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***Abstract*—This paper is concerned with global stabilization of discrete-time multiple integrators with bounded controls by utilizing the energy function based approach. First, a discrete-time double integrators system subject to input additive disturbances is stabilized by a bounded feedback whose formula involves a linear function of the state and a saturation function only. Next, this result is used to stabilize a discrete-time multiple integrators of arbitrary length by bounded control with the aid of a special canonical form. Compared with the existing results, the proposed controllers require fewer saturation functions, which allow a better use of the control energy. Moreover, some free parameters that are introduced into these controllers can help improve the transient performance of the closed-loop systems significantly. Finally, a numerical example is given to illustrate the effectiveness of the proposed method.**

Keywords: Asymptotic stabilization; discrete-time systems; bounded control; energy function; saturation function.

I. Introduction

Actuator saturation is an inherent phenomenon in practical circuits and systems and thereby many research efforts have been devoted to deal with it in the past decades [5], [6], [7], [8]. In particular, global stabilization of linear systems by bounded feedback has attracted considerable attention. It has been shown that this problem can be solved if and only if the considered system is stabilizable in the ordinary sense and is not exponentially unstable when the control is zero [16], [23]. Such a system is called asymptotically null controllable with bounded controls (ANCBC). Since the work [3] (see also [17]), it is well-known that for a multiple integrators system of dimension $n \geq 3$, which is an ANCBC system, global asymptotic stabilization cannot be achieved by saturated linear state feedback. Therefore, the global stabilization of general ANCBC systems requires the design of more complex control laws.

Multiple integrators systems, also called chains of integrators, have strong potential applications in engineering because they can approximate or be parts of models such as the inertia wheel pendulum and the vertical takeoff and landing aircraft [9], [24], [25], so that their study has a strong engineering motivation and thus attracted considerable attention in the literature [11], [13], [18]. For the global stabilization problem, A. Teel provided a pioneering control strategy by introducing a class of nonlinear controllers consisting of nested saturation functions [18]. This work was later successfully extended to general ANCBC continuous-time and discrete-time linear systems [16], [23]. Unfortunately, for high order systems and/or bigger initial conditions, these nonlinear methods always result in a deteriorated transient performance. Thereafter, a number of studies have been made to improve the transient performance. For instance, in [10] and [11], state-dependent functions were introduced into the controllers, which have been shown to significantly improve the transient performance of the closed-loop system. Later on, the results in [11] were successfully extended to the discrete-time multiple integrators system [12] and were further improved in [26]. Recently, the global stabilization problem of high order integrators subject to disturbance and saturated control was consider in [1], where an adaptive nested saturation feedback law was proposed. This method was also shown to improve the transient performance of the closed-loop system. When the multiple integrators system is subject to input saturation and time-delay, the corresponding global stabilization problem was for the first time investigated in [13] by extending Teel's forwarding approach [18]. Later on, the same problem was revisited in [29] by using delayed and bounded controls based on some special canonical forms.

In the studies of global stabilization problems mentioned above, two approaches, based respectively on Lyapunov functions and energy functions, are often adopted. When one applies the well-known Lyapunov function based approach, one needs to find a Lyapunov function whose derivative along the trajectories of the studied systems is negative definite [12], [14], [20], [30]. There is no request of this type in case of energy function based analyses: the energy function only has to eventually decrease to zero and can increase on some finite time intervals [18], [28]. In general, the energy function based approach leads to more difficult analyses than the Lyapunov function based approach. That is probably the reason why the energy function based approach has received little attention, with the notable exceptions: [2], [27], [28]. Let us observe that corresponding results for linear discrete-time systems by bounded control, as far as we know, have never been established.

In this paper, we investigate the problem of global stabilization of discrete-time integrators system with bounded controls and adopt an energy function based approach. The contribution of this paper and the significance of the obtained results can be summarized as follows. First, the global stabilization problem for a double integrators subject to input additive disturbances by bounded controls is addressed. Through an energy function based approach, we design a parameterized state feedback and give explicit conditions guaranteeing that all trajectories of the closed-loop system converge to a small neighborhood of the origin in finite steps and remain inside forever. We wish to emphasize that, by contrast with the analysis in [28], the proof for this result is far from trivial. We did our best to propose a simple proof but, due to the high difficulty of the problem, we did not obtain a short proof. Second, based on a new special canonical form (which is different from that used in [12] and [30]) the global stabilization problem of discrete-time multiple integrators with bounded controls is solved via a class of nonlinear control laws in which there are $[\frac{n+1}{2}]$ nested saturation functions only. In the control laws proposed in [12] and [30] there are n saturation functions so that the designed control laws have significantly less saturations. This is an advantage because this leads to better transient performances. Notice also that we introduce some free parameters in the formula of the

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feedback to make it possible to improve the transient performances further.

Notation. The notation used in this paper is standard. For two integers p and q with $p \leq q$, the symbol $\mathbf{I}[p,q]$ refers to the set $\{p, p+1, \ldots, q\}$. By I_m, we denote the identity matrix of dimension n. For a positive constant ε, $\sigma_\varepsilon(x) = \varepsilon \text{sign}(x) \min\{|x/\varepsilon|, 1\}$ denotes the standard saturation function. Moreover, we will denote $\sigma_1(x)$ by $\sigma(x)$ to simplify. The notation $|\cdot|$ refers to both the induced matrix 2 norm and the usual Euclidean vector norm. For a real number α, $[\alpha]$ refers to the integer part of α. For a pair of matrices (A, b) with $A \in \mathbf{R}^{n\times n}$ and $b \in \mathbf{R}^{n\times 1}$, $Q_c(A, b) = [b, Ab, \ldots, A^{n-1}b]$ denotes the controllability matrix of (A, b). We denote by $P > 0$ a real symmetric positive definite matrix.

II. Problem Formulation and Technical Preliminaries

Throughout the paper, we reconsider the following discrete-time integrators system [12], [30]

$$x(k+1) = A_o x(k) + b_o u(k), \tag{1}$$

with

$$A_o = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ & 1 & \ddots & \ddots & \vdots \\ & & \ddots & 1 & 0 \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix}, \; b_o = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{2}$$

where $x = [x_1, x_2, \ldots, x_n]^T \in \mathbf{R}^n$ and $u \in \mathbf{R}$ are state and control variables, respectively. The main task of this paper is to solve the following problem:

Problem 1: Constructing a state feedback u_f such that $|u_f| \leq 1$ for all $k \in \mathbb{N}$ and the system (1) in closed-loop with u_f is globally asymptotically and locally exponentially stable.

Problem 1 has been investigated by Marchand et al. in [12], in which a bounded nonlinear control law consisting of n saturation functions is proposed based on a special state representation (see Lemma 1 in [12]). To ensure stability, the constraints on the corresponding parameters (see Theorem 1 in [12]) may be conservative and may degrade the control performance. Recently, to solve Problem 1, we proposed in [30] some new classes of control laws by taking advantage of the state space representation of [12]. The expressions of these control laws have n saturation functions, and the new constrains on the corresponding parameters are less conservative than those in [12]. However, all the nonlinear controllers proposed in [12] and [30] require n saturation functions, which suggests that the actuator capacity is not fully utilized. Bearing this in mind, by virtue of a novel special state representation and a new crucial technique, we construct in the present paper a new family of control laws for solving Problem 1. These controllers only have $[\frac{n+1}{2}]$ saturation functions and they have some free parameters which are degrees of freedom and can be used to improve the transient performance, as illustrated by a numerical example in Section V.

We end this section with two technical lemmas borrowed from [28], which will play important roles in the proof of our model transformation.

Lemma 1: Let (A_i, b_i), $i = 1, 2$, be two pairs of matrices and let (A_1, b_1) be controllable. Then there exists an invertible matrix T such that $A_2 = TA_1T^{-1}$, $b_2 = Tb_1$, if and only if (A_2, b_2) is controllable and $\lambda(A_2) = \lambda(A_1)$. In this case, the unique transformation matrix T is given by $T = Q_c(A_2, b_2) Q_c^{-1}(A_1, b_1)$.

Lemma 2: Consider the matrices:

$$A_T = \begin{bmatrix} A_1 & b_1f_2 & \cdots & b_1f_{p-1} & b_1f_p \\ & A_2 & \cdots & b_2f_{p-1} & b_2f_p \\ & & \ddots & \vdots & \vdots \\ & & & A_{p-1} & b_{p-1}f_p \\ & & & & A_p \end{bmatrix}, b_T = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{p-1} \\ b_p \end{bmatrix},$$

where $b_i \in \mathbf{R}^{n_i\times 1}$, $A_i \in \mathbf{R}^{n_i\times n_i}$, $f_i \in \mathbf{R}^{1\times n_i}$ are arbitrary matrices and $n_1 + n_2 + \cdots + n_p = n$. If $\lambda(A_i - b_if_i) \cap \lambda(A_j) = \emptyset$, $j \in \mathbf{I}[1, i-1]$, $i \in \mathbf{I}[2, p]$, then (A_T, b_T) is controllable if and only if all the pairs (A_i, b_i), $i \in \mathbf{I}[1, p]$, are controllable.

III. The Key Result

In this section, we address Problem 1 when $n = 2$. The result will play a crucial role in the next section. To simplify the notation, let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{3}$$

Proposition 1: Consider the discrete-time system

$$\begin{cases} x(k+1) &= Ax(k) + bu(k), \quad x \in \mathbf{R}^2, \\ u(k) &= -\sigma_{\varepsilon_2}(F_\gamma x(k) + \sigma_{\varepsilon_1}(\varpi(k))), \end{cases} \tag{4}$$

where $\varepsilon_1 \geq 0$, $\varepsilon_2 > 0$ are given scalars, $\varpi(k)$ is an external signal,

$$F_\gamma = \left(1 + b^T P b\right)^{-1} b^T P A = \begin{bmatrix} \gamma^2 & 2\gamma \end{bmatrix}, \tag{5}$$

in which $\gamma \in (0, 1/2]$ and

$$P = \begin{bmatrix} \frac{\gamma^3}{(\gamma-1)^2} & \frac{\gamma^2}{(\gamma-1)^2} \\ \frac{\gamma^2}{(\gamma-1)^2} & \frac{1}{(\gamma-1)^2} - 1 \end{bmatrix}, \tag{6}$$

satisfies the discrete-time parametric Lyapunov equation (DPLE):

$$A^T P A - A^T P b(1 + b^T P b)^{-1} b^T P A = (1-\gamma) P. \tag{7}$$

If

$$\left(\sqrt{\frac{2-\gamma}{1-\gamma}} + 1\right) \varepsilon_1 < \varepsilon_2, \tag{8}$$

then there exists an integer $k^* > 0$ such that

$$|F_\gamma x(k) + \sigma_{\varepsilon_1}(\varpi(k))| \leq \varepsilon_2, \; k \geq k^*. \tag{9}$$

and then there holds $x(k+1) = (A - bF_\gamma) x(k) - b\sigma_{\varepsilon_1}(\varpi(k))$, $\forall k \geq k^*$.

The proof of this proposition is length and will be given in Section VI where we will use a new energy function based method.

Remark 1: For continuous time systems, there is a similar result in [28]. However, by contrast with the analysis in [28], the proof here is far from trivial. For instance, the state-space is first partitioned into three parts ($\mathbf{I}^+, \mathbf{II}$ and $\mathbf{I}^-$) in both the continuous case (see (38) in [28]) and the discrete case (see (30) in Section VI). In the continuous case, one can show that any trajectory starting at region $\mathbf{I}^+$ ($\mathbf{I}^-$) will return to $\mathbf{II}$ in finite time. However, in the discrete case, after finite steps, the trajectory starting in region $\mathbf{I}^+$ ($\mathbf{I}^-$) will enter region $\mathbf{II}$ directly, or jump over $\mathbf{II}$ and enter region $\mathbf{I}^-$ ($\mathbf{I}^+$) due to the discontinuity. This difficulty is inherent in the trajectory analysis. In addition, There exist many other cases, which are more challenging than those in [28] and need to be treated carefully, as shown in Section VI.

The following corollary is a direct consequence of Proposition 1:

Corollary 1: Consider the system

$$x(k+1) = Ax(k) + b\sigma_\varepsilon(u(k)), \; x \in \mathbf{R}^2, \tag{10}$$

where (A, b) is given by (3) and ε is a positive real number. Then for any $\gamma \in (0, 1/2]$, the control law

$$u(k) = -F_\gamma x(k), \tag{11}$$

where F_γ is given by (5), globally asymptotically stabilizes system (10).

When the input additive disturbance vanishes, a discrete-time double integrators with bounded controls described by

$$x(k+1) = Ax(k) + b\sigma(u(k)), \quad x \in \mathbf{R}^2, \tag{12}$$

where

$$u(k) = f_1 x_1(k) + f_2 x_2(k), \tag{13}$$

has been discussed in detail by Yang et al. in [21] and [22]. Different from the continuous time case that any linear state feedback control law, which locally stabilizes the double integrators, also globally stabilizes the double integrators when the control is bounded [15], [19], Yang et al. in [22] has pointed out that, the discrete-time double integrators has intrinsically different behavior, namely, even it is stabilized locally by some linear feedback, it can not be stabilized globally by the same controller when the control is bounded. Later, Yang et al. in [21] further studied the global stabilization of the discrete-time double integrator (12), and proved that the closed-loop system consisting of (12) and (13) is globally asymptotically stable if and only if

$$\frac{1}{2}f_1 - 2 < f_2 < \frac{3}{2}f_1 < 0. \tag{14}$$

Now, let us return to Corollary 1. Let $y(k) = [y_1(k), y_2(k)]^{\mathrm{T}} = x(k)/\varepsilon_2$, $v(k) = u(k)/\varepsilon_2$, $\hat{f}_1 = \gamma^2$ and $\hat{f}_2 = 2\gamma$. Then system (10) and (11) can be rewritten as

$$\begin{cases} y(k+1) &= Ay(k) + b\sigma(v(k)), \quad y \in \mathbf{R}^2, \\ v(k) &= -\hat{f}_1 y_1(k) - \hat{f}_2 y_2(k), \end{cases} \tag{15}$$

which is in the form (12)-(13) by setting $f_1 = -\hat{f}_1 = -\gamma^2$ and $f_2 = -\hat{f}_2 = -2\gamma$. Since $\gamma \in (0, 1/2]$, the constants $f_1 = -\gamma^2$ and $f_2 = -2\gamma$ satisfy (14). Thus, by using the results in [21], we conclude that system (15) (and also the closed-loop system consisting of (10) and (11)) is globally asymptotically stable, which is consistent with Corollary 1.

IV. The Solution to Problem 1

In this section, with the aid of Proposition 1, a class of nonlinear control laws will be constructed to solve Problem 1. To this end, we need a special state space description of system (1), as stated in the following lemma. For convenience, we let $p = [\frac{n+1}{2}]$,

$$\kappa = \begin{cases} I_2, & n \text{ is even}, \\ [0,1], & n \text{ is odd}, \end{cases} \quad y_1(k) \in \begin{cases} \mathbf{R}^2, & n \text{ is even}, \\ \mathbf{R}, & n \text{ is odd}, \end{cases} \tag{16}$$

and

$$y(k) = \begin{bmatrix} y_1^{\mathrm{T}}(k) & y_2^{\mathrm{T}}(k) & \cdots & y_p^{\mathrm{T}}(k) \end{bmatrix}^{\mathrm{T}}, \tag{17}$$

where $y_i(k) \in \mathbf{R}^2, i \in \mathbf{I}[2,p]$.

Lemma 3: Let $\gamma_i, i \in \mathbf{I}[2,p]$, be given positive constants. Consider the discrete-time system:

$$y(k+1) = A_p y(k) + b_p u(k), \tag{18}$$

where

$$A_p = \begin{bmatrix} \kappa A \kappa^{\mathrm{T}} & \kappa A_{\gamma_2} & \cdots & \kappa A_{\gamma_{p-1}} & \kappa A_{\gamma_p} \\ & A & \cdots & A_{\gamma_{p-1}} & A_{\gamma_p} \\ & & \ddots & \vdots & \vdots \\ & & & A & A_{\gamma_p} \\ & & & & A \end{bmatrix}, b_p = \begin{bmatrix} \kappa b \\ b \\ \vdots \\ b \\ b \end{bmatrix}, \tag{19}$$

in which (A, b) is given by (3), κ is given by (16), $y(k)$ is given by (17), and

$$A_{\gamma_i} = \begin{bmatrix} 0 & 0 \\ \gamma_i^2 & 2\gamma_i \end{bmatrix} \triangleq bF_{\gamma_i}, \ i \in \mathbf{I}[1,p]. \tag{20}$$

Let $T = Q_c(A_p, b_p) Q_c^{-1}(A_o, b_o)$, where (A_o, b_o) is given by (2) and (A_p, b_p) is given by (19). Then system (1) is transformed into (18) by the invertible transformation $y(k) = Tx(k)$.

Proof: By Lemma 1, we only need to show that (A_p, b_p) is controllable since $\lambda(A_o) = \lambda(A_p)$. Let us introduce the function $\mathrm{d}_i(\lambda) = |\lambda I_2 - (A - bF_{\gamma_i})|$, $i \in \mathbf{I}[2,p]$. Then we have $\mathrm{d}_i(1) = \gamma_i^2 \neq 0$, $i \in \mathbf{I}[2,p]$, which implies that$\lambda(A - bF_{\gamma_i}) \cap (\lambda(\kappa A \kappa^{\mathrm{T}}) \cup \lambda(A)) = \emptyset$, $i \in \mathbf{I}[2,p]$. On the other hand, it is easy to verify that both $(\kappa A \kappa^{\mathrm{T}}, \kappa b)$ and (A, b) are controllable. Therefore, by using Lemma 2, we conclude that (A_p, b_p) is controllable. This concludes the proof. ■

From Lemma 3 it follows that, if system (18) is globally stabilized by a controller $u(k) = u(y(k))$, then system (1) is also globally stabilized by $u(T^{-1}x(k))$. Thus, we only need to design stabilizing controller $u(y(k))$ for the system (18). By virtue of 1, we are ready to give the main result in this section.

Theorem 1: Let $\gamma_i, i \in \mathbf{I}[1,p]$, be given positive constants satisfying $\gamma_i \in (0, 1/2]$. Then Problem 1 is solved by the nonlinear controller $u(k) = -u_p(k)$, where

$$\begin{cases} u_i(k) &= \sigma_{\varepsilon_i}(F_{\gamma_i} y_i(k) + u_{i-1}(k)), \ i \in \mathbf{I}[2,p], \\ u_1(k) &= \sigma_{\varepsilon_1}(F_{\gamma_1} \kappa^{\mathrm{T}} y_1(k)), \end{cases} \tag{21}$$

in which $y_i(k), i \in \mathbf{I}[1,p]$, are defined in Lemma 3, κ is given by (16), $F_{\gamma_i} = [\gamma_i^2, 2\gamma_i], i \in \mathbf{I}[1,p]$, are given by (20), and $\varepsilon_i, i \in \mathbf{I}[1,p]$, are a series of positive scalars satisfying

$$\left(\sqrt{\frac{2-\gamma_i}{1-\gamma_i}} + 1\right)\varepsilon_{i-1} < \varepsilon_i, \ i \in \mathbf{I}[2,p], \quad \varepsilon_p \le 1. \tag{22}$$

Remark 2: Compared to the results in [12] and [30], the most significant advantage of the control law in Theorem 1 is that it needs less saturation functions, which implies a better use of the control capacity and can lead to better transient performances. This fact will be illustrated by a numerical example in Section V.

Proof: Let us observe from (21) that $|u_i(k)| \le \varepsilon_i$, $i \in \mathbf{I}[1,p]$. Now, we consider the p-th subsystem of the closed-loop system consisting of (18) and (21), namely,

$$y_p(k+1) = Ay_p(k) - b\sigma_{\varepsilon_p}(F_{\gamma_p} y_p(k) + u_{p-1}(k)). \tag{23}$$

Notice that system (23) is in the form of (4). Then, we deduce from Proposition 1 that if $(\sqrt{(2-\gamma_p)/(1-\gamma_p)} + 1)\varepsilon_{p-1} < \varepsilon_p$, namely, (22) is satisfied with $i = p$, there exists a finite integer $k_p \ge 0$ such that for all $k \ge k_p$, the inequality $|F_{\gamma_p} y_p + u_{p-1}| \le \varepsilon_p$ holds. Consequently, $u_p(k) = F_{\gamma_p} y_p(k) + u_{p-1}(k)$, $\forall k \ge k_p$. It follows that for all $k \ge k_p$, system (18) can be written as

$$\begin{cases} y_1(k+1) &= \kappa A \kappa^{\mathrm{T}} y_1(k) + \kappa \sum\limits_{i=2}^{p-1} A_{\gamma_i} y_i(k) - \kappa b u_{p-1}(k), \\ y_2(k+1) &= Ay_2(k) + \sum\limits_{i=3}^{p-1} A_{\gamma_i} y_i(k) - bu_{p-1}(k), \\ &\vdots \\ y_{p-1}(k+1) &= Ay_{p-1}(k) - bu_{p-1}(k), \\ y_p(k+1) &= (A - A_{\gamma_p}) y_p(k) - bu_{p-1}(k), \end{cases} \tag{24}$$

We next consider the y_{p-1}-system of (24), which is also in the form of (23). Then, arguing as we did above, we deduce that if $(\sqrt{(2-\gamma_{p-1})/(1-\gamma_{p-1})} + 1)\varepsilon_{p-2} < \varepsilon_{p-1}$, namely, (22) is satisfied with $i = p-1$, there exists an integer $k_{p-1} \ge k_p$ such that for all $k \ge k_{p-1}$,$u_{p-1}(k) = F_{\gamma_{p-1}} y_{p-1}(k) + u_{p-2}(k)$. The closed-loop system can be simplified accordingly. By repeating the above process for $y_i, i = p-2, p-3, \ldots, 2$, we arrive at the y_1-subsystem

$$y_1(k+1) = \begin{cases} Ay_1(k) - b\sigma_{\varepsilon_1}(F_{\gamma_1} y_1(k)), & n \text{ is even,} \\ y_1(k) - \sigma_{\varepsilon_1}(2\gamma_1 y_1(k)), & n \text{ is odd.} \end{cases} \quad (25)$$

If n is even, system (25) is in the form of (23) where the term $u_{p-1}(k)$ vanishes. Thus, since $0 < \varepsilon_1$, there exists an integer $k_1 \geq k_2$ such that$y_1(k+1) = (A - A_{\gamma_1}) y_1(k), \ \forall k \geq k_1$. If n is odd, since $0 < \varepsilon_1$, we deduce from Lemma 1 in [30] that there exists an integer $k_1 \geq k_2$ such that$y_1(k+1) = (1 - 2\gamma_1) y_1(k), \ \forall k \geq k_1$. To summarize, we have, for all $k \geq k_1$,

$$\begin{cases} y_1(k+1) = & \left(\kappa A \kappa^{\mathrm{T}} - \kappa A_{\gamma_1} \kappa^{\mathrm{T}}\right) y_1(k), \\ y_2(k+1) = & (A - A_{\gamma_2}) y_2(k) - A_{\gamma_1} \kappa^{\mathrm{T}} y_1(k), \\ \quad \vdots & \\ y_p(k+1) = & \left(A - A_{\gamma_p}\right) y_p(k) - \sum_{i=2}^{p-1} A_{\gamma_i} y_i(k) \\ & -A_{\gamma_1} \kappa^{\mathrm{T}} y_1(k). \end{cases}$$

Thus the closed-loop system is asymptotically stable if and only if the following series of systems

$$\begin{cases} \varphi_1(k+1) = \begin{cases} (A - A_{\gamma_1}) \varphi_1(k), & n \text{ is even,} \\ (1 - 2\gamma_1) \varphi_1(k), & n \text{ is odd,} \end{cases} \\ \varphi_i(k+1) = (A - A_{\gamma_i}) \varphi_i(k), \ i \in \mathbf{I}[2, p], \end{cases}$$

are all asymptotically stable, which is obvious since $\gamma_i \in (0, 1/2]$ for all $i \in \mathbf{I}[1, p]$. This concludes the proof. ■

V. A Numerical Example

We illustrate the main result of our work with system (1) when $n = 4$.

- **Control Law 1:** This control law is based on Theorem 1 and is given by

$$u_1 = -\sigma_{\varepsilon_2}(F_{\gamma_2} y_2 + \sigma_{\varepsilon_1}(F_{\gamma_1} y_1)), \quad (26)$$

where $F_{\gamma_i} = \left[\gamma_i^2, 2\gamma_i\right], \ \gamma_i \in (0, 1/2], i \in \mathbf{I}[1, 2], \ y_i \in \mathbf{R}^2, i \in \mathbf{I}[1, 2]$,

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \gamma_2^2 & 2\gamma_2 & 1 & 0 \\ 0 & \gamma_2^2 & 2\gamma_2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x,$$

and $\varepsilon_2 = 1, \ \varepsilon_1 = \eta\left(\sqrt{(1-\gamma_2)(2-\gamma_2)} - (1-\gamma_2)\right)\varepsilon_2, \ \eta < 1$. For the simulation purpose, we choose $\eta = 0.99$.
- **Control Law 2:** This control law is given by [30] and takes the form

$$u_4 = -\sigma_{\varepsilon_4}(\lambda_4 y_4 + \sigma_{\varepsilon_3}(\lambda_3 y_3 + \sigma_{\varepsilon_2}(\lambda_2 y_2 + \sigma_{\varepsilon_1}(\lambda_1 y_1)))), \quad (27)$$

where the variables $y_i \in \mathbf{R}, i \in \mathbf{I}[1, 4]$, are given by

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & 1 \\ 0 & t_{22} & t_{23} & 1 \\ 0 & 0 & t_{33} & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} x, \quad (28)$$

with $t_{11} = \lambda_2\lambda_3\lambda_4, t_{12} = \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4, t_{13} = \lambda_2 + \lambda_3 + \lambda_4, \ t_{22} = \lambda_3\lambda_4, \ t_{23} = \lambda_3 + \lambda_4$ and $t_{33} = \lambda_4, \ \varepsilon_i, i \in \mathbf{I}[1, 4]$, satisfy$\varepsilon_{i-1} < \varepsilon_i/2, i \in \mathbf{I}[2, 4], \ \varepsilon_4 \leq 1$, and $\lambda_i, i \in \mathbf{I}[1, 4]$, satisfy $0 < \lambda_i < 1, \ i \in \mathbf{I}[1, 4]$. Let η be a prescribed number and choose $\varepsilon_4 = 1, \ \varepsilon_3 = \eta/2, \ \varepsilon_2 = \eta^2/4, \ \varepsilon_1 = \eta^3/8, \ \eta < 1$. For the simulation purpose, we choose $\eta = 0.99$.
- **Control Law 3:** This control law is given in [12] and takes the form

$$u_5 = -\delta \sum_{i=1}^{4} \lambda_i \sigma_{M_i}\left(\frac{y_i}{\delta}\right), \quad (29)$$

where $y_i, i \in \mathbf{I}[1, 4]$, are given by (28), and $M_4 = 1, M_j = 1 + \alpha_j(M_{j+1} - |\sigma_{M_{j+1}}(y_{j+1}/\delta)|)\lambda_{j+1}/\lambda_j, \ j \in \mathbf{I}[1, 3]$, in which $\alpha_i \in [0, 1], i \in \mathbf{I}[1, 3]$, and $\delta = 1/(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)$ with $0 < \Sigma_{i=1}^{k}\lambda_i < \lambda_k < 1, k \in \mathbf{I}[2, 4]$. In order to achieve the fastest convergence of the states, we set $\alpha_i = 1, i \in \mathbf{I}[1, 3]$, and $\lambda_i = 0.5437^{4-i+1}, i \in \mathbf{I}[1, 4]$.

We first illustrate how to choose the parameters $\gamma_i, i \in \mathbf{I}[1, 2]$, in the controller (26). As in the analysis in [30], to achieve a satisfactory transient performance, a trial-and-error procedure via numerical simulation is necessary. That way one determines suitable parameters $\gamma_i, i \in \mathbf{I}[1, 2]$, for the controller (26). To this end, we let $\gamma_i = \gamma, i \in \mathbf{I}[1, 2]$. For a given initial condition $x(0) = [-5, -5, 5, 5]^{\mathrm{T}}$, the trajectories of the closed-loop system under Control Law 1 with different γ are plotted in Fig. 1. It follows that, for Control Law 1, the parameter $\gamma = 0.15$ leads to the best control performance. Let $\lambda_i = \lambda, i \in \mathbf{I}[1, 4]$, in (27). Similarly, we can also show that for Control Law 2, the parameter $\lambda = 0.1$ leads to the best control performance.

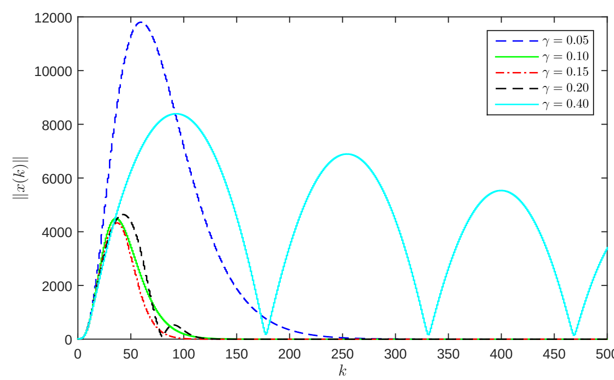

Fig. 1. Trajectories of the 4-th order discrete-time multiple integrators system under Control Law 1 with different γ.

Finally, we make a comparison of Control Laws 1-3. Let the parameters in Control Laws 1-3 be chosen as the optimal ones obtained above. The resulting 2-norms of the state trajectories of the closed-loop systems are recorded in Fig. 2. From the figure, we clearly see that Control Law 1 established in this paper leads to a much better transient performance than the other ones.

VI. Proof of Proposition 1

Let the state-space be partitioned as follows:

$$\begin{cases} \mathbf{I}^+ : F_\gamma x > \varepsilon_1 + \varepsilon_2, \\ \mathbf{II} : |F_\gamma x| \leq \varepsilon_1 + \varepsilon_2, \\ \mathbf{I}^- : F_\gamma x < -(\varepsilon_1 + \varepsilon_2), \end{cases} \quad (30)$$

which means that the x_1-x_2 plane is divided into three regions by two lines $F_\gamma x = \pm(\varepsilon_1 + \varepsilon_2)$ (see Fig.3 for an illustration). For future use, let us introduce: $\Omega_1 = \{x = [x_1, x_2]^{\mathrm{T}} \mid -\rho_1 \leq \gamma^2 x_2 \leq \rho_1\} \cap \mathbf{II}$ and $\Omega_2 = \{x = [x_1, x_2]^{\mathrm{T}} \mid V(x) \leq q\}$, where $\rho_1 = 2(\varepsilon_1 + \varepsilon_2) - 2\gamma\varepsilon_2 + \gamma^2\varepsilon_2, \ V(x) = x^{\mathrm{T}} P x$, and q is the positive constant:

$$q = \mu\varepsilon_1^2, \quad (31)$$

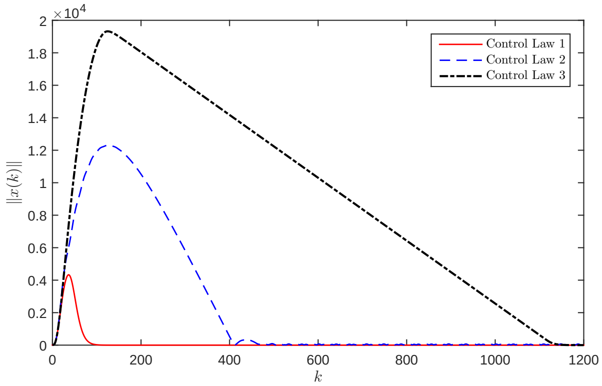

Fig. 2. Comparison of the trajectories of the 4-th order discrete-time multiple integrators system with different controllers.

where μ is a constant satisfying

$$\left\{\begin{array}{l}\frac{1}{\gamma(1-\gamma)^2} < \mu < \frac{4(1-\gamma)}{\gamma^3(1-\gamma)^2}, \\ \left(\sqrt{\gamma\left(1-\gamma\right)\left(2-\gamma\right)\mu}+1\right)\varepsilon_1 \le \varepsilon_2. \end{array}\right. \tag{32}$$

The existence of μ is due to (8).

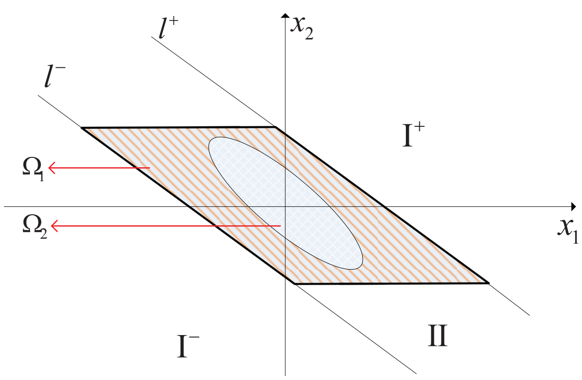

Fig. 3. Partitioning of the state space according to (30). The lines l^+ : $F_\gamma x = \varepsilon_1+\varepsilon_2$, and l^- : $F_\gamma x = -\left(\varepsilon_1+\varepsilon_2\right)$. $\Omega_1 = \{x=[x_1,x_2]^{\mathrm{T}} \mid -\rho_1 \le \gamma^2 x_2 \le \rho_1\} \cap \mathbf{II}$, $\Omega_2 = \{x=[x_1,x_2]^{\mathrm{T}} \mid V(x) \le q\}$.

It turns out that $\Omega_2 \subset \Omega_1$. This fact can be verified as follows: $\Omega_2 \subset \mathbf{II}$ if and only if

$$\left(\frac{F_\gamma}{\varepsilon_1+\varepsilon_2}\right)\left(\frac{P}{q}\right)^{-1}\left(\frac{F_\gamma^{\mathrm{T}}}{\varepsilon_1+\varepsilon_2}\right) < 1. \tag{33}$$

Combining (5), (6), (31) and (33) gives $\varepsilon_2 > \left(\sqrt{\gamma\left(1-\gamma\right)\left(2-\gamma\right)\mu}-1\right)\varepsilon_1$, which can be guaranteed by (32) and thus we have $\Omega_2 \subset \mathbf{II}$. We next show that $\Omega_2 \subset \Omega_1$. To this end, we only need to show that

$$\gamma^2 x_2 = \rho_1, x^{\mathrm{T}} P x = q, \tag{34}$$

has no solution. Substituting (6) and (31) into (34) gives $d_1 x_1^2 + d_2 x_1 + d_3 = 0$, where $d_1 = \gamma^3/(1-\gamma)^2$, $d_2 = 2\rho_1/(1-\gamma)^2$, $d_3 = (2-\gamma)\rho_1^2/(\gamma^3(1-\gamma)^2) - \mu\varepsilon_1^2$. Notice that from (32), $\gamma \in (0, 1/2]$ and $\rho_1 > 2\varepsilon_1$, we have

$$\begin{aligned} & d_2^2 - 4d_1 d_3 \\ =& -\frac{4}{\left(1-\gamma\right)^3}\rho_1^2 + \frac{4\gamma^3}{\left(1-\gamma\right)^2}\mu\varepsilon_1^2 \\ <& -\frac{4}{\left(1-\gamma\right)^3}\left(2\varepsilon_1\right)^2 + \frac{4\gamma^3}{\left(1-\gamma\right)^2}\frac{4\left(1-\gamma\right)}{\gamma^3\left(1-\gamma\right)^2}\varepsilon_1^2 \\ =& -\frac{16}{\left(1-\gamma\right)^3}\varepsilon_1^2 + \frac{16}{\left(1-\gamma\right)^3}\varepsilon_1^2 \\ =& 0, \end{aligned}$$

which implies that (34) has no solution. Therefore, we have $\Omega_2 \subset \Omega_1$.

Now consider the nonlinear system (4). We have the following claims:

1) For any initial state $x(0) \in \mathbf{I}^+$ or $\mathbf{I}^-$, there exists a finite integer $k > 0$ such that $x(k) \in \mathbf{II}$.
2) For any initial state $x(0) \in \mathbf{II}\backslash\Omega_1$, there exists a finite integer $k > 0$ such that $x(k) \in \Omega_1$.
3) For any initial state $x(0) \in \Omega_1\backslash\Omega_2$, if $x(1) \in \mathbf{II}$, then $x(1) \in \Omega_1$ and $V(x(1)) - V(x(0)) < -\upsilon_1$ where υ_1 is a suitable positive constant.
4) For any initial state $x(0) \in \Omega_1\backslash\Omega_2$, if $x(1) \in \mathbf{I}^+$ or $\mathbf{I}^-$, there exists a finite integer $k > 1$ such that $x(k) \in \Omega_1$ and $V(x(k)) - V(x(0)) < -\upsilon_2$ where υ_2 is a suitable positive constant independent of k.
5) If there exists an integer $k \ge 0$ such that $x(k) \in \Omega_2$, then $|F_\gamma x(k) + \sigma_{\varepsilon_1}(\varpi(k))| \le \varepsilon_2$ and $x(k+1) \in \Omega_2$.

We conclude from Claims 1-4 that for any initial condition $x(0) \in \mathbf{I}^+ \cup \mathbf{I}^- \cup \mathbf{II}$, there exists a finite integer $k^* \ge 0$ such that $x(k^*) \in \Omega_2$. From Claim 5, we know that if the state x enters the bounded set Ω_2, it will remain inside forever. Hence, we have $x(k) \in \Omega_2, \forall k \ge k^*$. Moreover, from Claim 5 we can also get $|F_\gamma x(k) + \sigma_{\varepsilon_1}(\varpi(k))| \le \varepsilon_2,\ \forall k \ge k^*$, which implies that $u(k)$ in (4) satisfies $u(k) = -F_\gamma x(k) - \sigma_{\varepsilon_1}(\varpi),\ \forall k \ge k^*$, which ends the proof of Proposition 1. In the following, we prove Claims 1-5.

Proof of Claim 1

Step 1.1: We show that any trajectory starting in $\mathbf{I}^+$ will come to $\mathbf{II} \cup \mathbf{I}^-$ after finite steps. Notice that in region $\mathbf{I}^+$, $u(k) = -\sigma_{\varepsilon_2}(F_\gamma x(k) + \sigma_{\varepsilon_1}(\varpi(k))) = -\varepsilon_2$, by which the nonlinear system (4) can be expressed as

$$x(k+1) = Ax(k) - b\varepsilon_2. \tag{35}$$

Assume that at the initial time $k = 0$, the state $x(0) = [x_1(0), x_2(0)]^{\mathrm{T}}$ is in region $\mathbf{I}^+$, namely,

$$F_\gamma x(0) = \gamma^2 x_1(0) + 2\gamma x_2(0) > \varepsilon_1 + \varepsilon_2. \tag{36}$$

By solving (35), we obtain

$$\begin{aligned} \left[\begin{array}{c} x_1(k) \\ x_2(k) \end{array}\right] &= A^k \left[\begin{array}{c} x_1(0) \\ x_2(0) \end{array}\right] - \sum_{i=0}^{k-1} A^i b\varepsilon_2 \\ &= \left[\begin{array}{c} x_1(0) + kx_2(0) - \frac{k(k-1)}{2}\varepsilon_2 \\ x_2(0) - k\varepsilon_2 \end{array}\right]. \end{aligned} \tag{37}$$

Now consider the equation $0 = F_\gamma x(k) - (\varepsilon_1+\varepsilon_2) = -\gamma^2\varepsilon_2 k^2/2 + (\gamma^2 x_2(0) + \gamma^2\varepsilon_2/2 - 2\gamma\varepsilon_2)k + \gamma^2 x_1(0) + 2\gamma x_2(0) - (\varepsilon_1+\varepsilon_2)$, which is quadratic and admits a unique positive solution since $\gamma^2 x_1(0) + 2\gamma x_2(0) - (\varepsilon_1+\varepsilon_2) > 0$ in view of (36). Since k should be a positive integer, it follows that x will enter $\mathbf{II} \cup \mathbf{I}^-$ after finite steps. The region $\mathbf{I}^-$ can be considered by symmetry.

Step 1.2: We show that if the trajectory of x skips back and forth between regions $\mathbf{I}^+$ and $\mathbf{I}^-$ (namely, $\mathbf{I}^+ \to \mathbf{I}^- \to \mathbf{I}^+ \to \mathbf{I}^- \to \cdots$), it will enter region $\mathbf{II}$ after finite steps. Assume that $x(0) \in \mathbf{I}^+$, namely, (36) holds. Without loss of generality, we let $x(1) \in \mathbf{I}^-$. Then, by virtue of (37), we get

$$\begin{aligned} F_\gamma x(1) &= \gamma^2 x_1(1) + 2\gamma x_2(1) \\ &= \gamma^2 x_1(0) + 2\gamma x_2(0) + \gamma^2 x_2(0) - 2\gamma\varepsilon_2 \end{aligned}$$

$$< -\left(\varepsilon_1+\varepsilon_2\right). \tag{38}$$

Combining (36) and (38) gives

$$f_1 < \gamma^2 x_1(0) + 2\gamma x_2(0) < f_2, \tag{39}$$

where

$$f_1 = \varepsilon_1+\varepsilon_2,\ f_2 = -\left(\varepsilon_1+\varepsilon_2\right) - \gamma^2 x_2(0) + 2\gamma\varepsilon_2. \tag{40}$$

Therefore,

$$\gamma^2 x_2(0) < -2\left(\varepsilon_1+\varepsilon_2\right) + 2\gamma\varepsilon_2 < 0, \tag{41}$$

where we have noticed that $\gamma \in (0, 1/2]$. It follows from (41) that if x skips from $\mathbf{I}^+$ to $\mathbf{I}^-$, then $x \in \Omega_3 \triangleq \{x = [x_1, x_2]^{\mathrm{T}} \mid \gamma^2 x_1 + 2\gamma x_2 > \varepsilon_1+\varepsilon_2,\ \gamma^2 x_2 < 0\}$. In region $\mathbf{I}^-$, we have $u(k) = -\sigma_{\varepsilon_2}(F_\gamma x(k) + \sigma_{\varepsilon_1}(\varpi(k))) = \varepsilon_2$. Let us use the notation: $y(k) = x(k+1)$. Then the nonlinear system (4) in $\mathbf{I}^-$ can be expressed as

$$\begin{aligned} y(k) &= A^k \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} + \sum_{i=0}^{k-1} A^i b\varepsilon_2 \\ &= \begin{bmatrix} x_1(0) + (k+1)x_2(0) + \frac{k(k-3)}{2}\varepsilon_2 \\ x_2(0) - \varepsilon_2 + k\varepsilon_2 \end{bmatrix}. \end{aligned} \tag{42}$$

From Step 1.1, we know that y will enter region $\mathbf{II}\cup\mathbf{I}^+$ after finite steps. If y enters region $\mathbf{I}^+$, there exists an integer $k_1 > 0$ such that

$$F_\gamma y(k_1) < -\left(\varepsilon_1+\varepsilon_2\right),\ F_\gamma y(k_1+1) > \left(\varepsilon_1+\varepsilon_2\right), \tag{43}$$

namely, $f_3(k_1) < \gamma^2 y_1(k_1) + 2\gamma y_2(k_1) < f_4$, where

$$f_3(k) = \left(\varepsilon_1+\varepsilon_2\right) - \gamma^2 y_2(k_1) - 2\gamma\varepsilon_2,\ f_4 = -\left(\varepsilon_1+\varepsilon_2\right). \tag{44}$$

Moreover, it follows from (42) and (43) that $f_5(k_1) < \gamma^2 x_1(0) + 2\gamma x_2(0) < f_6(k_1)$, where

$$\begin{cases} f_5(k_1) = & -\frac{1}{2}\gamma^2\varepsilon_2 k_1^2 + \left(\frac{1}{2}\gamma^2\varepsilon_2 - \gamma^2 x_2(0) - 2\gamma\varepsilon_2\right)k_1 \\ & -2\gamma^2 x_2(0) + \gamma^2\varepsilon_2 + \left(\varepsilon_1+\varepsilon_2\right), \\ f_6(k_1) = & -\frac{1}{2}\gamma^2\varepsilon_2 k_1^2 + \left(\frac{3}{2}\gamma^2\varepsilon_2 - \gamma^2 x_2(0) - 2\gamma\varepsilon_2\right)k_1 \\ & -\gamma^2 x_2(0) + 2\gamma\varepsilon_2 - \left(\varepsilon_1+\varepsilon_2\right). \end{cases} \tag{45}$$

When y skips from $\mathbf{I}^-$ to $\mathbf{I}^+$, we denote $z(k) = y(k+k_1+1)$. Then the equation of state in $\mathbf{I}^+$ can be expressed as

$$\begin{aligned} z(k) &= A^k \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} - \sum_{i=0}^{k-1} A^i b\varepsilon_2 \\ &= \begin{bmatrix} y_1(k_1) + (k+1)y_2(k_1) - \frac{k(k-3)}{2}\varepsilon_2 \\ y_2(k_1) + \varepsilon_2 - k\varepsilon_2 \end{bmatrix}. \end{aligned} \tag{46}$$

From Step 1.1 again we know that z will enter region $\mathbf{II}\cup\mathbf{I}^-$ after finite steps. If z enters region $\mathbf{I}^-$, there exists an integer $k_2 > 0$ such that

$$F_\gamma z(k_2) > \left(\varepsilon_1+\varepsilon_2\right),\ F_\gamma z(k_2+1) < -\left(\varepsilon_1+\varepsilon_2\right). \tag{47}$$

It follows from (46) and (47) that $f_7(k_2) < \gamma^2 y_1(k_1) + 2\gamma y_2(k_1) < f_8(k_2)$, where

$$\begin{cases} f_7(k_2) = & \frac{1}{2}\gamma^2\varepsilon_2 k_2^2 + \left(-\frac{3}{2}\gamma^2\varepsilon_2 - \gamma^2 y_2(k_1) + 2\gamma\varepsilon_2\right)k_2 \\ & -\gamma^2 y_2(k_1) - 2\gamma\varepsilon_2 + \left(\varepsilon_1+\varepsilon_2\right), \\ f_8(k_2) = & \frac{1}{2}\gamma^2\varepsilon_2 k_2^2 + \left(-\frac{1}{2}\gamma^2\varepsilon_2 - \gamma^2 y_2(k_1) + 2\gamma\varepsilon_2\right)k_2 \\ & -2\gamma^2 y_2(k_1) - \gamma^2\varepsilon_2 - \left(\varepsilon_1+\varepsilon_2\right). \end{cases} \tag{48}$$

To conclude, if the trajectory starting at $x(0) = [x_1(0), x_2(0)]^{\mathrm{T}} \in \mathbf{I}^+$ moves along the path $\mathbf{I}^+ \to \mathbf{I}^- \to \mathbf{I}^+ \to \mathbf{I}^-$, there must exist two integers $k_1 > 0$ and $k_2 > 0$ such that

$$f_1 < f_6(k_1),\ f_5(k_1) < f_2,\ f_3 < f_8(k_2),\ f_7(k_2) < f_4. \tag{49}$$

Denote $f_9 = -\left(\varepsilon_1+\varepsilon_2\right) + 4\gamma\varepsilon_2 - 2\gamma^2\varepsilon_2$, $f_{10}(k_1) = f_5(k_1) - 2\left(\varepsilon_1+\varepsilon_2\right)$, $f_{11} = \left(\varepsilon_1+\varepsilon_2\right) - 4\gamma\varepsilon_2 + 2\gamma^2\varepsilon_2$, $f_{12}(k_2) = f_8(k_2) + 2\left(\varepsilon_1+\varepsilon_2\right)$. Then it follows from (40), (44), (45) and (48) that

$$f_1 > f_9,\ f_5(k_1) > f_{10}(k_1),\ f_{11} > f_4,\ f_{12}(k_2) > f_8(k_2). \tag{50}$$

Then it follows from (49) and (50) that k_1 and k_2 will further satisfy the following inequalities: $f_9 < f_6(k_1)$, $f_{10}(k_1) < f_2$, $f_3 < f_{12}(k_2)$, $f_7(k_2) < f_{11}$, namely,

$$\begin{cases} \left(\frac{\gamma^2\varepsilon_2}{2}k_1 + \frac{\gamma^2\varepsilon_2}{2}\right)\left(k_1 - \left(\frac{-2\left(\gamma^2 x_2(0) + 2\gamma\varepsilon_2\right)}{\gamma^2\varepsilon_2} + 4\right)\right) < 0, \\ \left(\frac{\gamma^2\varepsilon_2}{2}k_1 + \frac{\gamma^2\varepsilon_2}{2}\right)\left(k_1 - \left(\frac{-2\left(\gamma^2 x_2(0) + 2\gamma\varepsilon_2\right)}{\gamma^2\varepsilon_2} + 2\right)\right) > 0, \\ \left(\frac{\gamma^2\varepsilon_2}{2}k_2 + \frac{\gamma^2\varepsilon_2}{2}\right)\left(k_2 - \left(\frac{2\left(\gamma^2 y_2(k_1) - 2\gamma\varepsilon_2\right)}{\gamma^2\varepsilon_2} + 2\right)\right) > 0, \\ \left(\frac{\gamma^2\varepsilon_2}{2}k_2 + \frac{\gamma^2\varepsilon_2}{2}\right)\left(k_2 - \left(\frac{2\left(\gamma^2 y_2(k_1) - 2\gamma\varepsilon_2\right)}{\gamma^2\varepsilon_2} + 4\right)\right) < 0. \end{cases} \tag{51}$$

Solving (51) gives

$$\begin{cases} \frac{-2\left(\gamma^2 x_2(0) + 2\gamma\varepsilon_2\right)}{\gamma^2\varepsilon_2} + 2 < k_1 < \frac{-2\left(\gamma^2 x_2(0) + 2\gamma\varepsilon_2\right)}{\gamma^2\varepsilon_2} + 4, \\ \frac{2\left(\gamma^2 y_2(k_1) - 2\gamma\varepsilon_2\right)}{\gamma^2\varepsilon_2} + 2 < k_2 < \frac{2\left(\gamma^2 y_2(k_1) - 2\gamma\varepsilon_2\right)}{\gamma^2\varepsilon_2} + 4. \end{cases}$$

Let

$$\begin{cases} k_1 = \frac{-2\left(\gamma^2 x_2(0) + 2\gamma\varepsilon_2\right)}{\gamma^2\varepsilon_2} + 2 + a_1, \\ k_2 = \frac{2\left(\gamma^2 y_2(k_1) - 2\gamma\varepsilon_2\right)}{\gamma^2\varepsilon_2} + 2 + a_2, \end{cases} \tag{52}$$

where $a_i \in (0, 2), i = 1, 2$. It follows from (42), (46) and (52) that

$$\begin{aligned} x_2(k_1+k_2+2) &= z_2(k_2) \\ &= y_2(k_1) + \varepsilon_2 - k_2\varepsilon_2 \\ &= x_2(0) + (k_1 - k_2)\varepsilon_2 \\ &= -x_2(0) - 2y_2(k_1) + (a_1 - a_2)\varepsilon_2 \\ &= \left(\frac{8}{\gamma} - (a_1 + a_2 + 2)\right)\varepsilon_2 + x_2(0) \\ &> 10\varepsilon_2 + x_2(0), \end{aligned} \tag{53}$$

where we have noticed that $\gamma \in (0, 1/2]$ and $a_i \in (0, 2), i = 1, 2$. Therefore, from (53), we can conclude that after finite steps k, $x(k)$ in region $\mathbf{I}^+$ satisfies $\gamma^2 x_2(k) \geq 0$, namely, $x(k) \notin \Omega_3$, which implies that the trajectory starting at $x(0) = [x_1(0), x_2(0)]^{\mathrm{T}}$ in $\mathbf{I}^+$ cannot come to $\mathbf{I}^-$, but enter $\mathbf{II}$ eventually. The region $\mathbf{I}^-$ can be considered by symmetry.

Proof of Claim 2

In region $\mathbf{II}$, we have

$$u(k) = -\sigma_{\varepsilon_2}\left(F_\gamma x(k) + \sigma_{\varepsilon_1}(\varpi(k))\right) = -a(k)\varepsilon_2,\ \forall k \geq 0. \tag{54}$$

where $-1 \leq a(k) \leq 1, \forall k \geq 0$. When $|F_\gamma x(k) + \sigma_{\varepsilon_1}(\varpi(k))| \leq \varepsilon_2$, it follows from (54) that $F_\gamma x(k) + \sigma_{\varepsilon_1}(\varpi(k)) = a(k)\varepsilon_2$, by which we get $a(k)\varepsilon_2 - \varepsilon_1 \leq \gamma^2 x_1(k) + 2\gamma x_2(k) \leq a(k)\varepsilon_2 + \varepsilon_1$. When $F_\gamma x(k) + \sigma_{\varepsilon_1}(\varpi(k)) > \varepsilon_2$, it follows from (54) that $a(k) = 1$. Then $x(k)$ satisfies $a(k)\varepsilon_2 - \varepsilon_1 \leq \varepsilon_2 - \sigma_{\varepsilon_1}(\varpi(k)) < \gamma^2 x_1(k) + 2\gamma x_2(k) \leq a(k)\varepsilon_2 + \varepsilon_1$, where we have noticed $F_\gamma x(k) \leq \varepsilon_2 + \varepsilon_1, \forall x(k) \in \mathbf{II}$. When $F_\gamma x(k) + \sigma_{\varepsilon_1}(\varpi(k)) < -\varepsilon_2$, it follows from (54) that $a(k) = -1$. Then $x(k)$ satisfies $a(k)\varepsilon_2 - \varepsilon_1 \leq \gamma^2 x_1(k) + 2\gamma x_2(k) < -\varepsilon_2 - \sigma_{\varepsilon_1}(\varpi(k)) \leq a(k)\varepsilon_2 + \varepsilon_1$, where we have noticed $F_\gamma x(k) \geq -\varepsilon_2 - \varepsilon_1, \forall x(k) \in \mathbf{II}$. In conclusion, the state $x(k)$ in region $\mathbf{II}$ satisfies

$$a(k)\varepsilon_2 - \varepsilon_1 \leq \gamma^2 x_1(k) + 2\gamma x_2(k) \leq a(k)\varepsilon_2 + \varepsilon_1,\ \forall k \geq 0, \tag{55}$$

where $a(k)$ is given by (54).

Step 2.1: We first show that for any initial state $x(0) \in \mathbf{II} \backslash \Omega_1$, then $x(1) \in \mathbf{I}^+$ or $\mathbf{I}^-$. Let us assume that $x(0) = [x_1(0), x_2(0)]^{\mathrm{T}} \in \mathbf{II}$. From (55), we have

$$a(0)\varepsilon_2 - \varepsilon_1 \le \gamma^2 x_1(0) + 2\gamma x_2(0) \le a(0)\varepsilon_2 + \varepsilon_1, \tag{56}$$

where $a(0) \in [-1, 1]$. Bearing in mind (54), in region $\mathbf{II}$, the nonlinear system (4) can be rewritten as $x(k+1) = Ax(k) - ba(k)\varepsilon_2, \ -1 \le a(k) \le 1, \ \forall k \ge 0$, by which we have

$$x(1) = \begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = \begin{bmatrix} x_1(0) + x_2(0) \\ x_2(0) - a(0)\varepsilon_2 \end{bmatrix}. \tag{57}$$

If $x(1) \in \mathbf{II}$, we have

$$-(\varepsilon_1 + \varepsilon_2) \le \gamma^2 x_1(1) + 2\gamma x_2(1) \le (\varepsilon_1 + \varepsilon_2). \tag{58}$$

Combining (57) and (58) gives

$$\begin{aligned} & -(\varepsilon_1 + \varepsilon_2) - \gamma^2 x_2(0) + 2a(0)\gamma\varepsilon_2 \\ \le & \gamma^2 x_1(0) + 2\gamma x_2(0) \\ \le & (\varepsilon_1 + \varepsilon_2) - \gamma^2 x_2(0) + 2a(0)\gamma\varepsilon_2. \end{aligned} \tag{59}$$

Therefore, from (56) and (59) we know that, if $x(0) \in \mathbf{II}$ and $x(1) \in \mathbf{II}$, then necessarily, $a(0)\varepsilon_2 - \varepsilon_1 \le (\varepsilon_1 + \varepsilon_2) - \gamma^2 x_2(0) + 2a(0)\gamma\varepsilon_2$ and $a(0)\varepsilon_2 + \varepsilon_1 \ge -(\varepsilon_1 + \varepsilon_2) - \gamma^2 x_2(0) + 2a(0)\gamma\varepsilon_2$, namely,

$$\begin{aligned} -\rho_1 < & -2\varepsilon_1 - \varepsilon_2 + a(0)(2\gamma - 1)\varepsilon_2 \\ & \le \gamma^2 x_2(0) \le 2\varepsilon_1 + \varepsilon_2 + a(0)(2\gamma - 1)\varepsilon_2 < \rho_1, \end{aligned} \tag{60}$$

where we have noticed $a(0) \in [-1, 1]$ and $\gamma \in (0, 1/2]$. Hence, $x(0) \in \Omega_1$. In conclusion, if $x(0) \in \mathbf{II}$ and $x(1) \in \mathbf{II}$, then necessarily $x(0) \in \Omega_1$. Therefore, $x(0) \in \mathbf{II} \backslash \Omega_1 \Rightarrow x(1) \in \mathbf{I}^+$ or $\mathbf{I}^-$.

Step 2.2: We show that for any initial condition $x(0) \in \mathbf{II} \backslash \Omega_1$, there exists an integer $k > 0$ such that $x(k) \in \Omega_1$. Without loss of generality, we let $x(0) \in \mathbf{II} \backslash \Omega_1, \ x(1) \in \mathbf{I}^+$. It follows that

$$\gamma^2 x_2(0) > \rho_1. \tag{61}$$

By virtue of Claim 1, we know that the state trajectory will return to $\mathbf{II}$ after finite steps $k > 1$, namely, $x(k) = [x_1(k), x_2(k)]^{\mathrm{T}} \in \mathbf{II}$. If $|x_2(k)| - |x_2(0)| \le -\kappa$ where κ is a suitable positive constant independent of k, we can conclude that eventually x will enter Ω_1. Thus, it remains to show that $|x_2(k)| - |x_2(0)| \le -\kappa$. To this end, we have the following statements, whose proofs are given in Appendix for clarity.

(1) For any initial state $x(0) \in \mathbf{II} \backslash \Omega_1$ with $x(1) \in \mathbf{I}^+$, if the state trajectory enters $\mathbf{II}$ along the path $\mathbf{II} \to \mathbf{I}^+ \to \mathbf{II}$ after finite steps k, then $|x_2(k)| - |x_2(0)| \le -\kappa$.

(2) For any initial state $x(0) \in \mathbf{II} \backslash \Omega_1$ with $x(1) \in \mathbf{I}^+$, if the state trajectory enters $\mathbf{I}^-$ along the path $\mathbf{II} \to \mathbf{I}^+ \to \mathbf{I}^-$ after finite steps k, then $|x_2(k-1)| - |x_2(0)| \le -\kappa$.

(3) For any initial state $x(0) \in \mathbf{I}^+$ with $x(1) \in \mathbf{I}^-$, if the state trajectory enters $\mathbf{II}$ along the path $\mathbf{I}^+ \to \mathbf{I}^- \to \mathbf{II}$ after finite steps k, then $|x_2(k)| - |x_2(0)| \le -\kappa$. By symmetry, for any state $x(0) \in \mathbf{I}^-$ with $x(1) \in \mathbf{I}^+$, if the state trajectory enters $\mathbf{II}$ along the path $\mathbf{I}^- \to \mathbf{I}^+ \to \mathbf{II}$ after finite steps k, the inequality $|x_2(k)| - |x_2(0)| \le -\kappa$ holds.

(4) For any initial state $x(0) \in \mathbf{I}^+$ with $x(1) \in \mathbf{I}^-$, if the state trajectory enters $\mathbf{I}^+$ along the path $\mathbf{I}^+ \to \mathbf{I}^- \to \mathbf{I}^+$ after finite steps k, then $|x_2(k-1)| - |x_2(0)| \le -\kappa$. By symmetry, for any initial state $x(0) \in \mathbf{I}^-$ with $x(1) \in \mathbf{I}^+$, if the state trajectory enters $\mathbf{I}^-$ along the path $\mathbf{I}^- \to \mathbf{I}^+ \to \mathbf{I}^-$ after finite steps k, then $|x_2(k-1)| - |x_2(0)| \le -\kappa$.

From (4), we get that $x_1(k+1) = x_1(k) + x_2(k)$, which implies that the state trajectory moves clockwise. With the aid of the above statements, and by letting $x_\varsigma = [x_1(\varsigma), x_2(\varsigma)]^{\mathrm{T}}$ with ς denoting $A, B, \ldots, J$, we have the following cases:

- $\mathbf{II} \backslash \Omega_1 \to \mathbf{I}^+ \to \mathbf{II}$ (see Fig. 4-I: $A(x_A) \to B(x_B) \to C(x_C) \to D(x_D)$). From statement (1), we have $|x_2(D)| - |x_2(A)| \le -\kappa$.
- $\mathbf{II} \backslash \Omega_1 \to \mathbf{I}^+ \to \mathbf{I}^- \to \mathbf{II}$ (see Fig. 4-II: $A(x_A) \to \cdots \to C(x_C) \to \cdots \to F(x_F)$). From statements (2)-(3) we have $|x_2(C)| - |x_2(A)| \le -\kappa$ and $|x_2(F)| - |x_2(C)| \le -\kappa$, which implies $|x_2(F)| - |x_2(A)| \le -\kappa$.
- $\mathbf{II} \backslash \Omega_1 \to \mathbf{I}^+ \to \mathbf{I}^- \to \mathbf{I}^+ \to \mathbf{II}$ (see FIG. 4-III: $A(x_A) \to \cdots \to C(x_C) \to \cdots \to E(x_E) \to \cdots \to H(x_H)$). From statements (2)-(4) we have $|x_2(C)| - |x_2(A)| \le -\kappa$, $|x_2(E)| - |x_2(C)| \le -\kappa$ and $|x_2(H)| - |x_2(E)| \le -\kappa$, which implies $|x_2(H)| - |x_2(A)| \le -\kappa$.
- $\mathbf{II} \backslash \Omega_1 \to \mathbf{I}^+ \to \mathbf{I}^- \to \mathbf{I}^+ \to \mathbf{I}^- \to \mathbf{II}$ (see Fig. 4-IV: $A(x_A) \to \cdots \to C(x_C) \to \cdots \to E(x_E) \to \cdots \to G(x_G) \to \cdots \to J(x_J)$). From statements (2)-(4) we have $|x_2(C)| - |x_2(A)| \le -\kappa$, $|x_2(E)| - |x_2(C)| \le -\kappa$, $|x_2(G)| - |x_2(E)| \le -\kappa$ and $|x_2(J)| - |x_2(G)| \le -\kappa$, which implies $|x_2(J)| - |x_2(A)| \le -\kappa$.
- The other cases can also be concluded from the statements (1)-(4).

In conclusion, for any initial state $x(0) \in \mathbf{II} \backslash \Omega_1$ with $x(1) \in \mathbf{I}^+$, there exists an integer $k > 0$ such that $x(k) \in \mathbf{II}$ and $x_2(k) - x_2(0) \le -\kappa$, where κ is a suitable positive constant independent of k. Thus, the state trajectory enters Ω_1 after finite steps. The case with any initial condition $x(0) \in \mathbf{II} \backslash \Omega_1$ with $x(1) \in \mathbf{I}^-$ can be considered by symmetry.

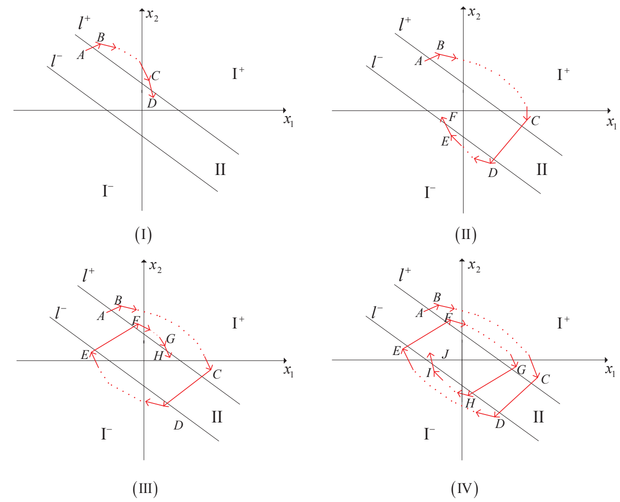

Fig. 4. Trajectory of the planar nonlinear discrete-time system (4) under the initial condition $x(0) \in \mathbf{II} \backslash \Omega_1$.

Proof of Claim 3

Step 3.1: We first show that $x(0) \in \Omega_1 \backslash \Omega_2$, $x(1) \in \mathbf{II} \Rightarrow x(1) \in \Omega_1$. As in the analysis of Claim 2, when $x(0) \in \Omega_1 \backslash \Omega_2$, $x(1) \in \mathbf{II}$, we get (57) and (60) with $a(0) \in [-1, 1]$. When $a(0) \in [0, 1]$, we have from (57) and (60) that $\gamma^2 x_2(1) = \gamma^2 (x_2(0) - a(0)\varepsilon_2) \le \gamma^2 x_2(0) < \rho_1$, and $\gamma^2 x_2(1) = \gamma^2 (x_2(0) - a(0)\varepsilon_2) \ge -2\varepsilon_1 - \varepsilon_2 + a(0)\left(2\gamma\varepsilon_2 - \varepsilon_2 - \gamma^2\varepsilon_2\right) \ge -\rho_1$. When $a(0) \in [-1, 0)$, we have from (57) and (60) that $\gamma^2 x_2(1) = \gamma^2 (x_2(0) - a(0)\varepsilon_2) \le 2\varepsilon_1 + \varepsilon_2 + a(0)\left(2\gamma\varepsilon_2 - \varepsilon_2 - \gamma^2\varepsilon_2\right) \le \rho_1$, and $\gamma^2 x_2(1) = \gamma^2 (x_2(0) - a(0)\varepsilon_2) > \gamma^2 x_2(0) > -\rho_1$. To sum up, for any $a(0) \in [-1, 1]$, we have $-\rho_1 \le \gamma^2 x_2(1) \le \rho_1$, which implies that $x(1) \in \Omega_1$.

Step 3.2: We show that the energy function $V(x) = x^{\mathrm{T}} P x$ is such that $V(x(1)) - V(x(0)) < 0$, when $x(0) \in \Omega_1 \backslash \Omega_2$ and $x(1) \in \Omega_1$. In region $\Omega_1 \backslash \Omega_2$, we consider the following system

$$x(k+1) = Ax(k) - b\sigma_{\varepsilon_2}(F_\gamma x(k) + \sigma_{\varepsilon_1}(\varpi(k))). \tag{62}$$

Notice that, for any $x \in \Omega_1 \backslash \Omega_2$, we have

$$\begin{aligned}&\sigma_{\varepsilon_2}(F_\gamma x(k) + \sigma_{\varepsilon_1}(\varpi(k)))\\ =&\varepsilon_2 \sigma\left(\frac{F_\gamma x(k) + \sigma_{\varepsilon_1}(\varpi(k))}{\varepsilon_2}\right)\\ \in&\varepsilon_2 \mathrm{coh}\left\{\frac{F_\gamma x(k) + \sigma_{\varepsilon_1}(\varpi(k))}{\varepsilon_2}, \frac{F_\gamma x(k)}{\varepsilon_1+\varepsilon_2}\right\}\\ =&\mathrm{coh}\left\{F_\gamma x(k) + \sigma_{\varepsilon_1}(\varpi(k)), \frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2} F_\gamma x(k)\right\},\end{aligned}$$

where $\mathrm{coh}\{\cdot\}$ denotes the convex hull [4]. Hence we only need to consider the following vertical systems of (62), namely,

$$x(k+1) = (A - bF_\gamma)x(k) - b\sigma_{\varepsilon_1}(\varpi(k)), \tag{63}$$

and

$$x(k+1) = \left(A - \frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2} bF_\gamma\right)x(k). \tag{64}$$

Denote $A_\mathrm{c} = A - bF_\gamma$. Then it is easy to verify that

$$A_\mathrm{c}^\mathrm{T} P A_\mathrm{c} - P = -\gamma P - F_\gamma^\mathrm{T} F_\gamma,\ b^\mathrm{T} P A_\mathrm{c} = F_\gamma. \tag{65}$$

It follows from (31), (32) and (65) that, the time-derivative of $V(x) = x^\mathrm{T} P x$ along the trajectories of system (63) satisfies

$$\begin{aligned}\nabla V(x)|_{(63)} =& x^\mathrm{T}\left(A_\mathrm{c}^\mathrm{T} P A_\mathrm{c} - P\right)x - 2x^\mathrm{T} A_\mathrm{c}^\mathrm{T} P b \sigma_{\varepsilon_1}(\varpi)\\ &+ \sigma_{\varepsilon_1}(\varpi) b^\mathrm{T} P b \sigma_{\varepsilon_1}(\varpi)\\ =& x^\mathrm{T}\left(-\gamma P - F_\gamma^\mathrm{T} F_\gamma\right)x - 2x^\mathrm{T} F_\gamma^\mathrm{T} \sigma_{\varepsilon_1}(\varpi)\\ &+ \sigma_{\varepsilon_1}(\varpi) b^\mathrm{T} P b \sigma_{\varepsilon_1}(\varpi)\\ \le& -\gamma x^\mathrm{T} P x + (\sigma_{\varepsilon_1}(\varpi))^2 + \sigma_{\varepsilon_1}(\varpi) b^\mathrm{T} P b \sigma_{\varepsilon_1}(\varpi)\\ <& -\gamma q + \varepsilon_1^2 + \left(\frac{1}{(1-\gamma)^2} - 1\right)\varepsilon_1^2\\ =& \left(-\gamma\mu + \frac{1}{(1-\gamma)^2}\right)\varepsilon_1^2\\ \triangleq& -\alpha\varepsilon_1^2,\ \forall x \in \Omega_1 \backslash \Omega_2,\end{aligned} \tag{66}$$

where $\alpha = \gamma\mu - 1/(1-\gamma)^2 > 0$. Similarly, the time-shift of $V(x) = x^\mathrm{T} P x$ along the trajectories of system (64) is

$$\nabla V(x)|_{(64)} = -x^\mathrm{T} R x, \tag{67}$$

where

$$\begin{aligned}R =& P - \left(A - \frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2} bF_\gamma\right)^\mathrm{T} P \left(A - \frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2} bF_\gamma\right)\\ =& \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix},\end{aligned}$$

with $r_{11} = \frac{\gamma^4\varepsilon_2}{(1-\gamma)^2(\varepsilon_1+\varepsilon_2)^2}((\gamma^2 - 2\gamma + 2)\varepsilon_2 + 2\varepsilon_1)$, $r_{22} = \frac{8\gamma^2\varepsilon_2}{(1-\gamma)^2(\varepsilon_1+\varepsilon_2)} + \frac{4(\gamma^4-2\gamma^3)\varepsilon_2^2}{(1-\gamma)^2(\varepsilon_1+\varepsilon_2)^2} - \frac{\gamma^3+2\gamma^2}{(1-\gamma)^2}$, $r_{12} = r_{21} = \frac{\gamma^3}{(1-\gamma)^2(\varepsilon_1+\varepsilon_2)^2}((2\gamma^2-4\gamma+3)\varepsilon_2^2+2\varepsilon_1\varepsilon_2-\varepsilon_1^2)$. One can prove easily that $\det(R) > \frac{\gamma^6}{(1-\gamma)^4(\varepsilon_1+\varepsilon_2)^2}(3(1-\gamma)\varepsilon_2 - \varepsilon_1)((1-\gamma)\varepsilon_2 - \varepsilon_1)$. Since $\gamma \in (0, 1/2]$, (8) ensures that $(1-\gamma)\varepsilon_2 - \varepsilon_1 > 0$, which implies that $R > 0$. Let $\lambda_{\min}(R)$ and $\lambda_{\max}(P)$ denote respectively the minimal eigenvalue of R and the maximal eigenvalue of P. From (67) we know that

$$\begin{aligned}\nabla V(x)|_{(64)} \le& -\lambda_{\min}(R)\|x\|^2\\ =& -\frac{\lambda_{\min}(R)}{\lambda_{\max}(P)}\left(\lambda_{\max}(P)\|x\|^2\right)\\ \le& -\frac{\lambda_{\min}(R)}{\lambda_{\max}(P)} V(x)\\ <& -\frac{\lambda_{\min}(R)}{\lambda_{\max}(P)} q,\ \forall x \in \Omega_1 \backslash \Omega_2,\end{aligned} \tag{68}$$

Hence, combining these two cases (66) and (68) gives $\nabla V(x)|_{(62)} < -\upsilon_1 < 0$, $\forall x \in \Omega_1 \backslash \Omega_2$, where $\upsilon_1 \triangleq \min\{\alpha, \lambda_{\min}(R) q/\lambda_{\max}(P)\}$, which concludes the proof of this claim.

Proof of Claim 4

Step 4.1: We first show that for any initial state $x(0) \in \Omega_1 \backslash \Omega_2$, if $x(1) \in \mathbf{I}^+$, then x will skip from $\mathbf{I}^+$ into $\Omega_1 \subset \mathbf{II}$ directly after finite steps. As in the analysis of Step 1.1, the state trajectory in $\mathbf{I}^+$ enters $\mathbf{II} \cup \mathbf{I}^-$ after finite steps k. If $x(k) \in \mathbf{I}^-$, namely, the state trajectory moves along the path $\Omega_1 \backslash \Omega_2 \to \mathbf{I}^+ \to \mathbf{I}^-$, then combining the analysis of statement (2) in Step 2.2, we have (110)-(112) and (122)-(124). It follows from (122) that

$$\gamma^2 y_2(k_4) < -2(\varepsilon_1+\varepsilon_2) + 2\gamma\varepsilon_2. \tag{69}$$

Moreover, by virtue of (112) and (124), we get

$$\begin{aligned}\gamma^2 y_2(k_4) =& \gamma^2(x_2(0) - a(0)\varepsilon_2 - k_4\varepsilon_2)\\ >& \left(a(0)\gamma^2 - 2\gamma^2 + 4\gamma\right)\varepsilon_2 - \gamma^2 x_2(0).\end{aligned} \tag{70}$$

Combining (69) with (70) gives $\gamma^2 x_2(0) > 2(\varepsilon_1+\varepsilon_2) + 2\gamma\varepsilon_2 + (a(0) - 2)\gamma^2\varepsilon_2 > \rho_1$, which is in contradiction with $x(0) \in \Omega_1$, where we have noticed $a(0) \in [-1, 1]$ and $\gamma \in (0, 1/2]$. Therefore, x enters region $\mathbf{II}$ along the path $\Omega_1 \backslash \Omega_2 \to \mathbf{I}^+ \to \mathbf{II}$ after finite steps k. Next, we show that $x(k) \in \Omega_1 \subset \mathbf{II}$. As in the analysis of statement (1) in Step 2.2, when x moves along the path $\Omega_1 \backslash \Omega_2 \to \mathbf{I}^+ \to \mathbf{II}$, we have (110)-(119), and

$$\gamma^2 x_2(0) \le \rho_1. \tag{71}$$

By (110) and (111), we have

$$\gamma^2 x_2(0) > (1 + a(0)(2\gamma - 1))\varepsilon_2 \ge 2\gamma\varepsilon_2, \tag{72}$$

where we have noticed $a(0) \in [-1, 1]$ and $\gamma \in (0, 1/2]$. On the one hand, it follows from (112), (119) and (71) that

$$\begin{aligned}\gamma^2 x_2(k_3+2) =& \gamma^2 y_2(k_3+1)\\ =& \gamma^2(x_2(0) - a(0)\varepsilon_2 - (k_3+1)\varepsilon_2)\\ =& -\gamma^2 x_2(0) + \gamma^2\left(a(0) - a_3 - 1 + \frac{4}{\gamma}\right)\varepsilon_2\\ \ge& -2(\varepsilon_1+\varepsilon_2) + 2\gamma\varepsilon_2 - \gamma^2\varepsilon_2 - 4\gamma^2\varepsilon_2 + 4\gamma\varepsilon_2\\ >& -2(\varepsilon_1+\varepsilon_2) + 2\gamma\varepsilon_2 - \gamma^2\varepsilon_2\\ =& -\rho_1,\end{aligned} \tag{73}$$

where we have noticed $a(0) \in [-1, 1]$, $a_3 \in (0, 2)$ and $\gamma \in (0, 1/2]$. On the other hand, it follows from (112), (119) and (72) that

$$\begin{aligned}\gamma^2 x_2(k_3+2) =& -\gamma^2 x_2(0) + \gamma^2\left(a(0) - a_3 - 1 + \frac{4}{\gamma}\right)\varepsilon_2\\ <& -2\gamma\varepsilon_2 + 4\gamma\varepsilon_2\\ <& 2(\varepsilon_1+\varepsilon_2) - 2\gamma\varepsilon_2 + \gamma^2\varepsilon_2\\ =& \rho_1,\end{aligned} \tag{74}$$

where we have noticed $a(0) \in [-1, 1]$, $a_3 \in (0, 2)$ and $\gamma \in (0, 1/2]$. Combining (73) and (74) gives $-\rho_1 < \gamma^2 x_2(k_3+2) < \rho_1$, which implies that $x(k) \in \Omega_1 \subset \mathbf{II}$ with $k = k_3 + 2$. The case with any initial condition $x(0) \in \Omega_1 \backslash \Omega_2$ with $x(1) \in \mathbf{I}^-$ can be considered by symmetry.

Step 4.2: We next show that $V(x(k)) - V(x(0)) < -\upsilon_2$ where υ_2 is a positive constant independent of k. Based on the analysis of statement (1) in Step 2.2, we have (110)-(119). From (110), (114) and (117), we know that the initial state $x(0) = [x_1(0), x_2(0)]^{\mathrm{T}}$ should satisfy

$$\left\{ \begin{array}{lll} f_{13} < & \gamma^2 x_1(0) + 2\gamma x_2(0) & \leq f_{17}, \\ f_{15}(k_3) < & \gamma^2 x_1(0) + 2\gamma x_2(0) & \leq f_{16}(k_3), \end{array} \right. \tag{75}$$

where f_{13} is given by (111), $f_{15}(k_3), f_{16}(k_3)$ are given by (115) and f_{17} is given by (116). For (75), there are four cases discussed as follows. For (75), there are four cases discussed as follows. For future use, we denote

$$\zeta = \frac{2\left(\gamma^2 x_2(0) - 2\gamma\varepsilon_2\right)}{\gamma^2 \varepsilon_2}. \tag{76}$$

Case 1: When $f_{13} < f_{15}(k_3),\ f_{16}(k_3) < f_{17}$, we have

$$\left\{ \begin{array}{l} k_3\left(k_3 - (\zeta + 1 - 2a(0))\right) > 0, \\ \left(\frac{\gamma^2\varepsilon_2}{2} k_3 + \gamma^2\varepsilon_2\right)\left(k_3 - (\zeta + 1 - 2a(0))\right) < 0. \end{array} \right. \tag{77}$$

Solving (77) gives

$$\hat{k}_1 < k_3 < \hat{k}_1, \tag{78}$$

where

$$\hat{k}_1 = \zeta + 1 - 2a(0), \tag{79}$$

which is incompatible.

Case 2: When $f_{15}(k_3) < f_{13},\ f_{17} < f_{16}(k_3)$, (78) also holds, which is incompatible.

Case 3: When $f_{13} \leq f_{15}(k_3) < f_{17} \leq f_{16}(k_3)$, we have

$$\left\{ \begin{array}{l} k_3\left(k_3 - (\zeta + 1 - 2a(0))\right) \geq 0, \\ \left(\frac{\gamma^2\varepsilon_2}{2} k_3 + \frac{\gamma^2\varepsilon_2}{2}\right)\left(k_3 - (\zeta + 2 - 2a(0))\right) < 0. \end{array} \right. \tag{80}$$

Solving (80) gives $\hat{k}_1 \leq k_3 < \hat{k}_2$, where $\hat{k}_1$ is given by (79) and

$$\hat{k}_2 = \zeta + 2 - 2a(0). \tag{81}$$

Case 4: When $f_{15}(k_3) \leq f_{13} < f_{16}(k_3) \leq f_{17}$, we have

$$\left\{ \begin{array}{l} k_3\left(k_3 - (\zeta + 1 - 2a(0))\right) \leq 0, \\ \left(\frac{\gamma^2\varepsilon_2}{2} k_3 + \frac{\gamma^2\varepsilon_2}{2}\right)\left(k_3 - (\zeta - 2a(0))\right) > 0. \end{array} \right. \tag{82}$$

Solving (82) gives $\hat{k}_3 < k_3 \leq \hat{k}_1$, where $\hat{k}_1$ is given by (79) and

$$\hat{k}_3 = \zeta - 2a(0). \tag{83}$$

To sum up, by (71)-(72) and the above four cases, we know that any initial state $x(0) \in \Omega_1 \backslash \Omega_2$ generating a solution that will enter region $\mathbf{I}^+$ and then return into Ω_1 should satisfy

$$\begin{aligned} 2\gamma\varepsilon_2 &\leq (1 + a(0)(2\gamma - 1))\varepsilon_2 && (84)\\ &< \gamma^2 x_2(0) && (85)\\ &\leq 2(\varepsilon_1 + \varepsilon_2) + 2\gamma\varepsilon_2 - 3\gamma^2\varepsilon_2, && (86) \end{aligned}$$

and

$$\left\{ \begin{array}{ll} f_{13} < \gamma^2 x_1(0) + 2\gamma x_2(0) \leq f_{16}(k_3), & \hat{k}_3 < k_3 \leq \hat{k}_1, \\ f_{15}(k_3) < \gamma^2 x_1(0) + 2\gamma x_2(0) \leq f_{17}, & \hat{k}_1 \leq k_3 < \hat{k}_2. \end{array} \right. \tag{87}$$

On the one hand, from (6) and (112) we have

$$\begin{aligned} & V(x(k_3 + 2)) - V(x(0)) \\ =& V(y(k_3 + 1)) - V(x(0)) \\ =& \frac{2\gamma^2 x_1(0)}{(\gamma - 1)^2} h_1 + h_2, \end{aligned} \tag{88}$$

where $h_i = h_i(k_3, x_2(0)), i = 1, 2$, and in particular, h_1 is given by

$$\begin{aligned} h_1 =& -\frac{1}{2}\gamma\varepsilon_2 k_3^2 + \left(\left(-\frac{1}{2} - a(0)\right)\gamma\varepsilon_2 + \gamma x_2(0) - \varepsilon_2\right) k_3 \\ & + 2\gamma x_2(0) - a(0)\gamma\varepsilon_2 - (a(0) + 1)\varepsilon_2. \end{aligned} \tag{89}$$

On the other hand, by (86), we have

$$\begin{aligned} & 2\gamma x_2(0) - a(0)\gamma\varepsilon_2 - (a(0) + 1)\varepsilon_2 \\ >& \frac{2\gamma(1 + a(0)(2\gamma - 1))\varepsilon_2}{\gamma^2} - a(0)\gamma\varepsilon_2 - (a(0) + 1)\varepsilon_2 \\ =& \frac{1}{\gamma}\left(-a(0)\left(\gamma^2 - 3\gamma + 2\right) - \gamma + 2\right)\varepsilon_2 \\ \geq& \frac{1}{\gamma}\left(2\gamma - \gamma^2\right)\varepsilon_2 \\ >& 0 \end{aligned} \tag{90}$$

and

$$\begin{aligned} h_1|_{k_3 = \hat{k}_2} =& (2 - \gamma) x_2(0) + (7 - 3a(0))\varepsilon_2 \\ & + \left(-\frac{13}{4} + 2a(0)\right)\gamma\varepsilon_2 \\ >& (2 - \gamma)\frac{2\gamma\varepsilon_2}{\gamma^2} + (7 - 3a(0))\varepsilon_2 \\ & + \left(-\frac{13}{4} + 2a(0)\right)\gamma\varepsilon_2 \\ =& -(3a(0) + 3\gamma - 2a(0)\gamma - 5)\varepsilon_2 \\ >& 0, \end{aligned} \tag{91}$$

where we have noticed that $a(0) \in [-1, 1]$ and $\gamma \in (0, 1/2]$. Moreover, (89) can be regarded as a quadratic equation of k_3. Then it follows from (90) and (91) that $h_1 > 0,\ \hat{k}_3 < k_3 < \hat{k}_2$, by which and (87), (88) we have

$$\begin{aligned} & V(x(k_3 + 2)) - V(x(0)) \\ =& \frac{2\gamma^2 x_1(0)}{(\gamma - 1)^2} h_1 + h_2 \\ \leq& \left\{ \begin{array}{ll} \frac{2(f_{16}(k_3) - 2\gamma x_2(0))}{(\gamma - 1)^2} h_1 + h_2, & \hat{k}_3 < k_3 \leq \hat{k}_1, \\ \frac{2(f_{17} - 2\gamma x_2(0))}{(\gamma - 1)^2} h_1 + h_2, & \hat{k}_1 \leq k_3 < \hat{k}_2. \end{array} \right. \end{aligned} \tag{92}$$

Next we will give a detailed discussion about $V(x(k_3+2)) - V(x(0))$ based on (92).

Case 1: When $\hat{k}_3 < k_3 \leq \hat{k}_1$, we can denote $k_3 = \hat{k}_3 + a_5$ with $a_5 \in (0, 1]$, where we have noticed (79) and (83). It follows from (92) and (115) that

$$V(x(k_3 + 2)) - V(x(0))|_{k_3 = \hat{k}_3 + a_5} \leq h_3 x_2^2(0) + h_4 x_2(0) + h_5, \tag{93}$$

where $h_3 = h_3(a_5, \gamma),\ h_4 = h_4(a(0), a_5, \gamma, \varepsilon_1, \varepsilon_2)$ and $h_5 = h_5(a(0), a_5, \gamma, \varepsilon_1, \varepsilon_2)$ satisfy

$$\left\{ \begin{array}{ll} h_3 & = \frac{\gamma^2(a_5 - 1)(\gamma - a_5\gamma + 2)}{(\gamma - 1)^2}, \\ h_4 & = \phi_1\varepsilon_1 + \left(\phi_2\gamma^3 + \phi_3\gamma^2 + \phi_4\right)\varepsilon_2, \end{array} \right. \tag{94}$$

with $\phi_1 = -2\gamma a_5 + 2\gamma + 4$, $\phi_2 = -a_5^3 + 2a(0)a_5^2 + (1 - 2a(0))a_5$, $\phi_3 = 5a_5^2 - (5 + 6a(0))a_5 + 6a(0) - 2$, $\phi_4 = -2\gamma a_5 + 14\gamma - 12$. Using

$$-1 \leq a(0) \leq 1,\ 0 < a_5 \leq 1,\ 0 < \gamma \leq 1/2, \tag{95}$$

we deduce from (94) and (8) that $h_3 \leq 0$ and $h_4 < 5\varepsilon_1 - \frac{23}{8}\varepsilon_2 < 0$. Thus, it follows from (86) and (93) that

$$\begin{aligned} & V(x(k_3 + 2)) - V(x(0))|_{k_3 = \hat{k}_3 + a_5} \\ <& h_3\left(\frac{2\gamma\varepsilon_2}{\gamma^2}\right)^2 + h_4\frac{2\gamma\varepsilon_2}{\gamma^2} + h_5 \\ =& \frac{\varepsilon_2}{4(\gamma - 1)^2}\left(h_6\varepsilon_1 + h_7\varepsilon_2\right), \end{aligned} \tag{96}$$

where $h_i = h_i(a(0), a_5, \gamma), i = 6, 7$, satisfy

$$h_6 = \phi_5\gamma + \phi_6,\ h_7 = \phi_7\gamma^3 + \phi_8\gamma^2 + \phi_9\gamma + \phi_{10}, \tag{97}$$

with $\phi_5 = -4a_5^2 + (8a(0) - 4)a_5$, $\phi_6 = 8a_5 - 24a(0) + 24$, $\phi_7 = -a_5^4 + (4a(0) - 2)a_5^3 + \left(-4a^2(0) + 4a(0) - 1\right)a_5^2$, $\phi_8 = (-8a(0) + 4)a_5^2 + \left(16a^2(0) - 16a(0)\right)a_5 - 4a^2(0) + 8a(0) - 4$, $\phi_9 = 12a_5^2 + (-24a(0) + 36)a_5 - 8a^2(0) + 8$, $\phi_{10} = -24a_5 + 40a(0) - 40$. In view of (95), we can compute from (97) that $h_6 > 0$. Therefore, from (96) we know that $V(x(k_3+2)) - V(x(0)) < -\frac{\gamma\varepsilon_2^2}{4(\gamma-1)^2}$ if $h_6\varepsilon_1 + h_7\varepsilon_2 < -\gamma\varepsilon_2$, namely,

$$\frac{\varepsilon_1}{\varepsilon_2} < -\frac{h_7+\gamma}{h_6}. \tag{98}$$

By using (95) and (97), we can compute that $-\frac{h_7+\gamma}{h_6} > 0.78$, from which we know that (98) can be guaranteed by $\varepsilon_1 < 0.78\varepsilon_2$, which can be further guaranteed by (8). Thus, we have

$$V(x(k_3+2)) - V(x(0)) < -\frac{\gamma\varepsilon_2^2}{4(\gamma-1)^2},\ \hat{k}_3 < k_3 \le \hat{k}_1. \tag{99}$$

Case 2: When $\hat{k}_1 \le k_3 < \hat{k}_2$, we can denote $k_3 = \hat{k}_3 + a_6$ with $a_6 \in [0,1)$, where we have noticed (79) and (81). It follows from (92) and (116) that

$$\begin{aligned} &V(x(k_3+2)) - V(x(0))|_{k_3=\hat{k}_1+a_6} \\ \le& h_8(x_2(0))^2 + h_9x_2(0) + h_{10}, \end{aligned} \tag{100}$$

where $h_8 = h_8(a_6,\gamma)$, $h_9 = h_9(a(0), a_6, \gamma, \varepsilon_1, \varepsilon_2)$ and $h_{10} = h_{10}(a(0), a_6, \gamma, \varepsilon_1, \varepsilon_2)$ satisfy

$$\begin{cases} h_8 &= \frac{a_6\gamma^2(a_6\gamma-2)}{(\gamma-1)^2}, \\ h_9 &= \frac{1}{(\gamma-1)^2}\left(\phi_{11}\varepsilon_1 + \left(\phi_{12}\gamma^3 + \phi_{13}\gamma^2 + \phi_{14}\right)\varepsilon_2\right), \end{cases} \tag{101}$$

with $\phi_{11} = 4 - 2a_6\gamma$, $\phi_{12} = 3a_6^2 + a_6^3 - 2a(0)a_6^2$, $\phi_{13} = 4a(0)a_6 - 9a_6 - 7a_6^2 - 2$, $\phi_{14} = 12\gamma + 14a_6\gamma - 12$. By using

$$-1 \le a(0) \le 1,\ 0 \le a_6 < 1,\ 0 < \gamma \le 1/2, \tag{102}$$

we can compute from (101) and (8) that $h_8 \le 0$ and $h_9 < 0$. Hence, we deduce from (86) and (100) that

$$\begin{aligned} &V(x(k_3+2)) - V(x(0))|_{k_3=\hat{k}_1+a_6} \\ <& h_8\left(\frac{2\gamma\varepsilon_2}{\gamma^2}\right)^2 + h_9\frac{2\gamma\varepsilon_2}{\gamma^2} + h_{10} \\ =& \frac{\varepsilon_2}{4(\gamma-1)^2}(h_{11}\varepsilon_1 + h_{12}\varepsilon_2), \end{aligned} \tag{103}$$

where $h_i = h_i(a(0), a_6, \gamma), i = 11, 12$, satisfy

$$h_{11} = \phi_{15}\gamma + \phi_{16},\ h_{12} = \phi_{17}\gamma^3 + \phi_{18}\gamma^2 + \phi_{19}\gamma + \phi_{20}, \tag{104}$$

with $\phi_{15} = -4a_6^2 + (8a(0) - 12)a_6 + 8a(0) - 8$, $\phi_{16} = 8a_6 - 24a(0) + 32$, $\phi_{17} = a_6^4 + (6 - 4a(0))a_6^3 + \left(4a^2(0) - 12a(0) + 9\right)a_6^2 - 4a^2(0) + 8a(0) - 4$, $\phi_{18} = -4a_6^3 + (12a(0) - 24)a_6^2 + \left(-8a^2(0) + 36a(0) - 40\right)a_6 + 12a^2(0) - 16a(0)$, $\phi_{19} = 12a_6^2 + (-24a(0) + 60)a_6 - 8a^2(0) - 24a(0) + 56$, $\phi_{20} = -24a_6 + 40a(0) - 64$. In view of (102), from (104) we can deduce that $h_{11} > 0$. Moreover, by (103) we know that $V(x(k_3+2)) - V(x(0)) < -\frac{\gamma\varepsilon_2^2}{4(\gamma-1)^2}$ if $h_{11}\varepsilon_1 + h_{12}\varepsilon_2 < -\gamma\varepsilon_2$, namely,

$$\frac{\varepsilon_1}{\varepsilon_2} < -\frac{h_{12}+\gamma}{h_{11}}. \tag{105}$$

By using (102) and (104), we can obtain $-\frac{h_{12}+\gamma}{h_{11}} \ge 1.28$, by which we know that (105) can be guaranteed by $\varepsilon_1 < 1.28\varepsilon_2$, which can be further guaranteed by (8). Thus, have

$$V(x(k_3+2)) - V(x(0)) < -\frac{\gamma\varepsilon_2^2}{4(\gamma-1)^2},\ \hat{k}_1 \le k_3 < \hat{k}_2. \tag{106}$$

Combining the inequalities in (99) and (106) in the above two cases gives $V(x(k_3+2)) - V(x(0)) < -\frac{\gamma\varepsilon_2^2}{4(\gamma-1)^2} \triangleq -\upsilon_2,\ \hat{k}_3 < k_3 < \hat{k}_2$, where $\hat{k}_i, i = 2, 3$, are respectively given by (81) and (83). The case with any initial condition $x(0) \in \Omega_1\backslash\Omega_2$ with $x(1) \in \mathbf{I}^-$ can be considered by symmetry. This ends the proof of the claim.

Proof of Claim 5

Step 5.1: We first show that if there exists an integer $k \ge 0$ such that $x(k) \in \Omega_2$, then $|F_\gamma x(k) + \sigma_{\varepsilon_1}(\varpi(k))| \le \varepsilon_2$. One can check easily that

$$F_\gamma^{\mathrm{T}}F_\gamma \le \gamma(1-\gamma)(2-\gamma)P, \tag{107}$$

where F_γ and P are given by (5) and (6), respectively. Then, it follows from (31), (32) and (107) that

$$\begin{aligned} &|F_\gamma x(k) + \sigma_{\varepsilon_1}(\varpi)| \\ \le& |F_\gamma x(k)| + \varepsilon_1 \\ =& \sqrt{x^{\mathrm{T}}(k)F_\gamma^{\mathrm{T}}F_\gamma x(k)} + \varepsilon_1 \\ \le& \sqrt{\gamma(1-\gamma)(2-\gamma)x^{\mathrm{T}}(k)Px(k)} + \varepsilon_1 \\ \le& \sqrt{\gamma(1-\gamma)(2-\gamma)q} + \varepsilon_1 \\ =& \left(\sqrt{\gamma(1-\gamma)(2-\gamma)\mu} + 1\right)\varepsilon_1 \\ \le& \varepsilon_2. \end{aligned} \tag{108}$$

Step 5.2: We next show that if there exists an integer $k \ge 0$ such that $x(k) \in \Omega_2$, then $x(k+1) \in \Omega_2$. It follows from (108) that in region Ω_2, the control input $u(k)$ can be simplified as $u(k) = -F_\gamma x(k) - \sigma_{\varepsilon_1}(\varpi(k))$. As a result, the system (4) can be rewritten as $x(k+1) = A_{\mathrm{c}}x(k) - b\sigma_{\varepsilon_1}(\varpi)$, by which and (31), (32), (65) we have

$$\begin{aligned} &V(x(k+1)) \\ =& x(k)^{\mathrm{T}}A_{\mathrm{c}}^{\mathrm{T}}PA_{\mathrm{c}}x(k) - 2x(k)^{\mathrm{T}}A_{\mathrm{c}}^{\mathrm{T}}Pb\sigma_{\varepsilon_1}(\varpi(k)) \\ &+ \sigma_{\varepsilon_1}(\varpi(k))b^{\mathrm{T}}Pb\sigma_{\varepsilon_1}(\varpi(k)) \\ =& (1-\gamma)V(x(k)) - x(k)^{\mathrm{T}}F_\gamma^{\mathrm{T}}F_\gamma x(k) \\ &- 2x(k)^{\mathrm{T}}F_\gamma^{\mathrm{T}}\sigma_{\varepsilon_1}(\varpi(k)) + \sigma_{\varepsilon_1}(\varpi(k))b^{\mathrm{T}}Pb\sigma_{\varepsilon_1}(\varpi(k)) \\ \le& (1-\gamma)V(x(k)) + \sigma_{\varepsilon_1}^2(\varpi(k)) \\ &+ \sigma_{\varepsilon_1}(\varpi(k))b^{\mathrm{T}}Pb\sigma_{\varepsilon_1}(\varpi) \\ =& (1-\gamma)V(x(k)) + \frac{1}{(1-\gamma)^2}\varepsilon_1^2 \\ \le& (1-\gamma)q + \frac{1}{(1-\gamma)^2}\varepsilon_1^2 \\ =& q - \gamma\mu\varepsilon_1^2 + \frac{1}{(1-\gamma)^2}\varepsilon_1^2 \\ <& q - \frac{\gamma}{\gamma(1-\gamma)^2}\varepsilon_1^2 + \frac{1}{(1-\gamma)^2}\varepsilon_1^2 \\ =& q, \end{aligned}$$

which implies that $x(k+1) \in \Omega_2$. This concludes the proof.

Remark 3: In view of the complexity of the proof of Claim 4, we give an example to further verify the correctness of Claim 4. Consider the following system

$$\begin{cases} x(k+1) &= Ax(k) + bu(k),\quad x \in \mathbf{R}^2, \\ u(k) &= -\sigma_{\varepsilon_2}(F_\gamma x(k) + \sigma_{\varepsilon_1}(-F_\gamma x(k))), \end{cases} \tag{109}$$

and the energy function $V(x(k)) = x^{\mathrm{T}}(k)Px(k)$, where (A, b) is given by (3), F_γ is given by (5), P is given by (6), $\varepsilon_i, i = 1, 2$, satisfy (8), and $\gamma \in (0, 1/2]$. For the simulations, we select $\gamma = 0.4$, $\varepsilon_2 = 1$, $\varepsilon_1 = 0.34$ and the initial condition $x(0) = [x_{10}(0), x_{20}(0)]^{\mathrm{T}} = [-45, 9.5]^{\mathrm{T}}$. Then $x(0) \in \Omega_1$. The state trajectory and energy function are respectively plotted in Figs. 5-6. From Fig. 5, we clearly see that the state trajectory starting at $x(0) = [-45, 9.5]^{\mathrm{T}} \in \Omega_1$ enters region $\mathbf{I}^+$, and then returns into Ω_1 directly after finite steps $k = 12$, which is consistent with Claim 4. Moreover, it follows from

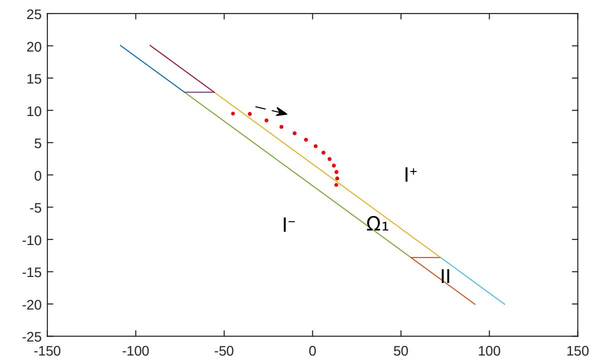

Fig. 5. The state trajectory of system (109) with $x(0) = [-45, 9.5]^{\mathrm{T}}$.

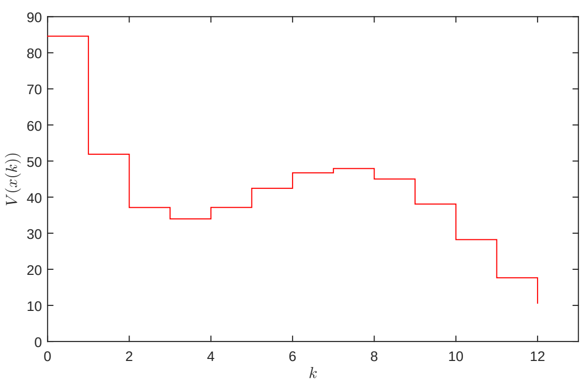

Fig. 6. The energy function $V(x(k)) = x^{\mathrm{T}}(k)Px(k)$.

Fig. 6 that $V(x(12)) - V(x(0)) < -60$, which is also consistent with Claim 4.

VII. Conclusion

In this paper, we have investigated the problem of global stabilization of discrete-time chains of integrators with length n by bounded controls. Based on an energy function analysis, the solution to the global stabilization of a discrete-time double integrators system in the presence of input additive disturbances by bounded controls was given first. Then, based on a specific linear transformation, a class of nonlinear controllers consisting of only $[\frac{n+1}{2}]$ nested saturation functions were constructed for globally asymptotically stabilizing a chain of discrete-time integrators by bounded controls. The main benefit of the designed controllers over those available in the literature is that the former one needs less saturation functions, which can allow much larger control energy and can thus lead to better transient performances. This fact is cooperated by simulations we performed.

Future works will be devoted to the design of globally asymptotically stabilizing feedbacks for general ANCBC discrete-time linear systems by bounded and delayed controls and extensions of these results to nonlinear feedforward systems.

Appendix

Proof of statement (1) of Step 2.2 When $x(0) \in \mathbf{II}$ and $x(1) \in \mathbf{I}^+$, it follows from (57) that

$$f_{13} < \gamma^2 x_1(0) + 2\gamma x_2(0) \leq f_{14}, \tag{110}$$

where

$$\begin{cases} f_{13} = (\varepsilon_1 + \varepsilon_2) - \gamma^2 x_2(0) + 2a(0)\gamma\varepsilon_2, \\ f_{14} = a(0)\varepsilon_2 + \varepsilon_1. \end{cases} \tag{111}$$

As in Step 1.2, in region $\mathbf{I}^+$ we denote $y(k) = x(k+1)$. Then the state equation can be expressed as $y(k+1) = Ay(k) - b\varepsilon_2$, with $y(0) = [y_1(0), y_2(0)]^{\mathrm{T}} = x(1) = [x_1(0) + x_2(0), x_2(0) - a(0)\varepsilon_2]^{\mathrm{T}}$, by which, we obtain

$$\begin{aligned} y(k) &= A^k \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} - \sum_{i=0}^{k-1} A^i b\varepsilon_2 \\ &= \begin{bmatrix} x_1(0) + (1+k)x_2(0) - \frac{k(k+2a(0)-1)}{2}\varepsilon_2 \\ x_2(0) - (a(0)+k)\varepsilon_2 \end{bmatrix}. \end{aligned} \tag{112}$$

By assumption, y will enter $\mathbf{II}$. Then there exists an integer $k_3 \geq 0$ such that

$$F_\gamma y(k_3) > \varepsilon_1 + \varepsilon_2, \ F_\gamma y(k_3+1) \leq \varepsilon_1 + \varepsilon_2. \tag{113}$$

Combining (112) and (113) gives

$$f_{15}(k_3) < \gamma^2 x_1(0) + 2\gamma x_2(0) \leq f_{16}(k_3), \tag{114}$$

with

$$\begin{cases} f_{15}(k_3) = & \frac{1}{2}\gamma^2\varepsilon_2 k_3^2 + c_1 k_3 + c_2, \\ f_{16}(k_3) = & \frac{1}{2}\gamma^2\varepsilon_2 k_3^2 + c_3 k_3 + c_4, \end{cases} \tag{115}$$

where $c_1 = (a(0) - \frac{1}{2})\gamma^2\varepsilon_2 - \gamma^2 x_2(0) + 2\gamma\varepsilon_2$, $c_2 = -\gamma^2 x_2(0) + (2a(0)\gamma + 1)\varepsilon_2 + \varepsilon_1$, $c_3 = (a(0) + \frac{1}{2})\gamma^2\varepsilon_2 - \gamma^2 x_2(0) + 2\gamma\varepsilon_2$ and $c_4 = -2\gamma^2 x_2(0) + \left(a(0)\gamma^2 + 2(a(0)+1)\gamma + 1\right)\varepsilon_2 + \varepsilon_1$. We conclude from (110) and (114) that, for any initial state $x(0) \in \mathbf{II}\backslash\Omega_1$ such that $x(1) \in \mathbf{I}^+$, if the state trajectory enters $\mathbf{II}$ along the path $\mathbf{II} \to \mathbf{I}^+ \to \mathbf{II}$, there must exist an integer $k_3 \geq 0$ such that $f_{13} < f_{16}(k_3)$, $f_{15}(k_3) < f_{14}$. Let

$$f_{17} = \varepsilon_1 + \varepsilon_2 + (1 - a(0))\gamma^2\varepsilon_2 - 2(1 - a(0))\gamma\varepsilon_2. \tag{116}$$

By (111) and (116) we have

$$f_{17} - f_{14} = (1 - a(0))(1 - \gamma)^2 \geq 0, \tag{117}$$

where we have noticed $a(0) \in [-1, 1]$. Hence, k_3 satisfies the following inequalities $f_{13} < f_{16}(k_3)$, $f_{15}(k_3) < f_{17}$, by which we have

$$\begin{cases} \left(\frac{\gamma^2\varepsilon_2}{2}k_3 + \frac{\gamma^2\varepsilon_2}{2}\right)(k_3 - (\zeta - 2a(0))) > 0, \\ \left(\frac{\gamma^2\varepsilon_2}{2}k_3 + \frac{\gamma^2\varepsilon_2}{2}\right)(k_3 - (\zeta + 2 - 2a(0))) < 0, \end{cases} \tag{118}$$

where ζ is given by (76). Solving (118) gives $\zeta - 2a(0) < k_3 < \zeta - 2a(0) + 2$. Let

$$k_3 = \zeta - 2a(0) + a_3, \tag{119}$$

where $a_3 \in (0, 2)$. Then from (112) and (119) we can compute

$$\begin{aligned} &|x_2(k_3+2)| - |x_2(0)| \\ =& |y_2(k_3+1)| - |x_2(0)| \\ =& |x_2(0) - a(0)\varepsilon_2 - (k_3+1)\varepsilon_2| - |x_2(0)| \\ =& \left| x_2(0) - \left(a(0) - a_3 - 1 + \frac{4}{\gamma}\right)\varepsilon_2 \right| - |x_2(0)| \\ \triangleq& |x_2(0) - \eta\varepsilon_2| - |x_2(0)|. \end{aligned} \tag{120}$$

Bearing in mind that $a(0) \in [-1, 1]$, $\gamma \in (0, 1/2]$ and $a_3 \in (0, 2)$, one can check easily that $\eta = a(0) - a_3 - 1 + \frac{4}{\gamma} > 4 > 0$. In view of (61), we have $x_2(0) > 0$. Next we consider (120). When $x_2(0) \geq \eta\varepsilon_2$, (120) gives $|x_2(k_3+2)| - |x_2(0)| = -\eta\varepsilon_2 < -4\varepsilon_2$. When $x_2(0) < \eta\varepsilon_2$, (120) gives $|x_2(k_3+2)| - |x_2(0)| = \frac{1}{\gamma^2}\left(\gamma^2\eta\varepsilon_2 - 2\gamma^2 x_2(0)\right) < \frac{1}{\gamma^2}\left(\gamma^2\eta\varepsilon_2 - 2\rho_1\right) < \frac{1}{\gamma^2}\left(-4\varepsilon_1 - 2\gamma^2\varepsilon_2\right)$, where we have noticed (61) and $a(0) \in [-1, 1]$, $\gamma \in (0, 1/2]$, $a_3 \in (0, 2)$. To sum up, we have

$$|x_2(k_3+2)| - |x_2(0)| < -\min\left\{4\varepsilon_2, \frac{1}{\gamma^2}\left(4\varepsilon_1 + 2\gamma^2\varepsilon_2\right)\right\}$$

$$\begin{aligned} &\leq -\min\left\{4\varepsilon_2, \frac{1}{\gamma^2}\left(4\varepsilon_1+\gamma^2\varepsilon_2\right)\right\} \\ &\triangleq -\kappa < 0. \end{aligned} \tag{121}$$

This concludes the proof of this step.

Proof of statement (2) of Step 2.2 As in the above analysis, when x skips from $\mathbf{II}$ to $\mathbf{I}^+$, we have (110)-(112). When y (here $y(k)=x(k+1)$) skips from $\mathbf{I}^+$ to $\mathbf{I}^-$, there exists an integer $k_4 \geq 0$ such that

$$F_\gamma y(k_4) > \varepsilon_1+\varepsilon_2,\ F_\gamma y(k_4+1) < -(\varepsilon_1+\varepsilon_2). \tag{122}$$

Combining (112) and (122) gives

$$f_{15}(k_4) < \gamma^2 x_1(0) + 2\gamma x_2(0) < f_{18}(k_4), \tag{123}$$

where $f_{18}(k_4) = f_{16}(k_4) - 2(\varepsilon_1+\varepsilon_2)$, and $f_{15}(k_4), f_{16}(k_4)$ are given by (115). It follows from (110) and (123) that k_4 should satisfy $f_{13} < f_{18}(k_4),\ f_{15}(k_4) < f_{14}$. Since $f_{18}(k_4) < f_{16}(k_4)$ and $f_{14} \leq f_{17}$, then k_4 also satisfies the following inequalities $f_{13} < f_{16}(k_4),\ f_{15}(k_4) < f_{17}$, by which we can deduce that

$$\zeta - 2a(0) < k_4 < \zeta - 2a(0) + 2, \tag{124}$$

where ζ is given by (76). Let

$$k_4 = \zeta - 2a(0) + a_4, \tag{125}$$

where $a_4 \in (0,2)$. Then it follows from (112) that

$$\begin{aligned} &|x_2(k_4+1)| - |x_2(0)| \\ =&|y_2(k_4)| - |x_2(0)| \\ =&|x_2(0) - a(0)\varepsilon_2 - k_4\varepsilon_2| - |x_2(0)| \\ =&\left|x_2(0) - \left(a(0) - a_4 + \frac{4}{\gamma}\right)\varepsilon_2\right| - |x_2(0)|. \end{aligned} \tag{126}$$

Since $a(0) \in [-1,1]$, $\gamma \in (0,1/2]$ and $a_4 \in (0,2)$, one can easily verify that $a(0) - a_4 + \frac{4}{\gamma} > 5 > 0$. Then, as in the proof of statement (1), we conclude from $a(0) \in [-1,1]$, $\gamma \in (0,1/2]$, $a_4 \in (0,2)$ and (61) that (126) can be reduced to $|x_2(k_4+1)| - |x_2(0)| < -\min\{5\varepsilon_2, \left(4\varepsilon_1+\gamma^2\varepsilon_2\right)/\gamma^2\} \leq -\kappa$, where κ is given by (121), which ends the proof.

Proof of statement (3) of Step 2.2 As in the analysis of Step 1.2, when x skips from $\mathbf{I}^+$ to $\mathbf{I}^-$, we have (39)-(42). When y (here $y(k)=x(k+1)$) skips from $\mathbf{I}^-$ to $\mathbf{II}$, there exists an integer $k_5 \geq 0$ such that

$$F_\gamma y(k_5) < -(\varepsilon_1+\varepsilon_2),\ F_\gamma y(k_5+1) \geq -(\varepsilon_1+\varepsilon_2). \tag{127}$$

Combining (42) and (127) gives

$$f_{19}(k_5) \leq \gamma^2 x_1(0) + 2\gamma x_2(0) < f_{20}(k_5), \tag{128}$$

where

$$\begin{cases} f_{19}(k_5) = & -\frac{1}{2}\gamma^2\varepsilon_2 k_5^2 + \left(\frac{1}{2}\gamma^2\varepsilon_2 - \gamma^2 x_2(0) - 2\gamma\varepsilon_2\right)k_5 \\ & -2\gamma^2 x_2(0) + \gamma^2\varepsilon_2 - (\varepsilon_1+\varepsilon_2), \\ f_{20}(k_5) = & -\frac{1}{2}\gamma^2\varepsilon_2 k_5^2 + \left(\frac{3}{2}\gamma^2\varepsilon_2 - \gamma^2 x_2(0) - 2\gamma\varepsilon_2\right)k_5 \\ & -\gamma^2 x_2(0) + 2\gamma\varepsilon_2 - (\varepsilon_1+\varepsilon_2). \end{cases} \tag{129}$$

It follows from (39) and (128) that k_5 should satisfy $f_1 < f_{20}(k_5),\ f_{19}(k_5) < f_2$. Since $f_1 > f_9$ (see (50)), k_5 also satisfies the following inequalities $f_9 < f_{20}(k_5)$, $f_{19}(k_5) < f_2$, by which we get

$$k_5 = \frac{-2\left(\gamma^2 x_2(0) + 2\gamma\varepsilon_2\right)}{\gamma^2\varepsilon_2} + 2 + a_5, \tag{130}$$

where $a_5 \in (0,2)$. By virtue of (42) and (130), we have

$$\begin{aligned} &|x_2(k_5+2)| - |x_2(0)| \\ =&|y_2(k_5+1)| - ||x_2(0)|| \\ =&|x_2(0) - \varepsilon_2 + (k_5+1)\varepsilon_2| - |y_2(k_4)| \\ =&\left|x_2(0) - \left(-\frac{4}{\gamma} + 2 + a_5\right)\varepsilon_2\right| - |x_2(0)| \\ \triangleq&|x_2(0) - \xi\varepsilon_2| - |x_2(0)|. \end{aligned} \tag{131}$$

Notice that, by using $\gamma \in (0,1/2]$ and $a_4 \in (0,2)$, one can check easily that $\xi = -\frac{4}{\gamma} + 2 + a_5 < -4 < 0$. In view of (41), we have $x_2(0) > 0$. Next we consider (131). When $x_2(0) \geq \xi\varepsilon_2$, (131) can be continued as $|x_2(k_5+2)| - |x_2(0)| = (2\gamma^2 x_2(0) - \gamma^2\xi\varepsilon_2)/\gamma^2 < (-4\varepsilon_1 + (-4+8\gamma)\varepsilon_2 - (2+a_5)\gamma^2\varepsilon_2)/\gamma^2 < (-4\varepsilon_1 - 2\gamma^2\varepsilon_2)/\gamma^2$. When $x_2(0) < \xi\varepsilon_2$, (131) gives $|x_2(k_5+2)| - |x_2(0)| = \xi\varepsilon_2 < -4\varepsilon_2$. To sum up, we have $|x_2(k_5+2)| - |x_2(0)| < -\min\{4\varepsilon_2, \frac{1}{\gamma^2}(4\varepsilon_1 + 2\gamma^2\varepsilon_2)\} \leq -\kappa$, where κ is given by (121), which ends the proof of this step.

Proof of statement (4) of Step 2.2 As in the analysis of Step 1.2, when x skips from $\mathbf{I}^+$ to $\mathbf{I}^-$ and then returns to $\mathbf{I}^+$, we have (38)-(45) with k_1 satisfying (52). By virtue of (42) and (52), we get

$$\begin{aligned} &|x_2(k_1+1)| - |x_2(0)| \\ =&|y_2(k_1)| - |x_2(0)| \\ =&|x_2(0) - \varepsilon_2 + k_1\varepsilon_2| - |x_2(0)| \\ =&\left|x_2(0) - \left(-\frac{4}{\gamma} + 1 + a_1\right)\varepsilon_2\right| - |x_2(0)|. \end{aligned} \tag{132}$$

Since $\gamma \in (0,1/2]$ and $a_1 \in (0,2)$, one can check easily that $-\frac{4}{\gamma} + 1 + a_1 < -5 < 0$. As in the proof of statement (3), we conclude from $\gamma \in (0,1/2]$, $a_1 \in (0,2)$ and (41) that (132) can be continued as $|x(k_1+1)| - |x_2(0)| < -\min\{5\varepsilon_2, \left(4\varepsilon_1+\gamma^2\varepsilon_2\right)/\gamma^2\} \leq -\kappa$, where κ is given by (121). This completes the proof.

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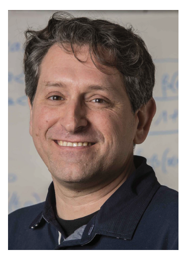

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