Stability and Observer Designs Using New Variants of Halanay's Inequality *

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Abstract

We provide a generalization of Halanay's inequality, where the decay rate is constant but the gain multiplying the delayed term is time varying. While the usual Halanay's conditions require the decay rate to be strictly larger than an upper bound on the gain, our less restrictive results allow times when the gain can exceed the decay rate. This allows us to prove asymptotic stability in significant cases that were not amenable to previous Lyapunov function constructions, and in cases that violate the contraction requirement that was needed to prove asymptotic stability in previous trajectory based results. We apply our work to stability problems for linear continuous time systems with switched delays, and to observers for nonlinear systems with discrete measurements.

Key words: Stability, delay systems, observers

1 Introduction

This paper continues the development (which was begun in Ahmed *et al.* (2018); Mazenc and Malisoff (2015); Mazenc *et al.* (2017, 2018)) of trajectory based and contractivity methods that can be used to prove asymptotic stability properties for control problems with delays and switching, in cases that may not lend themselves to standard Lyapunov functional methods. See for instance Mazenc *et al.* (2017, 2018) for applications to systems with discontinuous delays, and to switched systems for which some of the subsystems enjoy asymptotic stability properties while other subsystems may be unstable. One situation where trajectory based and contractivity methods have been useful is for systems whose vector field is not necessarily continuous that are encountered in many cases including systems that are asymptotically stabilized using piecewise constant feedbacks, systems with switched delays, and observers whose measurements are only available at discrete instants. For proving asymptotic stability of these systems, some available tools include extensions of Razumikhin's theorem (e.g., from Zhou and Egorov (2016)), as well as Halanay's approach (as in Halanay (1966)).

While there is a large and growing literature on constructing Lyapunov functions (such as Malisoff and Mazenc (2009); Zhou (2019); Zhou *et al.* (2020)), it is sometimes easier to find constants $\rho \in (0,1)$ and $T_* > 0$ such that every solution ζ of a system satisfies an inequality of the type $|\zeta(t)| \leq \rho \sup_{l\in[t-T_*,t]} |\zeta(l)|$ for all $t \geq T_*$. In such cases, ρ is called a contractivity constant, and we say that the solutions of the system satisfy a contractivity condition. Contractivity conditions can often be verified, by first proving that the solutions satisfy a Halanay type inequality of the form $\dot{V}(\zeta(t)) \leq -cV(\zeta(t)) + d(t)\sup_{t-T\leq\ell\leq t} V(\zeta(\ell))$ for some nonnegative valued function V, some positive constants c (called a decay rate) and T, and some nonnegative valued function $d(t)$ (called a gain); see, e.g., (Fridman, 2014, Lemma 4.2), (Selivanov and Fridman, 2015, Lemma 1), or (Selivanov and Fridman, 2016, Lemma 1) for the usual Halanay's inequality conditions, which ensure that V converges exponentially to 0 if $c > \sup_t d(t)$. However, if $c \leq \sup_t d(t)$, then the usual Halanay's inequality conditions cannot be used to prove exponential stability, and then standard contractivity conditions cannot be used to prove exponential stability. As we will see below, the usual Halanay's inequality in conjunction with contractivity can

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lead to conservative results.

Therefore, in the present paper, we improve on several stability conditions available in the literature, by providing a relaxed version of Halanay's inequality. We are motivated by the theoretical importance of Halanay's inequality and problems of convergence of observers with sampled data that were designed in the pioneering paper Karafyllis and Kravaris (2009). However, the present paper covers cases where the size of the sampling interval can violate the conditions in Karafyllis and Kravaris (2009), and where the contractivity conditions from Mazenc *et al.* (2017) cannot be satisfied. Our objectives differ significantly from other variants of Halanay's inequality, such as the notable works by Baker (2010) (which provides discrete time versions and nonlinear bounds) and Hien *et al.* (2015) (which uses integral conditions involving time varying decay rates and time varying gains, which we do not use in this work). We also cover systems with sampled outputs with scarce arbitrarily long sampling intervals in the sense of Mazenc (2019), but our results give more easily checked sufficient conditions than the integral condition in (Mazenc, 2019, Assumption A3). Our less restrictive results allow the sampling to be more frequent outside those intervals where violations of the usual Halanay's conditions occur. Therefore, we use the sampling to compensate for the failure of the usual Halanay's conditions to hold, to apply our less restrictive version of Halanay's conditions. This paper improves on our conference version Mazenc *et al.* (2020) by including proofs and an application to observers; the work Mazenc *et al.* (2020) only provides sketches of proofs and did not include the material on observers.

In Section 2, we motivate our work by illustrating why the contractivity condition from Mazenc *et al.* (2017) is conservative. In Sections 3-4, we provide our generalization of Halanay's inequality and applications to systems with switching delays, and to observers with sampled measurements where some intervals between the sampling times can be arbitrarily large. We conclude in Section 5 by summarizing our findings and suggestions for future research.

We use standard notation, which is simplified when no confusion would arise from the context, and where the dimensions of our Euclidean spaces are arbitrary unless otherwise noted. The standard Euclidean 2-norm, and the induced matrix norm, are denoted by $|\cdot|$, $|\cdot|_S$ is the supremum over any set S, and $|\cdot|_\infty$ is the usual sup norm. We define Ξ_t by $\Xi_t(s) = \Xi(t+s)$ for all Ξ, $s \le 0$, and $t \ge 0$ such that $t+s$ is in the domain of Ξ. We set $\mathbb{Z}_{\ge 0} = \{0, 1, \ldots\}$ and $\mathbb{N} = \mathbb{Z}_{\ge 0} \setminus \{0\}$. Throughout the paper, we consider sequences $t_i \in [0, +\infty)$ such that $t_0 = 0$ and such that there are two constants $\bar{T} > 0$ and $\underline{T} > 0$ such that

$$\underline{T} \le t_{i+1} - t_i \le \bar{T} \tag{1}$$

for all $i \in \mathbb{Z}_{\ge 0}$. For square matrices M_1 and M_2 of the same size, we use $M_1 \le M_2$ to mean that $M_2 - M_1$ is a nonnegative definite matrix, and I denotes the identity matrix in the dimension under consideration. For delay systems, our initial functions are assumed to be continuous.

2 Motivation: Limitation of contraction approach of Mazenc *et al.* (2017)

This subsection provides an example where a violation of the usual Halanay inequality condition may preclude the possibility of using contractivity arguments (such as those of Mazenc *et al.* (2017)) to prove asymptotic convergence results. Later (in Section 4.1), we show how to prove asymptotic convergence results in the setting of this subsection, using an alternative argument.

Let $T > 0$ be a constant and the sequence t_i satisfy the requirements from Section 1 with $\underline{T} > T$. Consider a function $v : [-T, +\infty) \to [0, +\infty)$ that satisfies

$$\dot{v}(t) = -v(t) + b(t) \sup_{\ell \in [t-T,t]} v(\ell) \tag{2}$$

for all $t \ge 0$, where $b : [0, +\infty) \to \{0, 2\}$ is defined by

$$b(t) = \begin{cases} 0, \text{ if } t \in \cup_{i \in \mathbb{Z}_{\ge 0}} [t_i + T, t_{i+1}) \\ 2, \text{ if } t \in \cup_{i \in \mathbb{Z}_{\ge 0}} [t_i, t_i + T). \end{cases} \tag{3}$$

The classical Halanay's result does not make it possible to conclude anything on the asymptotic behavior of the function v, because b takes values above the coefficient value 1 of the negative right side term in (2). On the other hand, we now show that without additional conditions on the sequence t_i, one cannot prove that the function $v(t)$ converges to zero via the trajectory based approach by simply integrating (2) over an interval $[t-g, t]$, where g is a positive constant that one can choose as in Mazenc *et al.* (2017).

Let us try to prove that $v(t)$ converges to zero by applying the trajectory based method from Mazenc *et al.* (2017). For any $i \in \mathbb{Z}_{\ge 0}$, we first integrate (2) over an interval $[t_i, t]$ with $t \in [t_i, t_{i+1})$ and obtain

$$\begin{aligned} v(t) &= e^{t_i - t} v(t_i) + \int_{t_i}^{t} e^{m-t} b(m) \sup_{\ell \in [m-T,m]} v(\ell) \mathrm{d}m \\ &\le e^{t_i - t} v(t_i) + \int_{t_i}^{t} e^{m-t} b(m) \mathrm{d}m \sup_{\ell \in [t_i - T, t]} v(\ell). \end{aligned} \tag{4}$$

As an immediate consequence it follows that for any $t \in [t_i, t_i + T)$, we have

$$\begin{aligned} v(t) &\le e^{t_i - t} v(t_i) + 2 \int_{t_i}^{t} e^{m-t} \mathrm{d}m \sup_{\ell \in [t_i - T, t]} v(\ell) \\ &= e^{t_i - t} v(t_i) + 2[1 - e^{t_i - t}] \sup_{\ell \in [t_i - T, t]} v(\ell) \\ &\le [2 - e^{t_i - t}] \sup_{\ell \in [t_i - T, t]} v(\ell). \end{aligned} \tag{5}$$

For any $t > t_i$, the inequality

$$2 - e^{t_i - t} > 1 \tag{6}$$

holds. It follows that one cannot deduce from (5) that the stability conditions of the usual contraction approach are satisfied, namely that there are a constant $g > 0$ and a constant $\rho \in [0, 1)$ such that

$$v(t) \le \rho \sup_{\ell \in [t-g,t]} v(\ell) \tag{7}$$

for all $t \geq g$. Thus, the trajectory based approach does not make it possible to prove a stability result for (2). In Section 4.1, we prove that v asymptotically converges to 0 under suitable conditions on the t_i's that ensure that $\underline{T}/T$ is large enough.

3 Improvement of Halanay's Inequality

This section provides an extension of Halanay's inequality, whose value lies in the fact that the analysis of switched systems with delays often leads to the study of generalized Halanay's inequalities of the type we consider in this section, as we will illustrate in Section 4.

3.1 Definitions and studied equation

Let t_i be a sequence of instants that satisfies the requirements of Section 1 for some constant $\underline{T} > 0$. Let

$$E = \cup_{i\in\mathbb{N}}[t_i, t_i + T) \tag{8}$$

where $T > 0$ is a constant such that

$$\underline{T} > 2T. \tag{9}$$

Condition (9) ensures that the intervals $[t_i, t_i + T)$ in the set E are disjoint. Let us introduce the constants

$$c > 0, \ \ \overline{\epsilon} \in [0, c), \text{ and } \overline{\varphi} > 0 \tag{10}$$

and the functions

$$\varphi(t) = \begin{cases} 0, \text{ if } t \notin E \\ \overline{\varphi}, \text{ if } t \in E \end{cases} \quad \text{and} \quad \epsilon(t) = \begin{cases} \overline{\epsilon}, \text{ if } t \notin E \\ 0, \text{ if } t \in E\ . \end{cases} \tag{11}$$

Consider a continuous and piecewise C^1 function $v : [-\underline{T}, +\infty) \to [0, +\infty)$ such that

$$\dot{v}(t) \ \leq \ -cv(t) + [\epsilon(t) + \varphi(t)]|v|_{[t-T,t]} \tag{12}$$

for all $t \geq 0$, where the derivatives in our differential inequalities should be understood in the Lebesgue almost everywhere sense, under the assumption:

Assumption 1 *Either*

$$\overline{\varphi} < c \tag{13}$$

or the inequality

$$\overline{\varphi}\left[e^{c(2T-\underline{T})} + \frac{\overline{\epsilon}}{c}\right]e^{2T\overline{\varphi}} + \frac{2T\overline{\varphi}}{\underline{T}} < c \tag{14}$$

is satisfied. □

3.2 Main result

We are ready to state and prove the following result (but see Remark 1 for constructions of the constants $\bar{C}_1$ and $\bar{C}_2$):

Theorem 1 *Let $v : [-\underline{T}, +\infty) \to [0, +\infty)$ be a continuous nonnegative valued solution of (12) under Assumption 1. Then we can construct positive constants $\bar{C}_1$ and $\bar{C}_2$ such that*

$$v(t) \leq \bar{C}_1 e^{-\bar{C}_2 t}|v|_{[-\underline{T},0]} \tag{15}$$

holds for all $t \geq 0$. □

Remark 1 *Basically, Assumption 1 means that no matter how large the constants $\overline{\varphi}$ and T are, v exponentially converges to zero, provided that $\underline{T}$ is sufficiently large and $\overline{\epsilon}$ is sufficiently small. The constant $\overline{\epsilon}$ can be interpreted to be the amount by which (12) differs from being a Lyapunov-like decay condition of the form*

$$\dot{v}(t) \leq -cv(t) \tag{16}$$

with decay rate $c > 0$ at times $t \notin E$. Using (Fridman, 2014, Lemma 4.2), we can show that the requirements of Theorem 1 are met with $\bar{C}_1 = e^{\underline{T}(2\delta+\bar{\varphi}/2+\max\{\bar{\varphi},\bar{\epsilon}\})}$ and $\bar{C}_2 = 2\delta$, where $\delta > 0$ is such that $\delta = \delta_0 - \delta_1 e^{2\delta\underline{T}}$, and $\delta_0 = 0.5(c - 2T\overline{\varphi}/\underline{T})$ and $\delta_1 = \frac{1}{2}\bar{\kappa}e^{2T\overline{\varphi}}$; see Appendix B below for details. □

Remark 2 *The intervals of E are still disjoint if we relax (9) to the assumption that $\underline{T} > T$. However, (9) is required in our proof of the theorem to ensure that $[t - T, t_j) \subseteq [t_{j-1} + T, t_j)$ holds for all $t \in [t_j, t_j + T)$ and $j \in \mathbb{N}$. We can extend Theorem 1 to an inequality of the type*

$$\dot{v}(t) \ \leq \ -cv(t) + [\epsilon(t) + \varphi(t)]|v|_{[t-r,t]} \tag{17}$$

with $r \in [0, T)$ because in this case (12) is satisfied. We can also extend this theorem to the case where $r \in (T, \underline{T}/2)$ and $\overline{\varphi} \geq \overline{\epsilon}$, by replacing the functions $\epsilon(t)$ and $\varphi(t)$ by functions $\epsilon_r(t)$ and $\varphi_r(t)$ defined by

$$\varphi_r(t) = \begin{cases} 0, \ \text{if } t \notin E_r \\ \overline{\varphi}, \ \text{if } t \in E_r \end{cases} \quad \text{and} \quad \epsilon_r(t) = \begin{cases} \overline{\epsilon}, \ \text{if } t \notin E_r \\ 0, \ \text{if } t \in E_r \end{cases} \tag{18}$$

with $E_r = \cup_{i\in\mathbb{N}}[t_i, t_i + r)$ because then any solution of (12) is a solution of (17). Our condition $r \in (T, \underline{T}/2)$ is more stringent than saying that T can be increased under the conditions of Theorem 1, since it does not allow $r \geq \underline{T}/2$. □

3.3 Proof of Theorem 1

Without loss of generality, we can assume that v is nonnegative valued and satisfies

$$\dot{v}(t) \ = \ -cv(t) + [\epsilon(t) + \varphi(t)]|v|_{[t-T,t]} \tag{19}$$

for all $t \geq 0$, because if this equality is not satisfied then we can prove the exponential convergence of the functions satisfying (12) with the help of a comparison system of the type of the equality (19); see the appendix below. Throughout the proof, we only consider the case where $\overline{\varphi} \geq c$, because the case $\overline{\varphi} < c$ is a consequence of the usual version of Halanay's inequality and our assumption that $\overline{\epsilon} < c$. We distinguish between two cases.

First case: $t \notin E$. Then (19) gives

$$\dot{v}(t) \ = \ -cv(t) + \overline{\epsilon}|v|_{[t-T,t]}. \tag{20}$$

Second case: $t \in E$ and $t \geq t_1$. Then, according to (19), there is a $j \in \mathbb{N}$ such that $t \in [t_j, t_j + T)$ and

$$\dot{v}(t) \ = \ -cv(t) + \overline{\varphi}|v|_{[t-T,t]}. \tag{21}$$

Then

$$\dot{v}(t) \ \leq \ -cv(t) + \overline{\varphi}|v|_{[t-T,t_j]} + \overline{\varphi}|v|_{[t_j,t]}. \tag{22}$$

Also, (21) gives $\dot{v}(t) \geq -cv(t) + \overline{\varphi}v(t) \geq 0$ for all $t \in [t_j, t_j + T)$ because $\overline{\varphi} \geq c$ and $v(t)$ is nonnegative for all $t \geq 0$. We deduce that $|v|_{[t_j,t]} = v(t)$ for all $t \in [t_j, t_j + T]$. Consequently, (22) gives

$$\dot{v}(t) \leq (\overline{\varphi} - c)v(t) + \overline{\varphi}|v|_{[t-T,t_j]}. \tag{23}$$

From (20), we deduce that for all $\ell \in [t_{j-1} + T, t_j)$ and $s \in [t_{j-1} + T, \ell]$, we have

$$v(\ell) = e^{c(s-\ell)}v(s) + \overline{\epsilon}\int_s^\ell e^{c(m-\ell)}|v|_{[m-T,m]}\mathrm{d}m. \tag{24}$$

Let $\ell \in [t - T, t_j)$. Then, according to (9), we have $t \geq t_j \geq t_{j-1} + \underline{T} > t_{j-1} + 2T$, so $\ell \in [t_{j-1} + T, t_j]$. On the other hand, (9) implies that $t - \underline{T} + T < t - T \leq \ell$. Also, we have $t - \underline{T} + T \geq t_j - \underline{T} + T \geq t_{j-1} + T$. Thus $t - \underline{T} + T \in [t_{j-1} + T, \ell)$. Also, (9) gives $t - T \geq t_j - T \geq t_{j-1} + \underline{T} - T \geq t_{j-1} + T$, so $[t - T, t_j) \subseteq [t_{j-1} + T, t_j)$. Thus, we can set $s = t - \underline{T} + T$ in (24) to get

$$\begin{aligned} v(\ell) &= e^{c(t-\underline{T}+T-\ell)}v(t - \underline{T} + T) \\ &\quad + \overline{\epsilon}\textstyle\int_{t-\underline{T}+T}^{\ell} e^{c(m-\ell)}|v|_{[m-T,m]}\mathrm{d}m \\ &\leq e^{c(2T-\underline{T})}v(t - \underline{T} + T) \\ &\quad + \overline{\epsilon}\textstyle\int_{t-\underline{T}+T}^{\ell} e^{c(m-\ell)}|v|_{[m-T,m]}\mathrm{d}m \end{aligned} \tag{25}$$

because $\ell \geq t - T$. We deduce from (9) that

$$\begin{aligned} v(\ell) &\leq e^{c(2T-\underline{T})}v(t - \underline{T} + T) \\ &\quad + \overline{\epsilon}\textstyle\int_{t-\underline{T}+T}^{\ell} e^{c(m-\ell)}\mathrm{d}m|v|_{[t-\underline{T},\ell]} \\ &\leq e^{c(2T-\underline{T})}v(t - \underline{T} + T) + \frac{\overline{\epsilon}}{c}|v|_{[t-\underline{T},\ell]} \\ &\leq \left[e^{c(2T-\underline{T})} + \frac{\overline{\epsilon}}{c}\right]|v|_{[t-\underline{T},\ell]} \end{aligned} \tag{26}$$

for all $\ell \in [t - T, t_j)$. As an immediate consequence,

$$|v|_{[t-T,t_j]} \leq \left[e^{c(2T-\underline{T})} + \tfrac{\overline{\epsilon}}{c}\right]|v|_{[t-\underline{T},t]}. \tag{27}$$

Combining the last inequality with (23), we obtain

$$\dot{v}(t) \leq (\overline{\varphi} - c)v(t) + \overline{\varphi}\left[e^{c(2T-\underline{T})} + \tfrac{\overline{\epsilon}}{c}\right]|v|_{[t-\underline{T},t]}. \tag{28}$$

General case. We deduce from (28) and (20) that

$$\dot{v}(t) \leq (\varphi(t) - c)v(t) + \overline{\kappa}|v|_{[t-\underline{T},t]} \tag{29}$$

with

$$\overline{\kappa} = \overline{\varphi}\left[e^{c(2T-\underline{T})} + \frac{\overline{\epsilon}}{c}\right] \tag{30}$$

for all $t \geq t_1$, because our condition $\overline{\varphi} \geq c$ implies that $\overline{\varphi}\frac{\overline{\epsilon}}{c} \geq \overline{\epsilon}$. Let us use (29) to prove the exponential convergence conclusion of the theorem.

To this end, first notice that

$$\tfrac{1}{\underline{T}}\textstyle\int_{t-\underline{T}}^{t}\int_{\ell}^{t}\varphi(m)\mathrm{d}m\mathrm{d}\ell \leq \int_{t-\underline{T}}^{t}\varphi(m)\mathrm{d}m \leq 2T\overline{\varphi} \tag{31}$$

for all $t \geq \underline{T}$, where the second inequality follows by the following argument. Let i be the largest index such that $t_i \leq t - \underline{T}$. If $t_{i+1} > t$, then the maximum interval $J \subseteq [t - \underline{T}, t]$ in which φ takes the value $\overline{\varphi}$ has length at most T. Otherwise, we have $t_i \leq t - \underline{T} < t_{i+1} \leq t \leq t_{i+2}$ (because $t_{i+2} - t_{i+1} \geq \underline{T}$), so $E \cap [t - \underline{T}, t]$ has length at most $2T$.

Hence, the time derivative of the function

$$\mu(t) = e^{-\frac{1}{\underline{T}}\int_{t-\underline{T}}^{t}\int_{\ell}^{t}\varphi(m)\mathrm{d}m\mathrm{d}\ell}v(t) \tag{32}$$

satisfies

$$\begin{aligned} \dot{\mu}(t) &= e^{-\frac{1}{\underline{T}}\int_{t-\underline{T}}^{t}\int_{\ell}^{t}\varphi(m)\mathrm{d}m\mathrm{d}\ell}\big[\dot{v}(t) - \varphi(t)v(t) \\ &\qquad + \tfrac{1}{\underline{T}}\textstyle\int_{t-\underline{T}}^{t}\varphi(m)\mathrm{d}mv(t)\big] \\ &\leq e^{-\frac{1}{\underline{T}}\int_{t-\underline{T}}^{t}\int_{\ell}^{t}\varphi(m)\mathrm{d}m\mathrm{d}\ell}\big[-cv(t) + \overline{\kappa}|v|_{[t-\underline{T},t]} \\ &\qquad + \tfrac{1}{\underline{T}}\textstyle\int_{t-\underline{T}}^{t}\varphi(m)\mathrm{d}mv(t)\big] \\ &\leq \left(\tfrac{2T\overline{\varphi}}{\underline{T}} - c\right)\mu(t) + \overline{\kappa}|v|_{[t-\underline{T},t]} \end{aligned} \tag{33}$$

for all $t \geq \underline{T}$. It follows from (31) that

$$\dot{\mu}(t) \leq \left(\tfrac{2T\overline{\varphi}}{\underline{T}} - c\right)\mu(t) + \overline{\kappa}e^{2T\overline{\varphi}}|\mu|_{[t-\underline{T},t]} \tag{34}$$

for all $t \geq \underline{T}$. Assumption 1 ensures that

$$\overline{\kappa}e^{2T\overline{\varphi}} < c - \frac{2T\overline{\varphi}}{\underline{T}}. \tag{35}$$

We deduce from the classical Halanay's result (e.g., (Fridman, 2014, Lemma 4.2)) that (34) and (35) imply that $\mu(t)$ converges exponentially to zero when t goes to $+\infty$. Since φ is nonnegative valued and upper bounded by $\overline{\varphi}$, the exponential convergence of μ implies exponential convergence of v. This allows us to conclude; see Appendix B below for a construction of the constant $\bar{C}_i$'s from (15).

4 Applications

We provide three applications of Theorem 1. Our first one will illustrate how Theorem 1 provides useful sufficient conditions for (2) to satisfy asymptotic stability conditions. Then we apply Theorem 1 to a class of systems whose delays can switch between small and large values. Finally, we apply Theorem 1 to an observer design problem with sampled outputs, in which there are scarce arbitrarily large sampling intervals in the same sense that scarce was used in Mazenc (2019). However, unlike Mazenc (2019) where the systems did not contain delays, the systems in our observer design application are allowed to have arbitrarily long delays, and our assumptions are less restrictive than those of Mazenc (2019).

4.1 System (2)

Consider the system (2) under the condition that

$$\underline{T} > 2T. \tag{36}$$

We apply Theorem 1. For the particular case we consider, we have

$$\overline{\varphi} = 2,\ \overline{\epsilon} = 0,\ \text{and } c = 1. \tag{37}$$

Then Assumption 1 gives the stability condition

$$2e^{6T-\underline{T}} + \tfrac{4T}{\underline{T}} < 1. \tag{38}$$

Then from Theorem 1, we conclude that $\lim_{t\to+\infty} v(t) = 0$ when (38) holds.

4.2 Systems with switching delays

Let t_i be a sequence as defined in Section 1 and τ_l and τ_s be two constants such that $\tau_l > \tau_s$ and

$$\underline{T} > 2(\tau_l + \tau_s). \tag{39}$$

Consider the family of systems

$$\dot{x}(t) = Mx(t) + Nx(t - \tau(t)) \tag{40}$$

where x valued in $\mathbb{R}^n$, τ is a time-varying piecewise continuous unknown delay such that

$$0 \le \tau(t) \le \tau_s \text{ if } t \notin E, \text{ and } 0 \le \tau(t) \le \tau_l \text{ if } t \in E \tag{41}$$

where E was defined by (8) for some constant $T \in (0, \underline{T}/2)$, and where $M \in \mathbb{R}^{n\times n}$ and $N \in \mathbb{R}^{n\times n}$ are constant matrices.

We introduce these two assumptions, the second of which is a largeness condition on $\underline{T}$ and a smallness condition on τ_s:

Assumption 2 *There are a symmetric positive definite matrix $Q \in \mathbb{R}^{n\times n}$ and a constant $q > 0$ such that*

$$Q(M + N) + (M + N)^\top Q \le -qQ \tag{42}$$

and

$$I \le Q \tag{43}$$

are satisfied. □

Assumption 3 *Either*

$$|N^\top QN| < \frac{q^2}{16} \tag{44}$$

or the inequality

$$|N^\top QN|\left[e^{q(2T-\underline{T})/2} + \tfrac{2L\tau_s^2}{q}\right]e^{\frac{16T|N^\top QN|}{q}} + \frac{2T|N^\top QN|}{\underline{T}} < \tfrac{q^2}{16} \tag{45}$$

with

$$L = \tfrac{2|N^\top QN|(|M|+|N|)^2}{q} \tag{46}$$

is satisfied. □

We prove the following proposition:

Proposition 1 *Let the system (40) satisfy Assumptions 2 and 3. Then its origin is a globally exponentially stable equilibrium point on $\mathbb{R}^n$.* □

Proof. For all $t \ge 0$, we have

$$\dot{x}(t) = (M + N)x(t) + N[x(t - \tau(t)) - x(t)]. \tag{47}$$

It follows from (42) that the time derivative of the positive definite function

$$U(x) = x^\top Qx \tag{48}$$

along all trajectories of (47) satisfies

$$\begin{aligned}\dot{U}(t) &\le -qU(x(t)) + \left\{2x(t)^\top QN\delta(x_t)\right\}\\ &\le -\tfrac{q}{2}U(x(t)) + \tfrac{2}{q}\left[\delta(x_t)\right]^\top N^\top QN\delta(x_t),\end{aligned} \tag{49}$$

where

$$\delta(x_t) = x(t - \tau(t)) - x(t) \tag{50}$$

and where we used the triangle inequality to get

$$\begin{aligned}&2\left|\sqrt{q/2}\sqrt{Q}x(t)\right|\left|\frac{\sqrt{Q}}{\sqrt{q/2}}N(x(t-\tau(t)) - x(t))\right|\\ &\le \tfrac{q}{2}U(x(t)) + \tfrac{2}{q}\left|\sqrt{Q}N(x(t-\tau(t)) - x(t))\right|^2\end{aligned} \tag{51}$$

to bound the quantity in curly braces. It follows that

$$\begin{aligned}\dot{U}(t) &\le -\tfrac{q}{2}U(x(t)) + \tfrac{2|N^\top QN|}{q}\left[|x(t-\tau(t))| + |x(t)|\right]^2\\ &\le -\tfrac{q}{2}U(x(t)) + \tfrac{8|N^\top QN|}{q}\sup_{l\in[t-\tau_l,t]} U(x(l))\end{aligned} \tag{52}$$

where the last inequality is a consequence of (43). On the other hand, the last inequality in (49) gives

$$\begin{aligned}\dot{U}(t) &\le -\tfrac{q}{2}U(x(t))\\ &\quad + \tfrac{2}{q}\left[\int_{t-\tau(t)}^t \dot{x}(s)\mathrm{d}s\right]^\top N^\top QN\int_{t-\tau(t)}^t \dot{x}(s)\mathrm{d}s\\ &\le -\tfrac{q}{2}U(x(t)) + \tfrac{2}{q}|N^\top QN|\left|\int_{t-\tau(t)}^t \dot{x}(s)\mathrm{d}s\right|^2\\ &\le -\tfrac{q}{2}U(x(t))\\ &\quad + \tfrac{2}{q}q_N\left|\int_{t-\tau(t)}^t [Mx(s) + Nx(s-\tau(s))]\mathrm{d}s\right|^2\\ &\le -\tfrac{q}{2}U(x(t))\\ &\quad + \tfrac{2}{q}q_N\left|\int_{t-\tau(t)}^t [|M|+|N|]\sup_{m\in[s-\tau_l,s]}|x(m)|\mathrm{d}s\right|^2\end{aligned} \tag{53}$$

for all $t \ge 0$, where $q_N = |N^\top QN|$. Consequently, we can use Jensen's inequality to get

$$\begin{aligned}&\dot{U}(t) \;\le\; -\tfrac{q}{2}U(x(t))\\ &+ \tfrac{2|N^\top QN|(|M|+|N|)^2\tau^2(t)}{q}\sup_{m\in[t-\tau_l-\tau(t),t]} U(x(m)).\end{aligned} \tag{54}$$

We deduce from the last inequality in (54) and the last inequality in (52) that

$$\dot{U}(t) \;\le\; -\tfrac{q}{2}U(x(t)) + L\tau_s^2\sup_{m\in[t-\tau_l-\tau_s,t]} U(x(m)) \tag{55}$$

with L defined in (46) when $t \notin E$, while

$$\dot{U}(t) \;\le\; -\tfrac{q}{2}U(x(t)) + \tfrac{8|N^\top QN|}{q}\sup_{l\in[t-\tau_l,t]} U(x(l)) \tag{56}$$

when $t \in E$. Assumption 3 ensures that Theorem 1 applies to $U(x(t))$ with $c = \frac{q}{2}$, $\overline{\epsilon} = L\tau_s^2$, and $\overline{\varphi} = 8|N^\top QN|/q$. It follows that $U(x(t))$ converges exponentially to zero. Since the function $U(x)$ is a positive definite quadratic function, we can conclude. □

4.3 Observer for systems with discrete measurements

In this section, we revisit Mazenc (2019), where continuous-time systems with discrete measurements were studied using the technique of Karafyllis and Kravaris (2009). The work Mazenc (2019) designed converging observers in cases where the lengths of some intervals between the measurements can exceed the upper bound that ensures convergence of the observer that is provided in (Karafyllis and

Kravaris, 2009, Equation (4.7)). This scarcity condition on the intervals in Mazenc (2019) is improved by the result that we give below, because our result below does not use the integral condition from (Mazenc, 2019, Assumption A3). Moreover, by contrast with Mazenc (2019), the system we consider has a delay.

4.3.1 Theoretical result

Let s_i be a strictly increasing sequence in $[0,+\infty)$ with $s_0 = 0$ such that there are two constants $s_\sharp > 0$ and $s_\pounds > s_\sharp$ such that $s_{i+1} - s_i \in [s_\sharp, s_\pounds]$ for all $i \in \mathbb{Z}_{\geq 0}$. We consider the system

$$\begin{cases} \dot{x}(t) \ = \ Hx(t) + Kx(t-\tau) + \Phi(Cx(t)) \ , \\ y(t) \ = \ Cx(s_i) \ \text{ if } \ t \in [s_i, s_{i+1}) \ \text{ and } \ i \in \mathbb{Z}_{\geq 0}. \end{cases} \tag{57}$$

where x is valued in $\mathbb{R}^n$, $K \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{q\times n}$ are nonzero constant matrices, $\tau > 0$ is a known constant delay, H is a Hurwitz matrix, and Φ is a nonlinear function. The assumption that H is Hurwitz is not restrictive. This is because for any system $\dot{x}(t) = Ax(t) + \phi(Cx(t))$ such that (A, C) is observable, there is a matrix L such that the matrix $A+LC$ is Hurwitz. Then the system $\dot{x}(t) = Ax(t)+\phi(Cx(t))$ can be rewritten as $\dot{x}(t) = Hx(t) + \Phi(Cx(t))$ with $H = A + LC$ and $\Phi(q) = \phi(q) - Lq$ and this system is of the type (57). Since the matrix H is Hurwitz, there are constants $c_1 > 0$, $p_1 > 0$, and $p_2 > 0$ and a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$PH + H^\top P \leq -2c_1 P \ \text{ and } \ p_1 I \leq P \leq p_2 I \ . \tag{58}$$

We fix a matrix P and positive constants c_1, p_1, and p_2 satisfying the preceding conditions (which can be selected as design choices) in the rest of this subsection and assume:

Assumption 4 *The function Φ is globally Lipschitz.* □

Assumption 5 *There is a sequence of instants t_j that satisfies the requirements of Section 1, and constants $T \in (0, \underline{T}/2)$ and $\underline{s} \in (0, \underline{T} - T)$, such that with the choice (8) of the set E, the following two conditions hold: (A) $\sup_{j\geq 0}(s_{j+1}-s_j) \leq T$ and (B) $\max\{s_{i+1}-s_i, s_i-s_{i-1}\} \leq \underline{s}$ for all $i \in \mathbb{N}$ such that $s_i \notin E$.* □

Our key assumption in this section will be that $\underline{s}$ is small enough as compared with the other parameters, which can be interpreted to mean that during each time interval $[t_k + T, t_{k+1})$ that is outside the union (8) that defines the set E, the sampling points s_i are close enough together, but this does not require any periodicity of the sampling interval lengths $s_{i+1} - s_i$. On the other hand, we allow $\underline{T}$ and so also T to be arbitrarily large, which is a scarcity condition as described in Mazenc (2019) that allows the s_i's to be further apart during the time intervals that define the set E; see Figure 1 below.

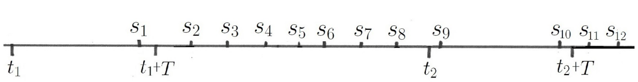

Fig. 1. Frequentness in the sampling points s_i outside the set $E = \cup_{i\in\mathbb{N}}[t_i, t_i + T)$ as required by our conditions.

To specify our requirements, we use the constants

$$\begin{aligned} &K^\sharp = 2|P^{\frac{1}{2}} K P^{-\frac{1}{2}}|, \quad B^\star = 4|C|k_\Phi, \\ &B^\dagger = \tfrac{\max\{|CK|^2, |CH|^2\}}{2|C|k_\Phi p_1}, \end{aligned} \tag{59}$$

where $k_\Phi > 0$ is a global Lipschitz constant for Φ,

$$\begin{aligned} &\overline{\beta} = K^\sharp + 4\left|P^{\frac{1}{2}}\right|^2 \tfrac{k_\Phi^2 B^\dagger}{c_1 B^\star}\left(e^{B^\star T} - 1\right) \ \text{ and} \\ &\underline{\beta} = K^\sharp + 4\left|P^{\frac{1}{2}}\right|^2 \tfrac{k_\Phi^2 B^\dagger}{c_1 B^\star}\left(e^{B^\star \underline{s}} - 1\right). \end{aligned} \tag{60}$$

Our final assumption is as follows, and can be viewed as smallness conditions on K and $\underline{s}$ and a largeness condition on $\underline{T}$:

Assumption 6 *Either $\overline{\beta} < c_1$ or the inequality*

$$\overline{\beta}\left[e^{c_1(2T-\underline{T})} + \frac{\underline{\beta}}{c_1}\right] e^{2T\overline{\beta}} + \frac{2T\overline{\beta}}{\underline{T}} < c_1 \tag{61}$$

holds. □

We use the dynamic extension

$$\begin{cases} \dot{\omega}(t) \ = \ CHz(t) + CKz(t-\tau) + C\Phi(\omega(t)) \\ \qquad \text{if } \ t \in [s_i, s_{i+1}) \ \text{ and } \ i \in \mathbb{Z}_{\geq 0} \\ \omega(s_i) \ = \ Cx(s_i) \ \text{ if } \ i \in \mathbb{Z}_{\geq 0} \\ \dot{z}(t) \ = \ Hz(t) + Kz(t-\tau) + \Phi(\omega(t)) \end{cases} \tag{62}$$

with ω valued in $\mathbb{R}^q$ and z valued in $\mathbb{R}^n$. This dynamic extension is similar to the one in Karafyllis and Kravaris (2009), but our allowing $\sup_i\{s_{i+1} - s_i\}$ to be arbitrarily large (by allowing T is arbitrarily large) puts our work outside the scope of Karafyllis and Kravaris (2009). We prove the following (whose proof will show that the convergence $\lim_{t\to\infty}(z(t) - x(t)) = 0$ is of exponential type):

Theorem 2 *Assume that the system (57) satisfies Assumptions 4 to 6. Then for all solutions $x(t)$ of (57) and all solutions (ω, z) of (62), we have $\lim_{t\to\infty}(z(t)-x(t)) = 0$.* □

Proof. We introduce the variables $e_\omega = \omega - Cx$ and $e_x = z - x$. Elementary calculations give

$$\begin{cases} \dot{e}_\omega(t) = CHe_x(t) + CKe_x(t-\tau) + C\Phi(\omega(t)) \\ \qquad - C\Phi(Cx(t)) \text{ if } t \in [s_i, s_{i+1}) \text{ and } i \in \mathbb{Z}_{\geq 0} \\ e_\omega(s_i) = 0 \ \text{ if } \ i \in \mathbb{Z}_{\geq 0} \\ \dot{e}_x(t) = He_x(t) + Ke_x(t-\tau) + \Phi(\omega(t)) \\ \qquad - \Phi(Cx(t)) \text{ if } t \in [s_i, s_{i+1}) \text{ and } i \in \mathbb{Z}_{\geq 0}. \end{cases} \tag{63}$$

Let us analyze (63) using the positive definite quadratic functions

$$V(e_x) = e_x^\top P e_x \text{ and } U(e_\omega) = \frac{1}{2}|e_\omega|^2. \tag{64}$$

The inequality (58) and Assumption 4 ensure that the time derivative of V along the trajectories of (63) satisfies

$$\begin{aligned} \dot{V}(t) \leq &-2c_1 V(e_x(t)) + 2e_x(t)^\top PKe_x(t-\tau) \\ &+ 2k_\Phi|e_x(t)^\top P||e_\omega(t)| \end{aligned}$$

and therefore also

$$\begin{aligned}\dot{V}(t) \leq& -2c_1 V(e_x(t))\\ &+2e_x(t)^\top P^{\frac{1}{2}}(P^{\frac{1}{2}}KP^{-\frac{1}{2}})P^{\frac{1}{2}}e_x(t-\tau)\\ &+2k_\Phi|e_x(t)^\top P^{\frac{1}{2}}||P^{\frac{1}{2}}||e_\omega(t)|\\ \leq& -2c_1V(e_x(t))\\ &+2|P^{\frac{1}{2}}KP^{-\frac{1}{2}}|\sqrt{V(e_x(t))}\sqrt{V(e_x(t-\tau))}\\ &+\left\{\sqrt{2c_1V(e_x(t))}\right\}\left\{k_\Phi\sqrt{2/c_1}|P^{1/2}||e_\omega(t)|\right\}.\end{aligned} \tag{65}$$

Here and in the rest of the proof, all equalities and inequalities are for all $t \geq 0$ unless otherwise noted.

Applying the triangle inequality to the terms in curly braces in (65) gives

$$\begin{aligned}&\dot{V}(t) \leq -c_1V(e_x(t))\\ &+K^\sharp\sqrt{V(e_x(t))}\sqrt{V(e_x(t-\tau))}+\tfrac{2k_\Phi^2}{c_1}\left|P^{\frac{1}{2}}\right|^2U(e_\omega(t))\end{aligned} \tag{66}$$

with $K^\sharp$ defined in (59). On the other hand, since

$$\sqrt{2U(e_\omega(t))} = |e_\omega(t)|, \tag{67}$$

we get

$$\begin{aligned}\dot{U}(t) \ \leq&\ 2|C|k_\Phi U(e_\omega(t))\\ &+\sqrt{2}|CK|\sqrt{U(e_\omega(t))}|e_x(t-\tau)|\\ &+\sqrt{2}|CH|\sqrt{U(e_\omega(t))}|e_x(t)|\\ \leq&\ 4|C|k_\Phi U(e_\omega(t))\\ &+\tfrac{\max\{|CK|^2,|CH|^2\}}{2|C|k_\Phi}\left(|e_x(t)|^2+|e_x(t-\tau)|^2\right)\\ \leq&\ B^\star U(e_\omega(t))+B^\dagger\left[V(e_x(t))+V(e_x(t-\tau))\right]\end{aligned} \tag{68}$$

for all $t \in [s_i, s_{i+1})$ and $i \in \mathbb{Z}_{\geq 0}$ with $B^\star$ and $B^\dagger$ defined in (59), using the triangle inequality to get the relation

$$\begin{aligned}&\sqrt{2}|CK|\sqrt{U(e_\omega(t))}|e_x(t-\tau)|\\ &=\left\{\sqrt{2|C|k_\Phi U(e_\omega(t))}\right\}\left\{\tfrac{|CK|}{\sqrt{k_\Phi|C|}}|e_x(t-\tau)|\right\}\\ &\leq |C|k_\Phi U(e_\omega(t))+\tfrac{1}{2}\tfrac{|CK|^2}{k_\Phi|C|}|e_x(t-\tau)|^2,\end{aligned} \tag{69}$$

and the same relation with K and $e_x(t-\tau)$ replaced by H and $e_x(t)$, respectively, and then using our condition on p_1 from (58).

By integrating the last inequality in (68) over the interval $[s_i, t)$ with $t \in [s_i, s_{i+1})$, and recalling that $e_\omega(s_i) = 0$ for all $i \in \mathbb{Z}_{\geq 0}$, we obtain

$$\begin{aligned}&U(e_\omega(t)) \leq\\ &B^\dagger\textstyle\int_{s_i}^t e^{B^\star(t-m)}\left[V(e_x(m))+V(e_x(m-\tau))\right]\mathrm{d}m.\end{aligned} \tag{70}$$

Combining (66) and (70), we obtain

$$\begin{aligned}\dot{V}(t) \ \leq&\ -c_1V(e_x(t))+K^\sharp\sqrt{V(e_x(t))}\sqrt{V(e_x(t-\tau))}\\ &+\tfrac{2}{c_1}\left|P^{\frac{1}{2}}\right|^2k_\Phi^2B^\dagger\textstyle\int_{s_i}^t e^{B^\star(t-m)}\left[V(e_x(m))\right.\\ &\left.+V(e_x(m-\tau))\right]\mathrm{d}m, \text{ and so also}\end{aligned}$$

$$\begin{aligned}\dot{V}(t) \ \leq&\ -c_1V(e_x(t))+K^\sharp\left\{\sqrt{\sup_{s\in[s_i,t]}V(e_x(s))}\right\}\\ &\times\left\{\sqrt{\sup_{s\in[s_i-\tau,t-\tau]}V(e_x(s))}\right\}\\ &+\tfrac{2}{c_1}\left|P^{\frac{1}{2}}\right|^2k_\Phi^2B^\dagger\int_{s_i}^t e^{B^\star(t-m)}\mathrm{d}m\\ &\times\left[\sup_{s\in[s_i,t]}V(e_x(s))+\sup_{s\in[s_i-\tau,t-\tau]}V(e_x(s))\right].\end{aligned} \tag{71}$$

It follows from applying the triangle inequality to the terms in curly braces in (71) that

$$\begin{aligned}\dot{V}(t) \ \leq&\ -c_1V(e_x(t))+\tfrac{K^\sharp}{2}\left[\sup_{s\in[s_i,t]}V(e_x(s))\right.\\ &\left.+\sup_{s\in[s_i-\tau,t-\tau]}V(e_x(s))\right]\\ &+\tfrac{2k_\Phi^2B^\dagger}{c_1}\left|P^{\frac{1}{2}}\right|^2\tfrac{e^{B^\star(t-s_i)}-1}{B^\star}\left[\sup_{s\in[s_i,t]}V(e_x(s))\right.\\ &\left.+\sup_{s\in[s_i-\tau,t-\tau]}V(e_x(s))\right]\\ \leq&\ -c_1V(e_x(t))\\ &+\left(K^\sharp+4\left|P^{\frac{1}{2}}\right|^2\tfrac{k_\Phi^2B^\dagger}{c_1B^\star}\left(e^{B^\star(t-s_i)}-1\right)\right)\\ &\times\sup_{s\in[s_i-\tau,t]}V(e_x(s))\end{aligned} \tag{72}$$

for all $t \in [s_i, s_{i+1})$ and $i \in \mathbb{Z}_{\geq 0}$.

Now, we distinguish between 2 cases:

First case. $t \in E$ and $t \geq \underline{T} + s_\pounds$. Choose $i \in \mathbb{Z}_{\geq 0}$ such that $s_i \leq t < s_{i+1}$. Thus

$$t - s_i < s_{i+1} - s_i \leq T \tag{73}$$

(where the last inequality is a consequence of Assumption 5). It follows that

$$\dot{V}(t) \ \leq\ -c_1V(e_x(t))+\overline{\beta}\sup_{s\in[s_i-\tau,t]}V(e_x(s)) \tag{74}$$

with $\overline{\beta}$ defined in (60).

Second case. $t \notin E$ and $t \geq \underline{T}+s_\pounds$. Then there is an $i \in \mathbb{Z}_{\geq 0}$ such that

$$s_i \leq t < s_{i+1} \tag{75}$$

and such that either

$$s_i \notin E \ \text{ or } \ s_{i+1} \notin E, \tag{76}$$

because $\underline{s} \in (0, \underline{T} - T]$, and because the distance between any two subintervals $[t_j, t_j + T)$ of E is at least $\underline{T} - T$. In either case, Assumption 5 gives

$$s_{i+1} - s_i \leq \underline{s}. \tag{77}$$

It follows that $t - s_i \leq \underline{s}$. We deduce that

$$\dot{V}(t) \ \leq\ -c_1V(e_x(t))+\underline{\beta}\sup_{s\in[s_i-\tau,t]}V(e_x(s)) \tag{78}$$

with $\underline{\beta}$ defined in (60).

Now, we can apply Theorem 1 with

$$c = c_1,\ \bar{\epsilon} = \underline{\beta},\ \text{and}\ \overline{\varphi} = \overline{\beta}, \tag{79}$$

because Assumption 6 ensures that Assumption 1 is satisfied. Then Theorem 1 ensures that

$$\lim_{t\to+\infty} V(e_x(t)) = 0, \tag{80}$$

which provides the desired result. □

4.3.2 Illustration

We illustrate Theorem 2 by applying it to a pendulum model with friction, building on the corresponding analysis for the pendulum without friction from Mazenc (2019). We can derive conditions on the constants $\underline{T} > 0$ and $\underline{s} > 0$ and on the ratio $k/m > 0$ such that the assumptions of Theorem 2 are satisfied for the pendulum dynamics with output

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\frac{g}{l}\sin(x_1(t)) - \frac{k}{m}x_2(t) \\ y(t) = x_1(s_i) \text{ if } t \in [s_i, s_{i+1}) \text{ and } i \in \mathbb{Z}_{\geq 0}, \end{cases} \tag{81}$$

where the positive constants g, k, l, and m represent gravity, friction, length, and mass, respectively.

To this end, we first rewrite the dynamics from (81) as

$$\dot{x}(t) = Hx(t) + \Phi(x_1(t)), \tag{82}$$

where

$$H = \begin{bmatrix} -2 & 1 \\ -1 & -\frac{k}{m} \end{bmatrix} \text{ and } \Phi(x_1) = \begin{bmatrix} 2x_1 \\ x_1 - \frac{g}{l}\sin(x_1) \end{bmatrix}. \tag{83}$$

Then Assumption 4 is satisfied with the global Lipschitz constant

$$k_\Phi = \sqrt{4 + \left(1 + \frac{g}{l}\right)^2} \tag{84}$$

for Φ. With the notation from Theorem 2, we now choose

$$P = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \text{ and } C = [1 \quad 0]. \tag{85}$$

Then our requirement

$$PH + H^\top P \leq -2c_1 P \tag{86}$$

is equivalent to the nonnegative definiteness of the matrix

$$M = \begin{bmatrix} 3 - 2c_1 & c_1 - 1 - \frac{k}{2m} \\ c_1 - 1 - \frac{k}{2m} & -2c_1 + 1 + \frac{2k}{m} \end{bmatrix}. \tag{87}$$

The preceding nonnegative definiteness condition will be satisfied if

$$\frac{k}{m} \in (0, 20.39) \tag{88}$$

and $c_1 > 0$ is small enough, because the matrix valued function

$$M_0(\ell) = \begin{bmatrix} 3 & -1 - \frac{\ell}{2} \\ -1 - \frac{\ell}{2} & 1 + 2\ell \end{bmatrix} \tag{89}$$

is positive definite for all $\ell \in (0, 20.39)$, and because of the continuity of eigenvalues of any matrix as functions of the entries of the matrix. However, the preceding bound depends on the choice of P in (85), which also affects the choice of c_1. Hence, it may be useful in practice to consider different choices of P to allow larger bounds on k/m.

Then we can choose

$$K = 0,\ p_1 = \frac{1}{2},\ p_2 = 1.5,\ K^\sharp = 0,\ |P^{1/2}| = 1.224, \tag{90}$$

and

$$B^* = 4\sqrt{4 + \left(1 + \frac{g}{l}\right)^2} \text{ and } B^\dagger = \frac{5}{\sqrt{4 + \left(1 + \frac{g}{l}\right)^2}}. \tag{91}$$

The preceding choices give the values

$$\begin{aligned} \overline{\beta} &= 4(1.224)^2 \frac{k_\Phi^2 B^\dagger}{c_1 B^\star}\left(e^{B^\star T} - 1\right) \\ &= \frac{7.49088}{c_1}\left(e^{4\sqrt{4+\left(1+\frac{g}{l}\right)^2}T} - 1\right) \end{aligned} \tag{92}$$

and

$$\begin{aligned} \underline{\beta} &= 4(1.224)^2 \frac{k_\Phi^2 B^\dagger}{c_1 B^\star}\left(e^{B^\star \underline{s}} - 1\right) \\ &= \frac{7.49088}{c_1}\left(e^{4\sqrt{4+\left(1+\frac{g}{l}\right)^2}\underline{s}} - 1\right) \end{aligned} \tag{93}$$

and Assumption 6 requires that $\overline{\beta} < c_1$ or

$$\overline{\beta}\left[e^{c_1(2T - \underline{T})} + \frac{\underline{\beta}}{c_1}\right]e^{2T\overline{\beta}} + \frac{2T\overline{\beta}}{\underline{T}} < c_1. \tag{94}$$

For each fixed c_1, the preceding formulas then show how our requirements from Assumption 6 will be satisfied if $\underline{s} > 0$ is small enough and $\underline{T}$ is large enough. Then Theorem 2 applies. Thus, with the constants we have selected,

$$\begin{cases} \dot{\omega}(t) = -2z_1(t) + z_2(t) + 2\omega(t), \\ \qquad \text{if } t \in [s_i, s_{i+1}) \text{ and } i \in \mathbb{Z}_{\geq 0} \\ \omega(t_i) = x_1(s_i) \text{ if } i \in \mathbb{Z}_{\geq 0} \\ \dot{z}_1(t) = -2z_1(t) + z_2(t) + 2\omega(t) \\ \dot{z}_2(t) = -z_1(t) - \frac{k}{m}z_2(t) + \omega(t) - \frac{g}{l}\sin(\omega(t)) \end{cases} \tag{95}$$

provides an asymptotic observer for the system (81), because for all solutions (ω, z) of (95) and all solutions of (81), we have $\lim_{t\to\infty}(z(t) - x(t)) = 0$, and the convergence is global (i.e., for all initial conditions) and of exponential type.

In Fig. 2, we plot the convergence of the components $z_1(t) - x_1(t)$ and $z_2(t) - x_2(t)$ of the estimation error for (81), which was generated from (81) and (95) using NDSolve in Mathematica. We used the initial state $x(0) = (1, 1)$ of (81), and with the initial states $\omega(0) = 0$ and $z(0) = (2, 1)$ (in red), $z(0) = (-2, -1)$ (in green), and $z(0) = (4, -2)$ (in blue) for the observer (95). We chose $l = 17$, $g = 9.8$, $c_1 = 2.5$, $T = 0.1$, $k = 1$, $m = 1$, $\underline{T} = 1$, $\underline{s} = 0.1$, and $s_j = 0.1j$ for all $j \in \mathbb{Z}_0$, which satisfy the preceding requirements, and then E is defined by the construction (8). Since the plot shows rapid convergence of the observation error to zero, it helps to validate our method, in the special case of the pendulum dynamics (81).

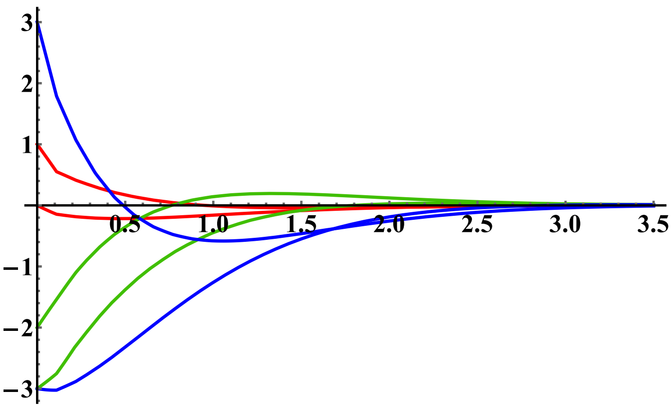

Fig. 2. Observer error $z(t)-x(t) = (z_1(t)-x_1(t), z_2(t)-x_2(t))$ converging to 0 with initial state $x(0) = (1, 1)$ for (81) and initial states $z(0) = (2, 1)$ (in red), $z(0) = (-2, -1)$ (in green), and $z(0) = (4, -2)$ (in blue) for observer.

5 Conclusion

The well known Halanay's inequality condition plays an important role in the analysis of dynamics with switching or delays, since it provides an alternative to the oftentimes difficult task of constructing Lyapunov functions. We proposed new stability analysis results which complement both the Halanay's and the trajectory based approach. This is significant, because our less restrictive conditions (which allow the gain on the delayed term to exceed the decay rate in Halanay's inequality) broadened the range of applicability of trajectory based approaches to proving asymptotic stability properties. We have shown the usefulness of our new approach, in the context of switched systems with delays, and observers for systems with discrete measurements. A key feature of our work is that it allows cases where some of the sampling intervals can be arbitrarily long, provided they occur in the scarce sense from the work Mazenc (2019) and as explained above. In future work, we hope to find methods to maximize the rates of convergence in our theorems.

Appendix: Comparison Lemma

Lemma 1 *Let* $v : [-T, +\infty) \to [0, +\infty)$ *be a nonnegative valued continuous solution of*

$$\dot{v}(t) \leq -cv(t) + \Lambda(t)|v|_{[t-T,t]} \tag{A.1}$$

where $T > 0$ *and* $c > 0$ *are constants, and where* Λ *is a piecewise constant function such that there is a constant* $\underline{\Lambda} > 0$ *such that* $\Lambda(t) > \underline{\Lambda}$ *for all* $t \geq 0$. *Let* w *be a nonnegative valued solution of*

$$\dot{w}(t) = -cw(t) + \Lambda(t)|w|_{[t-T,t]} \tag{A.2}$$

for all $t \geq 0$ *such that there is a constant* $t_0 \geq 0$ *such that*

$$v(m) < w(m) \text{ for all } m \in [t_0 - T, t_0]. \tag{A.3}$$

Then for all $t \geq t_0$, *the inequality* $v(t) < w(t)$ *is satisfied.* □

Proof. For any continuous function $w : [t_0 - T, t_0] \to [0, +\infty)$, the solution of (A.2) is continuous and uniquely defined on $[t_0-T, +\infty)$; see (Hale and Verduyn Lunel, 1993, Chapt. 2). Consider v and w such that (A.3) holds for all $t \in [t_0 - T, t_0]$. We proceed by contradiction. Suppose for the sake of obtaining a contradiction that the conclusion $v(t) < w(t)$ does not hold for all $t \geq t_0$. Then the continuity of v and w implies that there is a $t_c > t_0$ such that

$$v(m) < w(m) \text{ for all } m \in [t_0 - T, t_c) \tag{A.4}$$

and $v(t_c) = w(t_c)$. Also, (A.1) and (A.2) imply that for all $t \in [t_0, t_c)$, the function

$$\tilde{w}(t) = w(t) - v(t) \tag{A.5}$$

satisfies

$$\dot{\tilde{w}}(t) \geq -c\tilde{w}(t) + \Lambda(t)[|w|_{[t-T,t]} - |v|_{[t-T,t]}]. \tag{A.6}$$

Let

$$\varsigma(t) = e^{ct}\tilde{w}(t). \tag{A.7}$$

Then

$$\dot{\varsigma}(t) \geq e^{ct}\Lambda(t)[|w|_{[t-T,t]} - |v|_{[t-T,t]}], \tag{A.8}$$

which we can integrate over $[t, t_c]$ with $t \in [t_0, t_c)$ to get

$$\begin{aligned} &\varsigma(t_c) - \varsigma(t) \geq \\ &\int_t^{t_c} e^{cm}\Lambda(m)[|w|_{[m-T,m]} - |v|_{[m-T,m]}]\mathrm{d}m \end{aligned} \tag{A.9}$$

for all $t \in [t_0, t_c)$. Since $v(t_c) = w(t_c)$, we have $\varsigma(t_c) = 0$. It follows that

$$\begin{aligned} &\varsigma(t) \leq \\ &-\int_t^{t_c} e^{cm}\Lambda(m)[|w|_{[m-T,m]} - |v|_{[m-T,m]}]\mathrm{d}m \end{aligned} \tag{A.10}$$

for all $t \in [t_0, t_c)$. Since (A.4) and the continuity of v imply that $v(\ell) < |w|_{[m-T,m]}$ for all $\ell \in [m - T, m]$ and so also

$$|w|_{[m-T,m]} - |v|_{[m-T,m]} > 0 \tag{A.11}$$

for all $m \in [t_0, t_c)$, we deduce that

$$\varsigma(t) \leq -\underline{\Lambda}e^{ct}\int_t^{t_c}[|w|_{[m-T,m]} - |v|_{[m-T,m]}]\mathrm{d}m \tag{A.12}$$

for all $t \in [t_0, t_c)$. Thus,

$$\tilde{w}(t) \leq -\underline{\Lambda}\int_t^{t_c}[|w|_{[m-T,m]} - |v|_{[m-T,m]}]\mathrm{d}m < 0 \tag{A.13}$$

for all $t \in [t_0, t_c)$. Hence, $w(t) - v(t) < 0$ for all $t \in [t_0, t_c)$. This contradicts (A.4), allowing us to conclude. □

Appendix: Construction of $\bar{C}_1$ and $\bar{C}_2$ in (15)

To explicitly construct the constants $\bar{C}_1$ and $\bar{C}_2$ in our statement of Theorem 1, first note that by combining our decay estimate (34) on the function μ from (32) with our condition (35), it follows that we can apply (Fridman, 2014, Lemma 4.2) to the function μ with the choices $\delta_0 = 0.5(c - 2T\overline{\varphi}/\underline{T})$, $\delta_1 = \frac{1}{2}\bar{\kappa}e^{2T\overline{\varphi}}$, and $h = t_0 = \underline{T}$ to get

$$\mu(t) \le e^{-2\delta(t-\underline{T})}|\mu|_{[0,T]} \tag{B.1}$$

for all $t \ge \underline{T}$, where δ satisfies the requirements from Remark 1. Also, for all $t \ge 0$, our condition (12) gives

$$\dot{v}(t) \le \bar{\epsilon}^\sharp |v|_{[t-T,t]}, \tag{B.2}$$

where $\bar{\epsilon}^\sharp = \max\{\bar{\epsilon}, \overline{\varphi}\}$. Recalling that $\underline{T} > T$, it follows that

$$v(t) \le v(0) + \bar{\epsilon}^\sharp \int_0^t |v|_{[\ell-\underline{T},\ell]} \mathrm{d}\ell \tag{B.3}$$

for all $t \in [0, \underline{T}]$. Hence, for all $t \in [0, \underline{T}]$, the continuous function $v_s(\ell) = |v|_{[\ell-\underline{T},\ell]}$ satisfies

$$v_s(t) \le v_s(0) + \bar{\epsilon}^\sharp \int_0^t v_s(\ell)\mathrm{d}\ell \tag{B.4}$$

and so also

$$v(t) \le v_s(t) \le v_s(0)e^{\underline{T}\bar{\epsilon}^\sharp} \tag{B.5}$$

where (B.5) followed from Gronwall's inequality. Also, the nonnegative valuedness of φ and our formula (32) for μ gives

$$e^{-\overline{\varphi}\underline{T}/2}v(t) \le \mu(t) \le v(t) \tag{B.6}$$

for all $\in [0, \underline{T}]$. Combining (B.5)-(B.6) with (B.1) gives

$$v(t) \le e^{\overline{\varphi}\underline{T}/2} e^{-2\delta(t-\underline{T})} |v|_{[-\underline{T},0]} e^{\underline{T}\max\{\bar{\epsilon},\overline{\varphi}\}} \tag{B.7}$$

for all $t \ge 0$, which allows us to use the choices $\bar{C}_1 = e^{\underline{T}(2\delta+\bar{\varphi}/2+\max\{\bar{\varphi},\bar{\epsilon}\})}$ and $\bar{C}_2 = 2\delta$ as specified in Remark 1.

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