therefore, the two neutral chains are on the left of the imaginary axis.

If some parameters of $G_2(s)$ change slightly, the system might fail to satisfy the condition $\sum_{k=1}^{2}\beta_k r^k = 0$, and thus is no longer stable due to Corollary 5.2. This remark fits in the following system

$$G_2'(s) = (s + (-2s + s^{0.5} + 0.25)e^{-s} + (s - (1+\Delta)s^{0.5})e^{-2s})^{-1}.$$

If $\Delta \neq 0$, then $\sum_{k=1}^{2}\beta_k r^k \neq 0$, thus Corollary 5.2 states that the system has a chain of poles in the right half-plane.

We observe the chains of poles of $G_2(s)$ and $G_2'(s)$ with $\Delta = 0.01$ in Figure 3a and 3b. The unstable chain of G_2' crosses the imaginary axis from left to right.



(a) $G_2(s)$ (b) $G_2'(s)$ with $\Delta = 0.01$

Fig. 3: Neutral chains of poles

Proposition 5.5 shows that $G_2(s)$ is stable in the sense of H_∞-stability. Indeed, the system does not have unstable poles and is bounded on the imaginary axis (see Figure 4). Clearly, the system defined by $(s^{0.5}+1)G_2(s)$ is unstable since the order of the numerator is too high making the transfer function unbounded on the imaginary axis (see Figure 5).

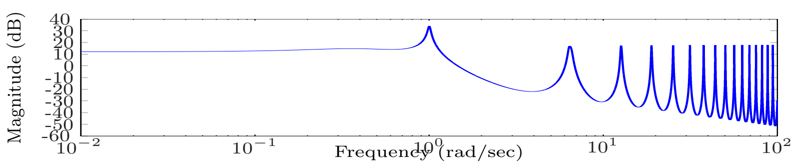


Fig. 4: Bode diagram for $G_2(s)$

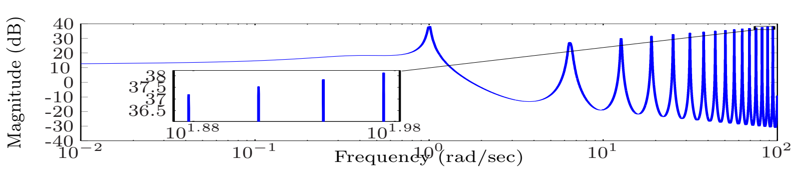


Fig. 5: Bode diagram for $(s^{0.5}+1)G_2(s)$

Example 3. We consider the system with the transfer function given by

$$G_3(s) = (s^{0.8} + (-3s^{0.8} + 3s^{0.4} + 1)e^{-s} + (3s^{0.8} - 5s^{0.4} + 2)e^{-2s} \\ +(-s^{0.8} + 2s^{0.4} + 3)s^{-3s})^{-1}.$$

Here, we see that $\mu = 0.4$, $\tau = 1$, and $\widetilde{c}_d(z) = 1 - 3z + 3z^2 - 1$ with a root $r = 1$ of multiplicity $m = 3$. Therefore, the chains of poles approach the imaginary axis.

Since $\sum_{k=1}^{3} \beta_k r^k = 0$, $\sum_{k=1}^{3} k\beta_k r^k = -1$, and $\sum_{k=1}^{3} \gamma_k r^k = 6$, then Theorem 5.3 is applied. More precisely, since $m = 3$, we obtain from (5.7) and (5.8) three values of $\nu_{n,1}$, which are $(0.2140 + \jmath 0.6585)/n^{0.2}$, $(-0.2140 - \jmath 0.6585)/n^{0.2}$, and $(-2.3272 + \jmath 1.6908)/n^{0.4}$. Therefore, the system has one chain of poles on the right and two chains on the left of the imaginary axis, which are shown in Figure 6. It is interesting to note that one chain approaches the imaginary axis faster than the other two, which is due to different orders of $\nu_{n,1}$.

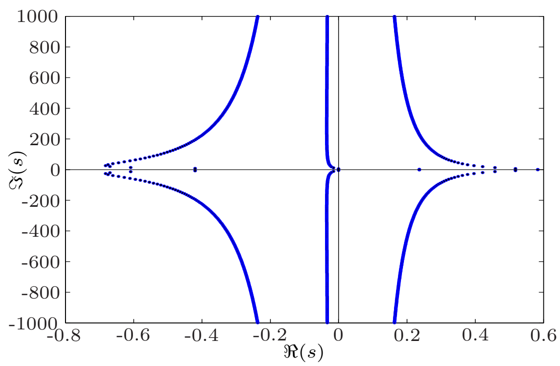


Fig. 6: Neutral chains of poles for $G_3(s)$

7. Conclusion. Fractional delay systems of neutral types where poles approach the imaginary axis is delicate for stability analysis. In this paper, we answer the stability question in the sense of H_∞-stability for a large class of systems, in particular systems with multiple chains asymptotic to the imaginary axis, and the necessary and sufficient conditions obtained are related not only to the location of poles w.r.t. the imaginary axis but also the relative order between the numerator and the denominator of the transfer function. These results will also be of use to decide on H_∞-stabilizability of several classes of fractional delay systems by rational or fractional controllers (with delays). The deployed method can be used for other cases which are not examined here. However, it requires time and effort for each particular system. With the aim of integrating the results into a computation software, we currently attempt to find approximations of neutral poles for general cases, both fractional and classical systems.

Some small changes in the fractional orders and the delays may lead to the class of systems with non-commensurate fractional orders and non-commensurate delays. The distribution of poles of these systems is complicated and different from the systems considered in this paper. For example, for integer-order systems with non-commensurate delays, poles are located in some vertical strips and no longer asymptotic to some vertical lines [2]. Therefore, other techniques will be needed to handle these systems.

Appendix A. Series expansion of the characteristic equation.

As $|s_n| \to \infty$, by using (2.2), we develop (3.1) as follows

$$\frac{d(s_n)}{p(s_n^\mu)} = 1 + \sum_{k=1}^{N} \left(\alpha_k + \frac{\beta_k}{s_n^\mu} + \frac{\gamma_k}{s_n^{2\mu}} + \frac{\delta_k}{s_n^{3\mu}} + \frac{\epsilon_k}{s_n^{4\mu}} + o(s_n^{-4\mu}) \right) e^{-ks_n\tau} = 0. \tag{A.1}$$

Note that $e^{-\lambda_n} = r$ and $e^{-k\nu_{n,i}} = 1 + \sum_{l=1}^{\left[\frac{4\mu}{y_i}\right]} \frac{(-1)^l \nu_i^l k^l}{l!} n^{-ly_i} + o(n^{-4\mu})$.

Thus when n is large enough, (A.1) becomes

$$1 + \sum_{k=1}^{N} \left(\alpha_k + \frac{\beta_k \tau^\mu}{(\jmath 2\pi)^\mu} n^{-\mu} \left(1 + O(n^{-1})\right) + \frac{\gamma_k \tau^{2\mu}}{(\jmath 2\pi)^{2\mu}} n^{-2\mu} \left(1 + O(n^{-1})\right) + \frac{\delta_k \tau^{3\mu}}{(\jmath 2\pi)^{3\mu}} n^{-3\mu} \right.$$
$$\left. + \frac{\epsilon_k \tau^{4\mu}}{(\jmath 2\pi)^{4\mu}} n^{-4\mu} + o(n^{-4\mu}) \right) r^k \prod_{i=1}^{M} \left(1 + \sum_{l=1}^{\left[\frac{4\mu}{y_i}\right]} \frac{(-1)^l \nu_i^l k^l}{l!} n^{-ly_i} + o(n^{-4\mu}) \right) = 0$$

and we obtain

$$1 + \sum_{k=1}^{N} \left(\alpha_k + \frac{\beta_k \tau^\mu}{(\jmath 2\pi)^\mu} n^{-\mu} \left(1 + O(n^{-1})\right) + \frac{\gamma_k \tau^{2\mu}}{(\jmath 2\pi)^{2\mu}} n^{-2\mu} \left(1 + O(n^{-1})\right) \right.$$
$$\left. + \frac{\delta_k \tau^{3\mu}}{(\jmath 2\pi n)^{3\mu}} n^{-3\mu} + \frac{\epsilon_k \tau^{4\mu}}{(\jmath 2\pi n)^{4\mu}} n^{-4\mu} + o(n^{-4\mu}) \right) r^k$$
$$\times \left(1 + \sum_{(l_1,\ldots,l_M) \in \mathcal{L}(4\mu)} \frac{(-1)^{\sum_{i=1}^M l_i} \left(\prod_{i=1}^M \nu_i^{l_i}\right) k^{\sum_{i=1}^M l_i}}{\left(\prod_{i=1}^M l_i!\right)} n^{-\sum_{i=1}^M l_i y_i} + o(n^{-4\mu}) \right) = 0$$

where $\mathcal{L}(x)$ is defined as in (3.6). After simple computations, we get (3.2).

Appendix B.

LEMMA A.1. *Let r be a root of multiplicity $m > 1$ of $f(z) = 1 + \sum_{k=1}^N \alpha_k z^k$, where $\alpha_k \in \mathbb{C}$. Then $\sum_{k=1}^N k^l \alpha_k r^k = 0$ for $l = 1, \ldots, m-1$ and $\sum_{k=1}^N k^m \alpha_k r^k \neq 0$.*

Proof. Since $z = r$ is a root of multiplicity m of $f(z) = 1 + \sum_{k=1}^N \alpha_k z^k$, then it is not difficult to see that $z = r$ is also a root of multiplicity m of $f_l(z) = z^l f(z)$ with $l = 1, \ldots, m-1$.

For $l = 1$, taking the derivative of $f_1(z) = z + \sum_{k=1}^N \alpha_k z^{k+1}$, we obtain $f_1'(z) = 1 + \sum_{k=1}^N \alpha_k z^k + \sum_{k=1}^N k\alpha_k z^k$. Since $f_1'(r) = 0$ and $1 + A_r^0 = 0$, then $A_r^1 = 0$.

Now, assume that $\sum_{k=1}^N k^l \alpha_k r^k = 0$ for $1 \leq l \leq a$ where $1 \leq a \leq m-1$.

For $l = a+1$, we have $f_{a+1}^{(a+1)}(z) = (a+1)! + \sum_{k=1}^N (k+1)(k+2)\ldots(k+a+1)\alpha_k z^k$. It is not difficult to see that $f_{a+1}^{(a+1)}(r)$ after being expanded contains the term $(a+1)!(1 + A_r^0)$, the terms $\sum_{k=1}^N k^l \alpha_k r^k$ for $1 \leq l \leq a$, which are zeros, and $\sum_{k=1}^N k^{a+1} \alpha_k r^k$. Since $f_{a+1}^{(a+1)}(r) = 0$, we derive $\sum_{k=1}^N k^{a+1} \alpha_k r^k = 0$.

For $l = m$, that is $a = m-1$, since $f_m^m(r) \neq 0$, then $\sum_{k=1}^N k^m \alpha_k r^k \neq 0$. ∎

Appendix C. Complete proof of Theorem 5.6.

Proof. Under the assumptions, the terms of highest order in n of g_1, g_2, and g_3, which are given by (3.3), (3.4), and (3.5), are -2μ, $-my_1$, and $\max\{-\mu -$

$2y_1, -2\mu - y_1\}$ respectively. Obviously, we just need to compare the first three orders as $-2\mu > -2\mu - y_1$. Hence, the following cases may occur for the highest order of the development of the denominator at s_n: (i) $2\mu = my_1 < \mu + 2y_1$, (ii) $2\mu = \mu + 2y_1 < my_1$, (iii) $my_1 = \mu + 2y_1 < 2\mu$, (iv) $2\mu = my_1 = \mu + 2y_1$. These cases are respectively equivalent to (i) $y_1 = 2\mu/m$ and $m < 4$, (ii) $y_1 = \mu/2$ and $m > 4$, (iii) $y_1 = \mu/(m-2)$ and $m > 4$, (iv) $y_1 = \mu/2$ and $m = 4$. Hence, by replacing these values of y_1 into (3.2) and noting that the term of highest order there vanishes, we obtain (5.17), (5.19), (5.20), and (5.18) respectively. □

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