SIR model with time dependent infectivity parameter : approximating the epidemic attractor and the importance of the initial phase.

Stefanella Boatto, Catherine Bonnet, Bernard Cazelles, Frédéric Mazenc

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Abstract We consider a SIR model with birth and death terms and time-varying infectivity parameter $\beta(t)$. In the particular case of a sinusoidal parameter, we show that the average Basic Reproduction Number $\bar{R}_o$, introduced in [Bacaër & Guernaoui, 2006], is not the only relevant parameter and we emphasize the rôle played by the initial phase, the amplitude and the period. For a (general) periodic infectivity parameter $\beta(t)$ a periodic orbit exists, as already proved in [Katriel, 2014]. In the case of a slowly varying $\beta(t)$ an approximation of such a solution is given, which is shown to be asymptotically stable under an extra assumption on the slowness of $\beta(t)$. For a non necessarily periodic $\beta(t)$, all the trajectories of the system are proved to be attracted into a tubular region around a suitable curve, which is then an approximation of the underlying attractor. Numerical simulations are given.

Keywords SIR model · Time-varying Transmission · Seasonality · Lyapunov function · Stability

Stefanella Boatto
Dep. de Matemática Aplicada, Instituto de Matemática, Universidade Federal de Rio de Janeiro, Av. Athos da Silveira Ramos 149, Centro de Tecnologia Bloco C, Cidade Universitária Ilha do Fundão, Caixa Postal 68530, 21941-909 , Rio de Janeiro, RJ, Brazil
& Inria, L2S, CentraleSupélec, 3 rue Joliot Curie, 91192 Gif-sur-Yvette, France
E-mail: boatto.stefanella@gmail.com
ORCID : orcid.org/0000-0001-5708-0281

Catherine Bonnet
Inria, L2S (CNRS) CentraleSupélec, 3 rue Joliot Curie, 91192 Gif-sur-Yvette, France
E-mail: catherine.bonnet@l2s.centralesupelec.fr

Bernard Cazelles
IBENS, UMR 8197 CNRS-ENS, Ecole Normale Supérieure, Paris, France and International Center for Mathematical and Computational Modeling of Complex Systems (UMMISCO), UPMC-IRD, Bondy, France
E-mail: cazelles@biologie.ens.fr

Frédéric Mazenc
Inria, L2S (CNRS) CentraleSupélec, 3 rue Joliot Curie, 91192 Gif-sur-Yvette, France
E-mail: Frederic.MAZENC@lss.supelec.fr

1 Introduction

Our world faces emerging and re-emerging diseases and it has been shown that this has intensified over the past 50 years, due, in part to climate change and to the development of intensive breeding, transportation and urbanization [15]. Mathematical models offer valuable tools for synthesizing information to understand observed epidemiological patterns, for testing different hypotheses on the underlying key mechanisms. Since Kermack and McKendrick initiated a famous SIR type of mathematical model in 1927 [19], mathematical modeling has become an important tool in the study of the dynamics of epidemic diseases in order to reveal the underlying mechanisms that influence the transmission of these diseases [4, 14, 18]. Moreover, mathematical models play a crucial role in infectious disease prevention by assessing the impact of different control measures, e.g. vaccination strategies [13].

Numerous factors contribute to the propagation of an emerging disease, including increased human connectivity and changeable human behavior, limited available economic resources for adequate vaccination campaigns, increasing antimicrobial resistance, evolution of the dominant strains, uncontrolled sylvatic transmission cycles and increasing parasite and vector resistance to the most widely used drugs and insecticides, etc. A key factor of this huge complexity is non-stationarity [7], meaning that the characteristics of the dynamical epidemiological processes evolve with time. One of the classic aspects of non-stationarity, is the seasonality of epidemiological dynamics, linked to climate but also to human behavior (e.g. school closure) [1, 11]. This is the case for instance, for flu [9], childhood infections as measles [23, 5], rotavirus[21], respiratory syncytial virus [25].

Nevertheless, few mathematical studies have taken into account this important aspect and each time they use a time-varying transmission rate (or a time-varying force of infection), mainly taking a sinusoidal function. Stone et al. [28] identified a new threshold effect determined by the populations susceptibility measured after the last outbreak. Then they give clear analytical conditions for predicting the occurrence of either a future epidemic outbreak, or a skip, a year in which an epidemic fails to initiate. Ponciano & Capistrn [26] analyzed the dynamics of seasonally forced SIRS model with the Liu-Hethcote-van den Driessches (LHD) incidence rate function [20]. They showed that the solution of the deterministic SIRS model with LHD incidence rate will reach either the disease free equilibrium or the endemic equilibrium depending on the initial conditions.

In 2006, Bacaer and Guernaoui [3] proposed a generalization of the definition of the Basic Reproduction Number for a vector-borne disease model with a periodic parameter as well as an algorithm to compute it. Bacaër and Gomes [2] studied a SIR model without birth and death terms for which they could describe the limit value of the state in terms of an average Basic Reproduction Number and indeed the initial condition on recovered people. Wang and Zhao introduced [30] for models in periodic environments the basic reproduction ratio (which may coincide with the Basic Reproduction Number of the time-averaged autonomous system) and proved it may represent a threshold for the stability of the disease-free periodic solution.

Here we established a full mathematical analysis of a SIR model with birth and death terms, and time-varying transmission rate β. The situation differs from the one

of [2] in the sense that this model possesses an unique equilibrium point (the trivial one), and, under suitable conditions, a periodic attractor.

This paper which aims at describing the asymptotic behaviour of the solutions is organized as follows. In Section 2, we consider the particular case of a constant infectivity parameter β and provide a complete stability analysis of the model. Those results are classical, however we propose here a new proof with a Lyapunov technique which will be of use in the more complicated case of a time-varying infectivity parameter. Section 3 considers the case of a periodic infectivity parameter β. We first analyse the stability if the equilibrium point of the system for a sinusoidal infectivity parameter, and enlighten the fact that the average Basic Reproduction Number is no longer the only parameter which characterizes an epidemics. Then in case of slowly-varying β, we are able to give an approximation of the periodic solution and under some extra assumptions on the slowness of β to prove the asymptotic stability of this periodic solution. The case of non necessarily periodic infectivity parameter β is studied in Section 4 and it is proved that all trajectories of the system enter in finite time and stay in a a tubular region around a prescribed curve. Those results are illustrated by several simulations and interpretations are given.

2 SIR model with birth and death and a constant infectivity parameter

We assume that for a single node the epidemic dynamics is modeled by the system

$$\frac{dS}{dt} = \mu(N-S) - \beta\frac{SI}{N}, \tag{1}$$

$$\frac{dI}{dt} = \beta\frac{SI}{N} - I(\gamma+\mu), \tag{2}$$

$$\frac{dR}{dt} = I\gamma - \mu R, \tag{3}$$

where $S(t), I(t), R(t)$ are, respectively, the numbers of healthy individuals (also called *susceptible* individuals), infected individuals and recovered individuals, at the time t. The parameter μ is a parameter of birth and death, γ is the recovery rate, N is the total population. These parameters are strictly positive.
The model is based on some simplifying assumptions such as:
– the population is well mixed (ignoring that some infected people might remain at home or in a clinic);
– all individuals are healthy at birth;
– all individuals have the same recovery rate γ;
– the total population is constant : $N = S(t)+I(t)+R(t),\ \forall t \geq 0$.
For these reasons $S, I, R \in [0,N]$ and the third equation of the system of equation above can be replaced by

$$R(t) = N - S(t) - I(t).$$

A consequence of this is that we can reduce the study to the populations of susceptible and infected individuals (S,I). For more details see, for instance, O. Diekmann & JAP. Heesterbeek [8], L. Edelstein-Keshet [10] and H. Weiss [31].

Epidemic interpretation of parameters. The parameter β is related to the probability that an healthy person becomes infected when encountering an infected one. In the case of diseases transmitted by a vector, and more specifically a mosquito, like in the case of Dengue Fever, Zika or Chikungunya, the parameter β is taking into account various factors which depend on the vector and on the environment/ways of leaving :
– the amount of mosquitoes (which might depend also on the climate),
– the lifestyle in each housing (such has having a net at the window, air conditioning, water reservoirs etc.) and the use of any kind of control to mosquitoes (such as repellent, insecticide, mosquito traps etc.),
– topography and climate.

Equilibrium states. One can check readily that the (1)-(2)-(3) system admits two equilibrium states when $\beta > \gamma+\mu$:

$$(S_1^*, I_1^*, R_1^*) = (N,0,0) \quad \text{and} \quad (S_2^*, I_2^*, R_2^*) = \left(\frac{N}{R_o}, \frac{\mu N(R_o-1)}{R_o(\gamma+\mu)}, \frac{\gamma N(R_o-1)}{R_o(\gamma+\mu)} \right)$$

where

$$R_o = \frac{\beta}{\gamma+\mu}.$$

It is worth noticing that the parameter R_o is important in epidemiology: it is the Basic Reproduction Number. For more details, see Appendix C.
It is then natural to inquire about the well-posedness of the problem and the long term behavior of solution. In other words: if the initial data are positive, does the solution remain positive for all time? What are the stability properties of the equilibrium states? For which values of the parameters is the endemic state stable? When is it asymptotically attracting?
The following proposition and theorem are answering these questions.

Proposition 1 *If $(S(0),I(0),R(0))$ belongs to the positive orthant, then the solution $(S(t),I(t),R(t))$ belongs to the positive orthant for all $t \geq 0$. Moreover, if $S(0) \leq N$ and $I(0) \leq N$, then $S(t) \leq N$ and $I(t) \leq N$ for all $t \geq 0$.*

For a proof see, for instance, [31].

Theorem 1 *1. If $R_o < 1$, the point $(N,0,0)$ is unique equilibrium of the system (1)-(2)-(3) and is globally attractive.*
2. If $R_o > 1$, the system (1)-(2)-(3) admits two equilibrium points $(N,0,0)$ and

$$(S^*, I^*, R^*) = \left(\frac{N}{R_o}, \frac{\mu N(R_o-1)}{R_o(\gamma+\mu)}, \frac{\gamma N(R_o-1)}{R_o(\gamma+\mu)} \right).$$

The first point is a saddle point and the second is attractive. Then if $I(0) > 0$, the number of invectives $I(t)$ approaches I^* as t goes to ∞.
3. If $R_o = 1$, the equilibrium point $(N,0,0)$ is globally attractive.

A proof of this theorem can be found, among others, in the lecture notes by H. Weiss [31]. Other references are, for instance, O. Diekmann & JAP. Heesterbeek [8] and L. Edelstein-Keshet [10]. However, in Appendix A, we propose here a new proof of the second item of Theorem 1, (i.e. the case when $R_o > 1$) for the equilibrium point (S^*, I^*, R^*) with $I^* > 0$. This new proof will help us setting up a new Lyapunov function, that we will denote U_3, which turns out to be particularly usefull later on in this article when considering the time periodic infectivity case.

3 Time-varing infection parameter I : a periodic infectivity parameter

We now assume that $\beta(t)$ is a positive continuous time-varying periodic function of period T. We will impose later a slowness condition on $\beta(t)$. Then Eqs (39) become

$$\begin{cases} \dfrac{ds}{dt}(t) = -\beta(t)\, s(t) i(t) + \mu[1 - s(t)], \\ \dfrac{di}{dt}(t) = \beta(t)\, s(t) i(t) - \delta\, i(t), \\ \dfrac{dr}{dt}(t) = \gamma i(t) - \mu\, r(t), \end{cases} \tag{4}$$

where $\delta = \mu + \gamma$, the state is $\mathscr{E} = [0,1] \times [0,1] \times [0,1]$ and, as before, we have the conservation law (40).

3.1 The equilibrium state, finite-time analysis and the importance of the initial phase

It is immediate to check that, if β is time-varying, the system (4) admits a unique equilibrium point $(s^*, i^*, r^*) = (1,0,0)$. The existence of $(s^*, i^*, r^*) = (1,0,0)$ is obvious. Now, suppose (s_e^*, i_e^*, r_e^*) is another equilibrium point. Since $\mu \neq 0$, we deduce that ncessarily $i_e^* \neq 0$. Then we deduce that if there is $t_1 > t_2$ such that $\beta(t_1) \neq \beta(t_2)$, then there is $s_e^* \in \mathbb{R}$ such that $\beta(t_1) s_e^* - \delta = \beta(t_2) s_e^* - \delta$. Consequently, $s_e^* = 0$. This would imply that $\delta = 0$ and $\mu = 0$, which is excluded.

Due to the conservation law (40), we can now restrict our attention to the subsystem

$$\frac{ds}{dt}(t) = -\beta(t)\, s(t) i(t) + \mu[1 - s(t)], \quad \frac{di}{dt}(t) = \beta(t)\, s(t) i(t) - \delta\, i(t), \tag{5}$$

and its equilibrium point $(s^*, i^*) = (1,0)$. Let us linearize (5) around $(s^*, i^*,) = (1,0)$. Writing $s(t) = s^* + \Delta s(t)$ and $i(t) = i^* + \Delta i(t)$, we obtain

$$\frac{d\Delta X}{dt} = A(t) \Delta X, \tag{6}$$

where $\Delta X = (\Delta s, \Delta i)$ and

$$A(t) = \begin{pmatrix} -\mu & -\beta(t) \\ 0 & \beta(t) - (\gamma + \mu) \end{pmatrix} \tag{7}$$

whose eigenvalues are

$$\lambda_1(t) = -\mu \quad \text{and} \quad \lambda_2(t) = \beta(t) - (\gamma+\mu). \tag{8}$$

It is immediate to check that

$$\Delta i(t) = \exp\left\{\int_0^t \lambda_2(\tau)\, d\tau\right\} \Delta i(0).$$

Remark 1 Observe that if $\lambda_2(t) > 0$ (resp. $\lambda_2(t) < 0$) for all $t \in [0,T]$ then $\Delta i(t)$ is increasing (resp. decreasing) in the interval $[0,T]$, while if $\lambda_2(t) = 0$ for some values of $t \in [0,T]$ then various cases are possible.

Now, let us consider the particular case of

$$\beta(t) = \beta_o \left[1 + \varepsilon \sin\left(\frac{2\pi}{T}t + \varphi_o\right)\right], \tag{9}$$

and, following Bacaër and Gomes [2], let us introduce the average Basic Reproduction Number (aBRN)

$$\bar{R}_o = \frac{\beta_o}{\gamma+\mu}. \tag{10}$$

Moreover, in what follows, it is also important to introduce the *instantaneous Basic Reproduction Number*

$$R_o(t) = \frac{\beta(t)}{\gamma+\mu}, \tag{11}$$

which allows us to re-write $\lambda_2(t)$ as

$$\lambda_2(t) = (\gamma+\mu)[R_o(t) - 1]. \tag{12}$$

From Remark 1 we can formulate the following :

Conjecture 1 Let us consider the nonlinear system (5). Its linearization around $X^* = (1,0)$ is given in (6) with $A(t+T) = A(t)$ defined as in (7) whose eigenvalues $\lambda_1(t)$ and $\lambda_2(t)$ are given in (8).

If $\lambda_2(t) \neq 0$ (i.e. $R_o(t) \neq 1$) for all $t \in [0,T]$ then during the period $[0,T]$:

a) An epidemics occurs if $\lambda_2(t) > 0$ (or equivalently $R_o(t) > 1$) for all $t \in [0,T]$, i.e. if

$$\bar{R}_o > 1 \quad \text{and} \quad 0 \leq \varepsilon < 1 - \frac{1}{\bar{R}_o}; \tag{13}$$

b) An epidemics will not occur if $\lambda_2(t) < 0$ (or equivalently $R_o(t) < 1$) for all $t \in [0,T]$, i.e. if

$$\bar{R}_o < 1 \quad \text{and} \quad 0 \leq \varepsilon < \frac{1}{\bar{R}_o} - 1. \tag{14}$$

Remark 2 When $\lambda_2(t) \neq 0$ for all t in $[0,T]$ then we can consider X^* to be the generalization of the *hyperbolic fixed point* of autonomous systems [24].
Furthermore, for the cases not contemplated in Conjecture 1 when $\lambda_2(t) = 0$ for some $t \in [0,T]$, then a more refined analysis has to be considered. More specifically, when decomposing $\beta(t)$ (and correspondingly $R_o(t)$) as $\beta(t) = \beta_o + \tilde{\beta}(t)$, i.e. the sum of its average β_o and its oscillatory part $\tilde{\beta}(t)$, its oscillatory part may play an important rôle in the initial growth (or decline) of the number of infected individuals $\Delta i(t)$, for some combinations of amplitude ε, period T and initial phase φ_o.

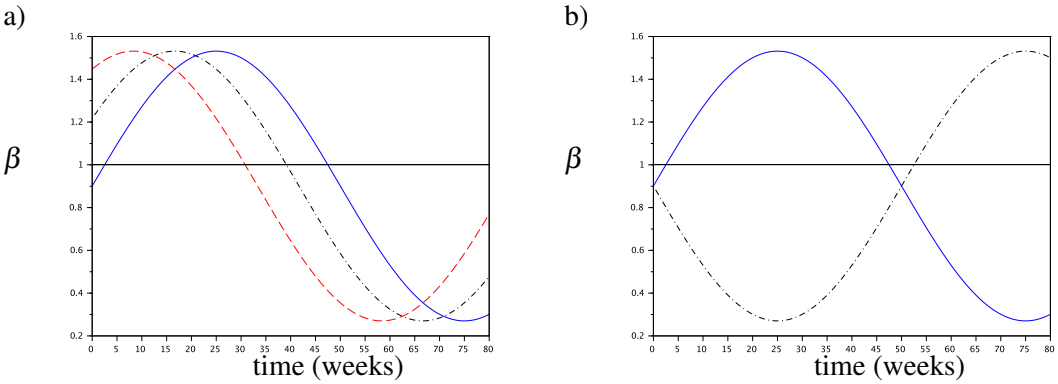

Fig. 1 $\beta(t)$ **as a function of time**. $\beta(t)$ as in (9) with $\varepsilon = 0.7$, $T = 100$ (weeks), and **a)** the initial phase $\varphi_o = 0$ (solid line (–), $\varphi_o = \pi/6$ (line (-.), $\varphi_o = \pi/3$ (line (- -); **b)** the initial phase $\varphi_o = 0$ (solid line (–), the initial phase $\varphi_o = \pi$ (line (-.).

In fact, despite the fact that the average value of $\beta(t)$ corresponds to a $\bar{R}_o < 1$ an epidemics may happen if one of the two situations occurs :

i) $\lambda_2(0) > 0$ (i.e. $R_o(0) > 1$)), $\lambda_2(t_1) = 0$, $\lambda_2(t) > 0$ for all $t \in [0,t_1)$, as shown in Figure 1.a) for $\varphi_o = \pi/6$, $\pi/3$, and the amplitude ε and the period T are such that during the time $[0,t_1)$ the number of infected $\Delta i(t)$ grows considerably, as shown in Figure 3 for $\varphi_o = \pi/6$ and $\varphi_o = \pi/3$;
ii) $\lambda_2(0) < 0$ (i.e. $R_o(0) < 1$), $\lambda_2(t_1) = \lambda_2(t_2) = 0$, $\lambda_2(t) > 0$ for all $t \in [t_1,t_2]$, with $t_1, t_2 \in [0,T/2]$, as shown in Figure 1 for $\varphi_o = 0$, and the amplitudeε and the period T are such that during the time $[t_1,t_2]$, the number of infected $\Delta i(t)$ grows considerably, as shown in Figure 3 for $\varphi_o = 0$;

then, during an initial finite period of time, the growth of the infected population is dominated by the oscillatory part in the .inearized equation.
Similarly, for the case when $\beta(t)$ corresponds to a $\bar{R}_o > 1$ if, for example, the initial phase is $\varphi_o = \pi$ as in Figure 1.b) we may not have an epidemics for suitable combinations of ε and T.

Remark 3 It is easy to prove (13) and (14) for the linearized system (6). We indeed have

$$\lambda_2(t) = (\gamma+\mu)[R_o(t)-1] \quad \text{where} \quad R_o(t) = \frac{\beta(t)}{\gamma+\mu}.$$

Then $\lambda_2(t) > 0$ if and only if $R_o(t) > 1$. We have

$$R_{omin} = \bar{R}_o[1-\varepsilon] \leq R_o(t) = \bar{R}_o\left[1+\varepsilon\sin\left(\frac{2\pi}{T}t+\varphi_o\right)\right] \leq R_{omax} = \bar{R}_o[1+\varepsilon],$$

so if $\bar{R}_o > 1$ and $\varepsilon < (1-1/\bar{R}_o)$ it follows that $R_o(t) > 1$ for all t. Now, if $R_o(t) > 1$ if follows that $\bar{R}_o > 1$ and $R_{omin} > 1$, and therefore $\varepsilon < 1-1/\bar{R}_o$.
Similarly, we can prove that $R_o(t) < 1$ if and only if $\varepsilon < \frac{1}{\bar{R}_o} - 1$ then $R_o(t) < 1$ for all t.

The proof of the conjecture is the subject of a forthcoming work. We currently illustrate it through numerical examples.

Remark 4 Bacaër and Gomes [2] also considered an infectious parameter with a periodic time dependence as in (9), but in the setting of the SIR equations without birth and death terms, i.e.

$$\frac{ds}{dt}(t) = -\beta(t)\,s(t)\,i(t), \quad \frac{di}{dt}(t) = [\beta(t)\,s(t) - \gamma(t)]\,i(t), \quad \frac{dr}{dt}(t) = \gamma(t)\,i(t),$$

where also $\gamma(t)$ is a periodic function of period T and, as $\beta(t)$, it can be decomposed in its averaged, $\bar{\gamma}$, and oscilating part, $\tilde{\gamma}$, i.e. $\gamma(t) = \bar{\gamma} + \tilde{\gamma}(t)$. Their model admits an infinite number of equilibrium points E given by $E = (1-r_o, 0, r_o)$ where $r_o \in]0,1[$. In their case, it is immediate to show that $s(t)$ and $r(t)$ are, respectively, always decreasing and increasing with respect to the time, and therefore no periodic orbit is expected. Their article is mainly devoted to the description of the asymptotic time, and therefore no periodic orbit is expected. Their article is mainly devoted to the description of the asymptotic value of $r(t)$ in terms of the initial condition $r(0) = r_o$ and to the generalization of the Basic Reproduction Number $\bar{R}_o$ (10).

3.1.1 Numerical examples

By means of a Scilab code, we are going to numerically integrate the fully non linear system (5), with $\beta(t)$ as in (9), for different values of the aBRN $\bar{R}_o$, the period T, amplitude ε and initial phase φ_o. We fix the values of the population $N = 10^6$, the recovery rate $\gamma = 1$ (weeks^{-1}) and of the birth/mortality rate $\mu = 0.001$ (weeks^{-1}), and we shall consider two different values of the average Basic Reproduction Number $\bar{R}_o$: $\bar{R}_o = 0.9$ and $\bar{R}_o = 1.5$.

1) Similarly to what is done in Troyo Guerrero [29], in Figure 2 we show the results of numerical simulations over a year period (52 weeks). Each point of the graph rapresents the total number of infected individuals over the period of a year, I_{Sum}, and we have color coded it according to the following : yellow if $I_{Sum} < N/1000$, green if $I_{Sum} < N/10$, blue if $I_{Sum} < N/5$, red if $I_{Sum} < N/3$ and black if $I_{Sum} > N/3$.

We observe the following :

1.a) Case $\lambda_2(t) \neq 0$ for all $t \in [0,T]$: numerical verification of the Conjecture 1. .
As shown in Figures 2.a)-b), for $\bar{R}_o = 0.9$, independently of the chosen initial phase φ_o, whenever $\lambda_2(t) < 0$ is negative for all $t \in [0,T]$ (i.e. when $0 \leq \varepsilon < 1/9$)

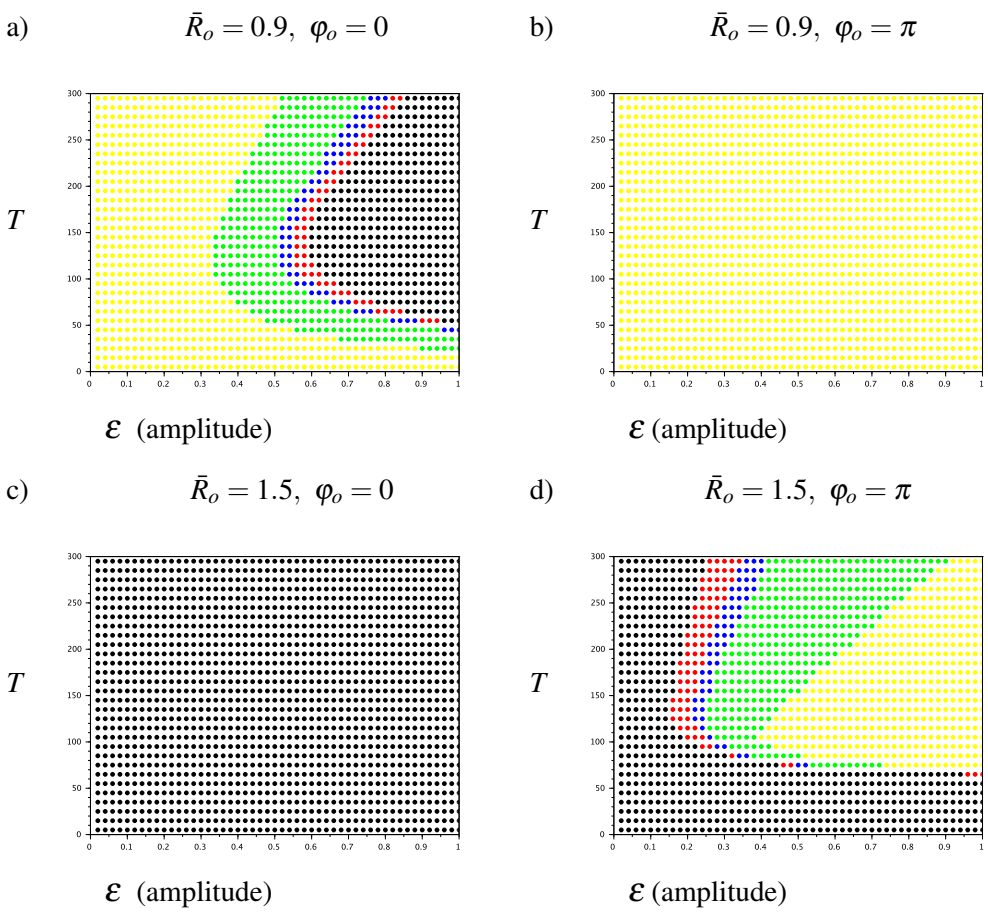

Fig. 2 Total sum of infected individuals over the period of a year, I_{Sum}.
$\beta(t)$ as in (9). Each point of the graph represents the total number of infected individual over a year period I_{Sum}, and we have color coded it according to the following : ● if $I_{Sum} < N * 10^{-3}$, ● if $I_{Sum} < N/10$, ● if $I_{Sum} < N/5$, ● if $I_{Sum} < N/3$ and ● if $I_{Sum} > N/3$.
a) $\bar{R}_o = 0.9$ and the initial phase $\varphi_o = 0$; **b)** $\bar{R}_o = 0.9$ and the initial phase $\varphi_o = \pi$; **c)** $\bar{R}_o = 1.5$ and the initial phase $\varphi_o = 0$; **d)** $\bar{R}_o = 1.5$ and the initial phase $\varphi_o = \pi$

we do not have an epidemics.
Similarly, As shown in Figures 2.c)-d), for $\bar{R}_o = 1.5$, independently of the chosen initial phase φ_o, whenever $\lambda_2(t) > 0$ for all $t \in [0,T]$ (i.e. when $0 \leq \varepsilon < 1/3$) we have an epidemics.

1.b) Case $\lambda_2(t) = 0$ for some $t \in [0,T]$: the relation between ε and T, and the importance of the phase
Observe that, as shown in Figure 2.a) for the initial phase $\varphi_o = 0$, that even even if $\bar{R}_o < 1$ we can have strong epidemics – even more than one third of the population getting infected– for an appropriate combination of period T and amplitude ε. On the other hand, in Figure 2.b) for $\bar{R}_o = 0.9$ we have chosen the initial phase $\varphi_o = \pi$, and it is manifest that for all combinations of period T and amplitude ε we

will not have an epidemics. We can see in Figure 3.a)-b) the behavior of $I(t)$ for some particular cases. Analougously, Figures 2.c)-d) show an even, perhaps, more

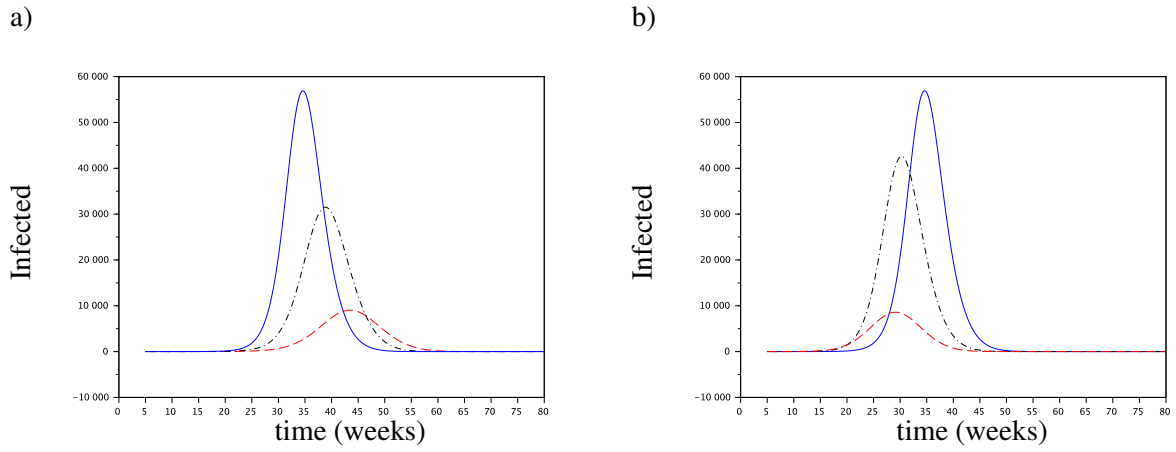

Fig. 3 Infected individuals as a function of time. $\beta(t)$ as in (9) with $\bar{R}_o = 0.9$, $T = 100$ (weeks). **a)** For $\varphi_o = 0$ we consider different values of the amplitude ε : $\varepsilon = 0.7$ (solid blue $-$), $\varepsilon = 0.6$ (black $-.$), $\varepsilon = 0.5$ (red $--$); **b)** for $\varepsilon = 0.7$ we consider different values of the initial phase φ_o : $\varphi_o = 0$ (solid blue $-$), $\varphi_o = \pi/6$ (black $-.$), $\varphi_o = \pi/3$ (red $--$).

surprising result : even when $\bar{R}_o$ is well above one ($\bar{R}_o = 1.5$), for appropriate combinations of amplitude ε and period T, we might not have an epidemics if the initial phase φ_o is not favorable, as commented in Remark 2. Notice that in our model the initial phase φ_o represents the arrival of the infected individuals : roughly speaking they can arrive in a better or worse moment for them to be sucessful in spreading an epidemics.

2) The impact of the initially recovered individuals $R(0)$.

We also observe an impact of the number of initial recovered individuals $R(0)$ over the occurence of an epidemics as shown in Figure 4 for a period $T = 52$ (weeks). In Figure 4.a) where the number of initially recovered individuals $R(0)$ has been fixed at 80% of the total population N, we can observe that, depending of the amplitude ε a rather high value of $\bar{R}_o$ is necessary in order to have a moderate epidemics. In Figure 4.b), where we have fixed $\bar{R}_o = 2.5$, we show for which values of the initial recovered individuals $R(0)$ we do have an epidemics. Observe as above 60% there is no epidemics.

Real data in the cities of Rio de Janeiro and Recife (Brazil) show that the recovered individuals are 80% of the resident population [6]. The arrival of tourists during Carneval (which increases the population of Rio by 30% and the one of Recife by 100%) changes significantly the number of recovered individuals and, therefore, affects the spreading of the Dengue dynamics.

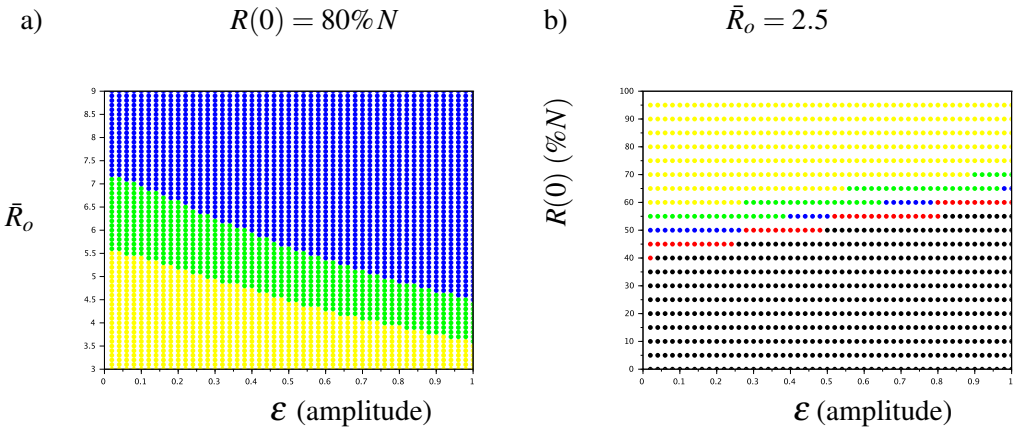

Fig. 4 Total sum of infected individuals over the period of a year, I_{Sum}.
$\beta(t)$ as in (9). Each point of the graph represents the total number of infected individual over a year period I_{Sum}, and we have color coded it according to the following : ● if $I_{Sum} < N * 10^{-3}$, ● if $I_{Sum} < N/10$, ● if $I_{Sum} < N/5$, ● if $I_{Sum} < N/3$ and ● if $I_{Sum} > N/3$. We fix $T = 1\,year$.
a) for $\varphi_o = 0$ and $R(0) = 80\%N$, we vary the amplitude ε and the aBRN $\bar{R}_o$; **b)** for aBRN $\bar{R}_o = 2.5$ and $\varphi_o = \pi$, we vary the amplitude ε and initial number of recovered individuals $R(0)$.

3.2 Approximation of the periodic solution

In Katriel [17] it is proved that whenever $\bar{R}_o > 1$ there exists at least one T-periodic solution $(s(t), i(t), r(t))$ of (4). In this section we prove that under additional assumptions an approximation of such a periodic solution can be given. From an epidemic point of view, such a periodic orbit represents an endemic state. As in the previous section we assume a conservation law $(s(t) + i(t) + r(t) = 1)$ which, as before, allows us to restrict our attention to the first two equations. Moreover, since we shall restrict only to the case of positive $i(t)$, we can perform the change of variable given in (41) so that the system to be analyzed is

$$\begin{cases} \dot{s}(t) = -\beta(t)s(t)e^{x(t)} + \mu - \mu s(t), \\ \dot{x}(t) = \beta(t)s(t) - \delta. \end{cases} \tag{15}$$

Since β is continuous, positive and periodic of period T,

$$\beta_* = \inf_{m\in[0,T]} \{\beta(m)\} \tag{16}$$

and

$$\beta^* = \sup_{m\in[0,T]} \{\beta(m)\} \tag{17}$$

are well-defined positive constants. We impose the assumptions:

Assumption A2. *The inequality*

$$\delta < \beta_* \tag{18}$$

is satisfied.

Observe that Assumption A2 is equivalent to say that $R_o(t) > 1$ for all time $t \in [o,T]$ is a stronger assumption than $\bar{R}_o > 1$.

Assumption A3. *The inequality*

$$\delta^2 \leq \beta_* \frac{9\mu}{20} \tag{19}$$

is satisfied.

The following result shows that we can determine an approximation of the periodic solution, whose accuracy depends on the slowness of the variations of β.

Theorem 2 *Consider the system (15) under Assumptions A2 and A3, and a sufficiently slowing varying $\beta(t)$, i.e. there exists $\beta^\dagger > 0$ such that, for all $t \geq 0$, $|\dot{\beta}(t)| \leq \beta^\dagger$ with $0 < \beta^\dagger < \beta_o^\dagger$. Then the function $X_*(t) = (s_*(t), x_*(t))$, $t \geq 0$ with*

$$s_*(t) = \frac{\delta}{\beta(t)}\,, \quad x_*(t) = \ln\left(\frac{\mu - \mu s_*(t)}{\beta(t) s_*(t)}\right) \tag{20}$$

is such that there exists a constant $K > 0$ such that for all $t > 0$

$$|X_*(t) - X_\triangle(t)| < K\beta^\dagger, \tag{21}$$

where $X_\triangle = (s_\triangle, x_\triangle)$ denotes a positive periodic solution of (15) of period T.

Proof : see Appendix B.

3.3 Stability of the periodic solution

Katriel [17] proved the existence of a periodic orbit $X_\triangle(t) = (s_\triangle(t), x_\triangle(t))$. In what follows we establish the asymptotic stability of the periodic solution under the same extra assumptions on the slowness of the function $\beta(t)$ as in the previous section.

Theorem 3 *Consider the system (15) under Assumptions A2 and A3, and a sufficiently slowing varying $\beta(t)$, i. e. there exists a constant $\beta^\dagger > 0$ such that $|\dot{\beta}(t)| \leq \beta^\dagger$ with $0 < \beta^\dagger < \beta_o^\dagger$. Then the periodic orbit $X_\triangle = (s_\triangle(t), x_\triangle(t))$, defined in [17], is asymptotically stable, i. e. all solution $X(t) = (s(t), x(t))$ of the system (15) satisfy*

$$\lim_{t\to\infty} |X(t) - X_\triangle(t)| = 0.$$

□

Proof of Theorem 3. Let $\tilde{s}(t) = s(t) - s_\triangle(t)$, $\tilde{x}(t) = x(t) - x_\triangle(t)$.

From Equations (15) and (60) we have

$$\begin{cases} \dot{\tilde{s}}(t) = \beta(t)\left[s_\triangle(t)e^{x_\triangle(t)} - s(t)e^{x(t)}\right] - \mu\tilde{s}(t), \\ \dot{\tilde{x}}(t) = \beta(t)\tilde{s}(t) \end{cases}. \tag{22}$$

It follows that

$$\begin{cases} \dot{\tilde{s}}(t) = \beta(t)\left[s_\triangle(t)e^{x_\triangle(t)} - (\tilde{s}(t)+s_\triangle(t))e^{x(t)}\right] - \mu\tilde{s}(t), \\ \dot{\tilde{x}}(t) = \beta(t)\tilde{s}(t) \end{cases}, \tag{23}$$

or, equivalently,

$$\begin{cases} \dot{\tilde{s}}(t) = -\left[\omega_\triangle(t)e^{\tilde{x}(t)} + \mu\right]\tilde{s}(t) + \Omega_\triangle(t)\left[1 - e^{\tilde{x}(t)}\right], \\ \dot{\tilde{x}}(t) = \beta(t)\tilde{s}(t) \end{cases}. \tag{24}$$

where $\omega_\triangle(t) = \beta(t)e^{x_\triangle(t)}$ and $\Omega_\triangle(t) = \beta(t)s_\triangle(t)e^{x_\triangle(t)}$.

Now, recall that, roughly speaking, when $\dot{\beta}(t)$ is small, then $\dot{s}_*(t)$ and $\dot{x}_*(t)$ are small as well. This, in combination with (21) and (65), implies that $\dot{s}_\triangle(t)$, $\dot{x}_\triangle(t)$ are small as well.
Bearing this in mind, consider now:

$$U_4(t,\tilde{s},\tilde{x}) = k\frac{\beta(t)}{2\Omega_\triangle(t)}\tilde{s}^2 + k\left(e^{\tilde{x}} - \tilde{x} - 1\right) + \tilde{s}\left(e^{\tilde{x}} - 1\right) \tag{25}$$

with

$$k \geq \frac{2\sqrt{\Omega_\triangle(t)\delta}}{\sqrt{\mu(\beta(t)-\delta)}} \tag{26}$$

which is inspired by U_3 defined in (53).
Then,

$$\dot{U}_4(t) = -\mathfrak{W}(t,\tilde{s}(t),\tilde{x}(t)) + k\frac{\dot{\beta}(t)\Omega_\triangle(t) - \beta(t)\dot{\Omega}_\triangle(t)}{2\Omega_\triangle(t)^2}\tilde{s}(t)^2, \tag{27}$$

with

$$\mathfrak{W}(t,\tilde{s},\tilde{x}) = \frac{k}{2\Omega_\triangle(t)}\left(\omega_\triangle(t)e^{\tilde{x}} + \mu\right)\beta(t)\tilde{s}^2 + \frac{\Omega_\triangle(t)}{2}\left[1 - e^{\tilde{x}}\right]^2. \tag{28}$$

Through lengthy calculations, one can determine a function σ of class $\mathscr{K}$ such that

$$\dot{U}_4(t) \leq -\mathfrak{W}(t,\tilde{s}(t),\tilde{x}(t)) + \sigma(\beta^\dagger)\mathfrak{W}(t,\tilde{s}(t),\tilde{x}(t)). \tag{29}$$

We deduce from (29) that if $\beta^\dagger$ is sufficiently small, then the global asymptotic stability holds. Notice in particular that the uniqueness of the periodic nonnegative solution is a consequence of this stability property. □

3.4 **Numerical examples : a sinus-like infection** $\beta(t)$

As in Bacaër and Gomes [2] and Troyo Guerrero [29], we consider a sinus-like infectivity parameter (see (9)),

$$\beta(t) = \beta_o + \tilde{\beta}(t) = \beta_o \left[1 + \varepsilon \sin\left(\frac{2\pi}{T}t + \varphi_o\right)\right],$$

where β_o is the average value of $\beta(t)$, $\varepsilon\beta_o$ is the amplitude of the fluctuating part of $\tilde{\beta}(t)$. The above equation can be rewritten as

$$\beta(t) = \bar{R}_o(\gamma+\mu)\left[1 + \varepsilon \sin\left(\frac{2\pi}{T}t + \varphi_o\right)\right]. \tag{30}$$

In the case of Dengue fever, the recovery period is estimated to be on average of one week therefore $\gamma = 1\,(weeks^{-1})$. We are then left with five parameters : R_o, μ, ε, T and φ_o.

Example 1: *Existence of periodic globally attracting orbit.*

In Figure 5 we have an example of periodic attracting orbit for $\beta(t)$ as in (30) for the case $T = 1\,years$, $R_o = 1.2$, $\varepsilon = 0.1$, $\mu = 0.1$, $\varphi_o = 0$. Observe that for different initial conditions the corresponding orbits converge to a limit cycle.
Analogously, for $R_o = 2.2$, $\mu = 0.1$, $\varphi_o = 0$, in Figures 6 and 7, respectively, for $T = 0.5\,years$, $\varepsilon = 0.01$ and $T = 5\,years$, $\varepsilon = 0.2$, different orbits converge to the same limit cycle.

Example 2: *Approximating the periodic orbit.*

In [17] Theorem 1 gives us conditions for the existence of a periodic orbit (i.e. an epidemic cycle) but not its formula. This is why Theorem 2 can be useful in giving us an estimation of the region of the plane (S,I) where the periodic orbit can be found, by providing us with the approximation $(S_*(t), I_*(t)) = (Ns_*(t), Ni_*(t))$ for the periodic orbit, as illustrated in the example of Figure 8. Let us remember that among the hypotheses of the theorem we have the condition

$$|\dot{\beta}| < \beta^\dagger.$$

Observe that it follows from (30) that

$$\sup\{\dot{\beta}(t)\} = 2\pi R_o(\gamma+\mu)\frac{\varepsilon}{T}.$$

Then the condition $|\dot{\beta}(t)| < \beta^\dagger$ in Theorem 2 translates into

$$R_o(\gamma+\mu)\varepsilon\frac{2\pi}{T} < \beta^\dagger \qquad \text{or} \qquad \frac{\varepsilon}{T} < \frac{\beta^\dagger}{2\pi R_o(\gamma+\mu)} \qquad \text{or} \qquad T > \left[\frac{2\pi R_o(\gamma+\mu)}{\beta^\dagger}\right]\varepsilon.$$

Therefore, as in the example shown in Figure 8 (orbit in the plane (S,I) and in Figure 9 (infected individuals as a function of time) as $\dot{\beta}$ decreases, for increasing T, the quality of the approximation improves, as expected.

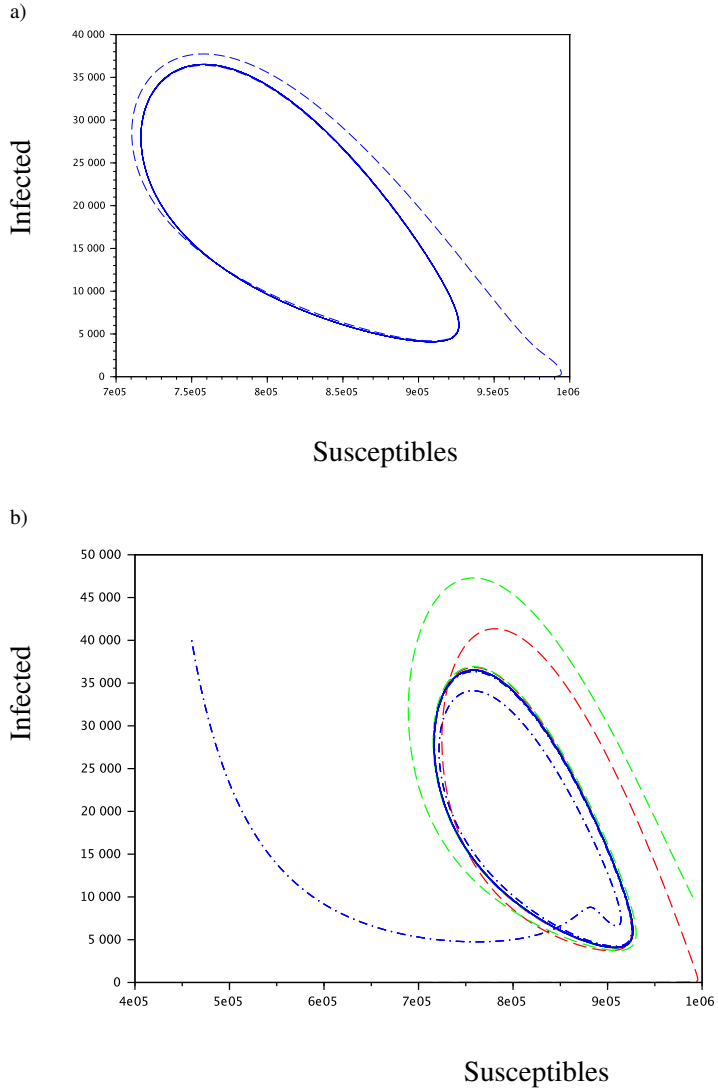

Fig. 5 Periodic attractor. T=1 years. Periodic infection parameter $\beta(t)$ as in (9). $N = 10^6$, $R_o = 1.2$, $\mu = 0.1$, $\varepsilon = 0.1$, and $S(0) = N - I(0) - R(0)$. Orbits with initial conditions : a) $I(0) = 1, R(0) = 3\%N$. b) $I(0) = 4\%N, R(0) = 50\%N$ (–blue); $I(0) = 1, R(0) = 30\%N$ (-.red); $I(0) = 1\%N, R(0) = 0$ (–green).

4 Time-varying infection parameters II : the non-periodic case

We now assume that $\beta(t)$ is a positive continuous time-varying function with bounded variations, but we do not impose on it to be periodic. More specifically, we impose :

Assumption A4. *There are constants $\beta_* > 0$ and $\beta^\dagger > 0$ such that*

$$\beta_* \leq \beta(t) \leq \beta^\dagger, \tag{31}$$

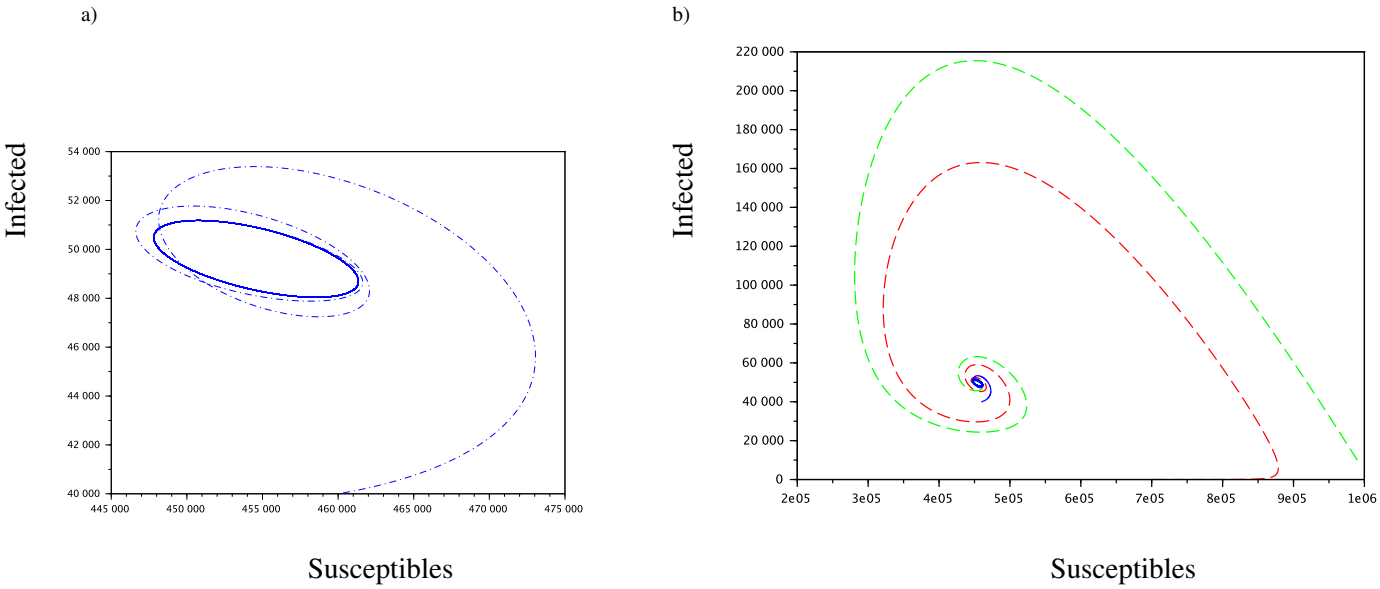

Fig. 6 Periodic attractor. T=0.5 years. Periodic infection parameter $\beta(t)$ as in (9). $N = 10^6$, $R_o = 2.2$, $\mu = 0.1$, $\varepsilon = 0.01$,, and $S(0) = N - I(0) - R(0)$. Orbits with initial conditions : a) $I(0) = 4\%N$, $R(0) = 50\%N$; b) $I(0) = 4\%N$, $R(0) = 50\%N$ (–blue); $I(0) = 1$, $R(0) = 30\%N$ (-.red); $I(0) = 1\%N$, $R(0) = 0$ (–green).

for all $t \geq 0$.

As already commented at the beginning of Section 3, it is immediate to see that the system of equations (4) admits only one equilibrium $(s^*, i^*, r^*) = (1, 0, 0)$. As before, we consider the change of variable $x = \ln(i)$ and consider the behavior of solutions $(s(t), i(t))$ of the system (42). Nevertheless this time, differently from the previous periodic case, we shall not prove directly the existence a periodic orbit. Instead, we shall prove that asymptotically all trajectories of the system (42) are within a tubular region around a given bounded function.

Theorem 4 *Let the system (15) satisfy Assumptions A2, A4 and let* $\beta(t)$ *be a sufficiently slowing varying function, i.e. a function such that there exists a constant* $\beta^\dagger > 0$ *such that* $|\dot{\beta}(t)| \leq \beta^\dagger$ *with* $0 < \beta^\dagger < \beta_o^\dagger$. *Then the function* $X_*(t) = (s_*(t), x_*(t))$, $t \geq 0$ *defined in (20), i.e.*

$$s_*(t) = \frac{\delta}{\beta(t)}, \quad x_*(t) = \ln\left(\frac{\mu - \mu s_*(t)}{\beta(t) s_*(t)}\right)$$

is close to an attractor of the system in the following sense: for each trajectory $X(t)$ *of the system (15), there exist constants* $K > 0$ *and* $\tau \geq 0$ *(which may depend on the considered trajectory) such that for all* $t > \tau$

$$|X_*(t) - X(t)| < K\beta^\dagger.$$

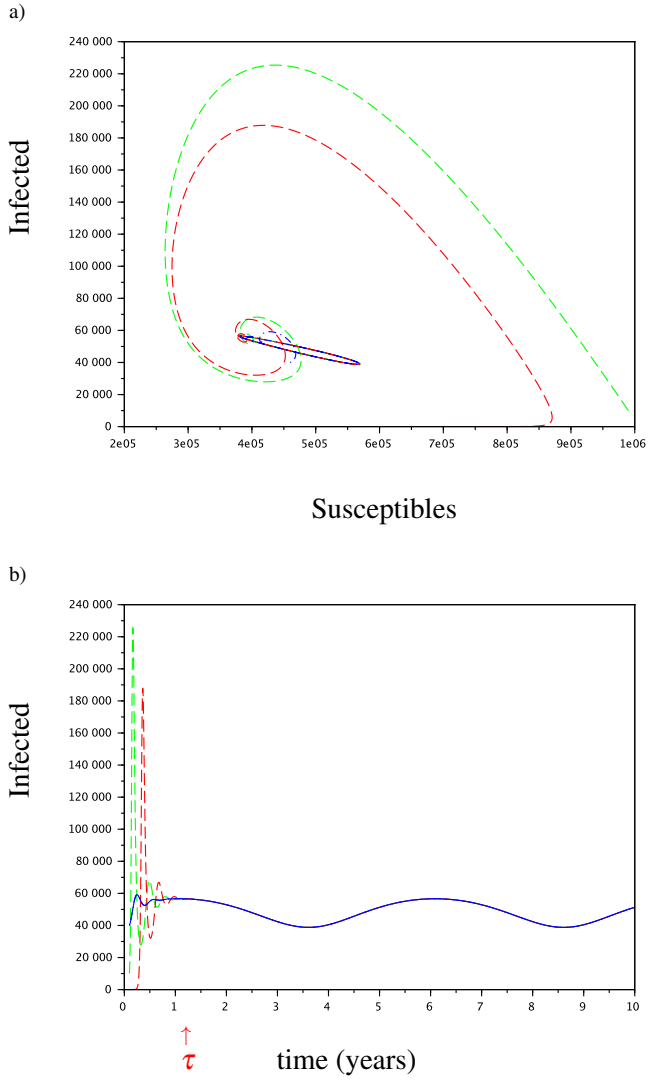

Fig. 7 Epidemic attractor. T=5 years. Periodic infection parameter $\beta(t)$ as in (9). $N = 10^6$, $R_o = 2.2$, $\mu = 0.1$, $\varepsilon = 0.2$, and $S(0) = N - I(0) - R(0)$. Orbits with initial conditions : $I(0) = 4\%N$, $R(0) = 50\%N$ (–blue); $I(0) = 1$, $R(0) = 30\%N$ (-.red); $I(0) = 1\%N$, $R(0) = 0$ (–green). **a)** orbit in the *SI* plane; **b)** infected as a function of time.

Proof In what follows, we establish that each solution $X(t)$ of (15) is close to $X_*(t)$ for all $t > \tau$. More precisely, for $t > \tau$, all $X(t) = (s(t), x(t))$ is contained in a tubular neighborhood of $X_* = (s_*(t), x_*(t))$, i. e. there exists a constant $K > 0$ such that for all $t > \tau$ the inequality $|X(t) - X_*(t)| < K\beta^\dagger$ holds. To do so let us introduce the functions

$$\tilde{s} = s(t) - s_*(t), \qquad \tilde{x}(t) = x(t) - x_*(t).$$

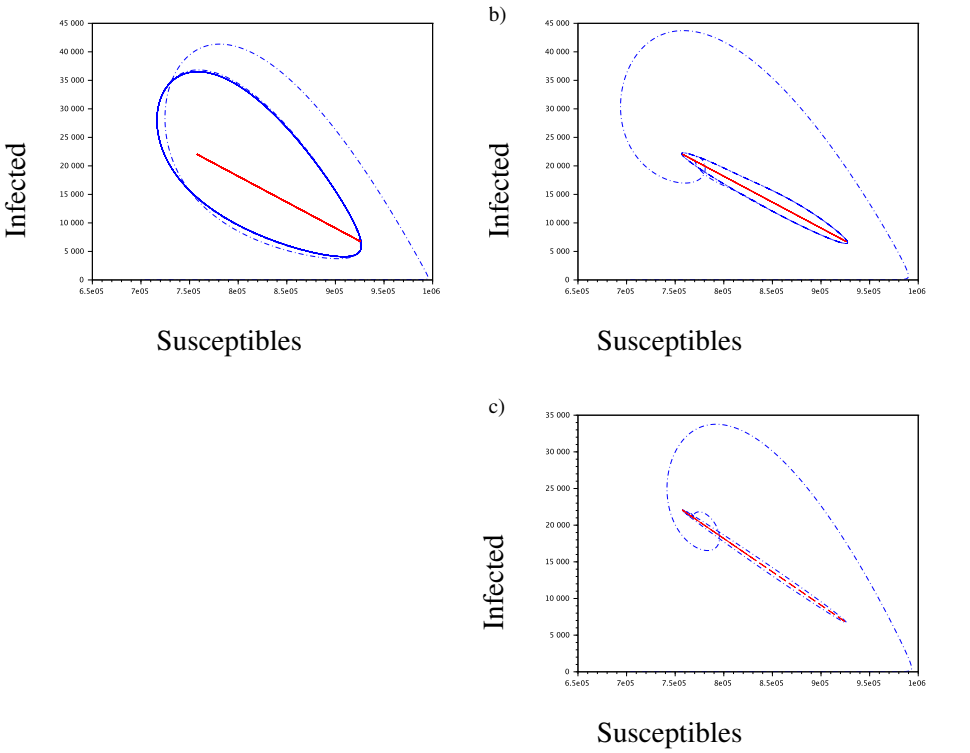

Fig. 8 Approximating the periodic attractor : susceptible versus infected individuals. Periodic infection parameter $\beta(t)$ as in (30). $N = 10^6$, $R_o = 1.2$, $\varepsilon = 0.1$ e $\mu = 0.1$, $S(0 = N - I(0) - R(0)$. Orbits with initial conditions : $I(0) = 1$, $R(0) = 30\%N$ (-.blue) and approximation of the periodic orbit (20) (–red), i.e. $I_* = N\mu\,(N - S_*)/(\gamma + \mu)$.
As $\dot{\beta}$ decreases for increasing period T, the approximation (S_*, I_*) given in (20), of the periodic orbit $(S_\triangle, I_\triangle)$ is a better approximation when increasing the period T, as shown for
a) $T = 1\,years$; **b)** $T = 5\,years$; **c)** $T = 15\,years$.

We shall prove this by formulating the problem in a similar way than in (46) and by proving that, for a sufficiently slowly varying function $\beta(t)$, the function,

$$U_5(t, \tilde{s}, \tilde{x}) = \frac{k}{2e^{x_*(t)} s_*(t)} \tilde{s}^2 + k\left(e^{\tilde{x}} - \tilde{x} - 1\right) + \tilde{s}\left(e^{\tilde{x}} - 1\right), \tag{32}$$

that is deduced from U_3 introduced in (53), is a strict Lyapunov function of the trajectory $\tilde{X} = (\tilde{s}(t), \tilde{x}(t)) = (s(t) - s_*(t), x(t) - x_*(t))$.

One can readily check that $X_*(t) = (s_*(t), x_*(t))$, defined in (20) verifies

$$\begin{cases} \dot{s}_*(t) = -\frac{\delta}{\beta(t)^2} \dot{\beta}(t), \\ \dot{x}_*(t) = \frac{1}{\beta(t) - \delta} \dot{\beta}(t) \end{cases}. \tag{33}$$

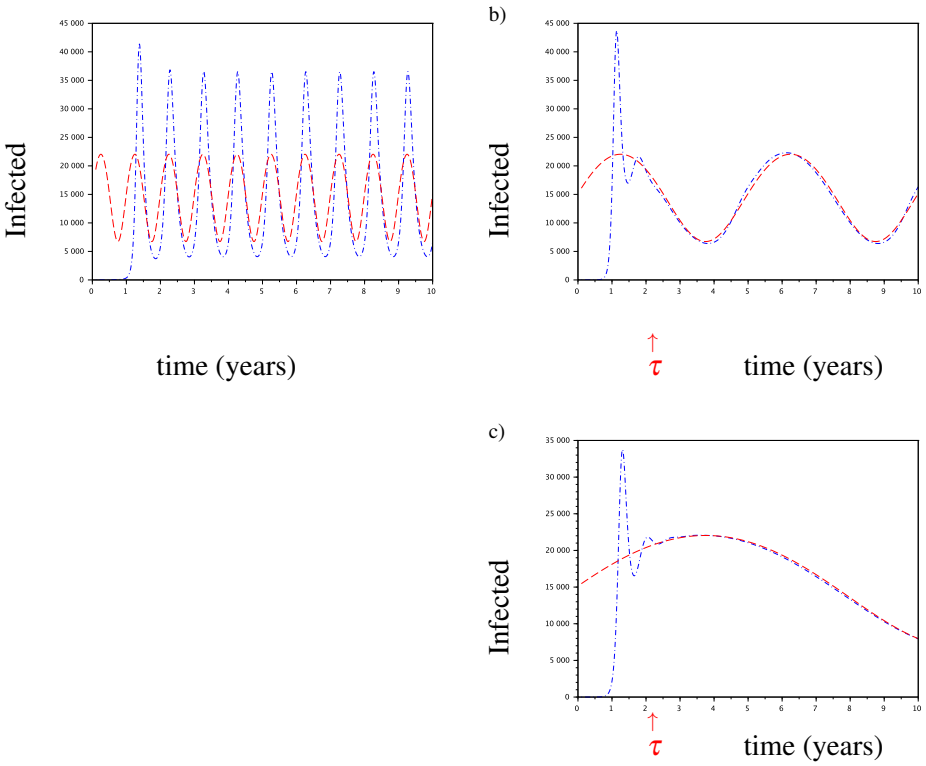

Fig. 9 Approximating the periodic attractor : Infected individuals versus time. Periodic infection parameter $\beta(t)$ as in (30). $N = 10^6$, $R_o = 1.2$, $\varepsilon = 0.1$ e $\mu = 0.1$, $S(0 = N - I(0) - R(0)$. Orbits with initial conditions : $I(0) = 1$, $R(0) = 30\%N$ (-.blue) and approximation of the periodic orbit Eq. (20) (–red), i.e. $I_* = N\mu\,(N - S_*)/(\gamma + \mu)$.
As $\dot{\beta}$ decreases for increasing period T, the approximation (S_*, I_*), Eq. (20), of the periodic orbit $(S_\triangle, I_\triangle)$ is a better approximation increasing the period T, as shown for
a) $T = 1\,years$; b) $T = 5\,years$; c) $T = 15\,years$.

Recall that Assumption A2 ensures that $\frac{1}{\beta_* - \delta}$ is well-defined and positive and that $|\dot{s}_*(t)| \leq \frac{\delta}{\beta_*^2}|\dot{\beta}(t)|$ and $|\dot{x}_*(t)| \leq \frac{1}{\beta_* - \delta}|\dot{\beta}(t)|$.

Now, observe that $-\beta(t)s_*(t)e^{x_*(t)} + \mu(1 - s_*(t)) = -\beta(t)\frac{\delta}{\beta(t)}\left(\frac{\mu - \mu s_*(t)}{\beta(t)s_*(t)}\right) + \mu - \mu\frac{\delta}{\beta(t)} = 0$ and $\beta(t)s_*(t) - \delta = 0$. It follows that

$$\begin{cases} \dot{s}_*(t) = -\beta(t)s_*(t)e^{x_*(t)} + \mu(1 - s_*(t)) - \frac{\delta}{\beta(t)^2}\dot{\beta}(t), \\ \dot{x}_*(t) = \beta(t)s_*(t) - \delta + \frac{1}{\beta(t) - \delta}\dot{\beta}(t) \end{cases}. \tag{34}$$

Then

$$\begin{cases} \dot{\tilde{s}}(t) = \beta(t)\left[s_*(t)e^{x_*(t)} - s(t)e^{x(t)}\right] - \mu\tilde{s}(t) + \frac{\delta}{\beta(t)^2}\dot{\beta}(t), \\ \dot{\tilde{x}}(t) = \beta(t)\tilde{s}(t) - \frac{1}{\beta(t) - \delta}\dot{\beta}(t) \end{cases}, \tag{35}$$

or, equivalently,

$$\begin{cases} \dot{\tilde{s}}(t) = -\left[\omega(t)e^{\tilde{x}(t)} + \mu\right]\tilde{s}(t) + \Omega(t)\left[1 - e^{\tilde{x}(t)}\right] + \frac{\delta}{\beta(t)^2}\dot{\beta}(t), \\ \dot{\tilde{x}}(t) = \beta(t)\tilde{s}(t) - \frac{1}{\beta(t)-\delta}\dot{\beta}(t), \end{cases} \tag{36}$$

with $\omega(t) = \beta(t)e^{x_*(t)}$, $\Omega(t) = \beta(t)e^{x_*(t)}s_*(t)$.

The rest of the proof is similar to the one of Theorem 2 and is omitted here.

4.1 Numerical examples : quasi periodic infection $\beta(t)$

Example 3: *Existence of globally attracting orbit. Multi-periodic case with the annual and El Ninõforcing*

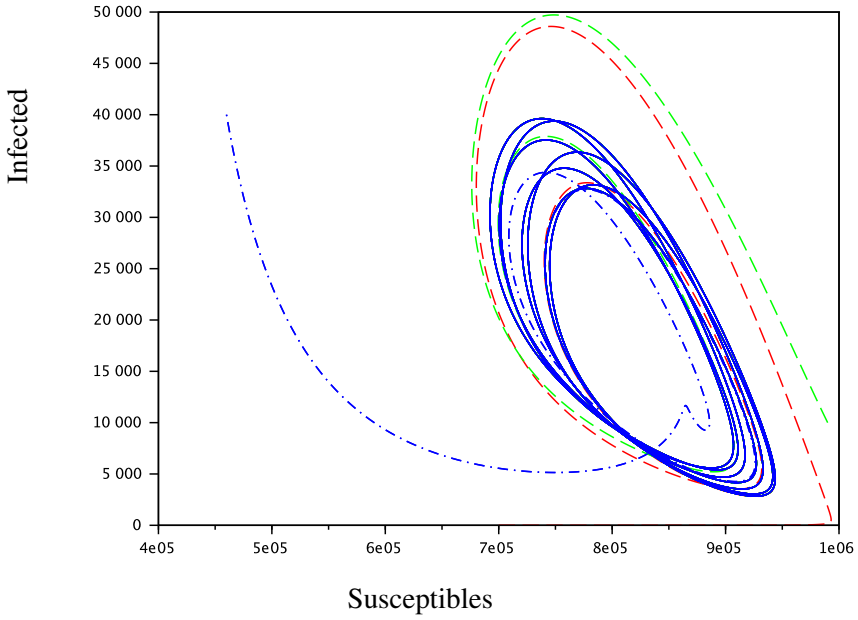

Fig. 10 Multi-periodic infection parameter : two frequencies case. Multi-periodic infection parameter $\beta(t)$ as in Eq. (37). $N = 10^6$, $\beta_o = 1.32, T_1 = 1\,years$, $T_2 = 3.5\,years$, $\varepsilon_1 = 0.1$, $\varepsilon_2 = 0.03$ e $\mu = 0.1$, $S(0 = N - I(0) - R(0)$. Orbits with initial conditions : a) $I(0) = 4\%N$, $R(0) = 50\%N$ (–blue) and approximation of the attractor in red Eq. (20);) $I(0) = 4\%N$, $R(0) = 50\%N$ (–blue); $I(0) = 1$, $R(0) = 30\%N$ (-.red); $I(0) = 1\%N$, $R(0) = 0$ (–green).

We considered a Multi-periodic infection parameter, as

$$\beta(t) = \beta_o\left[1 + \varepsilon_1 \sin\left(\frac{2\pi}{T_1}t + \varphi_{o1}\right) + \varepsilon_2 \sin\left(\frac{2\pi}{T_2}t + \varphi_{o2}\right)\right], \tag{37}$$

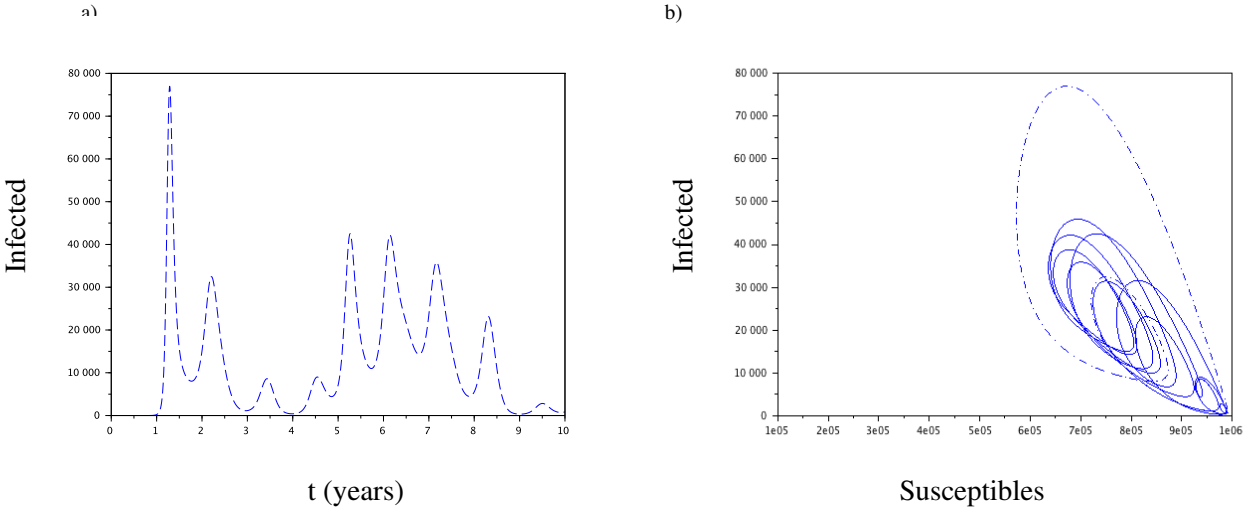

Fig. 11 Multi-periodic infection parameter : two frequencies case. Multi-periodic infection parameter $\beta(t)$ as in Eq. (37). $N = 10^6$, $\beta_o = 1.32, T_1 = 1\,years$, $T_2 = 5.5$, $\varepsilon_1 = 0.1$, $\varepsilon_2 = 0.05 * \pi$, $\varphi_{o1} = 0$, $\varphi_{o2} = \pi/6$ e $\mu = 0.1$, $S(0 = N - I(0) - R(0)$. Orbits with initial conditions : $I(0) = 1$, $R(0) = 85\%N$.
a) Plot of the infected individuals as a function of time; b) Orbit in the *SI* plane.

where T_1/T_2 is a rational number. In the example shown in Figure 10, for slowlying varying $\beta(t)$ we observe the existence of an attracting orbit. Furthermore, observe that if the number of initially recuperated individuals, $R(0)$, is high, as in the example shown in Figure 11, the temporal behaviour of the infected individuals could seem rather irregular on a given interval of time (Figure 11.a)) even if the underline attractor is a quasi-periodic orbit (Figure 11.b)).

Example 4: *Dependence upon the number of recovered individuals $R(0)$. Multi-periodic case with three sinudosal forcings*

We considered a multi-periodic infection parameter with three sinudosal forcings, i.e.

$$\beta(t) = \beta_o \left[1 + \varepsilon_1 \sin\left(\frac{2\pi}{T_1}t + \varphi_{o1}\right) + \varepsilon_2 \sin\left(\frac{2\pi}{T_2}t + \varphi_{o2}\right) + \varepsilon_3 \sin\left(\frac{2\pi}{T_3}t + \varphi_{o3}\right)\right]. \quad (38)$$

In particular, we consider a β with forcings that models the annual seasonality ($T_1 = 1\,year$), El Ninõ ($T_2 = 5.5\,years$) and the bi-annual rain season ($T_3 = 0.5\,year$). Notice that even in this case we can have a global attractor as shown in Figure 12. Furthermore, as shown in Figure 13, as mentioned in Theorem 4, for $\beta(t)$ varying sufficiently slowly, we can can still approximate the attractor by $(S^*(t), I^*(t)) = (Ns^*(t), Ni^*(t))$, with $(s^*(t), i^*(t))$ defined in (20).

Conclusions

In our changing world it is crucial to account for the non-stationarity characteristics of observed epidemics [6]. To allow for this, in this paper, we have considered an

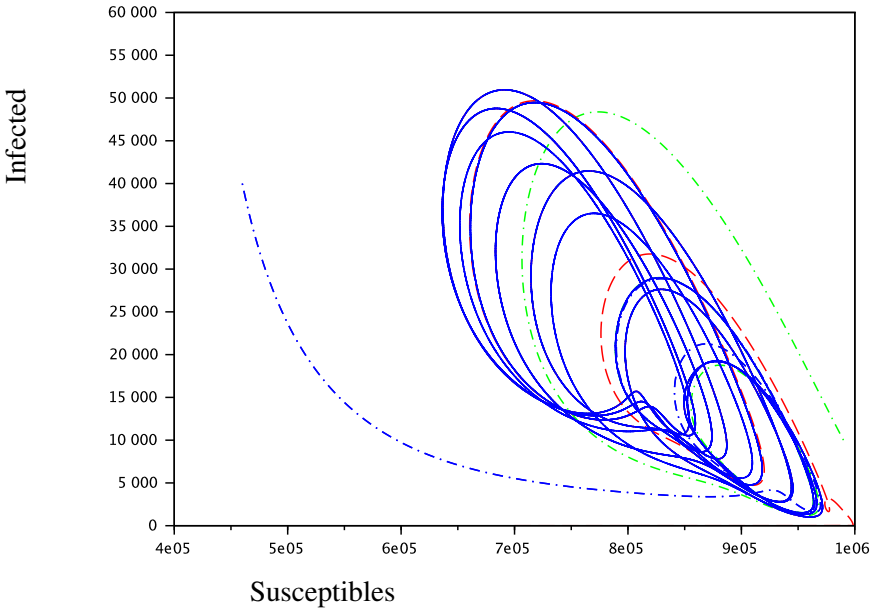

Fig. 12 Attractor. Multi-periodic infection parameter $\beta(t)$ as in Eq. (37). $N = 10^6$, $\beta_o = 1.32, T_1 = 1\,years$, $T_2 = 3.5\,years$, $\varepsilon_1 = 0.1$, $\varepsilon_2 = 0.03$ e $\mu = 0.1$, $S(0 = N - I(0) - R(0)$. Orbits with initial conditions : a) $I(0) = 4\%N, R(0) = 50\%N$ (–blue) and approximation of the attractor in red Eq. (20);) $I(0) = 4\%N, R(0) = 50\%N$ (–blue); $I(0) = 1, R(0) = 30\%N$ (-.red); $I(0) = 1\%N, R(0) = 0$ (–green).

SIR model with a time-varying infectivity parameter $\beta(t)$ and proposed new results about its stability and, more important, one way to approximating it. This model is applicable to numerous epidemics for which the seasonal climate forcing is crucial, such as influenza or arbovirus diseases for instance. Estimation of the parameters of the models of these diseases (specially their Basic Reproduction Number), as well as the final size of the epidemics, are of critical importance.

Our results enlighten that the average Basic Reproduction Number $\bar{R}_0$ is not the only relevant parameter: the initial phase, the amplitude and the period of the $\beta(t)$ time evolution have an important influence on the stability of the dynamics. In particular by considering periodic, multi-periodic and quasi-periodic $\beta(t)$, we provide a simple approximation of the underlying attractor.

Finally, by means of numerical experiments, we point out that the initial number of recovered individual $R(0)$, together with the initial phase φ_o, plays an important role in the epidemic dynamics. Real data in the cities of Rio de Janeiro and Recife (Brazil) show that the recovered individuals are 80% of the resident population [6]. The arrival of tourists during Carneval (which increases the population of Rio by 30% and the one of Recife by 100% exactly in the season of highest density of Aedes mosquitoes) changes significantly the number of recovered individuals and, therefore,

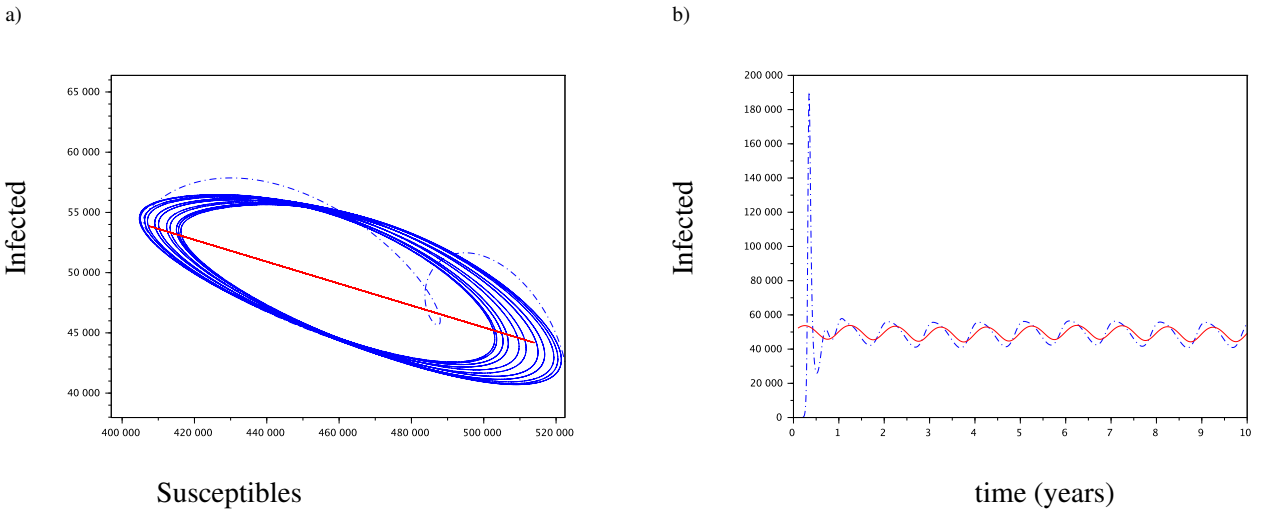

Fig. 13 Approximating the attractor : three frequencies case. Multi-periodic infection parameter $\beta(t)$ as in Eq. (38). $N = 10^6$, $\beta_o = 2.42$, $\varepsilon_1 = 0.1$, $\varepsilon_2 = 0.005\,\pi$, $\varepsilon_3 = 0.001$, $T_1 = 1\,years$, $T_2 = 5.5\,years$, $T_3 = 0.5\,years$, $\mu = 0.1$, $S(0 = N - I(0) - R(0)$. Orbits with initial conditions : $I(0) = 1$, $R(0) = 30\%N$ (-.blue) and approximation of the periodic orbit Eq. (20) (–red), i.e. $I_* = N\mu\,(N - S_*)/(\gamma + \mu)$.
a) orbit in the plane SI plane ; b) infected individuals as a function of time.

affects the spreading of the Dengue dynamics. This might show the importance of some regulation policies.

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A Proof Theorem 1

□

Proof of item 2 of Theorem 1. Using the change of coordinates $s = \frac{S}{N}$, $i = \frac{I}{N}$ and $r = \frac{R}{N}$ and the notation $\delta = \mu + \gamma$, we rewrite the system (1)-(2)-(3) as

$$\begin{cases} \dfrac{ds}{dt}(t) &= -\beta s(t) i(t) + \mu - \mu s(t), \\ \dfrac{di}{dt}(t) &= \beta s(t) i(t) - \delta i(t), \\ \dfrac{dr}{dt}(t) &= \gamma i(t) - \mu r(t). \end{cases} \tag{39}$$

The state space we consider is $\mathscr{E} = [0,1]\times[0,1]\times[0,1]$. Bearing in mind the conservation law

$$s(t)+i(t)+r(t)=1, \tag{40}$$

we restrict our attention to the first two equations. Moreover, we consider positive solutions only. To ease the analysis, we perform the change of variable:

$$x=\ln(i). \tag{41}$$

It gives

$$\begin{cases} \dot{s}(t) &= \mu-\beta s(t)e^{x(t)}-\mu s(t), \\ \dot{x}(t) &= \beta s(t)-\delta. \end{cases} \tag{42}$$

Evidently, when $R_o>1$, the system (42) admits one and only one equilibrium point: (s^*,x^*) with

$$s^*=\frac{\delta}{\beta}\ , \quad x^*=\ln\left(\frac{\mu(1-s^*)}{\beta s^*}\right)=\ln\left(\frac{\mu\,(\beta-\delta)}{\beta\,\delta}\right). \tag{43}$$

To analyze the stability properties of the equilibrium point (s^*,x^*), we introduce the variables:

$$\tilde{s}=s-s^*\ ,\ \tilde{x}=x-x^*, \tag{44}$$

which satisfy the following equations

$$\begin{cases} \dot{\tilde{s}}(t) &= \mu-[(\tilde{s}(t)+s^*](\beta e^{\tilde{x}+x^*}+\mu)=-[\omega e^{\tilde{x}(t)}+\mu]\tilde{s}(t)-\Omega e^{\tilde{x}(t)}+\mu(1-s^*), \\ \dot{\tilde{x}}(t) &= \beta\tilde{s}(t), \end{cases} \tag{45}$$

with $\omega=\beta e^{x^*}$ and $\Omega=\beta s^* e^{x^*}$, or

$$\begin{cases} \dot{\tilde{s}}(t) &= -\left[\omega e^{\tilde{x}(t)}+\mu\right]\tilde{s}(t)+\Omega\left[1-e^{\tilde{x}(t)}\right], \\ \dot{\tilde{x}}(t) &= \beta\tilde{s}(t). \end{cases} \tag{46}$$

Now, we construct a Lyapunov function for the system (46) using a technique explained for instance in [22]. First, we consider the positive definite and radially unbounded function:

$$U_1(\tilde{s},\tilde{x})=\frac{\beta}{2\Omega}\tilde{s}^2+e^{\tilde{x}}-\tilde{x}-1. \tag{47}$$

Its derivative along the trajectories of (46) satisfies

$$\begin{aligned} \dot{U}_1(t) &= -\tfrac{1}{\Omega}\left(\omega e^{\tilde{x}(t)}+\mu\right)\beta\tilde{s}(t)^2+\left(1-e^{\tilde{x}(t)}\right)\beta\tilde{s}(t)+\left(e^{\tilde{x}(t)}-1\right)\beta\tilde{s}(t), \\ &= -\tfrac{1}{\Omega}\left(\omega e^{\tilde{x}(t)}+\mu\right)\beta\tilde{s}(t)^2\le 0. \end{aligned} \tag{48}$$

To design a function with a negative definite Lie derivative along the trajectories of (46), we introduce the auxiliary function (see [22] for explanations for this choice):

$$U_2(\tilde{s},\tilde{x})=\tilde{s}\left(e^{\tilde{x}}-1\right). \tag{49}$$

Its derivative along the trajectories of (46) satisfies

$$\begin{aligned} \dot{U}_2(t) &= -\left(\omega e^{\tilde{x}(t)}+\mu\right)\left(e^{\tilde{x}}-1\right)\tilde{s}(t)-\Omega\left(1-e^{\tilde{x}(t)}\right)^2+\beta\tilde{s}(t)^2\left(e^{\tilde{x}(t)}-1\right), \\ &\le -\tfrac{\Omega}{2}\left(1-e^{\tilde{x}(t)}\right)^2+\left[\tfrac{1}{2\Omega}\left(\omega e^{\tilde{x}(t)}+\mu\right)^2+\beta\left(e^{\tilde{x}(t)}-1\right)\right]\tilde{s}(t)^2, \\ &\le -\tfrac{\Omega}{2}\left(1-e^{\tilde{x}(t)}\right)^2+\left(\tfrac{\omega^2}{\Omega}e^{2\tilde{x}(t)}+\tfrac{\mu^2}{\Omega}+\beta e^{\tilde{x}(t)}-\beta\right)\tilde{s}(t)^2. \end{aligned} \tag{50}$$

Since $\tilde{x}(t)\in[-x_*,N-x_*]$, the inequality

$$\dot{U}_2(t)\le -\tfrac{\Omega}{2}\left[1-e^{\tilde{x}(t)}\right]^2+\left[\left(\tfrac{\omega^2 e^{N-x_*}}{\Omega}+\beta\right)e^{\tilde{x}(t)}+\tfrac{\mu^2}{\Omega}-\beta\right]\tilde{s}(t)^2\ issatisfied. \tag{51}$$

From (43), it follows that

$$\dot{U}_2(t) \leq -\frac{\Omega}{2}\left[1-e^{\tilde{x}(t)}\right]^2+\left[\left(\frac{\omega^2}{\Omega}\frac{\beta\delta e^N}{\mu(\beta-\delta)}+\beta\right)e^{\tilde{x}(t)}+\frac{\mu^2}{\Omega}-\beta\right]\tilde{s}(t)^2. \tag{52}$$

Now, we introduce a new function :

$$U_3(\tilde{s},\tilde{x}) = kU_1(\tilde{s},\tilde{x})+U_2(\tilde{s},\tilde{x}) \tag{53}$$

where $k>0$ such that

$$k \geq 2\max\left\{\frac{\mu^2-\Omega\beta}{\mu\beta}, \frac{\omega\delta e^N}{\mu(\beta-\delta)}+\frac{\Omega}{\omega}, \frac{\sqrt{\delta\Omega}}{\sqrt{2\mu(\beta-\delta)}}\right\}: \tag{54}$$

Observe that for all $\mathfrak{c}>0$, it follows from (49) that

$$|U_2(\tilde{s},\tilde{x})| \leq \frac{\mathfrak{c}}{2}\tilde{s}^2+\frac{1}{2\mathfrak{c}}\left(e^{\tilde{x}}-1\right)^2 \leq \frac{\mathfrak{c}}{2}\tilde{s}^2+\frac{1}{2\mathfrak{c}}\max\left\{1,\frac{\delta\beta}{\mu(\beta-\delta)}\right\}\left(e^{\tilde{x}}-\tilde{x}-1\right) \tag{55}$$

for all $(\tilde{s},\tilde{x}) \in [-s_*, 1-s_*]\times(-\infty,-x_*)$. Consequently, with $\mathfrak{c}=\frac{k\beta}{2\Omega}$, we obtain

$$|U_2(\tilde{s},\tilde{x})| \leq \frac{k\beta}{4\Omega}\tilde{s}^2+\frac{\delta\Omega}{k\mu(\beta-\delta)}\left(e^{\tilde{x}}-\tilde{x}-1\right) \tag{56}$$

As an immediate consequence,

$$U_3(\tilde{s},\tilde{x}) \geq \frac{k\beta}{4\Omega}\tilde{s}^2+\left[k-\frac{\delta\Omega}{k\mu(\beta-\delta)}\right]\left(e^{\tilde{x}}-\tilde{x}-1\right). \tag{57}$$

Now, since $k \geq \frac{2\sqrt{\Omega\delta}}{\sqrt{\mu(\beta-\delta)}}$,

$$U_3(\tilde{s},\tilde{x}) \geq \frac{k}{2}U_1(\tilde{s},\tilde{x}) \tag{58}$$

for all $(\tilde{s},\tilde{x}) \in [-s^*, 1-s^*]\times(-\infty,-x^*)$. It follows that the function U_3 is positive definite on the state space. Then we deduce from (48) and (51) that

$$\begin{aligned}\dot{U}_3(t) &\leq -\tfrac{k}{\Omega}\left(\omega e^{\tilde{x}(t)}+\mu\right)\beta\tilde{s}(t)^2-\tfrac{\Omega}{2}\left[1-e^{\tilde{x}(t)}\right]^2\\ &\quad+\left[\left(\tfrac{\omega^2}{\Omega}\tfrac{\beta\delta e^N}{\mu(\beta-\delta)}+\beta\right)e^{\tilde{x}(t)}+\tfrac{\mu^2}{\Omega}-\beta\right]\tilde{s}(t)^2,\\ &\leq -W(\tilde{s}(t),\tilde{x}(t)) = -\tfrac{k}{2\Omega}\left(\omega e^{\tilde{x}(t)}+\mu\right)\beta\tilde{s}(t)^2-\tfrac{\Omega}{2}\left[1-e^{\tilde{x}(t)}\right]^2,\end{aligned} \tag{59}$$

where the last inequality is a consequence of (54).
Noting that (59) implies that all the solutions of (46) are bounded, we conclude that they all converge to the origin. Thus all solutions $(s(t),x(t))$ of (42), with initial conditions in the positively invariant domain $(-s^*,s^*)\times(-\infty,0)$, converge asymptotically to the origin, which in addition is locally exponentially stable. □

B Proof Theorem 2

□

Proof of Theorem 2. According to the definition of $(s_\triangle, x_\triangle)$, we have

$$\begin{cases}\dot{s}_\triangle(t) = -\beta(t)s_\triangle(t)e^{x_\triangle(t)}+\mu(1-s_\triangle(t))\\ \dot{x}_\triangle(t) = \beta(t)s_\triangle(t)-\delta.\end{cases} \tag{60}$$

In what follows we prove that $X_*(t) = (s_*(t), x_*(t))$ is *close* to the periodic solution we are looking for in the sense of the inequality (21). In other words, $X_*(t) = (s_*(t), x_*(t))$ is contained in a tubular neighborhood of $(s_\triangle(t), x_\triangle(t))$. We prove it by formulating the problem in a similar way than in (46).

One can readily check that $X_*(t) = (s_*(t), x_*(t))$, defined in (20) verifies

$$\begin{cases} \dot{s}_*(t) &= -\frac{\delta}{\beta(t)^2}\dot{\beta}(t) \\ \dot{x}_*(t) &= \frac{1}{\beta(t)-\delta}\dot{\beta}(t) \end{cases} \tag{61}$$

Observe for later use that Assumption A2 ensures that $\frac{1}{\beta_*-\delta}$ is well-defined and positive and $|\dot{s}_*(t)| \leq \frac{\delta}{\beta_*^2}|\dot{\beta}(t)|$ and $|\dot{x}_*(t)| \leq \frac{1}{\beta_*-\delta}|\dot{\beta}(t)|$.

Now, observe that $-\beta(t)s_*(t)e^{x_*(t)} + \mu(1-s_*(t)) = -\beta(t)\frac{\delta}{\beta(t)}\left(\frac{\mu-\mu s_*(t)}{\beta(t)s_*(t)}\right) + \mu - \mu\frac{\delta}{\beta(t)} = 0$ and $\beta(t)s_*(t) - \delta = 0$. It follows that

$$\begin{cases} \dot{s}_*(t) &= -\beta(t)s_*(t)e^{x_*(t)} + \mu(1-s_*(t)) - \frac{\delta}{\beta(t)^2}\dot{\beta}(t) \\ \dot{x}_*(t) &= \beta(t)s_*(t) - \delta + \frac{1}{\beta(t)-\delta}\dot{\beta}(t) \end{cases} \tag{62}$$

Now, let us introduce the functions:

$$s_r = s_\triangle - s_* \ , \ x_r = x_\triangle - x_* \tag{63}$$

Then

$$\begin{cases} \dot{s}_r(t) &= \beta(t)\left[s_*(t)e^{x_*(t)} - s_\triangle(t)e^{x_\triangle(t)}\right] - \mu s_r(t) + \frac{\delta}{\beta(t)^2}\dot{\beta}(t) \\ \dot{x}_r(t) &= \beta(t)s_r(t) - \frac{1}{\beta(t)-\delta}\dot{\beta}(t) \end{cases} \tag{64}$$

or, equivalently,

$$\begin{cases} \dot{s}_r(t) &= -\left[\omega(t)e^{x_r(t)} + \mu\right]s_r(t) + \Omega(t)\left[1 - e^{x_r(t)}\right] + \frac{\delta}{\beta(t)^2}\dot{\beta}(t) \\ \dot{x}_r(t) &= \beta(t)s_r(t) - \frac{1}{\beta(t)-\delta}\dot{\beta}(t) \end{cases} \tag{65}$$

with $\omega(t) = \beta(t)e^{x_*(t)}$, $\Omega(t) = \beta(t)e^{x_*(t)}s_*(t)$.

Observe that the system of equations (65) is the system (46) plus time-varying extra terms which are "small" when $\beta^\dagger$ is small. Therefore in analogy with the case of constant β, we consider, as a candidate for a Lyapunov function, the function U_3 defined in (53), i. e.

$$U_3(s_r(t), x_r(t)) = kU_1(s_r(t), x_r(t)) + U_2(s_r(t), x_r(t)), \tag{66}$$

with $U_1(s_r, x_r)$ and $U_2(s_r, x_r)$ defined, respectively, as in (47) and (49).

Next, we have

$$\dot{U}_3(t) \leq -W(s_r(t), x_r(t)) + \frac{\partial U_3}{\partial s_r}(s_r(t), x_r(t))\frac{\delta}{\beta(t)^2}\dot{\beta}(t) - \frac{\partial U_3}{\partial x_r}(s_r(t), x_r(t))\frac{1}{\beta(t)-\delta}\dot{\beta}(t) \tag{67}$$

where $W(s_r, x_r)$ is the positive definite function defined in (59). Through simples calculations, one can prove that

$$\begin{aligned} \left|\frac{\partial U_3}{\partial s_r}(s_r, x_r)\right|^2 &\leq \frac{4}{\Omega}\max\left\{1, \frac{k\beta}{\mu}\right\}W(s_r, x_r) \\ \left|\frac{\partial U_3}{\partial x_r}(s_r, x_r)\right|^2 &\leq 4\max\left\{\frac{\Omega}{k\omega\beta}, \frac{k^2}{\Omega}\right\}W(s_r, x_r) \end{aligned} \tag{68}$$

It follows that

$$\begin{aligned} \dot{U}_3(t) &\leq -W(s_r(t), x_r(t)) + 2\left[\sqrt{\frac{1}{\Omega}\max\left\{1, \frac{k\beta}{\mu}\right\}}\frac{\delta}{\beta_*^2} + \sqrt{\max\left\{\frac{\Omega}{k\omega\beta}, \frac{k^2}{\Omega}\right\}}\frac{1}{\beta_*-\delta}\right] \\ &\quad \beta^\dagger\sqrt{W(s_r, x_r)}, \\ &\leq -\tfrac{1}{2}W(s_r(t), x_r(t)) + g_1, \end{aligned} \tag{69}$$

with

$$g_1 = 2\left[\sqrt{\frac{1}{\Omega}\max\left\{1,\frac{k\beta}{\mu}\right\}}\frac{\delta}{\beta_*^2}+\sqrt{\max\left\{\frac{\Omega}{k\omega\beta},\frac{k^2}{\Omega}\right\}}\frac{1}{\beta_*-\delta}\right]^2\beta_\dagger^2 \tag{70}$$

Now, through lengthy but simple calculations, one can prove that

$$2g_2U_3(s_r,x_r)\leq W(s_r,x_r) \tag{71}$$

with

$$g_2=\frac{1}{4\max\left\{\frac{1}{\mu},\frac{ke^3}{\Omega}+\frac{1}{2k\beta}\right\}} \tag{72}$$

It follows that

$$\dot{U}_3(t)\ \leq\ -g_2U_3(s_r(t),x_r(t))+g_1 \tag{73}$$

By integrating this inequality, we obtain for all $t\geq t_c+T$,

$$e^{g_2t}U_3(s_r(t),x_r(t))-e^{g_2(t-T)}U_3(s_r(t-T),x_r(t-T))\leq\frac{g_1}{g_2}\left(e^{g_2t}-e^{g_2(t-T)}\right) \tag{74}$$

Consequently

$$\left(e^{g_2t}-e^{g_2(t-T)}\right)U_3(s_r(t),x_r(t))\leq\frac{g_1}{g_2}\left(e^{g_2t}-e^{g_2(t-T)}\right) \tag{75}$$

This inequality simplifies as

$$U_3(s_r(t),x_r(t))\leq\frac{g_1}{g_2} \tag{76}$$

From (58), it follows that

$$\frac{\beta}{2\Omega}s_r^2+e^{x_r}-x_r-1=U_1(s_r,x_r)\leq\frac{2g_1}{kg_2} \tag{77}$$

Since $|x_r|\leq 1$, it follows that

$$\frac{\beta}{2\Omega}s_r^2+\frac{1}{2e}x_r^2=U_1(s_r,x_r)\leq\frac{2g_1}{kg_2}, \tag{78}$$

and we deduce that

$$s_r^2+x_r^2\leq\frac{4g_1}{kg_2}\max\left\{\frac{\Omega}{\beta},e\right\} \tag{79}$$

Now, observe that

$$\begin{aligned}\frac{g_1}{g_2}\ =\ &8\left[\sqrt{\frac{1}{\Omega}\max\left\{1,\frac{k\beta}{\mu}\right\}}\frac{\delta}{\beta_*^2}+\sqrt{\max\left\{\frac{\Omega}{k\omega\beta},\frac{k^2}{\Omega}\right\}}\frac{1}{\beta_*-\delta}\right]^2\\ &\max\left\{\frac{1}{\mu},\frac{ke^3}{\Omega}+\frac{1}{2k\beta}\right\}\beta_\dagger^2\end{aligned} \tag{80}$$

We deduce that $\mathrm{K}=4\sqrt{2}\sqrt{\frac{1}{k}\max\left\{\frac{\Omega}{\beta},e\right\}}\left[\sqrt{\frac{1}{\Omega}\max\left\{1,\frac{k\beta}{\mu}\right\}}\frac{\delta}{\beta_*^2}+\sqrt{\max\left\{\frac{\Omega}{k\omega\beta},\frac{k^2}{\Omega}\right\}}\frac{1}{\beta_*-\delta}\right]$
$\sqrt{\max\left\{\frac{1}{\mu},\frac{ke^3}{\Omega}+\frac{1}{2k\beta}\right\}}$. □

C The basic reproduction number R_o

In epidemiology, a very fundamental problem is the one of estimating how many new infected individuals a single infected one could generate. More specifically one considers the impact of a single infected individual on a population of all healthy individuals, also called *susceptible* individuals. Then the question that one would like to answer is then the following: *how many infected individuals, on average, can a single infected one generate at each generation*?

When addressing such a question, the main issue is not to be specific about the exact number at each

generation – the epidemic dynamics is varying in its intensity according with the number of susceptible – but to describe an asymptotic behavior, or mean behavior, over a long time[1]. When considering a discrete linearized process, where $t_n = n\Delta t$, this can be translated into the fact that the epidemic dynamics can be modeled by a map, an "averaged" map, of the type $I(t_{n+1}) = R_o I(t_n)$, where I_n and I_{n+1} are respectively the number of infected R_o, called the *Basic Reproduction Number*, corresponds to the average number of secondary cases produced in his entire contagious history. The *Basic Reproduction Number* R_o is a famous concept in mathematical biology and is used in several epidemic models, in order to calculate a threshold for the existence (or not) of an epidemic outbreak. Some good references of R_o theory can be found in the work of O. Diekmann and J.A.P Heesterbeek [12].
For the continue time dynamics the Basic reproduction number coincides with a adimensional parameter which is naturally defined in the linear analysis when perturbing the equilibrium state where the whole population is healthy. More specifically, when considering the model of Eqs. (1)-(2), it is immediate to verify that $(S^*, I^*) = (N, 0)$ is an equilibrium state. Moreover when linearizing about the equilibrium

$$S(t) = S^* + \Delta S(t) = N + \Delta S(t) \qquad \text{and} \qquad I(t) = I^*$$

the linearized equation for the infected population is

$$\frac{d(\Delta I)}{dt} = [\beta - (\gamma+\mu)]\Delta I = (\gamma+\mu)[R_o - 1]\Delta I, \qquad \Delta I(0) = 1$$

where

$$R_o = \frac{\beta}{\gamma+\mu}.$$

Therefore $\Delta I(t) = e^{(\gamma+\mu)[R_o-1]t}$, and we can conclude that $\Delta I(t)$ decreases if $R_o < 1$.

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[1] *Over a long time* is mathematically translated as *in the limit of* $t \to \infty$

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