New Finite-Time and Fast Converging Obs. with a Single Delay

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Abstract— **We provide new reduced order observer designs for a key class of nonlinear dynamics. When continuous output measurements are available, we prove that our observers converge in a fixed finite time in the absence of perturbations, and we prove a robustness result under uncertainties in the output measurements and in the dynamics, which bounds the observation error in terms of bounds on the uncertainties. The observers contain a dynamic extension with only one pointwise delay, and they use the observability Gramian to eliminate an invertibility condition that was present in earlier finite time observer designs. We also provide analogs for cases where the measurements are only available at discrete times, where we prove exponential input-to-state stability. We illustrate the advantages of our new observers using a DC motor dynamics.**

I. Introduction

Finite and fixed time observers present an obvious advantage by providing estimates of the state variables of systems in finite time [5]. Fixed time observers are special cases of finite time observers where the finite convergence time is independent of the initial state. Several types of fixed time observers are available. Some use discontinuous dynamic extensions, such as [12]. Others use time-varying high gains [3]. Others use delays to solve output feedback stabilization problems or homogeneity conditions [14].

In earlier works, e.g., [2], [7], and [13], fixed time observers are designed using dynamic extensions and a delay τ. The designs rely on the invertibility of a matrix which can be problematic because it is not invertible for all τ's and because, when it exists, the inverse can contain big terms when the delays are close to values where it is not invertible. We refer to such delays as artificial delays, because although they are not present in the given dynamics, they occur in the observers. The work [11] provides an exact calculation of state variables using a formula with several delays and the inversion of a matrix, which can also be problematic because it may be noninvertible for some delay values.

To overcome these shortcomings, we revisit the problem of estimating the state variables of a system in finite time using an artificial delay. For a family of unperturbed systems that are affine in the unmeasured state, we propose a new family of observers that converge in fixed time when continuous output measurements are available. The observers only estimate unmeasured variables, and so are reduced order. A key aspect of the observer design we propose is that it relies on the introduction of only one pointwise delay, which can be arbitrarily chosen. The delay is the fixed convergence time. Our fundamental tool is the observability Gramian. We also establish a robustness result for the observers with respect to additive disturbances on the output measurement and on the dynamics. We then provide an analog for cases where the measurements are only available at discrete instants, using the key ideas of the pioneering paper [4]. In this case, the exponential convergence rate is proportional to the logarithm of the size of the largest sampling interval.

We use standard notation, which we simplify when no confusion would arise. The dimensions of our Euclidean spaces are arbitrary, unless we indicate otherwise. The standard Euclidean 2-norm, and its induced matrix norm, are denoted by $|\cdot|$, $|\cdot|_\infty$ is the usual $\mathcal{L}_\infty$ sup norm, and $|\cdot|_S$ denotes the essential supremum over any set S. We use I to denote an identity matrix of arbitrary dimension.

II. Studied System

We consider the class of continous-time systems

$$\left\{\begin{array}{rcl}\dot{\chi}(t) & = & M\chi(t)+\Psi(N\chi(t),t)+\delta_1(t)\\ Y(t) & = & N\chi(t)+\delta_2(t)\end{array}\right. \tag{1}$$

where χ is valued in $\mathbb{R}^n$, the output Y is valued in $\mathbb{R}^q$, the time dependence in Ψ can represent the effects of a control, and the locally essentially bounded measurable functions δ_1 and δ_2 represent disturbances. We assume that (1) is forward complete, and that Ψ is locally Lipschitz. We also assume that the pair (M,N) is observable and that N has full rank.

From [6, pp. 304-306], we can deduce that there is a linear change of coordinates which yields the system

$$\left\{\begin{array}{rcl}\dot{\xi}_1(t) & = & A_1\xi_1(t)+F_1(Y(t)-\epsilon_3(t),t)+\epsilon_1(t)\\ \dot{\xi}_2(t) & = & A_2\xi_1(t)-k\xi_2(t)+F_2(Y(t)-\epsilon_3(t),t)\\ & & +\,\epsilon_2(t)\\ Y(t) & = & \xi_2(t)+\epsilon_3(t)\end{array}\right. \tag{2}$$

which is affine in the unmeasured variable ξ_1, where ξ_1 is valued in $\mathbb{R}^{n-q}$, ξ_2 is valued in $\mathbb{R}^q$, $A_1\in\mathbb{R}^{(n-q)\times(n-q)}$, $A_2\in\mathbb{R}^{q\times(n-q)}$, the pair (A_1,A_2) is observable, and $k>0$ is a constant such that A_1+kI is invertible. Then F_1 and F_2 are locally Lipschitz, and the measurable locally essentially bounded functions ϵ_i represent disturbances. Although the $-k\xi_2(t)$ term can be incorporated into the function F_2 in (2), we keep it separate to facilitate the analysis that follows, and we write ξ_2(t) as $Y(t)-\epsilon_3(t)$ in the F_i's in (2) to facilitate our study of the key special case where ϵ_3 is the zero function.

Changing the tuning parameter k can be done by changing F_2. One can always choose it such that A_1+kI is invertible, by taking k larger than the spectral radius of A_1.

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III. Continuous Measurement Cases

A. Assumptions and Statement of Theorem

In this section, we construct an observer for the system (2) which will provide our fixed time convergence, under the following assumption:

Assumption 1: Either (i) there are two constants $K_1 \geq 0$ and $K_2 \geq 0$ such that

$$|F_i(a,t) - F_i(b,t)| \leq K_i|a-b| \text{ for } i = 1,2 \tag{3}$$

for all $t \geq 0$ and a and b in $\mathbb{R}^q$ or (ii) $\epsilon_3(t) = 0$ for all $t \geq 0$.

Let us introduce any positive constant τ and the function $\lambda : \mathbb{R} \to \mathbb{R}^{q\times(n-q)}$ defined by

$$\lambda(r) = A_2(A_1 + kI)^{-1}\left[I - e^{(A_1+kI)r}\right] \tag{4}$$

which is well-defined because we choose $k > 0$ such that the matrix $A_1 + kI$ is invertible. We also use the matrix

$$\mathcal{S} = \int_{-\tau}^{0} \lambda(m)^\top \lambda(m)\mathrm{d}m \in \mathbb{R}^{(n-q)\times(n-q)}. \tag{5}$$

In the appendix below, we prove that since the pair (A_1, A_2) is observable, $\mathcal{S}$ is invertible. Then we define the matrices

$$\begin{array}{l}\mathcal{N} = \int_{-\tau}^{0} \lambda(m)^\top \mathrm{d}m \in \mathbb{R}^{(n-q)\times q},\\ \mathcal{R} = \mathcal{S}^{-1}\mathcal{N} \in \mathbb{R}^{(n-q)\times q}, \text{ and}\\ \mathcal{H} = ((A_1+kI)^{-1})^\top A_2^\top \in \mathbb{R}^{(n-q)\times q}\end{array} \tag{6}$$

and we introduce the dynamic extension

$$\left\{\begin{array}{rcl}\dot{\hat{\xi}}_1(t) &=& A_1\hat{\xi}_1(t) + F_1(Y(t),t)\\ \dot{\hat{\xi}}_2(t) &=& A_2\hat{\xi}_1(t) - k\hat{\xi}_2(t) + F_2(Y(t),t)\\ \dot{\psi}_1(t) &=& -k\psi_1(t) + \mathcal{H}[Y(t) - \hat{\xi}_2(t)]\\ \dot{\psi}_2(t) &=& -(A_1^\top + 2kI)\psi_2(t) + \mathcal{H}[Y(t) - \hat{\xi}_2(t)]\end{array}\right. \tag{7}$$

where $\hat{\xi}_1$ is valued in $\mathbb{R}^{n-q}$, $\hat{\xi}_2$ is valued in $\mathbb{R}^q$, and ψ_1 and ψ_2 are valued in $\mathbb{R}^{n-q}$. Finally, in terms of the functions

$$\begin{array}{l}\Delta_*(p) = e^{A_1 p} - e^{-kpI} \text{ and}\\ \Delta_{**}(p,q) = \int_p^q e^{A_1(p-\ell)}\epsilon_1(\ell)\mathrm{d}\ell\end{array} \tag{8}$$

where k is from (2), we let $\epsilon_\ddagger$ be the $\mathbb{R}^{n-q}$-valued function

$$\begin{array}{l}\epsilon_\ddagger(t) = \mathcal{S}^{-1}\int_{t-\tau}^{t} \lambda(s-t)^\top \mathcal{H}^\top \Delta_*(t-s)\Delta_{**}(s,t)\mathrm{d}s\\ -\mathcal{S}^{-1}\int_{t-\tau}^{t} \lambda(s-t)^\top\Big[-\int_s^t e^{k(m-t)}A_2\Delta_{**}(m,s)\mathrm{d}m\\ +\int_s^t e^{k(\ell-t)}\epsilon_2(\ell)\mathrm{d}\ell\Big]\,\mathrm{d}s.\end{array} \tag{9}$$

In terms of the preceding notation and the functions

$$\epsilon_i^\sharp(m) = K_i|\epsilon_3(m)| + |\epsilon_i(m)| \tag{10}$$

for $i = 1,2$ and the constants

$$\begin{array}{l}\overline{S} = |\mathcal{S}^{-1}| \text{ and } c_\triangle(\tau) = \tau\overline{S}|A_2|e^{|A_1|\tau}\\ +\overline{S}\left|A_2(A_1+kI)^{-1}\right|\left[e^{|A_1|\tau}+1\right]e^{|A_1|\tau},\end{array} \tag{11}$$

our first theorem is then as follows:

Theorem 1: Let (2) satisfy Assumption 1. Then, with the preceding notation, when ϵ_3 is the zero function, we have

$$\xi_1(t) = \xi_e(t) + \epsilon_\ddagger(t) \tag{12}$$

for all $t \geq \tau$, where

$$\begin{array}{rcl}\xi_e(t) &=& \hat{\xi}_1(t) + \mathcal{R}(\xi_2(t) - \hat{\xi}_2(t))\\ && +\mathcal{S}^{-1}\left[e^{-k\tau}\psi_1(t-\tau) - \psi_1(t)\right]\\ && +\mathcal{S}^{-1}\left[\psi_2(t) - e^{-(A_1^\top+2kI)\tau}\psi_2(t-\tau)\right].\end{array} \tag{13}$$

Also, if F_1 and F_2 satisfy (3) and $\epsilon_3 \neq 0$, then

$$\xi_1(t) = \xi_e(t) + \epsilon_\bigstar(t) \tag{14}$$

holds for all $t \geq \tau$ where $\epsilon_\bigstar$ is a function such that

$$\begin{array}{rcl}|\epsilon_\bigstar(t)| &\leq& c_\triangle(\tau)\int_{t-\tau}^{t}|\lambda(s-t)|\int_s^t \epsilon_1^\sharp(m)\mathrm{d}m\mathrm{d}s\\ && +\,\overline{S}\int_{t-\tau}^{t}|\lambda(s-t)|\int_s^t \epsilon_2^\sharp(m)\mathrm{d}m\mathrm{d}s\\ && +\,\overline{S}|\mathcal{H}|\left(1+e^{|A_1^\top+2kI|\tau}\right)\int_{t-\tau}^{t}|\epsilon_3(s)|\mathrm{d}s\end{array} \tag{15}$$

for all $t \geq \tau$,

Remark 1: A key feature of the observer ξ_e we propose is that it incorporates only one delay τ. This delay can be any positive value because for any $\tau > 0$, $\mathcal{S}$ is invertible.

Remark 2: Since

$$|\lambda(r)| \leq |A_2(A_1+kI)^{-1}|\left[1 + e^{|A_1+kI||r|}\right] \tag{16}$$

for all $r \in \mathbb{R}$, we deduce that there are two constants $c_\natural \geq 0$ and $c_\diamondsuit \geq 0$ such that $|\epsilon_\ddagger(t)| \leq c_\natural|(\epsilon_1,\epsilon_2)|_{[t-\tau,t]}$ and $|\epsilon_\bigstar(t)| \leq c_\diamondsuit|(\epsilon_1,\epsilon_2,\epsilon_3)|_{[t-\tau,t]}$ for all $t \geq \tau$.

B. Proof of Theorem 1

We introduce the variables $y = Y - \hat{\xi}_2$ and

$$\begin{array}{l}\Delta_i(t) = F_i(Y(t)-\epsilon_3(t),t) - F_i(Y(t),t) \text{ and}\\ x_i(t) = \xi_i(t) - \hat{\xi}_i(t) \text{ for } i = 1,2.\end{array} \tag{17}$$

Then simple calculations based on (2) and (7) give

$$\left\{\begin{array}{rcl}\dot{x}_1(t) &=& A_1x_1(t) + \Delta_1(t) + \epsilon_1(t)\\ \dot{x}_2(t) &=& A_2x_1(t) - kx_2(t) + \Delta_2(t) + \epsilon_2(t)\\ y(t) &=& x_2(t) + \epsilon_3(t).\end{array}\right. \tag{18}$$

Here and in the sequel, all equalities and inequalities hold for all $t \geq 0$, unless otherwise indicated.

By applying variation of parameters to (18), we obtain

$$\begin{array}{l}x_1(t) = e^{A_1(t-s)}x_1(s)\\ +\int_s^t e^{A_1(t-m)}[\Delta_1(m) + \epsilon_1(m)]\mathrm{d}m\end{array} \tag{19}$$

and

$$\begin{array}{l}x_2(t) - e^{-k(t-s)}x_2(s) = \rho_1(t,s)\\ +A_2(A_1+kI)^{-1}\left[e^{A_1(t-s)} - e^{-k(t-s)I}\right]x_1(s)\end{array} \tag{20}$$

for all $s \geq 0$ and $t \geq s$, where

$$\begin{array}{l}\rho_1(t,s) = \int_s^t e^{k(\ell-t)}[\Delta_2(\ell) + \epsilon_2(\ell)]\mathrm{d}\ell\\ +\int_s^t e^{k(m-t)}A_2\int_s^m e^{A_1(m-\ell)}[\Delta_1(\ell) + \epsilon_1(\ell)]\mathrm{d}\ell\mathrm{d}m,\end{array} \tag{21}$$

and where we used the fact that

$$\begin{array}{l}\int_s^t e^{k(m-t)}A_2e^{A_1(m-s)}x_1(s)\mathrm{d}m\\ = e^{k(s-t)}A_2\int_s^t e^{(A_1+kI)(m-s)}\mathrm{d}mx_1(s)\\ = e^{k(s-t)}A_2(A_1+kI)^{-1}\left(e^{(A_1+kI)(t-s)} - I\right)x_1(s).\end{array} \tag{22}$$

According to (19), we have

$$\begin{array}{l} x_1(s) = \\ e^{A_1(s-t)}x_1(t) - \int_s^t e^{A_1(s-m)}[\Delta_1(m) + \epsilon_1(m)]\mathrm{d}m, \end{array} \quad (23)$$

and our formulas (4) and (8) give $\lambda(s-t) = \mathcal{H}^\top \Delta_*(t-s)e^{A_1(s-t)}$. Hence, we can substitute (23) into (20) to obtain

$$\lambda(s-t)x_1(t) = x_2(t) - e^{-k(t-s)}x_2(s) + \rho_2(t,s) \quad (24)$$

$$\begin{array}{l} \text{where } \rho_2(t,s) = -\rho_1(t,s) \\ +\mathcal{H}^\top \Delta_*(t-s)\int_s^t e^{A_1(s-m)}[\Delta_1(m) + \epsilon_1(m)]\mathrm{d}m. \end{array} \quad (25)$$

By left multiplying both sides of (24) by $\lambda(s-t)^\top$, we obtain

$$\begin{array}{l} \lambda(s-t)^\top\lambda(s-t)x_1(t) = \lambda(s-t)^\top\rho_2(t,s) \\ +\lambda(s-t)^\top x_2(t) - e^{-k(t-s)}\lambda(s-t)^\top x_2(s). \end{array} \quad (26)$$

By integrating (26) with respect to s over $[t-\tau, t]$, we obtain

$$\begin{array}{l} \int_{t-\tau}^t \lambda(s-t)^\top\lambda(s-t)\mathrm{d}s x_1(t) \\ = \int_{t-\tau}^t \lambda(s-t)^\top \mathrm{d}s x_2(t) \\ - \int_{t-\tau}^t e^{-k(t-s)}\lambda(s-t)^\top x_2(s)\mathrm{d}s \\ + \int_{t-\tau}^t \lambda(s-t)^\top\rho_2(t,s)\mathrm{d}s \text{ for all } t \geq \tau. \end{array} \quad (27)$$

Hence, our choices in (5)-(6), and the invertibility of $\mathcal{S}$, give

$$\begin{array}{rcl} x_1(t) & = & \mathcal{R}x_2(t) + \mathcal{S}^{-1}\int_{t-\tau}^t \lambda(s-t)^\top\rho_2(t,s)\mathrm{d}s \\ & & -\mathcal{S}^{-1}\int_{t-\tau}^t e^{-k(t-s)}\lambda(s-t)^\top x_2(s)\mathrm{d}s. \end{array} \quad (28)$$

Using the formula for λ from (4), we obtain

$$\begin{array}{l} x_1(t) = \mathcal{R}x_2(t) + \mathcal{S}^{-1}\int_{t-\tau}^t \lambda(s-t)^\top\rho_2(t,s)\mathrm{d}s \\ -\mathcal{S}^{-1}\int_{t-\tau}^t e^{-k(t-s)}\left[I - e^{(A_1^\top + kI)(s-t)}\right]\mathcal{H}x_2(s)\mathrm{d}s \end{array} \quad (29)$$

with $\mathcal{H}$ defined in (6). This equality can be rewritten as

$$\begin{array}{rcl} x_1(t) & = & \mathcal{R}x_2(t) + \mathcal{S}^{-1}\int_{t-\tau}^t \lambda(s-t)^\top\rho_2(t,s)\mathrm{d}s \\ & & -\mathcal{S}^{-1}\int_{t-\tau}^t e^{-k(t-s)}\mathcal{H}x_2(s)\mathrm{d}s \\ & & +\mathcal{S}^{-1}\int_{t-\tau}^t e^{-(A_1^\top + 2kI)(t-s)}\mathcal{H}x_2(s)\mathrm{d}s. \end{array} \quad (30)$$

Since $\xi_2(s) = Y(s) - \epsilon_3(s)$, we have $Y - \hat{\xi}_2 = x_2 + \epsilon_3$, so we deduce from (7) and (30) that

$$\begin{array}{l} x_1(t) = \mathcal{R}x_2(t) - \mathcal{S}^{-1}\left[\psi_1(t) - e^{-k\tau}\psi_1(t-\tau)\right] \\ +\mathcal{S}^{-1}\left[\psi_2(t) - e^{-(A_1^\top + 2kI)\tau}\psi_2(t-\tau)\right] + \epsilon_\star(t), \end{array} \quad (31)$$

$$\begin{array}{rcl} \text{where } \epsilon_\star(t) & = & \mathcal{S}^{-1}\Big[\int_{t-\tau}^t \lambda(s-t)^\top\rho_2(t,s)\mathrm{d}s \\ & & + \int_{t-\tau}^t e^{-k(t-s)}\mathcal{H}\epsilon_3(s)\mathrm{d}s \\ & & - \int_{t-\tau}^t e^{-(A_1^\top + 2kI)(t-s)}\mathcal{H}\epsilon_3(s)\mathrm{d}s\Big]. \end{array} \quad (32)$$

Hence, (13) and (17) give $\xi_1 = \xi_e + \epsilon_\star$ for all $t \geq \tau$.

Recalling the formula for ρ_2 in (25) and (17) and our Lipshitz condition (3) on the F_i's, it follows that (32) satisfies

$$\begin{array}{rcl} |\epsilon_\star(t)| & \leq & \overline{S}\int_{t-\tau}^t |\lambda(s-t)|\rho_1(t,s)|\mathrm{d}s \\ & & +\overline{S}\int_{t-\tau}^t |\lambda(s-t)||\mathcal{H}^\top|J_1(t,s)\mathrm{d}s \\ & & +\overline{S}|\mathcal{H}|\int_{t-\tau}^t e^{-k(t-s)}|\epsilon_3(s)|\mathrm{d}s \\ & & +\overline{S}|\mathcal{H}|\int_{t-\tau}^t e^{|A_1^\top + 2kI|(t-s)}|\epsilon_3(s)|\mathrm{d}s \end{array} \quad (33)$$

and so also

$$\begin{array}{rcl} |\epsilon_\star(t)| & \leq & \overline{S}\int_{t-\tau}^t |\lambda(s-t)|\rho_1(t,s)|\mathrm{d}s \\ & & +\overline{S}\int_{t-\tau}^t |\lambda(s-t)||\mathcal{H}^\top|J_2(t,s)\mathrm{d}s \\ & & +\overline{S}|\mathcal{H}|\int_{t-\tau}^t J_3(t-s)|\epsilon_3(s)|\mathrm{d}s, \text{ where} \end{array} \quad (34)$$

$$\begin{array}{l} J_1(t,s) \\ = |\Delta_*(t-s)|\int_s^t e^{|A_1|(m-s)}[|\Delta_1(m)| + |\epsilon_1(m)|]\mathrm{d}m \end{array} \quad (35)$$

and the function Δ_* was defined in (8), and where

$$\begin{array}{l} J_2(t,s) = \\ \left[e^{|A_1|\tau} + 1\right]\int_s^t e^{|A_1|(m-s)}[K_1|\epsilon_3(m)| + |\epsilon_1(m)|]\mathrm{d}m \end{array} \quad (36)$$

and $J_3(r) = e^{-kr} + e^{|A_1^\top + 2kI|r}$, and where $\overline{S}$ is defined in (11). Also, when $\epsilon_3 = 0$, we can use (17) to get $\Delta_1 = \Delta_2 = 0$, so our formula (9) for $\epsilon_\ddagger$ and (21) and (25) give $\epsilon_\star = \epsilon_\ddagger$ when $\epsilon_3 = 0$. Hence, since the right side of (15) is an upper bound for the right side of (34), this allows us to conclude.

IV. Discrete Measurements Cases

A. Assumptions and Statement of Theorem

Although the observer from the preceding section enjoys fixed time convergence and robustness properties, it requires continuous measurements of the output, which might not always be available in practice. Therefore, in this section, we consider the case where the variables are only measured at discrete instants. We propose an observer which combines the one of Theorem 1 and the technique of [4].

Let t_j be a sequence such that $t_0 = 0$ and such that there are two constants $\underline{T} > 0$ and $\overline{T} > \underline{T}$ such that

$$\underline{T} \leq t_{j+1} - t_j \leq \overline{T} \text{ for all } j \geq 0. \quad (37)$$

We continue the notation from Section III except we consider

$$\left\{\begin{array}{rcl} \dot{\xi}_1(t) & = & A_1\xi_1(t) + F_1(\xi_2(t), t) + \epsilon_1(t) \\ \dot{\xi}_2(t) & = & A_2\xi_1(t) - k\xi_2(t) + F_2(\xi_2(t), t) \\ & & + \epsilon_2(t) \\ Y(t_j) & = & \xi_2(t_j) + \epsilon_3(t_j) \text{ for all } j \geq 0, \end{array}\right. \quad (38)$$

under the assumption that F_1 and F_2 satisfy (3) and where k is selected as in Section II. We also use these constants:

$$\begin{array}{rcl} \varsigma_1 & = & \frac{\bar{S}|A_2|e^{|A_1|\tau}}{k^2}\int_{-\tau}^0 |\lambda(s)|(e^{ks} - sk - 1)\mathrm{d}s \\ \varsigma_2 & = & \frac{\overline{S}}{k}\int_{-\tau}^0 |\lambda(s)|(1 - e^{ks})\mathrm{d}s \\ \varsigma_3 & = & \overline{S}\tau e^{|A_1|\tau}|\mathcal{H}^\top|\int_{-\tau}^0 |\lambda(s)|(e^{|A_1|\tau} + e^{-ks})\mathrm{d}s \\ \varsigma_4 & = & \overline{S}|\mathcal{H}|\left(\frac{1}{k}(1 - e^{-k\tau}) + \tau e^{|A_1^\top + 2kI|\tau}\right) \end{array} \quad (39)$$

where $\mathcal{H}$ is from (6) as before, and τ satisfies the requirements from Section II. We introduce the dynamic extension

$$\left\{\begin{array}{rcl} \dot{\hat{\xi}}_1(t) & = & A_1\hat{\xi}_1(t) + F_1(\omega(t), t) \\ \dot{\hat{\xi}}_2(t) & = & A_2\hat{\xi}_1(t) - k\hat{\xi}_2(t) + F_2(\omega(t), t) \\ \dot{\psi}_1(t) & = & -k\psi_1(t) + \mathcal{H}[\omega(t) - \hat{\xi}_2(t)] \\ \dot{\psi}_2(t) & = & -(A_1^\top + 2kI)\psi_2(t) + \mathcal{H}[\omega(t) - \hat{\xi}_2(t)] \\ \dot{\omega}(t) & = & A_2\xi_e(t) - k\omega(t) + F_2(\omega(t), t) \\ & & \text{for all } t \in [t_j, t_{j+1}) \text{ and } j \geq 0 \\ \omega(t_j) & = & Y(t_j) \text{ for all } j \geq 0 \end{array}\right. \quad (40)$$

where ξ_e is defined as in (13). We prove:

Theorem 2: Let the constant $\overline{T}$ in (37) be such that

$$\overline{T}\mu < 1, \text{ where} \tag{41}$$

$$\mu = |A_2|q_1 + k + K_2 \text{ and } q_1 = (\varsigma_1+\varsigma_3)K_1 + \varsigma_2 K_2 + \varsigma_4 \tag{42}$$

using the constants (39). Then we can find positive constants a_1 and a_2 such that all solutions of (38) and (40) satisfy

$$\begin{array}{l} |\xi_1(t) - \xi_e(t)| \ \leq \ a_1|\xi_2 - \omega|_{[r-2\tau-\overline{T},r]} e^{\frac{\ln(\overline{T}\mu)}{\tau+\overline{T}}(t-r)} \\ +a_2 \sup_{\ell\in[r-\overline{T}-2\tau,t]} [|\epsilon_1(\ell)| + |\epsilon_2(\ell)| + |\epsilon_3(\ell)|] \end{array} \tag{43}$$

for all $r \geq 2\tau + \overline{T}$ and $t \geq r + \tau$.

Remark 3: The inequality (43) is of ISS type because

$$\frac{\ln(\overline{T}\mu)}{\tau+\overline{T}} < 0, \tag{44}$$

by (41). Moreover, since μ is independent of $\overline{T}$, the left side of (44) converges to $-\infty$ as $\overline{T} \to 0^+$. Therefore, we can have arbitrarily large rates of convergence of the estimation error $|\xi_1(t) - \xi_e(t)|$ to 0 when the ϵ_i's are zero, by choosing the sample times t_i such that $\overline{T}$ is small enough.

B. Proof of Theorem 2

Our proof will use the variables

$$\begin{array}{l} \tilde{\omega}(t) = \xi_2(t) - \omega(t), \text{ and } x_i(t) = \xi_i(t) - \hat{\xi}_i(t) \\ \text{and } \kappa_i(t) = F_i(\xi_2(t), t) - F_i(\omega(t), t) \text{ for } i = 1, 2. \end{array} \tag{45}$$

Then $\omega - \hat{\xi}_2 = \xi_2 - \tilde{\omega} - \hat{\xi}_2 = x_2 - \tilde{\omega}$, so we obtain

$$\left\{ \begin{array}{rcl} \dot{x}_1(t) & = & A_1 x_1(t) + \kappa_1(t) + \epsilon_1(t) \\ \dot{x}_2(t) & = & A_2 x_1(t) - k x_2(t) + \kappa_2(t) + \epsilon_2(t) \\ \dot{\psi}_1(t) & = & -k\psi_1(t) + \mathcal{H}x_2(t) - \mathcal{H}\tilde{\omega}(t) \\ \dot{\psi}_2(t) & = & -(A_1^\top + 2kI)\psi_2(t) + \mathcal{H}x_2(t) - \mathcal{H}\tilde{\omega}(t) \\ \dot{\tilde{\omega}}(t) & = & A_2[\xi_1(t) - \xi_e(t)] - k\tilde{\omega}(t) + \kappa_2(t) + \epsilon_2(t) \\ & & \text{for all } t \in [t_j, t_{j+1}) \text{ and } j \geq 0 \\ \tilde{\omega}(t_j) & = & -\epsilon_3(t_j) \text{ for all } j \geq 0. \end{array} \right. \tag{46}$$

We also use the $\mathbb{R}^{n-q}$-valued variables

$$\begin{array}{l} \gamma_1(t) = -\mathcal{S}^{-1} \int_{t-\tau}^t \lambda(s-t)^\top \mathcal{H}^\top \Delta_*(t-s) J_a(t,s) \mathrm{d}s \\ -\mathcal{S}^{-1} \int_{t-\tau}^t \lambda(s-t)^\top \int_s^t e^{k(m-t)} A_2 J_a(s,m) \mathrm{d}m \mathrm{d}s \\ -\mathcal{S}^{-1} \int_{t-\tau}^t \lambda(s-t)^\top \int_s^t e^{k(\ell-t)} [\kappa_2(\ell) + \epsilon_2(\ell)] \mathrm{d}\ell \mathrm{d}s, \end{array} \tag{47}$$

$$\begin{array}{rcl} \gamma_2(t) & = & -\mathcal{S}^{-1} \int_{t-\tau}^t e^{-k(t-s)} \mathcal{H}\tilde{\omega}(s) \mathrm{d}s \\ & & + \mathcal{S}^{-1} \int_{t-\tau}^t e^{(A_1^\top + 2kI)(-t+s)} \mathcal{H}\tilde{\omega}(s) \mathrm{d}s, \end{array} \tag{48}$$

and

$$\begin{array}{rcl} x_a(t) & = & \mathcal{R}x_2(t) - \mathcal{S}^{-1} \left[\psi_1(t) - e^{-k\tau} \psi_1(t-\tau) \right] \\ & & + \mathcal{S}^{-1} \left[\psi_2(t) - e^{-(A_1^\top + 2kI)\tau} \psi_2(t-\tau) \right], \end{array} \tag{49}$$

$$\text{where } J_a(s,m) = \int_s^m e^{A_1(m-\ell)} [\kappa_1(\ell) + \epsilon_1(\ell)] \mathrm{d}\ell. \tag{50}$$

The rest of the proof of the theorem consists of two steps. In the first step, we prove that the preceding variables satisfy

$$\xi_1(t) = \xi_e(t) + \gamma_1(t) + \gamma_2(t) \tag{51}$$

for all $t \geq \tau$. In the second step, we bound $\gamma_1(t) + \gamma_2(t)$ by the right side of (43) for suitable values of r and t.

First Step. Since the (x_1, x_2)-dynamics of (46) agree with the first two equations of (18) except with the Δ_i's replaced by the κ_i's, the same reasoning that led to (30) gives

$$\begin{array}{l} x_1(t) = \mathcal{R}x_2(t) - \mathcal{S}^{-1} \int_{t-\tau}^t e^{-k(t-s)} \mathcal{H}x_2(s) \mathrm{d}s \\ +\mathcal{S}^{-1} \int_{t-\tau}^t e^{-(A_1^\top + 2kI)(t-s)} \mathcal{H}x_2(s) \mathrm{d}s + \gamma_1(t). \end{array} \tag{52}$$

Also, by applying the method of variation of parameters separately to the ψ_1 and ψ_2 dynamics in (46), we obtain

$$\begin{array}{l} \int_{t-\tau}^t e^{-k(t-s)} \mathcal{H}x_2(s) \mathrm{d}s = \\ \psi_1(t) - e^{-k\tau} \psi_1(t-\tau) + \int_{t-\tau}^t e^{-k(t-s)} \mathcal{H}\tilde{\omega}(s) \mathrm{d}s \end{array} \tag{53}$$

and

$$\begin{array}{l} \int_{t-\tau}^t e^{-(A_1^\top + 2kI)(t-s)} \mathcal{H}x_2(s) \mathrm{d}s = \\ \psi_2(t) - e^{-(A_1^\top + 2kI)\tau} \psi_2(t-\tau) \\ + \int_{t-\tau}^t e^{(A_1^\top + 2kI)(-t+s)} \mathcal{H}\tilde{\omega}(s) \mathrm{d}s \end{array} \tag{54}$$

for all $t \geq \tau$. By combining (52)-(54), we obtain

$$\begin{array}{rcl} x_1(t) & = & \mathcal{R}x_2(t) - \mathcal{S}^{-1} \big[\psi_1(t) - e^{-k\tau} \psi_1(t-\tau) \\ & & + \int_{t-\tau}^t e^{-k(t-s)} \mathcal{H}\tilde{\omega}(s) \mathrm{d}s \big] + \gamma_1(t) \\ & & + \mathcal{S}^{-1} \Big[\psi_2(t) - e^{-(A_1^\top + 2kI)\tau} \psi_2(t-\tau) \\ & & + \int_{t-\tau}^t e^{(A_1^\top + 2kI)(-t+s)} \mathcal{H}\tilde{\omega}(s) \mathrm{d}s \Big] \end{array} \tag{55}$$

for all $t \geq \tau$ Hence, our choices (47)-(49), and our choice of x_1 in (45) and our formula (13) for ξ_e, give

$$x_1 = x_a + \gamma_1 + \gamma_2 \text{ and } x_1 - x_a = \xi_1 - \xi_e. \tag{56}$$

Therefore, (51) follows from combining the relations (56).

Second Step. From (51), it follows that

$$\begin{array}{l} |\xi_1(t) - \xi_e(t)| \leq \\ \overline{S} \int_{t-\tau}^t |\lambda(s-t)| \int_s^t |A_2| e^{k(m-t)} J_4(s,m) \mathrm{d}m \mathrm{d}s \\ + \overline{S} \int_{t-\tau}^t |\lambda(s-t)| \int_s^t e^{k(\ell-t)} [|\kappa_2(\ell)| + |\epsilon_2(\ell)|] \mathrm{d}\ell \mathrm{d}s \\ + \overline{S} \int_{t-\tau}^t |\lambda(s-t)| |\mathcal{H}^\top| \left[e^{|A_1|\tau} + e^{k(s-t)} \right] J_4(s,t) \mathrm{d}s \\ + \overline{S} \int_{t-\tau}^t \left(e^{k(s-t)} + e^{|A_1^\top + 2kI|\tau} \right) |\mathcal{H}| |\tilde{\omega}(s)| \mathrm{d}s \end{array} \tag{57}$$

for all $t \geq \tau$, where

$$J_4(s,m) = \int_s^m e^{|A_1|\tau} [|\kappa_1(\ell)| + |\epsilon_1(\ell)|] \mathrm{d}\ell. \tag{58}$$

Moreover, our choices of the κ_i's in (45) and (3) give $|\kappa_2(\ell)| \leq K_2 |\tilde{w}(\ell)|$ when $s \leq \ell \leq t$, and so also

$$J_4(s,m) \leq (m-s) e^{|A_1|\tau} (K_1 |\tilde{w}|_{[s,m]} + |\epsilon_1|_{[s,m]}) \tag{59}$$

when $s \leq m \leq t$. Therefore, by upper bounding the right side of (57) and then collecting the coefficients of $|\tilde{w}|_{[t-\tau,t]}$ and $|\epsilon_i|_{[t-\tau,t]}$ for $i = 1, 2$ in the result, it follows from our choices of the constants ς_i in (39) that

$$\begin{array}{l} |\xi_1(t) - \xi_e(t)| \leq \\ q_1 |\tilde{w}|_{[t-\tau,t]} + q_2 (|\epsilon_1|_{[t-\tau,t]} + |\epsilon_2|_{[t-\tau,t]}) \text{ for all } t \geq \tau, \end{array} \tag{60}$$

where q_1 is from (42) and $q_2 = \max\{\varsigma_1 + \varsigma_3, \varsigma_2\}$.

By combining (46) and (60), and recalling (3), we get

$$\begin{array}{rcl}|\dot{\tilde{\omega}}(t)| & \leq & |A_2||\xi_1(t)-\xi_e(t)|+k|\tilde{\omega}(t)|+|\epsilon_2(t)| \\ & & +|F_2(\xi_2(t),t)-F_2(\omega(t),t)| \\ & \leq & (|A_2|q_1+k+K_2)|\tilde{w}|_{[t-\tau,t]}+\epsilon_{\mathcal{L}}(t)\end{array} \quad (61)$$

for all $t \in [t_j, t_{j+1})$ and all $j \geq 0$ when $t \geq \tau$, where $\epsilon_{\mathcal{L}}(t) = |A_2|q_2(|\epsilon_1|_{[t-\tau,t]} + |\epsilon_2|_{[t-\tau,t]}) + |\epsilon_2(t)|$. Since

$$\tilde{\omega}(t) = \tilde{\omega}(t_j) + \int_{t_j}^{t} \dot{\tilde{\omega}}(\ell)\mathrm{d}\ell \quad (62)$$

for all $t \in [t_j, t_{j+1})$, we deduce that, for all $t \geq \overline{T} + \tau$,

$$\begin{array}{rl}|\tilde{\omega}(t)| & \leq |\tilde{\omega}(t_j)| + \overline{T}(|A_2|q_1+k+K_2)|\tilde{w}|_{[t-\overline{T}-\tau,t]} \\ & \quad +\overline{T}|\epsilon_{\mathcal{L}}|_{[t-\overline{T},t]} \\ & \leq \overline{T}\mu|\tilde{\omega}|_{[t-\tau-\overline{T},t]} + \overline{T}^{\sharp}\sum_{j=1}^{3}|\epsilon_i|_{[t-\overline{T}-\tau,t]}\end{array} \quad (63)$$

for all $t \in [t_j, t_{j+1})$ and $j \geq 0$ when $t \geq r$ and $r \geq \overline{T} + \tau$, where $\bar{T}^{\sharp} = \bar{T}(|A_2|q_2 + 1) + 1$, the last inequality in (63) used the fact that $|\tilde{\omega}(t_j)| = |\epsilon_3(t_j)|$ for all $j \geq 0$, and μ is from (42). Using (41), it follows from applying the trajectory based approach from [10, Lemma 1] to the function $w_0(t) = |\tilde{\omega}(t + r)|$ that

$$|\tilde{\omega}(t)| \quad \leq \quad |\tilde{\omega}|_{[r-\tau-\overline{T},r]} e^{\frac{\ln(\overline{T}\mu)}{\tau+\overline{T}}(t-r)} + \mathcal{T}_\epsilon(t,r) \quad (64)$$

for all $t \geq r$ with

$$\mathcal{T}_\epsilon(t,r) = \frac{\overline{T}^{\sharp}}{1-\overline{T}\mu}\sum_{j=1}^{3}|\epsilon_i|_{[t-\overline{T}-\tau,t]}. \quad (65)$$

The theorem now follows by using (64) to upper bound the first right side term in (60).

V. Illustrations

A. Illustration of Theorem 1

Consider this model for a single-link direct-drive manipulator actuated by a permanent magnet DC brush motor [1]:

$$\begin{array}{l} M\ddot{q} + B\dot{q} + N\sin(q) = \mathcal{I} \ \text{ and} \\ L\dot{\mathcal{I}} = V_e - R\mathcal{I} - K_B\dot{q}, \\ \text{where } M = \frac{J}{K_\tau} + \frac{mL_0^2}{3K_\tau} + \frac{M_0L_0^2}{K_\tau} + \frac{2M_0R_0^2}{5K_\tau}, \\ N = \frac{mL_0G}{2K_\tau} + \frac{M_0L_0G}{K_\tau}, \text{ and } B = \frac{B_0}{K_\tau}\end{array} \quad (66)$$

and where m is the mass of the link, J is the rotor inertia, L_0 is the length of the link, M_0 is the mass of the load, B_0 is the viscous friction coefficient at the joint, R_0 is the radius of the load, G is the gravitational constant, $q(t)$ is the position of the load (which is the angular motor position), $\mathcal{I}(t)$ is the motor armature current, the coefficient K_τ characterizes the electromagnetic conversion of armature current to torque, R is the armature resistance, L is the armature inductance, K_B is the back-emf coefficient, and V_e is the input current voltage. All constants in (66) are positive, and we assume that perturbed values of q are available for measurement.

The preceding model has been studied extensively. For instance, see [9] for continuous-discrete observers for (66), and [8] for full order observers for (66) with sampling and input delays. However, we believe that the problem we solve in this subsection of building reduced order observers for (66) with arbitrarily small fixed convergence times τ under continuous measurements was open.

By also allowing additive uncertainties in the model (66) and in the measurements, we obtain the dynamics

$$\left\{\begin{array}{rcl}\dot{\chi}_1(t) & = & \chi_2(t) + \delta_{1,1}(t) \\ \dot{\chi}_2(t) & = & b_1\chi_3(t) - a_1\sin(\chi_1(t)) - a_2\chi_2(t) \\ & & + \delta_{1,2}(t) \\ \dot{\chi}_3(t) & = & b_0u(t) - a_3\chi_2(t) - a_4\chi_3(t) + \delta_{1,3}(t) \\ Y(t) & = & \chi_1(t) + \delta_2(t)\end{array}\right. \quad (67)$$

where $\chi_1 = q$, $\chi_2 = \dot{q}$, $\chi_3 = \mathcal{I}$, $a_1 = N/M$, $a_2 = B/M$, $a_3 = K_B/L$, $a_4 = R/L$, $b_0 = 1/L$, and $b_1 = 1/M$, and $u = V_e$ is the control. As in [7], we choose $b_0 = 40$, $b_1 = 15$, $a_1 = 35$, $a_2 = 1$, $a_3 = 36.4$ and $a_4 = 200$.

Adopting the notation $\xi_2 = \chi_1$, $\xi_{1,1} = \chi_2$, $\xi_{1,2} = \chi_3$, $\varepsilon_{1,1}(t) = \delta_{1,2}(t)$, $\varepsilon_{1,2}(t) = \delta_{1,3}(t)$, $\varepsilon_2(t) = \delta_{1,1}(t)$, and $\varepsilon_3(t) = \delta_2(t)$, we can rewrite the system (67) as

$$\left\{\begin{array}{rcl}\dot{\xi}_{1,1}(t) & = & -a_2\xi_{1,1}(t) + b_1\xi_{1,2}(t) \\ & & -a_1\sin(Y(t) - \epsilon_3(t)) + \varepsilon_{1,1}(t) \\ \dot{\xi}_{1,2}(t) & = & -a_3\xi_{1,1}(t) - a_4\xi_{1,2}(t) + b_0u(t) \\ & & +\varepsilon_{1,2}(t) \\ \dot{\xi}_2(t) & = & \xi_{1,1}(t) - k\xi_2(t) + k[Y(t) - \epsilon_3(t)] + \varepsilon_2(t) \\ Y(t) & = & \xi_2(t) + \varepsilon_3(t)\end{array}\right. \quad (68)$$

for a constant $k > 0$ that will be specified.

Then the notation of Sections II-III produces the choices

$$A_1 = \begin{bmatrix} -a_2 & b_1 \\ -a_3 & -a_4\end{bmatrix}, \ A_2 = [1 \quad 0], \quad (69)$$

$F_1(s,t) = (-a_1\sin(s), b_0u(t))$, $F_2(s,t) = ks$, $K_1 = a_1$ and $K_2 = k$. With the preceding parameter choices, $A_1 + kI$ is invertible when $k^2 - 201k + 746 \neq 0$. Thus we can take $k = 1$. Then the dynamic extension (7) of Section III is

$$\left\{\begin{array}{rcl}\dot{\hat{\xi}}_{1,1}(t) & = & -\hat{\xi}_{1,1}(t) + 15\hat{\xi}_{1,2}(t) - 35\sin(Y(t)) \\ \dot{\hat{\xi}}_{1,2}(t) & = & -36.4\hat{\xi}_{1,1}(t) - 200\hat{\xi}_{1,2}(t) + 40u(t) \\ \dot{\hat{\xi}}_2(t) & = & \hat{\xi}_{1,1}(t) - \hat{\xi}_2(t) + Y(t) \\ \dot{\psi}_1(t) & = & -\psi_1(t) - \frac{1}{546}\begin{bmatrix}199 \\ 15\end{bmatrix}[Y(t) - \hat{\xi}_2(t)] \\ \dot{\psi}_2(t) & = & \begin{bmatrix}-1 & 36.4 \\ -15 & 198\end{bmatrix}\psi_2(t) \\ & & -\frac{1}{546}\begin{bmatrix}199 \\ 15\end{bmatrix}[Y(t) - \hat{\xi}_2(t)].\end{array}\right. \quad (70)$$

Then, Theorem 1 provides the exact value

$$\begin{array}{rcl}\xi_1(t) & = & \hat{\xi}_1(t) + \mathcal{R}(\xi_2(t) - \hat{\xi}_2(t)) + \epsilon_\star(t) \\ & & +\mathcal{S}^{-1}\left[e^{-\tau}\psi_1(t-\tau) - \psi_1(t)\right] \\ & & +\mathcal{S}^{-1}\left[\psi_2(t) - e^{-(A_1^\top + 2I)\tau}\psi_2(t-\tau)\right]\end{array} \quad (71)$$

for all $t \geq \tau$ for a function $\epsilon_\star$ satisfying (15). The preceding observer contrasts significantly with the fixed time observer for (67) that was presented in [7, Section 5.2], whose fixed convergence time τ is required to be such that $e^{-H\tau} - e^{-\tau A}$ is invertible where $H = A + L_AC$ for a suitable matrix L_A, and where C is from the representation $y = Cx$ of the output in terms of the state x. Moreover, [7, Section 5.2] produces large coefficients in the final estimation error under discrete time measurements for small $\tau > 0$ values. Hence, we believe that the observer designs from this work offer potential advantages over previously available observers.

B. Illustration of Theorem 2

We next illustrate the sampled output case using the following pendulum dynamics that was studied, e.g., in [1]:

$$\begin{cases} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\sin(x_1(t)), \\ y(t_j) &= x_1(t_j) + \epsilon_3(t_j) \end{cases} \tag{72}$$

with $x_1(t)$ and $x_2(t)$ both valued in $\mathbb{R}$. This has the form (38) with $\xi_1 = x_2$, $\xi_2 = x_1$, $A_1 = 0$, $F_1(s,t) = -\sin(s)$, $A_2 = 1$, $F_2(s,t) = ks$, $\epsilon_1 = \epsilon_2 = 0$, and any positive constant k, by subtracting and adding $kx_1(t)$ in the x_1 dynamics. Hence, Theorem 2 applies for any positive values k and τ if the sup $\overline{T}$ of the sampling intervals satisfies $\overline{T} < 1/\mu$, where μ was defined by (42). For example, by choosing $k = 1.2$ and $\tau = 0.6$, we get the bound $\overline{T} < 1/\mu = 0.0159168$. On the other hand, smaller values of $\overline{T}$ produce faster convergence rates in the exponential input-to-state stability estimate in the theorem; see Remark 3 above. The upper bound $1/\mu$ depends on the choices of k and τ. In Figure 2, we used Mathematica to plot $1/\mu$ on the vertical axis, as a function of k and τ for the preceding values of the other parameters. Our figure shows how smaller k values for given choices

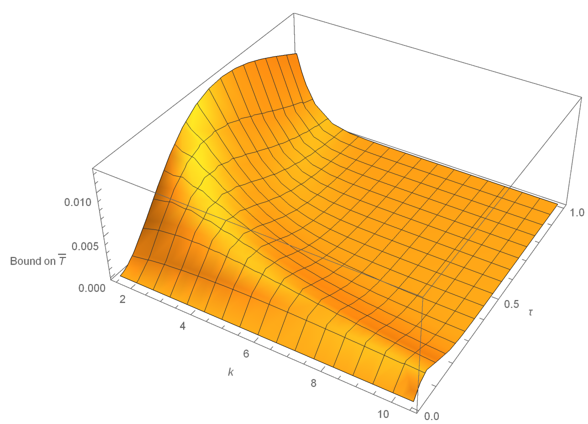

Fig. 1. Bound $1/\mu$ on $\overline{T}$ for Values $k \in [1.5, 10.5]$ and $\tau \in [0.01, 1]$.

of τ lead to larger values of the upper bound for $\overline{T}$. This illustrates the tradeoffs between the choices of τ and k and the sample interval bounds $\overline{T}$.

VI. Conclusion

We solved significant observer design problems for systems that are linear in the unmeasured variables and with continuous but perturbed measurements, using a novel Gramian approach which eliminates an invertibility condition on the fixed convergence time from prior works such as [7], while only requiring a single delay in the observer. We also provided an analog for cases where only discrete measurements are available and where we instead get arbitrarily large exponential convergence rates. Our examples illustrated the potential advantages of our new approaches and a tradeoff between sample rates in the measurements and the parameter choices in our observer. We aim to obtain extensions for systems with state delays or measurement delays.

Appendix: Invertibility of the Matrix $\mathcal{S}$

Let us prove that the matrix $\mathcal{S}$ defined in (5) is invertible, which was needed for the observer designs from our theorems. Let $V \in \mathbb{R}^{n-q}$ be a vector such that $\mathcal{S}V = 0$. Then

$$\int_{-\tau}^{0} V^\top \lambda(m)^\top \lambda(m) V \mathrm{d}m = 0. \tag{A.1}$$

As an immediate consequence, we get $\lambda(m)V = 0$ for all $m \in [-\tau, 0]$. It follows that for all integers $j > 0$, $\lambda^{(j)}(0)V = 0$. Also, simple calculations give $\lambda^{(j)}(0) = -A_2(A_1 + kI)^{j-1}$. Thus $A_2(A_1 + kI)^l V = 0$ for all integers $l \geq 0$. Using the fact that these equalities are equivalent to

$$A_2 \sum_{j=0}^{l} C_l^j A_1^j V = 0 \tag{A.2}$$

for suitable nonzero integers C_l^j, we deduce that $A_2 A_1^l V = 0$ for all integers $l \geq 0$, by induction on l. Since (A_1, A_2) is observable, it follows that $V = 0$. This allows us to conclude.

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