ISS Inequalities for Vector Versions of Halanay's Inequality and of the Trajectory-Based Approach

Frederic Mazenc, Michael Malisoff

ISS Inequalities for Vector Versions of Halanay's Inequality and of the Trajectory-Based Approach

Frédéric Mazenc Michael Malisoff

Abstract— **Halanay's inequality is a powerful and widely used tool to prove asymptotic convergence properties of functions that arise in the study of systems with delays or continuous-discrete features. In its standard form, it applies to scalar valued functions that satisfy decay conditions, with overshoots depending on suprema of the functions over suitable intervals. Then it provides an exponential decay estimate on the scalar functions. Here, we provide vector versions of Halanay's inequality, and of the so-called trajectory based approach, both yielding input-to-state stability (or ISS) inequalities. Our proofs of the inequalities use the theory of positive systems. We apply our results to prove ISS for interval observers and other cases.**

I. Introduction

Halanay's inequality result, which was introduced in [7], is a well-known powerful tool for the stability analysis of systems with delays [4, Chapter 4]. It has been used in many papers. In the last three decades, it has been revisited in some works, notably in the context of time-varying inequalities; see for instance [2], [8], [9], [14], [19], [21], and [23]. In its basic form, it assumes that w is a nonnegative valued C^1 real valued scalar function that admits constants $T > 0$, $c > 0$ (called a decay rate), and $d \in (0, c)$ (called a gain) such that $\dot{w}(t) \leq -cw(t) + d\sup_{t-T\leq\ell\leq t} w(\ell)$ holds for all $t \geq 0$. Then the usual conclusion is that w satisfies an exponential decay condition. More general versions with time dependent functions c and d allow values t where $d(t) > c(t)$.

The recent 'trajectory based approach' is another stability analysis technique. It is efficient for analyzing complex interconnected systems, notably systems with delay and continuous-discrete features. It was developed in works such as [13], [16], and [17]. The trajectory based approach applies to continuous nonnegative valued scalar valued functions w that are assumed to admit constants $\rho \in (0, 1)$ (called the contractivity constant) and $T > 0$ such that $w(t) \leq \rho\sup_{\ell\in[t-T,t]} w(\ell)$ for all $t \geq T$. Then the usual conclusion is that w exponentially converges to zero as $t \to +\infty$.

On the other hand, it is well-known that the notion of input-to-state-stability (or ISS), introduced by E. Sontag, has been exceptionally fruitful for control design and stability analysis of dynamical systems; see, e.g., [22]. The ISS property was developed in many contributions and has been used to solve many theoretical as well as applied problems.

Key Words: Stability, delay, positive systems, Halanay's inequality.

F. Mazenc is with Inria EPI DISCO, L2S-CNRS-CentraleSupélec, 3 rue Joliot Curie, 91192, Gif-sur-Yvette, France (e-mail: frederic.mazenc@l2s.centralesupelec.fr).

M. Malisoff is with Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803-4918, USA (e-mail: malisoff@lsu.edu).
Supported by NSF Grants 1711299 and 2009659.

ISS inequalities are especially useful when one analyzes the stability of interconnected systems [17] and when one wants to estimate the impact of disturbances.

Only a few papers on Halanay's inequality and the trajectory based approach propose results of ISS type. In [16] and [19], ISS inequalities are established, but these results do not directly extend to vector inequalities. In [15], we provided necessary and sufficient stability conditions for functions satisfying vector inequalities of Halanay's type. However, the proofs in [15] do not lead to ISS inequalities when additive disturbances are present. In [1], a vector version of the trajectory-based approach is developed, but no ISS inequality is given under additive disturbances. This is a shortcoming of these theories because vector inequalities arise in many circumstances, so there is a strong motivation to cover them.

These remarks motivate the present paper. In a first part, we determine an ISS inequality for functions satisfying a vector version of Halanay's inequality. The first part uses methods from the second part, where we consider vector inequalities enabling us to apply the trajectory based approach, and where we determine ISS inequalities when additive disturbances are present. These results complement [1], [15] and [16, Lemma 1]. Our proofs of the main results use the theory of positive systems, to obtain explicit constructions of the comparison functions in the ISS estimates.

We use standard notation, which is simplified when no confusion would arise, and where the dimensions of our Euclidean spaces are arbitrary, and the entries of our matrices are real values. Let $||\cdot||$ be the usual Euclidean norm of matrices and vectors, 0 is the zero matrix, and I is the identity matrix. Inequalities between matrices are meant entry-wise, so $M_1 \leq M_2$ (resp. $M_1 < M_2$) for matrices M_1 and M_2 of the same size means each entry of M_1 is at most (resp. strictly less than) the corresponding entry of M_2. Let $\mu(M)$ denote the spectral radius of a square matrix M. A square matrix M is called Schur stable provided $\mu(M) < 1$. A square matrix is called Metzler provided all of its off-diagonal entries are nonnegative. A function $f : \mathbb{R} \to \mathbb{R}^n$ is called nondecreasing provided each of its components f_i for $i = 1, \ldots, n$ are nondecreasing. We write diag$\{s_1, \ldots, s_n\}$ to denote an $n \times n$ diagonal matrix whose arguments give its diagonal elements starting from its upper left entry s_1.

In the next section, we state our main theorems that produce ISS results under vector Halanany's and vector trajectory-based inequalities. We defer their proofs to Section IV. Their proofs are based on key lemmas from Section III. We illustrate the value of our new approaches in Section V. Possible extensions are suggested in Section VI.

II. Statement of Main Results

A. Vector Halanay's Inequality with ISS

Given real constants $\tau > 0$ and $h > 0$, a diagonal matrix $D \in \mathbb{R}^{n\times n}$ whose diagonal entries are all positive, a matrix $P \in \mathbb{R}^{n\times n}$ such that $P > 0$, and $\delta : [-\tau - h, +\infty) \to [0, +\infty)^n$ whose components δ_i are piecewise continuous, let $W : [-\tau, +\infty) \to [0, +\infty)^n$ be a C^1 function such that

$$\begin{aligned}\dot{W}(t) \;\leq\; & -DW(t) + \delta(t) \\ & + P\left[\sup_{\ell\in[t-\tau,t]} w_1(\ell), \ldots, \sup_{\ell\in[t-\tau,t]} w_n(\ell)\right]^\top\end{aligned} \tag{1}$$

for all $t \geq 0$, where w_i is the ith component of W for all i (but see Section V-B for an illustration of how we can relax the requirement that the matrix D is diagonal). In (1) and other places in this work, we use the sup notation instead of the $\mathcal{S}(w_t)$ notation from [15] to indicate the interval over which the sup is computed (which will sometimes be different intervals later in this paper). Also, for our fixed choice of h, we assume without loss of generality that the δ_i's are nondecreasing and such that $\delta(t) = \delta(0)$ for all $t \in [-\tau - h, 0]$. We consider the case where the matrix $-D + P$ is Hurwitz. This condition is imposed because it is necessary for the GAS of (1) when $\delta = 0$ and when (1) holds with equality; see [15, Theorem 1]. When $-D + P$ is Hurwitz, the main result of [15] ensures that W exponentially converges to the origin when δ is not present. We also use

$$M = e^{-Dh} + \int_{-h}^{0} e^{D\ell}\mathrm{d}\ell P. \tag{2}$$

Since the matrix M satisfies $M > 0$, the Perron-Frobenius theorem [10] provides a vector $U \in [1, +\infty)^n$ such that

$$MU = cU \tag{3}$$

where $c = \mu(M)$. In the next section, we will prove that M is Schur stable. Using this fact, it follows that $c \in (0, 1)$. Let us introduce the function $\chi : [-\tau, +\infty) \to \mathbb{R}^n$ defined by

$$\begin{aligned}&\chi(t) = \\ &-(-I+M)^{-1}D^{-1}\left(I-e^{-Dh}\right)\begin{bmatrix}\sup\limits_{\ell\in[t-h,t]} \delta_1(\ell) \\ \vdots \\ \sup\limits_{\ell\in[t-h,t]} \delta_n(\ell)\end{bmatrix}\end{aligned} \tag{4}$$

which is well-defined because the entries of D are positive and M is Schur stable; see below. Our first main result is:

Theorem 1: With the preceding definitions and notation,

$$W(t) \leq e^{-\frac{\ln(c)}{\tau+h}h} \sum_{j=1}^{n} \sup_{\ell\in[-\tau,h]} w_j(\ell) e^{\frac{\ln(c)}{\tau+h}t} U + \chi(t) \tag{5}$$

is satisfied for all $t \geq h$. □

For a proof of the preceding theorem, see Section IV-A.

Remark 1: Note that (5) provides an $N \in \mathbb{R}^{n\times n}$ such that

$$\begin{aligned}W(t) \;\;\leq\;\; & e^{-\frac{\ln(c)}{\tau+h}h} \sum_{j=1}^{n} \sup_{\ell\in[-\tau,h]} w_j(\ell) e^{\frac{\ln(c)}{\tau+h}t} U \\ & + N\left[\sup\nolimits_{\ell\in[0,t]} \delta_1(\ell), \ldots, \sup\nolimits_{\ell\in[0,t]}\right]^\top\end{aligned} \tag{6}$$

for all $t \geq h$. From this inequality, we can deduce that

$$\begin{aligned}||W(t)|| \;\;\leq\;\; & e^{-\frac{\ln(c)}{\tau+h}h} n||U|| e^{\frac{\ln(c)}{\tau+h}t} \sup_{\ell\in[-\tau,h]} ||W(\ell)|| \\ & + ||N|| n \sup_{\ell\in[0,t]} ||\delta(\ell)||\end{aligned} \tag{7}$$

holds for all $t \geq 0$, which is the desired ISS property. □

B. Vector Trajectory-Based ISS Result

Our proof of Theorem 1 in Section IV-A uses the theorem of this section, which is of independent interest. It provides a vector trajectory-based ISS estimate for a continuous function $\omega : [-T, +\infty) \to [0, +\infty)^n$ that we assume satisfies

$$\omega(t) \leq S \begin{bmatrix}\sup\limits_{\ell\in[t-T,t]} \omega_1(\ell) \\ \vdots \\ \sup\limits_{\ell\in[t-T,t]} \omega_n(\ell)\end{bmatrix} + \Delta(t), \tag{8}$$

for all $t \geq 0$, where $T > 0$ is a given constant, each component Δ_i for $i = 1, \ldots, n$ of Δ is piecewise continuous, and the matrix $S > 0$ is Schur stable. Without loss of generality, we assume that each function $\Delta_i(t)$ is nondecreasing. Moreover, we let Δ be defined over $[-T, +\infty)$ by $\Delta(t) = \Delta(0)$ for all $t \in [-T, 0]$.

By again using the Perron-Frobenius Theorem [10], we can fix a vector $V \in \mathbb{R}^n$ and a constant $q \in (0, 1)$ such that

$$SV = qV, \tag{9}$$

with each entry of V in $[1, +\infty)$, and we will see below that $-I + S$ is invertible. In terms of the vector valued function

$$\rho(t) = -(-I + S)^{-1}\Delta(t), \tag{10}$$

our vector valued ISS trajectory-based result is:

Theorem 2: With the preceding definitions and notation,

$$\omega(t) \leq \sum_{j=1}^{n} \sup_{\ell\in[-T,0]} \omega_j(\ell) e^{\frac{\ln(q)}{T}t} V + \rho(t) \tag{11}$$

holds for all $t \geq 0$. □

For a proof of the preceding theorem, see Section IV.

Remark 2: The inequality (11) implies that

$$\begin{aligned}||\omega(t)|| \;\;\leq\;\; & n||V|| \sup_{\ell\in[-T,0]} ||\omega(\ell)|| e^{\frac{\ln(q)}{T}t} \\ & + ||(-I + S)^{-1}|| ||\Delta(t)||\end{aligned} \tag{12}$$

which gives

$$\begin{aligned}||\omega(t)|| \;\;\leq\;\; & n||V|| \sup_{\ell\in[-T,0]} ||\omega(\ell)|| e^{\frac{\ln(q)}{T}t} \\ & + ||(-I + S)^{-1}|| \sup_{\ell\in[0,t]} ||\Delta(\ell)||,\end{aligned} \tag{13}$$

which is our desired ISS estimate. □

In the next section, we provide key lemmas that we will use to prove our theorems.

III. Key Lemmas

We first recall the following well known equivalences, e.g, from [20, Proposition 1]:

Lemma 1: Let $\mathcal{M} \in \mathbb{R}^{n\times n}$ be a Metzler matrix. Then the following three statements are equivalent: (i) $\mathcal{M}$ is Hurwitz, (ii) $\mathcal{M}V < 0$ for some vector $V > 0$, and (iii) $\mathcal{M}$ is invertible and $\mathcal{M}^{-1} \leq 0$. □

We also use:

Lemma 2: Let $P \in \mathbb{R}^{n\times n}$ satisfy $P > 0$, and $D \in \mathbb{R}^{n\times n}$ be a diagonal matrix whose diagonal entries are all positive. Let $h > 0$ be a given positive real value. If the matrix

$$-D + P \tag{14}$$

is Hurwitz, then the matrix (2) is Schur stable. □

Proof: Assume that $-D+P$ is Hurwitz. Let $d > 0$ be such that $dI - D + P > 0$. The Perron-Frobenius theorem [10] ensures that there are $V > 0$ and $k > 0$ such that

$$(dI - D + P)V = kV. \tag{15}$$

It follows that with the choice $c = d - k$, we have

$$(-D + P)V = -cV. \tag{16}$$

Since $-D + P$ is Hurwitz, necessarily $c > 0$. Since the diagonal entries of D are positive, one can prove easily that the matrix M in (2) can be written as

$$M = I + D^{-1}\left(I - e^{-Dh}\right)(-D + P). \tag{17}$$

This equality and (16) give

$$MV = \left[I - cD^{-1}\left(I - e^{-Dh}\right)\right]V. \tag{18}$$

Since $-cD^{-1}\left(I - e^{-Dh}\right)$ is diagonal with negative diagonal entries, there is a constant $p \in (0,1)$ such that $MV \leq pV$. Since the matrix M is nonnegative, the lemma now follows from [6, Chapter 2, Theorem 2.22]. ■

The next lemma follows from (9), because $q \in (0,1)$, and is included for reference because it is key for our analysis:

Lemma 3: Let $T > 0$ be a given real value, and the matrix S, the vector V, and the constant $q \in (0,1)$ be such that (9) holds. Then

$$e^{\frac{\ln(q)}{T}t}V = e^{\frac{\ln(q)}{T}(t-T)}SV = S\sup_{m\in[t-T,t]}\left(e^{\frac{\ln(q)}{T}m}\right)V \tag{19}$$

holds for all $t \geq 0$. □

We also use:

Lemma 4: Let $S > 0$ be a Schur stable matrix in $\mathbb{R}^{n\times n}$ and $T > 0$ be a constant. Let $\Delta : [-T,+\infty) \to [0,+\infty)^n$ be such that each of its components Δ_i for $i = 1,\ldots,n$ is piecewise continuous. Then the function $\rho : [-T,+\infty) \to \mathbb{R}^n$ given by (10) is nonnegative valued and nondecreasing, and it satisfies

$$\rho(t) = S\rho(t) + \Delta(t) \tag{20}$$

for all $t \geq -T$. □

Proof: Since $S > 0$, the matrix $-I+S$ is Metzler. Also, by the Perron-Frobenius theorem, there are $V > 0$ and $q \in (0,1)$ such that $SV \leq qV$ because S is positive and Schur stable. It follows that $(-I+S)V \leq (q-1)V$. Since $-I+S$ is Metzler, this inequality implies that $-I+S$ is Hurwitz, by Lemma 1. Then, according to Lemma 1, $(-I+S)^{-1} \leq 0$. Consequently, the function ρ is well-defined nonnegative and nondecreasing. Then, one can prove easily that (20) holds. ■

Our final lemma is as follows:

Lemma 5: Consider a Schur stable matrix $S > 0$, a constant $T > 0$, and a continuous function $\Omega : [-T,+\infty) \to [0,+\infty)^n$ such that

$$\Omega(t) \leq S\begin{bmatrix}\sup\limits_{\ell\in[t-T,t]}\Omega_1(\ell)\\ \vdots \\ \sup\limits_{\ell\in[t-T,t]}\Omega_n(\ell)\end{bmatrix}, \tag{21}$$

for all $t \geq 0$. Let $q \in (0,1)$ and $V \in [1,+\infty)^n$ be such that $SV = qV$. Choose the functions

$$\beta_0(t) = \kappa_0 e^{\frac{\ln(q)}{T}t}V \tag{22}$$

and

$$\kappa_0 = \sum_{j=1}^{n}\sup_{\ell\in[-T,0]}\Omega_j(\ell). \tag{23}$$

Then

$$\Omega(t) \leq \beta_0(t) \tag{24}$$

holds for all $t \geq -T$. □

Proof: Let us introduce the functions

$$\beta_\epsilon(t) = \kappa_\epsilon e^{\frac{\ln(q)}{T}t}V, \tag{25}$$

where

$$\kappa_\epsilon = \kappa_0 + \epsilon, \tag{26}$$

and where $\epsilon > 0$ is any positive constant. Since $\ln(q) < 0$,

$$\beta_\epsilon(m) \geq (\kappa_0 + \epsilon)V \tag{27}$$

for all $m \in [-T,0]$. We deduce that the inequality

$$\Omega(m) \leq k_0 V < \beta_\epsilon(m) \tag{28}$$

holds for all $m \in [-T,0]$, since $V \in [1,\infty)^n$. Bearing in mind that Ω is continuous, we proceed by contradiction: we assume that there is a $t_c > 0$ such that

$$\Omega(t) < \beta_\epsilon(t) \tag{29}$$

for all $t \in [0,t_c)$ and $\Omega(t_c) \not< \beta_\epsilon(t_c)$. Then the continuity of Ω ensures that there is an $i \in \{1,...,n\}$ such that

$$\Omega_i(t_c) = \beta_{\epsilon,i}(t_c). \tag{30}$$

From Lemma 3, we deduce that

$$\beta_\epsilon(t) = S\begin{bmatrix}\sup\limits_{\ell\in[t-T,t]}\beta_{\epsilon,1}(\ell)\\ \vdots \\ \sup\limits_{\ell\in[t-T,t]}\beta_{\epsilon,n}(\ell)\end{bmatrix} \tag{31}$$

for all $t \geq 0$. Then, with $S = [s_{ij}]$, condition (21) and (30)-(31) give

$$\begin{aligned}&\sum_{j=1}^{n}s_{i,j}\sup_{\ell\in[t_c-T,t_c]}\beta_{\epsilon,j}(\ell) = \Omega_i(t_c)\\ &\leq \sum_{j=1}^{n}s_{i,j}\sup_{\ell\in[t_c-T,t_c]}\Omega_j(\ell).\end{aligned} \tag{32}$$

The last inequality is equivalent to

$$\sum_{j=1}^{n} s_{i,j}\left(\sup_{\ell\in[t_c-T,t_c]}\beta_{\epsilon,j}(\ell)-\sup_{\ell\in[t_c-T,t_c]}\Omega_j(\ell)\right)\leq 0. \quad (33)$$

On the other hand, the definition of t_c implies that for all $j\in\{1,...,n\}$, we have

$$\sup_{\ell\in[t_c-T,t_c]}\beta_{\epsilon,j}(\ell)-\sup_{\ell\in[t_c-T,t_c]}\Omega_j(\ell)\geq 0. \quad (34)$$

Since $S>0$, we deduce from (33)-(34) that

$$\sup_{\ell\in[t_c-T,t_c]}\beta_{\epsilon,j}(\ell)=\sup_{\ell\in[t_c-T,t_c]}\Omega_j(\ell) \quad (35)$$

for $j=1,\dots,n$. Since each function $\beta_{\epsilon,j}$ is nonincreasing, these equalities are equivalent to

$$\beta_{\epsilon,j}(t_c-T)=\sup_{\ell\in[t_c-T,t_c]}\Omega_j(\ell) \quad (36)$$

for all $j\in\{1,...,n\}$. Since Ω is continuous, for any $j\in\{1,...,n\}$, there is $t_{j,\star}\in[t_c-T,t_c]$ such that

$$\sup_{\ell\in[t_c-T,t_c]}\Omega_j(\ell)=\Omega_j(t_{j,\star}). \quad (37)$$

It follows that

$$\beta_{\epsilon,j}(t_c-T)=\Omega_j(t_{j,\star}). \quad (38)$$

According to the definition of t_c, the inequality $\beta_{\epsilon,j}(t_c-T)>\Omega_j(t_c-T)$ is satisfied. Therefore (38) implies that $t_{j,\star}\in(t_c-T,t_c]$. We deduce from (38) that

$$\beta_{\epsilon,j}(t_{j,\star})<\beta_{\epsilon,j}(t_c-T)=\Omega_j(t_{j,\star}). \quad (39)$$

However according to the definition of t_c and the fact that $t_{j,\star}\in[t_c-T,t_c]$, the inequality $\beta_{\epsilon,j}(t_{j,\star})\geq\Omega_j(t_{j,\star})$ is satisfied. This yields a contradiction. We conclude that $\Omega(t)<\beta_\epsilon(t)$ for all $t\geq -T$. By continuity of β_ϵ as a function of ϵ, we deduce that (24) holds for all $t\geq -T$. ■

Remark 3: Notice that [1, Lemma 1] ensures that under the assumptions of the preceding lemma, Ω exponentially goes to the origin. The preceding lemma complements [1] by determining an upper bounding function which exponentially converges to the origin. □

IV. PROOFS OF THEOREMS 1-2

A. Proof of Theorem 1

A variation of parameters argument and the nonnegtivity of e^{-Dt} give

$$\begin{aligned} W(t) \;\leq\; & e^{-Dh}W(t-h) \\ & +\int_{t-h}^{t} e^{-D(t-\ell)}\mathrm{d}\ell P \begin{bmatrix} \sup_{\ell\in[t-\tau-h,t]} w_1(\ell) \\ \vdots \\ \sup_{\ell\in[t-\tau-h,t]} w_n(\ell)\end{bmatrix} \\ & +\int_{t-h}^{t} e^{-D(t-\ell)}\mathrm{d}\ell \begin{bmatrix} \sup_{\ell\in[t-h,t]} \delta_1(\ell) \\ \vdots \\ \sup_{\ell\in[t-h,t]} \delta_n(\ell)\end{bmatrix}, \end{aligned} \quad (40)$$

for all $t\geq h$. It follows from our choice of M in (2) that

$$\begin{aligned} W(t) \;\leq\; & M \begin{bmatrix} \sup_{\ell\in[t-\tau-h,t]} w_1(\ell) \\ \vdots \\ \sup_{\ell\in[t-\tau-h,t]} w_n(\ell)\end{bmatrix} \\ & +D^{-1}\left(I-e^{-Dh}\right) \begin{bmatrix} \sup_{\ell\in[t-h,t]} \delta_1(\ell) \\ \vdots \\ \sup_{\ell\in[t-h,t]} \delta_n(\ell)\end{bmatrix} \end{aligned} \quad (41)$$

for all $t\geq h$. Let $W_\triangle(s)=W(s+h)$. Then (41) ensures that for all $s\geq 0$,

$$\begin{aligned} W_\triangle(s) \;\leq\; & M \begin{bmatrix} \sup_{\ell\in[s-\tau-h,s]} w_{\triangle,1}(\ell) \\ \vdots \\ \sup_{\ell\in[s-\tau-h,s]} w_{\triangle,n}(\ell)\end{bmatrix} \\ & +D^{-1}\left(I-e^{-Dh}\right) \begin{bmatrix} \sup_{\ell\in[s,s+h]} \delta_1(\ell) \\ \vdots \\ \sup_{\ell\in[s,s+h]} \delta_n(\ell)\end{bmatrix}. \end{aligned} \quad (42)$$

By Lemma 2, M is Schur stable. Moreover, $M>0$. Then we can apply Theorem 2 to $W_\triangle$ and obtain:

$$W_\triangle(s)\leq\sum_{j=1}^{n}\sup_{\ell\in[-\tau-h,0]}\omega_{\triangle,j}(\ell)e^{\frac{\ln(c)}{\tau+h}s}U+\chi(s+h). \quad (43)$$

Thus

$$W(s+h)\leq\sum_{j=1}^{n}\sup_{\ell\in[-\tau,h]}\omega_j(\ell)e^{\frac{\ln(c)}{\tau+h}s}U+\chi(s+h). \quad (44)$$

Taking $t=s+h$,

$$W(t)\leq\sum_{j=1}^{n}\sup_{\ell\in[-\tau,h]}\omega_j(\ell)e^{\frac{\ln(c)}{\tau+h}(t-h)}U+\chi(t) \quad (45)$$

which is (5).

B. Proof of Theorem 2

We use the function $\tilde{\omega}:[-T,+\infty)\to[0,+\infty)^n$ that is defined by

$$\tilde{\omega}(t)=\begin{bmatrix}\max\{0,\omega_1(t)-\rho_1(t)\} \\ \vdots \\ \max\{0,\omega_n(t)-\rho_n(t)\}\end{bmatrix}. \quad (46)$$

Let $\Psi:\mathbb{R}\to\mathbb{R}$ be the function defined by $\Psi(m)=1$ if $m>0$ and $\Psi(m)=0$ if $m\leq 0$. Then

$$\tilde{\omega}(t)=\Lambda(t)\begin{bmatrix}\omega_1(t)-\rho_1(t) \\ \vdots \\ \omega_n(t)-\rho_n(t)\end{bmatrix} \quad (47)$$

with

$$\Lambda(t)=\mathrm{diag}\big\{\Psi(\omega_1(t)-\rho_1(t)),....,\Psi(\omega_n(t)-\rho_n(t))\big\}. \quad (48)$$

On the other hand, (8) and Lemma 4 imply that

$$\begin{aligned}\omega(t)-\rho(t) &\leq S\begin{bmatrix}\sup_{\ell\in[t-T,t]}\omega_1(\ell)\\ \vdots\\ \sup_{\ell\in[t-T,t]}\omega_n(\ell)\end{bmatrix}+\Delta(t)\\ &\quad -S\rho(t)-\Delta(t) \qquad (49)\\ &= S\begin{bmatrix}\sup_{\ell\in[t-T,t]}\omega_1(\ell)\\ \vdots\\ \sup_{\ell\in[t-T,t]}\omega_n(\ell)\end{bmatrix}-S\rho(t)\end{aligned}$$

for all $t \geq 0$. Since $S \geq 0$, it follows that

$$\begin{aligned}&\omega(t)-\rho(t)\\ &\leq S\begin{bmatrix}\sup_{\ell\in[t-T,t]}(\omega_1(\ell)-\rho_1(\ell)+\rho_1(\ell))-\rho_1(t)\\ \vdots\\ \sup_{\ell\in[t-T,t]}(\omega_n(\ell)-\rho_n(\ell)+\rho_n(\ell))-\rho_n(t)\end{bmatrix}\\ &\leq S\begin{bmatrix}\sup_{\ell\in[t-T,t]}(\omega_1(\ell)-\rho_1(\ell))+\sup_{\ell\in[t-T,t]}\rho_1(\ell)-\rho_1(t)\\ \vdots\\ \sup_{\ell\in[t-T,t]}(\omega_n(\ell)-\rho_n(\ell))+\sup_{\ell\in[t-T,t]}\rho_n(\ell)-\rho_n(t)\end{bmatrix}\end{aligned}$$

for all $t \geq 0$. Since Lemma 4 implies that ρ is nondecreasing, we deduce that

$$\omega(t)-\rho(t) \leq S\begin{bmatrix}\sup_{\ell\in[t-T,t]}(\omega_1(\ell)-\rho_1(\ell))\\ \vdots\\ \sup_{\ell\in[t-T,t]}(\omega_n(\ell)-\rho_n(\ell))\end{bmatrix}. \qquad (50)$$

Since Λ is nonnegative, bearing in mind (47), we can left multiply (50) through by $\Lambda(t)$ and obtain

$$\tilde{\omega}(t) \leq \Lambda(t)S\begin{bmatrix}\sup_{\ell\in[t-T,t]}(\omega_1(\ell)-\rho_1(\ell))\\ \vdots\\ \sup_{\ell\in[t-T,t]}(\omega_n(\ell)-\rho_n(\ell))\end{bmatrix} \qquad (51)$$

for all $t \geq 0$. As an immediate consequence of (46), we get

$$\tilde{\omega}(t) \leq \Lambda(t)S\tilde{\omega}^\sharp(t) \qquad (52)$$

for all $t \geq 0$, where the $\mathbb{R}^n$-valued function $\tilde{\omega}^\sharp$ is defined by $\tilde{\omega}^\sharp(t) = [\sup_{\ell\in[t-T,t]}\tilde{\omega}_1(\ell),\ldots,\sup_{\ell\in[t-T,t]}\tilde{\omega}_n(\ell)]^\top$. Since $\Lambda(t) \leq I$ holds for all $t \geq -T$, we deduce that

$$\tilde{\omega}(t) \leq S\tilde{\omega}^\sharp(t). \qquad (53)$$

Since S is Schur stable and positive and since $\tilde{\omega}(t) \geq 0$ for all $t \geq -T$, Lemma 5 implies that with the choice

$$\nu = \sum_{j=1}^n \sup_{\ell\in[-T,0]} \max\{0, \omega_j(\ell)-\rho_j(\ell)\}, \qquad (54)$$

we have

$$\tilde{\omega}(t) \leq \nu e^{\frac{\ln(q)}{T}t}V \qquad (55)$$

for all $t \geq -T$. This inequality can be rewritten as

$$\begin{bmatrix}\max\{0,\omega_1(t)-\rho_1(t)\}\\ \vdots\\ \max\{0,\omega_n(t)-\rho_n(t)\}\end{bmatrix} \leq \nu e^{\frac{\ln(q)}{T}t}V. \qquad (56)$$

As an immediate consequence,

$$\begin{bmatrix}\omega_1(t)-\rho_1(t)\\ \vdots\\ \omega_n(t)-\rho_n(t)\end{bmatrix} \leq \nu e^{\frac{\ln(q)}{T}t}V. \qquad (57)$$

This inequality can be rewritten as:

$$\begin{aligned}\omega(t) &\leq \rho(t)\\ &\quad + \sum_{j=1}^n \sup_{\ell\in[-T,0]} \max\{0,\omega_j(\ell)-\rho_j(\ell)\}e^{\frac{\ln(q)}{T}t}V.\end{aligned}$$

Next, note that $\max\{0,\omega_j(\ell)-\rho_j(\ell)\} \leq \max\{0,\omega_j(\ell)\} = \omega_j(\ell)$ because both ρ and ω are nonnegative, by Lemma 4.

We deduce that

$$\omega(t) \leq \rho(t) + \sum_{j=1}^n \sup_{\ell\in[-T,0]} \omega_j(\ell)e^{\frac{\ln(q)}{T}t}V. \qquad (58)$$

This concludes the proof of Theorem 2.

V. Illustrations

We next apply the new approach from the preceding sections to the two main examples from [15]. Unlike [15], where the vector Halanay's inequalities did not lend themselves to proving ISS estimates, here we establish ISS results and therefore improve on previously reported results.

A. Dynamics from [3]

We revisit the first illustration of the contribution [15] by addressing the case where an additive disturbance is present. We consider the n-dimensional dynamics

$$\left\{\begin{aligned}\dot{z}_1(t) &= -r_1z_1(t)+z_2(t-\tau_1)+\delta_1(t)\\ \dot{z}_2(t) &= -r_2z_2(t)+z_3(t-\tau_2)+\delta_2(t)\\ &\vdots\\ \dot{z}_{n-1}(t) &= -r_{n-1}z_{n-1}(t)+z_n(t-\tau_{n-1})\\ &\quad +\delta_{n-1}(t)\\ \dot{z}_n(t) &= -r_nz_n(t)+\chi(t)+\delta_n(t)\end{aligned}\right. \qquad (59)$$

with the input χ, which occurs in [3, Lemma 2] (in the context of stabilizing linear strict-feedback systems having delayed integrators), where the constants r_i and τ_i are positive for $i = 1, 2, \ldots, n-1$. We have added nonnegative and nondecreasing functions δ_i, which may represent disturbances, and which are assumed to be piecewise continuous. As in [15], we assume that χ takes the form

$$\chi(t) = g_1z_1(t-\tau_{n1}) + \ldots + g_nz_n(t-\tau_{nn}) \qquad (60)$$

for constants $\tau_{ij} \geq 0$ and $g_i > 0$ for $i = 1, 2, \ldots, n$. For such a function χ, the system (59) is nonnegative (by [5, Proposition 3.1]). Hence, we focus our attention on the nonnegative solutions of (59), but an extension of the result below to the general case can easily be obtained.

Choosing any $\tau > \max\{\tau_1, \dots, \tau_{n-1}, \tau_{n1}, \dots, \tau_{nn}\}$, it follows that all C^1 solutions $Z : [-\tau, +\infty) \to [0, +\infty)^n$ of (59) satisfy (1) with $D = \mathrm{diag}\{r_1, \dots, r_n\}$ and

$$P = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ g_1 & g_2 & g_3 & \dots & g_{n-1} & g_n \end{bmatrix} \quad (61)$$

with $w_i = z_i$ for all i. Then we can find conditions on the r_i's and g_i's to ensure that the corresponding matrix $-D+P$ is Hurwitz. Then Theorem 1 can be applied.

For the sake of illustration, let us consider the special case where $n = 2$ with $g_1 = 1$, $g_2 = 2$, $r_1 = 4$, and $r_2 = 6$. Then one can easily check that the matrix $-D + P$ is Hurwitz. Next, for any $h > 0$, the corresponding matrix M is

$$\begin{aligned} M &= I + D^{-1}\left(I - e^{-Dh}\right)(-D + P) \\ &= I + \mathrm{diag}\left\{\tfrac{1-e^{-4h}}{4}, \tfrac{1-e^{-6h}}{6}\right\}\begin{bmatrix} -4 & 1 \\ 1 & -4 \end{bmatrix} \\ &= \begin{bmatrix} e^{-4h} & \frac{1-e^{-4h}}{4} \\ \frac{1-e^{-6h}}{6} & \frac{1+2e^{-6h}}{3} \end{bmatrix}, \end{aligned} \quad (62)$$

by our formula (17) for M. Choose $h = \frac{\ln(2)}{2}$. Then

$$M = \begin{bmatrix} \frac{1}{4} & \frac{3}{16} \\ \frac{7}{48} & \frac{5}{12} \end{bmatrix} \quad (63)$$

whose larger eigenvalue is $\frac{1}{3} + \frac{\sqrt{79}}{48}$. Simple calculations give

$$M \begin{bmatrix} 1 \\ \frac{4+\sqrt{79}}{9} \end{bmatrix} = \left[\frac{1}{3} + \frac{\sqrt{79}}{48}\right] \begin{bmatrix} 1 \\ \frac{4+\sqrt{79}}{9} \end{bmatrix}. \quad (64)$$

Then Theorem 1 ensures that

$$\begin{aligned} &Z(t) \leq \chi(t) \\ &+ e^{\frac{\ln\left(\frac{1}{3} + \frac{\sqrt{79}}{48}\right)}{\tau + \frac{\ln(2)}{2}}(t-h)} \sum_{j=1}^{2} \sup_{\ell \in \left[-\tau, \frac{\ln(2)}{2}\right]} z_j(\ell) \begin{bmatrix} 1 \\ \frac{4+\sqrt{79}}{9} \end{bmatrix} \end{aligned} \quad (65)$$

holds for all $t \geq h$, where

$$\begin{aligned} \chi(t) &= -(-I + M)^{-1} D^{-1}\left(I - e^{-Dh}\right)\delta^\sharp(t) \\ &= -\begin{bmatrix} -\frac{3}{4} & \frac{3}{16} \\ \frac{7}{48} & -\frac{7}{12} \end{bmatrix}^{-1} \mathrm{diag}\left\{\tfrac{1-e^{-4h}}{4}, \tfrac{1-e^{-6h}}{6}\right\}\delta^\sharp(t) \\ &= \frac{1}{15}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\delta^\sharp(t) \end{aligned}$$

and

$$\delta^\sharp(t) = \left[\sup_{\ell \in [t-h,t]} \delta_1(\ell), \sup_{\ell \in [t-h,t]} \delta_2(\ell)\right]^\top. \quad (66)$$

B. Interval Observer Design

Consider the system

$$\dot{X}(t) = AX(t) + \sum_{i=1}^{p} B_i X(t - \tau_i(t)) + \delta(t) \quad (67)$$

whose state X is valued in $\mathbb{R}^n$, where A and the B_i's are constant matrices, p is any positive integer, A is Hurwitz (but not necessarily Metzler), and $\tau_i : [0, +\infty) \to [0, \overline{\tau}]$ for all i are piecewise continuous delays having a known bound $\overline{\tau} > 0$ for which solutions of (67) are uniquely defined on $[0, +\infty)$ for each initial condition. This agrees with the dynamics from [15], except we added the unknown bounded function δ having piecewise continuous components and representing uncertainty. This provides challenges because the stability conditions for (67) in [15] were not based on standard Lyapunov function conditions. We provide conditions that are independent of the delays τ_i's which ensure that one can build an interval observer whose existence implies that (67) is ISS; see [12] for background on interval observers.

Our analysis is more complex than standard treatments of linear systems, and is called for because of the mildness of our requirements on the coefficient matrices and on the delays. However, as in [15], a key ingredient will be the proof of [11, Theorem 2] which provides a time-varying change of variables formula for linear systems having Hurwitz matrices A in their drift term, which is defined by a bounded C^1 function $Q : [0, +\infty) \to \mathbb{R}^{n \times n}$ having a bounded inverse and a constant Metzler matrix M_0 such that

$$\dot{Q}(t)Q(t)^{-1} + Q(t)AQ(t)^{-1} = M_0 \quad (68)$$

for all $t \geq 0$, and such that all diagonal elements of M_0 are negative real values.

Fixing a Q that satisfies the requirements of the preceding paragraph, it follows that $Z(t) = Q(t)X(t)$ satisfies

$$\dot{Z}(t) = M_0 Z(t) + \sum_{i=1}^{p} L_i(t) Z(t - \tau_i(t)) + Q(t)\delta(t), \quad (69)$$

where $L_i(t) = Q(t) B_i Q(t - \tau_i(t))^{-1}$ for all i and $t \geq \overline{\tau}$. Our analysis will use the interval observer

$$\left\{ \begin{aligned} \dot{\overline{Z}}(t) &= M_0 \overline{Z}(t) + \sum_{i=1}^{p} (L_i(t))^+ \overline{Z}(t - \tau_i(t)) \\ &\quad - \sum_{i=1}^{p} (L_i(t))^- \underline{Z}(t - \tau_i(t)) + (Q(t)\delta(t))^+ \\ \dot{\underline{Z}}(t) &= M_0 \underline{Z}(t) + \sum_{i=1}^{p} (L_i(t))^+ \underline{Z}(t - \tau_i(t)) \\ &\quad - \sum_{i=1}^{p} (L_i(t))^- \overline{Z}(t - \tau_i(t)) - (Q(t)\delta(t))^- \end{aligned} \right. \quad (70)$$

where $C^+ = [\max\{c_{i,j}, 0\}]$ and $C^- = C^+ - C$ for matrices $C = [c_{i,j}]$. The change of coordinates $Z_\ddagger(t) = -\underline{Z}(t)$ yields

$$\left\{ \begin{aligned} \dot{\overline{Z}}(t) &= M_0 \overline{Z}(t) + \sum_{i=1}^{p} (L_i(t))^+ \overline{Z}(t - \tau_i(t)) \\ &\quad + \sum_{i=1}^{p} (L_i(t))^- Z_\ddagger(t - \tau_i(t)) + (Q(t)\delta(t))^+ \\ \dot{Z}_\ddagger(t) &= M_0 Z_\ddagger(t) + \sum_{i=1}^{p} (L_i(t))^+ Z_\ddagger(t - \tau_i(t)) \\ &\quad + \sum_{i=1}^{p} (L_i(t))^- \overline{Z}(t - \tau_i(t)) + (Q(t)\delta(t))^-. \end{aligned} \right. \quad (71)$$

Then (71) is cooperative, because M_0 is Metzler, by the reasoning used in [15], which also implies ISS of (71) if all positive valued solutions of (71) satisfy an ISS estimate.

We therefore focus on positive solutions of (71). To this end, notice that the variable

$$\tilde{Z}(t) = \overline{Z}(t) + Z_{\ddagger}(t) \tag{72}$$

satisfies

$$\begin{aligned}\dot{\tilde{Z}}(t) \; &= \; M_0\tilde{Z}(t) + \sum_{i=1}^{p}[L_i(t)^+ + L_i(t)^-]\tilde{Z}(t-\tau_i(t)) \\ &\quad + |Q(t)\delta(t)|.\end{aligned} \tag{73}$$

Setting $\mathcal{S}_{\bar{\tau}}(\tilde{Z}_t) = \sup_{\ell\in[t-\bar{\tau},t]} \tilde{Z}(\ell)$, we then have

$$\begin{aligned}\dot{\tilde{Z}}(t) \; &\leq \; M_0\tilde{Z}(t) + \sum_{i=1}^{p}\left[L_i(t)^+ + L_i(t)^-\right]\mathcal{S}_{\bar{\tau}}(\tilde{Z}_t) \\ &\quad + |Q(t)\delta(t)| \\ &\leq \; D_{M_0}\tilde{Z}(t) + \overline{L}\mathcal{S}_{\bar{\tau}}(\tilde{Z}_t) + |Q(t)\delta(t)|\end{aligned} \tag{74}$$

where D_{M_0} is diagonal matrix whose entries are the diagonal elements of M_0, and the matrix $\overline{L} > 0$ is such that

$$\sum_{i=1}^{p}[L_i(t)^+ + L_i(t)^-] + M_0 - D_{M_0} \leq \overline{L} \tag{75}$$

for all $t \geq 0$. By choosing the diagonal matrix $D = -D_{M_0}$ (whose entries are all positive, by our choice of M_0), it follows that, if we can choose $\overline{L}$ such that $D_{M_0} + \overline{L}$ is Hurwitz, then we can apply Theorem 1 to conclude that the solutions of (73) satisfy an ISS estimate. Using the facts that $\overline{Z}$ and $Z_{\ddagger}$ are nonnegative valued, it now follows from (72) that (71) is also ISS, so the origin of (70) is also ISS.

Moreover, the formulas $L_i = L_i^+ - L_i^-$ for each i and the argument we used in our proof of cooperativity of (71) imply that cooperativity of the dynamics for $(Z_+, Z_-) = (\overline{Z} - Z, Z - \underline{Z})$. This implies that $\overline{Z}(t) \geq Z(t) \geq \underline{Z}(t)$ for all $t \geq 0$ provided we choose our initial functions for $\overline{Z}$ and $\underline{Z}$ such that $\overline{Z}(t) \geq Z(t) \geq \underline{Z}(t)$ for all $t \in [-\bar{\tau}, 0]$. Hence, (70) provides an interval observer for (69) which gives the ISS property of the $Z(t)$ dynamics. Finally, the bounds $|X(t)| \leq |Q(t)^{-1}||Z(t)|$ for all $t \geq 0$ and the boundedness of $Q(t)^{-1}$ let us conclude that the X dynamics satisfy ISS.

Remark 4: Our sufficient condition for the ISS of (67) is the same as the sufficient condition for global exponential stability to 0 for the $\delta = 0$ case in [15]. As noted in [15], we can often take Q to be constant, e.g., when all eigenvalues of A are real, by choosing Q such that $QAQ^{-1} = M_0$ is A's Jordan canonical form. When A is Metzler, we can choose $Q = I$ and then our sufficient condition says that $A + \sum_{i=1}^{p}[B_i^+ + B_i^-]$ is Hurwitz. See [11, Section 4.3] for an example of a Hurwitz matrix A having a conjugate pair of complex (nonreal) eigenvalues that calls for a nonconstant choice of the matrix valued function Q. □

VI. Conclusion

We developed the theories of the Halanay's inequality and the trajectory based approaches, by providing ISS inequalities when vector inequalities are satisfied and additive disturbances are present. This overcame an obstacle to proving ISS analogs of works like [15] with respect to additive uncertainties. Our main tools entailed positive systems methods, based on novel applications of Schur stable and positive matrices. We illustrated our method in an interval observer design with unknown delays. Extensions to cases where the coefficient matrices in our inequalities are time varying are expected.

References

[1] S. Ahmed, F. Mazenc, and H. Ozbay, "Dynamic output feedback stabilization of switched linear systems with delay via a trajectory based approach, *Automatica*, vol. 93, pp. 92-97, 2018.

[2] C. Baker and E. Buckwar, "Exponential stability in pth mean of solutions, and of convergent Euler-type solutions, of stochastic delay differential equations, *Journal of Computational and Applied Mathematics*, vol. 184, pp. 404-427, 2005.

[3] N. Bekiaris-Liberis and M. Krstic, "Stabilization of linear strict-feedback systems with delayed integrators," *Automatica*, vol. 46, pp. 1902-1910, 2010.

[4] E. Fridman, *Introduction to Time-Delay Systems*. Boston, MA: Birkhauser, 2014.

[5] W. Haddad and V. Chellaboina, "Stability theory for nonnegative and compartmental dynamical systems with time delay," *Systems & Control Letters*, vol. 51, pp. 355-361, 2004.

[6] W. Haddad, V. Chellaboina, and Q. Hui, *Nonnegative and Compartmental Dynamical Systems*. Princeton, NJ: Princeton University Press, 2010.

[7] A. Halanay, *Differential Equations: Stability, Oscillation, Time Lags*. New York, NY: Academic Press, 1966.

[8] L. Hien, V. Phat, and H. Trinh, "New generalized Halanay inequalities with applications to stability of nonlinear non-autonomous time-delay systems," *Nonlinear Dynamics*, vol. 82, no. 1-2, pp. 563-575, 2015.

[9] R. Katz and E. Fridman, "Finite-dimensional control of the heat equation: Dirichlet actuation and point measurement," *European Journal of Control*, vol. 62, pp. 158-164, 2021.

[10] B. Lemmens and R. Nussbaum, *Nonlinear Perron-Frobenius Theory*. Cambridge, UK: Cambridge University Press, 2012.

[11] F. Mazenc and O. Bernard, "Interval observers for linear time-invariant systems with disturbances," *Automatica*, vol. 47, no. 1, pp. 140-147, 2011.

[12] F. Mazenc and O. Bernard, "ISS interval observers for nonlinear systems transformed into triangular systems," *International Journal of Robust and Nonlinear Control*, vol. 24, no. 7, pp. 1241-1261, 2014.

[13] F. Mazenc and M. Malisoff, "Trajectory based approach for the stability analysis of nonlinear systems with time delays," *IEEE Transactions on Automatic Control*, vol. 60, no. 6, pp. 1716-1721, June 2015.

[14] F. Mazenc, M. Malisoff, and M. Krstic, "Stability analysis using generalized sup-delay inequalities," *IEEE Control Systems Letters*, vol. 5, No. 4, pp. 1411-1416, Oct. 2021.

[15] F. Mazenc, M. Malisoff, and M. Krstic, "Vector extensions of Halanay's inequality," *IEEE Transactions on Automatic Control*, vol. 67, no. 3, pp. 1453-1459, March 2022.

[16] F. Mazenc, M. Malisoff, and S.-I. Niculescu, "Stability and control design for time-varying systems with time-varying delays using a trajectory based approach," *SIAM Journal on Control and Optimization*, vol. 55, no. 1, pp. 533-556, 2017.

[17] F. Mazenc, M. Malisoff, and H. Ozbay, "Stability and robustness analysis for switched systems with time-varying delays," *SIAM Journal on Control and Optimization*, vol. 56, no. 1, pp. 158-182, 2018.

[18] S. Mohamad and K. Gopalsamy, "Continuous and discrete Halanay-type inequalities," *Bulletin of the Australian Mathematical Society*, vol. 61, pp. 371-385, 2000.

[19] P. Pepe, "A nonlinear version of Halanay's inequality for the uniform convergence to the origin," *Mathematical Control and Related Fields*, to be published, doi: http://dx.doi.org/10.3934/mcrf.2021045.

[20] A. Rantzer, "Distributed control of positive systems," *European Journal of Control*, vol. 24, pp. 72-80, 2015.

[21] A. Selivanov and E. Fridman, "Distributed event-triggered control of transport-reaction systems," *IFAC-PapersOnLine*, vol. 48, no. 11, pp. 593-597, 2015.

[22] E. Sontag, "The ISS philosophy as a unifying framework for stability-like behavior," in *Nonlinear Control in the Year 2000 Volume 2*, A. Isidori, F. Lamnabhi-Lagarrigue, and W. Respondek, Ed. New York, NY: Springer, 2000, pp. 443-467.

[23] L. Wen, Y. Yu, and W. Wang, "Generalized Halanay inequalities for dissipativity of Volterra functional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 347, pp. 169-178, 2008.