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Abstract

We prove a new robust stabilization theorem for time-varying systems that have sampled measurements and time-varying disturbances, and we provide new observer designs, using novel bounds for fundamental matrices for systems with disturbances. Our main tools use properties of Metzler matrices and positive systems.

Key words: Stabilization, sampling, positive system

1 Introduction

This paper is devoted to the problem of constructing matrix valued interval observers for fundamental matrices of time-varying linear systems with unknown coefficient matrices, and to demonstrating the usefulness of these constructions for feedback stabilization under sampling and for observer design. Time-varying systems can be difficult to study because in general, no explicit formulas for fundamental solutions (i.e., state transition matrices) can be obtained and because they can be difficult to estimate when the vector fields defining the systems are uncertain. An added difficulty arises when one must find globally asymptotically stabilizing feedback controls when only sampled state measurements are available. While sampling problems can be expressed as delay compensation problems (such as those of [4] and [6]) with time-varying sawtooth shaped delay functions, delay compensation methods generally require exact knowledge, or close approximations, of vector fields defining the given system, or can lead to bounds on the allowable delays (and so also bounds on the allowable sampling rates) that may be too small for applications. See [11], [12], and [20] for motivation for analyzing the effects of sampling in control systems.

To help us address these challenges, this paper presents a new construction of functions that are expressed in terms of fundamental solutions and which upper and lower bound the entries of the state transition matrix of a linear system whose bounded vector field is uncertain, where the inequalities are componentwise. Since the fundamental solutions in the bounds are those of time-varying systems with known vector fields (which can in turn be computed using a dynamical extension), this helps overcome the challenge of estimating the effects of uncertain vector fields. We apply this result to construct robust stabilizing controls for linear systems that have discrete measurements. Our control formula is the one in [9, Chapter 2, Section 2.9], which is based on the controllability Gramian and which is applied in [9] to systems whose state is available for continuous measurement. By contrast, here we apply the approach from [9] to systems whose state variables are only available for measurement at discrete instants. This makes it possible for us to cover situations in which the sup norm of an uncertainty in the dynamics or the sample rates are arbitrarily large, which we believe is a novel and significant contribution. Our proofs are based on comparison systems which share common features with interval observers (as defined, e.g., in [5], [7], and [17]), but our approach is outside the scope of earlier interval observer methods that did not take sampling and unknown vector fields into account.

Our main strategy entails novel applications of properties of positive systems and Metzler matrices. Although positive systems have been used in aerospace engineering, mathematical biology, and other fields, we believe that our work is the first one to use them for sample data feedback stabilization when there are uncertain vector fields, and where the inter-sample intervals or sup norms of the uncertainty can be arbitrarily large. This paper improves on our preliminary conference version [15] by allowing time-varying coefficients and additional uncertainties in the vector fields defining the system that were not considered in [15] (in the B coefficient that multiplies the controls) and

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by applying our fundamental solution estimation approach to discrete time systems and to design new observers [3], using novel applications of the Gramian matrix. This contrasts with [15], which was confined to time-invariant coefficients and which did not include observer designs or discrete time systems or the uncertainty δ_B below.

After providing the required definitions and notation, we provide our main feedback stabilization theorem in Section 3. In Section 4, we state our new estimation theorems, which are of independent interest because they help address challenges of estimating fundamental matrices for uncertain dynamics, and which we use to prove our main stabilization theorem in Section 5. In Section 6, we provide a discrete time analog of our stabilization theorem, whose proof uses discrete time versions of our theorems on estimating fundamental solutions for uncertain dynamics. In Section 7, we use our approach to provide new observer designs. We illustrate potential advantages of our approach in Section 8, and we close in Section 9 with a summary of our contributions and our suggestions for future research.

2 Definitions and Notation

We use the following definitions and notation thoughout this paper, where the dimensions of our systems are arbitrary unless we indicate otherwise. The notation will be simplified when no confusion could arise given the context, and all matrices in this paper are assumed to have only real entries. A square matrix is called Metzler provided all of its off-diagonal entries are nonnegative. A matrix is called nonnegative provided all of its entries are nonnegative. A square matrix is called Schur stable provided all of its eigenvalues have norm strictly less than one. For vectors $V_1 = (v_{1,1} \; .... v_{1,n})^\top$ and $V_2 = (v_{2,1} \; .... v_{2,n})^\top$, we write $V_1 < V_2$ provided for all $i \in \{1, ..., n\}$, $v_{1,i} < v_{2,i}$; and $V_1 \leq V_2$ provided for all $i \in \{1, ..., n\}$, $v_{1,i} \leq v_{2,i}$. We use analogous notation for matrices. Let $|| \cdot ||$ denote the standard Euclidean 2-norm of vectors and matrices. We also use the notation $|M|_\infty = [||m_{ij}||_\infty]$ when $M = [m_{ij}]$ is a bounded matrix valued function, where $||f||_\infty$ for a function f is the usual sup norm, and which we denote by $|M|$ when M is a constant matrix, so $|M| = [|m_{ij}|]$.

We set $G^+ = [\max\{g_{ij}, 0\}]$ and $G^- = G^+ - G$. Notice for later use that if M_1 and M_2 are matrices of the same size and $M_1 \leq M_2$, then the ordering properties $M_1^+ \leq M_2^+$ and $M_2^- \leq M_1^-$ and satisfied; the preceding two properties follow by separately considering the possible signs of the (i, j) entries of M_1^+, M_2^+, M_1^-, and M_2^- for all pairs (i, j). Let I be the identity matrix of any dimension, and 0 denote the matrix of any dimensions whose entries are all zero. Two square matrices M_1 and M_2 of the same size are called similar provided there is an invertible matrix P so that $M_1 = P^{-1} M_2 P$. For any continuous function $\mathcal{F} : [0, +\infty) \to \mathbb{R}^{n \times n}$, the fundamental (or state transition) matrix solution $\Phi_\mathcal{F}$ is defined to be the unique matrix valued function satisfying

$$\frac{\partial \Phi_\mathcal{F}}{\partial t}(t, t_0) = \mathcal{F}(t)\Phi_\mathcal{F}(t, t_0), \;\; \Phi_\mathcal{F}(t_0, t_0) = I \qquad (1)$$

for all real t_0 and t. We use basic properties of input-to-state stability (or ISS, which we also use to mean input-to-state stable), e.g., from [19], and we set $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$.

3 Stabilization Theorem

We study systems of the form

$$\dot{X}(t) = \big(M(t) - \delta_1(t)\big)X(t) + \big(B(t) + \delta_B(t)\big)U(t) + \delta_2(t), \quad (2)$$

with X valued in $\mathbb{R}^n$, the control U valued in $\mathbb{R}^p$, $M : [0, +\infty) \to \mathbb{R}^{n \times n}$ and $B : [0, +\infty) \to \mathbb{R}^{n \times p}$ being continuous matrix valued functions, and $\delta_1 : [0, +\infty) \to \mathbb{R}^{n \times n}$, $\delta_B : [0, +\infty) \to \mathbb{R}^{n \times p}$, and $\delta_2 : [0, +\infty) \to \mathbb{R}^n$ being piecewise continuous locally bounded functions. The output measurement Y is defined by $Y(t) = X(t_i)$ for all $t \in [t_i, t_{i+1})$, where $t_0 = 0$ and the t_i's are times when new state measurements become available, and we define $\sigma : [0, +\infty) \to [0, +\infty)$ by $\sigma(t) = t_i$ for all $t \in [t_i, t_{i+1})$. We assume the following conditions (but see Remark 2 for non-Metzler cases and a discussion of the class of time-varying systems that can be transformed into the form (2)):

Assumption 1 *There are known constant matrices $\overline{\delta} \geq 0$ and $\overline{\delta}_B \geq 0$ such that for all $t \geq 0$, we have*

$$0 \leq \delta_1(t) \leq \overline{\delta} \;\; \text{and} \;\; |\delta_B(t)| \leq \overline{\delta}_B. \qquad (3)$$

Also, the matrix $M(t)$ is Metzler for each $t \geq 0$, M and B both have some period $p_0 > 0$, and there are known positive constants η and ν such that $\eta \leq t_{i+1} - t_i \leq \nu$ for all $i \geq 0$ and such that with the choice

$$\mathcal{S} = \{(r, s) \in \mathbb{R}^2 : r \in [0, p_0], \eta \leq s - r \leq \nu\}, \qquad (4)$$

the system $\dot{z}(t) = M(t)z(t) + B(t)u(t)$ is controllable on $[\sigma, \tau]$ for all $(\sigma, \tau) \in \mathcal{S}$. □

We then use the functions and matrices

$$\begin{aligned} &\overline{\xi}(r, s) = \frac{\Phi_{M+\overline{\delta}}(r,s) + \Phi_{M-\overline{\delta}}(r,s)}{2} - \Phi_M(r, s), \\ &\underline{\xi}(r, s) = \frac{\Phi_{M-\overline{\delta}}(r,s) - \Phi_{M+\overline{\delta}}(r,s)}{2}, \;\; S(r) = B(r)B^\top(r), \\ &\text{and} \;\; \chi(r, s) = \int_r^s \Phi_M(r, \ell)S(\ell)\Phi_M^\top(r, \ell)\mathrm{d}\ell. \end{aligned} \qquad (5)$$

Then our controllability assumption from Assumption 1 is equivalent to the existence of the inverse $\chi^{-1}(r, s)$ for all $(r, s) \in \mathcal{S}$, e.g., by [18, Theorem 5]. Then, for all $(r, s) \in \mathcal{S}$, and with

$$\begin{aligned} &J_\delta(r, s) = \\ &(s-r)e^{(|M|_\infty + |\overline{\delta}|)(s-r)}\overline{\delta}_B|B^\top|_\infty e^{|M^\top|_\infty(s-r)}|\chi^{-1}(r, s)|, \end{aligned} \qquad (6)$$

the following matrices are well defined:

$$\begin{aligned} \overline{\mathcal{F}}_c(r, s) \; &= \; \int_r^s \Big[\overline{\xi}(s, \ell)\left(S(\ell)\Phi_M^\top(r, \ell)\chi^{-1}(r, s)\right)^- \\ &\quad - \underline{\xi}(s, \ell)\left(S(\ell)\Phi_M^\top(r, \ell)\chi^{-1}(r, s)\right)^+\Big]\mathrm{d}\ell \\ &\quad + \overline{\xi}(s, r) + J_\delta(r, s) \\ \underline{\mathcal{F}}_c(r, s) \; &= \; \int_r^s \Big[\underline{\xi}(s, \ell)\left(S(\ell)\Phi_M^\top(r, \ell)\chi^{-1}(r, s)\right)^- \\ &\quad - \overline{\xi}(s, \ell)\left(S(\ell)\Phi_M^\top(r, \ell)\chi^{-1}(r, s)\right)^+\Big]\mathrm{d}\ell \\ &\quad + \underline{\xi}(s, r) - J_\delta(r, s) \end{aligned} \qquad (7)$$

Using the preceding definitions, and the entrywise supremum notation $\sup_{v \in \mathcal{S}} \mathcal{G}(v) = [\sup_{v \in \mathcal{S}} \mathcal{G}_{ij}(v)]$ for matrix valued functions $\mathcal{G}(v) = [\mathcal{G}_{ij}(v)]$, our stabilization theorem is:

Theorem 1 *Let (2) satisfy Assumption 1, and choose p_0, ν, η, and $\mathcal{S}$ to satisfy the requirements from Assumption 1. Then (2), in closed-loop with the control defined by*

$$U(t) = -B^\top(t)\Phi_M^\top(t_i,t)\chi^{-1}(t_i,t_{i+1})X(t_i) \tag{8}$$

for all $\in [t_i, t_{i+1})$ and $i \geq 0$, is ISS with respect to $\delta_2(t)$ if the matrix

$$\Gamma = \max_{(r,s)\in\mathcal{S}} \begin{bmatrix} \overline{\mathcal{F}}_c^+(r,s) & \underline{\mathcal{F}_c}^-(r,s) \\ \underline{\mathcal{F}_c}^-(r,s) & \overline{\mathcal{F}}_c^+(r,s) \end{bmatrix} \tag{9}$$

is Schur stable. □

Before presenting our fundamental matrix estimation theorems (and then our proof of Theorem 1, in Section 5), we provide comments on the novelty and value of Theorem 1.

Remark 1 *A key aspect of Theorem 1 is that one can check whether Γ is Schur stable because the matrices (7) are expressed in terms of known functions, while the state transition matrix of $M - \delta_1(t)$ cannot be found because δ_1 is an unknown uncertainty. The periodicity in Assumption 1 makes it possible to express the maximum in the Γ formula over the compact set $\mathcal{S}$ (where the maximum exists by the continuity of the entries of the matrix in (9) as functions of $(r,s) \in \mathcal{S}$), and then the eigenvalues of Γ can be computed numerically to check the Schur stability condition, e.g., using the command* `Eigenvalues` *in Mathematica. Moreover, in the important special case where M is constant, Φ_M is a matrix exponential, so in that case we can express $\bar{\mathcal{F}}_c^+(r,s)$ and $\underline{\mathcal{F}_c}^-(r,s)$ as functions of $s-r$. Then the maximum of the matrix in (9) can be computed over the interval $[\eta,\nu]$. This includes the special case where the sample times are $t_i = i\nu$ for all $i \in \mathbb{Z}_0$, in which case we can take $\eta = \nu$ and $\mathcal{S} = \{(r, r+\nu) \in \mathbb{R}^2 : r \in [0, p_0]\}$. Although fundamental solutions for the known matrices M and $M \pm \overline{\delta}$ are used in the formula for the control and in (9), they can be computed from solving matrix valued differential equations, e.g., using the method from [10]. In fact, as noted in [10], we can compute $\Phi_A(t,s)$ for any piecewise continuous locally bounded square matrix valued function A by writing $\Phi_A(t,s) = \alpha_A(t)\beta_A(s)$ where α_A and β_A are the unique solutions of the matrix differential equations*

$$\dot{\alpha}_A(t) = A(t)\alpha_A(t) \text{ and } \dot{\beta}_A(s) = -\beta_A(s)A(s) \tag{10}$$

that satisfy $\alpha_A(0) = \beta_A(0) = I$. When $\delta_2(t)$ is the zero function, our proof of Theorem 1 will imply that the origin of the closed-loop system from Theorem 1 is a globally exponentially stable equilibrium on $\mathbb{R}^n$. For methods to compute $\overline{\mathcal{F}}_c$ and $\underline{\mathcal{F}}_c$ when M and B are constant, see [1, p.34]. □

Remark 2 *For any system of the form*

$$\dot{x}(t) = M_0(t)x(t) + B(t)U(t) \tag{11}$$

that is controllable on $[\sigma,\tau]$ for all $(\sigma,\tau) \in \mathcal{S}$ with state space $\mathbb{R}^n$ with bounded continuous matrix valued coefficients $M_0(t)$ and $B(t)$, we can find a matrix $\bar{\Delta}_M \in \mathbb{R}^{n\times n}$ such that for any continuous time-varying matrix valued function $\Delta_M(t)$ that satisfies $|\Delta_M(t)| \leq \bar{\Delta}_M$ for all $t \geq 0$, the system

$$\dot{X}(t) = (M_0(t) + \Delta_M(t))X(t) + B(t)U(t) \tag{12}$$

is also controllable on these intervals. This follows from the continuity of fundamental solutions for systems as functions of the entries of the matrix defining the system and standard characterizations of controllability, e.g., from [18, Chapter 3]. In the preceding situation, we can write $M_0(t) + \Delta_M(t) = M(t) - \delta_1(t)$, where $M(t) = M_0(t) + (\Delta_M(t))^+$ and $\delta_1(t) = (\Delta_M(t))^-$. If, in addition, $M_0(t)$ is Metzler for all $t \geq 0$, then this defines a class of time-varying systems (2) that satisfy the requirements of Assumption 1 with $\overline{\delta} = \bar{\Delta}_M$, because $0 \leq (\Delta_M(t))^- \leq (\Delta_M(t))^+ + (\Delta_M(t))^- = |\Delta_M(t)| \leq \bar{\Delta}_M$ for all $t \geq 0$. See, e.g., [18, Theorem 5] for easily checked sufficient conditions for controllability of (11), based on using the invertibility of the Gramian.

Moreover, we can often use a preliminary change of coordinates to ensure that the vector field in the drift term has the required Metzler property. For instance, for any controllable pair (A_0, B_0) of constant matrices for which all eigenvalues of A_0 are real, we can transform a linear system of the form $\dot{x}(t) = A_0x(t) + B_0U_0(t)$ into a new system $\dot{z}(t) = M_0z(t) + BU(t)$ that is controllable and for which M_0 is Metzler. This is done by choosing $M_0 = P_aA_0P_a^{-1}$, $B = P_aB_0$, and $z = P_ax$, where P_a is chosen so that M_0 is the Jordan canonical form of A_0, which makes M_0 Metzler. See also Section 7.2 where systems satisfying the requirements of Assumption 1 are useful for observer design.

By using Theorem 3 from Section 4.2 below, we can generalize Theorem 1 to cases where the matrices $M(t)$ are not Metzler, but where they are similar to Metzler matrices, by first using a change of coordinates to transform (2). The control (8) is the control from [9, Chapter 2, Section 2.9]. One can find matrices δ_* such that Γ is Schur stable when

$$0 \leq \overline{\delta} \leq \delta_* \text{ and } |\delta_B| \leq \delta_*, \tag{13}$$

because $\underline{\xi}$ and $\overline{\xi}$ are continuous and equal zero when $\overline{\delta} = 0$. This facilitates verifying the assumptions of Theorem 1. □

4 Estimation of Fundamental Matrices

This section provides our theorems on estimating fundamental matrices. In Section 5, we show how these estimation theorems can be used to prove our Theorem 1, and we use them for observer designs in Section 7.

4.1 Metzler Case

First, we address the problem of estimating fundamental solutions, when there is a known Metzler valued term in the vector field defining the dynamics; see Section 4.2 for a method for relaxing this Metzler requirement to cases where the corresponding matrix only needs to be similar to a Metzler matrix at each time t. Throughout this subsection, we assume that the given function Δ satisfies:

Assumption 2 *The function $\Delta : [0,+\infty) \to \mathbb{R}^{n\times n}$ is piecewise continuous, and there is a known constant nonnegative matrix $\overline{\Delta} \in \mathbb{R}^{n\times n}$ such that $0 \leq \Delta(t) \leq \overline{\Delta}$ holds for all $t \geq 0$.* □

In Appendix A2 below, we prove:

Theorem 2 *Let $\mathcal{M} : [0, +\infty) \to \mathbb{R}^{n\times n}$ be a matrix valued function such that $\mathcal{M}(t)$ is Metzler for all $t \geq 0$, and Δ be a function satisfying Assumption 2. Choose the functions*

$$\begin{aligned} &\overline{\Phi}_{\mathcal{M},\overline{\Delta}}(r, s) = \tfrac{\Phi_{\mathcal{M}-\overline{\Delta}}(r,s)+\Phi_{\mathcal{M}+\overline{\Delta}}(r,s)}{2} \text{ and} \\ &\underline{\Phi}_{\mathcal{M},\overline{\Delta}}(r, s) = \Phi_{\mathcal{M}}(r, s) + \tfrac{\Phi_{\mathcal{M}-\overline{\Delta}}(r,s)-\Phi_{\mathcal{M}+\overline{\Delta}}(r,s)}{2}. \end{aligned} \tag{14}$$

Then the inequalities

$$\underline{\Phi}_{\mathcal{M},\overline{\Delta}}(t, t_0) \leq \Phi_{\mathcal{M}-\Delta}(t, t_0) \leq \overline{\Phi}_{\mathcal{M},\overline{\Delta}}(t, t_0) \tag{15}$$

hold for all $t \geq t_0$ and $t_0 \geq 0$. □

Remark 3 *We believe that the preceding result is new, even in the special case where $\mathcal{M}$ is constant, in which case the fundamental solutions in the formulas (14) have the form*

$$\Phi_{\mathcal{M}\pm\overline{\Delta}}(r, s) = e^{(\mathcal{M}\pm\overline{\Delta})(r-s)}. \tag{16}$$

For cases where $\mathcal{M}$ is time-varying, the Metzler requirements from Theorem 2 are satisfied if $\mathcal{M}$ has the form $\mathcal{M}(t) = \mathcal{M}_0+\delta_{\mathcal{M}}(t)$, when the constant Metzler matrix $\mathcal{M}_0$ has positive off-diagonal entries, and when the sup norm of $\delta_{\mathcal{M}}$ is small enough. Another significant special case is when Δ and $\mathcal{M}$ are both constant, i.e., $\Delta(t) = \Delta_c$ for all $t \geq 0$, in which case $\Phi_{\mathcal{M}-\Delta}(t, t_0) = e^{(\mathcal{M}-\Delta_c)(t-t_0)}$. □

Remark 4 *A different approach to estimating fundmanental solutions of uncertain matrices is as follows. For any square matrices $\mathcal{A}$ and $\mathcal{B}$ having the same dimension, the inequality*

$$\left|\left|e^{\mathcal{A}+\mathcal{B}} - e^{\mathcal{A}}\right|\right| \leq ||\mathcal{B}||e^{\sup_{\ell\in[0,1]} ||\mathcal{A}+\ell\mathcal{B}||} \tag{17}$$

is satisfied; this follows by applying the Mean Value Theorem to the function $f(\ell) = e^{\mathcal{A}+\ell\mathcal{B}}$ on $[0, 1]$. Taking $\Delta = \Delta_c$, it follows from (17) that in the special case where $\mathcal{M}$ is constant and Δ is a constant matrix Δ_c, we have

$$\begin{aligned} &||\Phi_{\mathcal{M}-\Delta}(t, t_0) - \Phi_{\mathcal{M}}(t, t_0)|| \\ &\leq (t - t_0)||\Delta_c||e^{(t-t_0)\sup_{\ell\in[-1,1]} ||\mathcal{M}+\ell\Delta_c||} \end{aligned} \tag{18}$$

for all $t \geq t_0 \geq 0$. By contrast, in the special case where $\mathcal{M}$ and Δ are constant matrices, (15) gives

$$\underline{\varrho}(t - t_0) \leq \Phi_{\mathcal{M}-\Delta}(t, t_0) - \Phi_{\mathcal{M}}(t, t_0) \leq \overline{\varrho}(t - t_0) \tag{19}$$

where

$$\begin{aligned} &\overline{\varrho}(\ell) = \tfrac{e^{(\mathcal{M}-\Delta_c)\ell}-e^{\mathcal{M}\ell}+e^{(\mathcal{M}+\Delta_c)\ell}-e^{\mathcal{M}\ell}}{2} \text{ and} \\ &\underline{\varrho}(\ell) = \tfrac{e^{(\mathcal{M}-\Delta_c)\ell}-e^{\mathcal{M}\ell}+e^{\mathcal{M}\ell}-e^{(\mathcal{M}+\Delta_c)\ell}}{2} \end{aligned} \tag{20}$$

which gives (18), because of (17). The main advantage of using Theorem 2 in this time invariant case (instead of using the estimates that can be obtained directly from (17)) is that it produces estimates for each entry of $\Phi_{\mathcal{M}-\Delta}(t, t_0)$; see Section 8 for an example that illustrates this advantage of using our approach from Theorem 2. □

4.2 More General Case

Although Theorem 2 only applies when $\mathcal{M}(t)$ is Metzler for each t, we can relax this Metzler requirement to cover a broader family of matrices that are similar to Metzler matrices at each time t. The price to pay for this generalization is that the upper and lower bounding functions have more complicated forms. In the special case where $\mathcal{M}$ is constant, this generalization is motivated in part by the fact that Jordan canonical forms of matrices with real eigenvalues are Metzler. Using the theorem from this subsection, we can also relax the Metzler condition from Assumption 1 so that we only require $\mathcal{M}$ to be similar to a Metzler matrix at each time $t \geq 0$. In Appendix A3 below, we prove the following:

Theorem 3 *Let $A : [0, +\infty) \to \mathbb{R}^{n\times n}$ be a matrix valued function that admits an invertible matrix $P \in \mathbb{R}^{n\times n}$ and a matrix valued function $F : [0, +\infty) \to \mathbb{R}^{n\times n}$ such that $F(t)$ is Metzler for each $t \geq 0$ and such that $PA(t)P^{-1} = F(t)$ for each $t \geq 0$. Let the piecewise continuous function $\rho : [0, +\infty) \to \mathbb{R}^{n\times n}$ and the matrices $\overline{\rho} \in \mathbb{R}^{n\times n}$ and $\underline{\rho} \in \mathbb{R}^{n\times n}$ be such that*

$$\underline{\rho} \leq \rho(t) \leq \overline{\rho} \tag{21}$$

for all $t \geq 0$. Choose the functions

$$\begin{aligned} \Phi_{\pounds}(r, s) \;=\; & \tfrac{(P^{-1})^+}{2} \left(\Phi_{M-\tilde{\varsigma}}(r, s) + \Phi_{M+\tilde{\varsigma}}(r, s)\right) P^+ \\ & -(P^{-1})^- \left(\Phi_M(r, s) + \mathcal{F}(r, s)\right) P^+ \\ & -(P^{-1})^+ \left(\Phi_M(r, s) + \mathcal{F}(r, s)\right) P^- \\ & +\tfrac{(P^{-1})^-}{2} \left(\Phi_{M-\tilde{\varsigma}}(r, s) + \Phi_{M+\tilde{\varsigma}}(r, s)\right) P^- \\ \Phi_*(r, s) \;=\; & (P^{-1})^+ \left(\Phi_M(r, s) + \mathcal{F}(r, s)\right) P^+ \\ & -\tfrac{(P^{-1})^-}{2} \left(\Phi_{M-\tilde{\varsigma}}(r, s) + \Phi_{M+\tilde{\varsigma}}(r, s)\right) P^+ \\ & -\tfrac{(P^{-1})^+}{2} \left(\Phi_{M-\tilde{\varsigma}}(r, s) + \Phi_{M+\tilde{\varsigma}}(r, s)\right) P^- \\ & +(P^{-1})^-(\Phi_M(r, s)+\mathcal{F}(r, s))P^- \end{aligned} \tag{22}$$

where

$$\mathcal{F}(r, s) = \tfrac{\Phi_{M-\tilde{\varsigma}}(r,s)-\Phi_{M+\tilde{\varsigma}}(r,s)}{2} \tag{23}$$

and

$$\begin{aligned} &\tilde{\varsigma} = \\ &\left(\left[P^+\overline{\rho} - P^-\underline{\rho}\right](P^{-1})^+ - \left[P^+\underline{\rho} - P^-\overline{\rho}\right](P^{-1})^-\right)^+ \\ &- \left[P^+\underline{\rho} - P^-\overline{\rho}\right](P^{-1})^+ + \left[P^+\overline{\rho} - P^-\underline{\rho}\right](P^{-1})^- \end{aligned} \tag{24}$$

and

$$\begin{aligned} M(t) \;=\; & F(t) + \left(\left[P^+\overline{\rho} - P^-\underline{\rho}\right](P^{-1})^+ \right. \\ & \left. + \left[P^-\overline{\rho} - P^+\underline{\rho}\right](P^{-1})^-\right)^+ \end{aligned} \tag{25}$$

for all $t \geq 0$. Then the inequalities

$$\Phi_*(t, t_0) \leq \Phi_{A+\rho}(t, t_0) \leq \Phi_{\pounds}(t, t_0) \tag{26}$$

hold for all $t \geq t_0$ and $t_0 \geq 0$. □

5 Proof of Theorem 1

This section uses Theorem 2 to prove Theorem 1. The proof consists of three main parts. In the first part, we derive the dynamics for the discrete time variables $x_k = X(t_k)$, which is a sample data system that corresponds to the closed loop system from Theorem 1. In the second part, we prove an ISS estimate for this discrete time system that is based on our Schur stability assumption on (9) from the theorem. In the last part, we use the ISS property for the discrete time system to obtain the required ISS property for the system from the conclusion of Theorem 1.

First Step: Deriving the Discrete Time System. Let $i \geq 0$ be an integer. By applying the method of variation of parameters to the system (2) on the interval $[t_i, t]$ (e.g., [18, Formula (C.26)]), with $t \in [t_i, t_{i+1}]$, we get

$$\begin{aligned} X(t) \;=\; & \Phi_{M-\delta_1}(t,t_i)X(t_i) \\ & + \textstyle\int_{t_i}^{t} \Phi_{M-\delta_1}(t,\ell)B^\sharp(\ell)U(\ell)\mathrm{d}\ell \\ & + \textstyle\int_{t_i}^{t} \Phi_{M-\delta_1}(t,\ell)\delta_2(\ell)\mathrm{d}\ell, \end{aligned} \tag{27}$$

where $B^\sharp(\ell) = B(\ell) + \delta_B(\ell)$.

Then the semigroup property of fundamental solutions gives $\Phi_M(t,t_i)\Phi_M(t_i,\ell) = \Phi_M(t,\ell)$ and therefore also

$$\begin{aligned} & X(t) = \textstyle\int_{t_i}^{t} \Phi_{M-\delta_1}(t,\ell)\delta_2(\ell)\mathrm{d}\ell \\ & + \left\{\Phi_M(t,t_i)\left[X(t_i) + \textstyle\int_{t_i}^{t} \Phi_M(t_i,\ell)B^\sharp(\ell)U(\ell)\mathrm{d}\ell\right]\right\} \\ & + [\Phi_{M-\delta_1}(t,t_i) - \Phi_M(t,t_i)]X(t_i) \\ & + \textstyle\int_{t_i}^{t}[\Phi_{M-\delta_1}(t,\ell) - \Phi_M(t,\ell)]B^\sharp(\ell)U(\ell)\mathrm{d}\ell. \end{aligned} \tag{28}$$

Also, the feedback (8) gives

$$\begin{aligned} X(t) \;=\; & \textstyle\int_{t_i}^{t} \Phi_{M-\delta_1}(t,\ell)\delta_2(\ell)\mathrm{d}\ell + \mathcal{D}(t,t_i) + \mathcal{H}(t,t_i) \\ & + [\Phi_{M-\delta_1}(t,t_i) - \Phi_M(t,t_i)]X(t_i) \\ & + \textstyle\int_{t_i}^{t}[\Phi_M(t,\ell) - \Phi_{M-\delta_1}(t,\ell)]S_\ell^\sharp\mathrm{d}\ell X(t_i) \\ =\; & \left[-\textstyle\int_{t_i}^{t} \Phi_D(t,\ell)S_\ell^\sharp\mathrm{d}\ell + \Phi_D(t,t_i)\right]X(t_i) \\ & + \textstyle\int_{t_i}^{t} \Phi_{M-\delta_1}(t,\ell)\delta_2(\ell)\mathrm{d}\ell + \mathcal{D}(t,t_i) \\ & + \mathcal{H}(t,t_i), \end{aligned} \tag{29}$$

where $\mathcal{D}(t,t_i)$ is the quantity in curly braces in (28),

$$S_\ell^\sharp = S(\ell)\left(\Phi_M(\sigma(\ell),\ell)\right)^\top \chi_i^{-1}, \tag{30}$$

$$\begin{aligned} & \mathcal{H}(t,t_i) = \\ & -\textstyle\int_{t_i}^{t} \Phi_D(t,\ell)\delta_B(\ell)B^\top(\ell)\Phi_M^\top(t_i,\ell)\mathrm{d}\ell\chi_i^{-1}X(t_i), \end{aligned} \tag{31}$$

and

$$\Phi_D(a,b) = \Phi_{M-\delta_1}(a,b) - \Phi_M(a,b), \tag{32}$$

and where $\chi_i = \chi(t_i, t_{i+1})$. Moreover, our control (8) gives

$$\begin{aligned} & \mathcal{D}(t_{i+1},t_i) = \\ & \Phi_M(t_{i+1},t_i)\Big[X(t_i) + \textstyle\int_{t_i}^{t_{i+1}} \Phi_M(t_i,\ell)B(\ell)U(\ell)\mathrm{d}\ell \\ & + \textstyle\int_{t_i}^{t_{i+1}} \Phi_M(t_i,\ell)\delta_B(\ell)U(\ell)\mathrm{d}\ell\Big] \\ & = -\textstyle\int_{t_i}^{t_{i+1}} \Phi_M(t_{i+1},\ell)\delta_B^\sharp(\ell)\Phi_M^\top(t_i,\ell)\mathrm{d}\ell\chi_i^{-1}X(t_i) \end{aligned} \tag{33}$$

where $\delta_B^\sharp(\ell) = \delta_B(\ell)B^\top(\ell)$.

Specializing (29) to the case where $t = t_{i+1}$ therefore gives

$$\begin{aligned} X(t_{i+1}) \;=\; & W(i)X(t_i) \\ & + \textstyle\int_{t_i}^{t_{i+1}} \Phi_{M-\delta_1}(t_{i+1},\ell)\delta_2(\ell)\mathrm{d}\ell, \end{aligned} \tag{34}$$

where

$$\begin{aligned} & W(i) = \Phi_D(t_{i+1},t_i) - \textstyle\int_{t_i}^{t_{i+1}} \Phi_D(t_{i+1},\ell)S_\ell^\sharp\mathrm{d}\ell \\ & -\textstyle\int_{t_i}^{t_{i+1}} \Phi_{M-\delta_1}(t_{i+1},\ell)\delta_B^\sharp(\ell)\Phi_M^\top(\sigma(\ell),\ell)\mathrm{d}\ell\chi_i^{-1}, \end{aligned} \tag{35}$$

where the last term in (35) was obtained by adding (31) and (33). Equation (34) defines the desired closed loop discrete time system.

Second Step: Stability Analysis for Discrete Time System (34). Theorem 2 ensures that for all $a \geq b$, we have

$$\underline{\xi}(a,b) \leq \Phi_D(a,b) \leq \overline{\xi}(a,b), \tag{36}$$

where $\overline{\xi}$ and $\underline{\xi}$ are defined in (5). We deduce that, for any integer $i \geq 0$, we have

$$\begin{aligned} & \underline{\xi}(t_{i+1},\ell)\left(S(\ell)(\Phi_M(\sigma(\ell),\ell))^\top\chi_i^{-1}\right)^+ \\ & \leq \Phi_D(t_{i+1},\ell)\left(S(\ell)(\Phi_M(\sigma(\ell),\ell))^\top\chi_i^{-1}\right)^+ \\ & \leq \overline{\xi}(t_{i+1},\ell)\left(S(\ell)(\Phi_M(\sigma(\ell),\ell))^\top\chi_i^{-1}\right)^+ \end{aligned} \tag{37}$$

and

$$\begin{aligned} & \underline{\xi}(t_{i+1},\ell)\left(S(\ell)(\Phi_M(\sigma(\ell),\ell))^\top\chi_i^{-1}\right)^- \\ & \leq \Phi_D(t_{i+1},\ell)\left(S(\ell)(\Phi_M(\sigma(\ell),\ell))^\top\chi_i^{-1}\right)^- \\ & \leq \overline{\xi}(t_{i+1},\ell)\left(S(\ell)(\Phi_M(\sigma(\ell),\ell))^\top\chi_i^{-1}\right)^- \end{aligned} \tag{38}$$

for all $\ell \in (t_i, t_{i+1})$. By multiplying (38) through by -1 and then adding the result to (37), we get

$$\begin{aligned} & \underline{\xi}(t_{i+1},\ell)(S_\ell^\sharp)^+ - \overline{\xi}(t_{i+1},\ell)(S_\ell^\sharp)^- \\ & \leq \Phi_D(t_{i+1},\ell)S_\ell^\sharp \leq \overline{\xi}(t_{i+1},\ell)(S_\ell^\sharp)^+ - \underline{\xi}(t_{i+1},\ell)(S_\ell^\sharp)^- \end{aligned} \tag{39}$$

for all $\ell \in (t_i, t_{i+1})$, where we used the fact that $S_\ell^\sharp = (S_\ell^\sharp)^+ - (S_\ell^\sharp)^-$.

Also, by noting that

$$|\Phi_Q^\top(t,s)| \leq e^{|Q^\top|_\infty|t-s|} \text{ and } |\Phi_Q(t,s)| \leq e^{|Q|_\infty|t-s|} \tag{40}$$

hold for any bounded piecewise continuous matrix valued function $Q : [0,+\infty) \to \mathbb{R}^{n\times n}$ and all $s \geq 0$ and $t \geq 0$ (e.g., because of the Peano-Baker formula from [18, Appendix C]), we obtain the bound

$$\begin{aligned} & |\Phi_{M-\delta_1}(t_{i+1},\ell)\delta_B^\sharp(\ell)\Phi_M^\top(\sigma(\ell),\ell)\chi_i^{-1}| \\ & \leq e^{(|M|_\infty+|\delta_1|_\infty)(t_{i+1}-t_i)}\overline{\delta}_B|B^\top|_\infty e^{|M^\top|_\infty(t_{i+1}-t_i)}|\chi_i^{-1}| \end{aligned} \tag{41}$$

for all $\ell \in [t_i, t_{i+1}]$. We deduce from (36) that

$$\underline{\mathcal{F}}_c(t_i,t_{i+1}) \leq W(i) \leq \overline{\mathcal{F}}_c(t_i,t_{i+1}), \tag{42}$$

where (41) was used to bound the last term in (35) and to obtain the expression for the function J_δ that we used in (6) and our formulas (7) for $\underline{\mathcal{F}}_c$ and $\overline{\mathcal{F}}_c$.

In order to prove our ISS property for (34), we first use our bounds (42) to study the system

$$Z(t_{i+1}) = W(i)Z(t_i) \tag{43}$$

that corresponds to the $\delta_2 = 0$ case in (34). By rewriting (43) as $Z(t_{i+1}) = (W(i)^+ - W(i)^-)Z(t_i)$, we conclude that for any solution $Z(t_i)$ of (43), the function $\Xi(t_i) = [(Z(t_i))^\top, (-Z(t_i))^\top]^\top$ is a solution of

$$\begin{cases} \zeta(t_{i+1}) \;=\; W(i)^+\zeta(t_i) + W(i)^-\psi(t_i) \\ \psi(t_{i+1}) \;=\; W(i)^-\zeta(t_i) + W(i)^+\psi(t_i). \end{cases} \tag{44}$$

Moreover, (44) is a positive system (which can be checked by induction, using the fact that $W(i)^+$ and $W(i)^-$ are nonnegative matrices for each index i).

Next, let $[(\zeta_p(t_i))^\top, (\psi_p(t_i))^\top]^\top$ and $[(\zeta_n(t_i))^\top, (\psi_n(t_i))^\top]^\top$ denote the solutions of (44) with initial conditions

$$\begin{pmatrix} \zeta_p(t_0) \\ \psi_p(t_0) \end{pmatrix} = \begin{pmatrix} Z(t_0)^+ \\ Z(t_0)^- \end{pmatrix} \tag{45}$$

and

$$\begin{pmatrix} \zeta_n(t_0) \\ \psi_n(t_0) \end{pmatrix} = \begin{pmatrix} Z(t_0)^- \\ Z(t_0)^+ \end{pmatrix} \tag{46}$$

respectively. By the positivity of the system (44), these two solutions are nonnegative valued. Then, noticing that

$$\begin{aligned} \Xi(t_0) &= \begin{pmatrix} Z(t_0)^+ - Z(t_0)^- \\ Z(t_0)^- - Z(t_0)^+ \end{pmatrix} \\ &= \begin{pmatrix} \zeta_p(t_0) \\ \psi_p(t_0) \end{pmatrix} - \begin{pmatrix} \zeta_n(t_0) \\ \psi_n(t_0) \end{pmatrix}, \end{aligned} \tag{47}$$

we deduce from the uniqueness of solutions property for the discrete time system (44) that

$$\Xi(t_i) = \begin{pmatrix} \zeta_p(t_i) \\ \psi_p(t_i) \end{pmatrix} - \begin{pmatrix} \zeta_n(t_i) \\ \psi_n(t_i) \end{pmatrix} \tag{48}$$

for all $i \geq 0$.

Next consider the system

$$\begin{cases} \overline{\zeta}(t_{i+1}) = \overline{\mathcal{F}}_c^+(t_i, t_{i+1})\overline{\zeta}(t_i) + \underline{\mathcal{F}}_c^-(t_i, t_{i+1})\overline{\psi}(t_i) \\ \overline{\psi}(t_{i+1}) = \underline{\mathcal{F}}_c^-(t_i, t_{i+1})\overline{\zeta}(t_i) + \overline{\mathcal{F}}_c^+(t_i, t_{i+1})\overline{\psi}(t_i), \end{cases} \tag{49}$$

and let $[(\overline{\zeta}_p(t_i))^\top, (\overline{\psi}_p(t_i))^\top]^\top$ and $[(\overline{\zeta}_n(t_i))^\top, (\overline{\psi}_n(t_i))^\top]^\top$ denote the solutions of (49) satisfying

$$\begin{aligned} &\begin{pmatrix} \overline{\zeta}_p(t_0) \\ \overline{\psi}_p(t_0) \end{pmatrix} = \begin{pmatrix} Z(t_0)^+ \\ Z(t_0)^- \end{pmatrix} \text{ and} \\ &\begin{pmatrix} \overline{\zeta}_n(t_0) \\ \overline{\psi}_n(t_0) \end{pmatrix} = \begin{pmatrix} Z(t_0)^- \\ Z(t_0)^+ \end{pmatrix} \end{aligned} \tag{50}$$

respectively. By (42) and the order preserving properties from Section 2, we have

$$W(i)^+ \leq \bar{\mathcal{F}}_c^+(t_i, t_{i+1}) \text{ and } W(i)^- \leq \underline{\mathcal{F}}_c^-(t_i, t_{i+1}), \tag{51}$$

so by the comparison principle, we have

$$\begin{aligned} &0 \leq \begin{pmatrix} \zeta_p(t_i) \\ \psi_p(t_i) \end{pmatrix} \leq \begin{pmatrix} \overline{\zeta}_p(t_i) \\ \overline{\psi}_p(t_i) \end{pmatrix} \text{ and} \\ &0 \leq \begin{pmatrix} \zeta_n(t_i) \\ \psi_n(t_i) \end{pmatrix} \leq \begin{pmatrix} \overline{\zeta}_n(t_i) \\ \overline{\psi}_n(t_i) \end{pmatrix} \end{aligned} \tag{52}$$

for all $i \geq 0$; this follows by noting that the initial conditions in the right sides of (45)-(46) agree with those of the right sides of (52), and that (51) implies that the states with the bars are growing faster. As an immediate consequence,

$$\begin{aligned} -\Gamma^i \begin{pmatrix} \overline{\zeta}_n(t_0) \\ \overline{\psi}_n(t_0) \end{pmatrix} &\leq - \begin{pmatrix} \overline{\zeta}_n(t_i) \\ \overline{\psi}_n(t_i) \end{pmatrix} \leq \Xi(t_i) \\ &\leq \begin{pmatrix} \overline{\zeta}_p(t_i) \\ \overline{\psi}_p(t_i) \end{pmatrix} \leq \Gamma^i \begin{pmatrix} \overline{\zeta}_p(t_0) \\ \overline{\psi}_p(t_0) \end{pmatrix} \end{aligned} \tag{53}$$

for all $i \geq 0$, where Γ is the Schur stable matrix from (9), and where we used (48), (52), and the periodicity assumption in Assumption 1 (which gives $\bar{\mathcal{F}}_c(r + kp_0, s + kp_0) = \bar{\mathcal{F}}_c(r, s)$ and $\underline{\mathcal{F}}_c(r + kp_0, s + kp_0) = \underline{\mathcal{F}}_c(r, s)$ for all integers $k \geq 0$ and all $r \geq 0$ and $s \geq r$). Consequently,

$$-\Gamma^i \begin{pmatrix} Z(t_0)^- \\ Z(t_0)^+ \end{pmatrix} \leq \begin{pmatrix} Z(t_i) \\ -Z(t_i) \end{pmatrix} \leq \Gamma^i \begin{pmatrix} Z(t_0)^+ \\ Z(t_0)^- \end{pmatrix}, \tag{54}$$

by our choice of Ξ. Thus,

$$\begin{aligned} -[I, 0]\Gamma^i \begin{pmatrix} Z(t_0)^- \\ Z(t_0)^+ \end{pmatrix} &\leq Z(t_i) \\ &\leq [I, 0]\Gamma^i \begin{pmatrix} Z(t_0)^+ \\ Z(t_0)^- \end{pmatrix}. \end{aligned} \tag{55}$$

By our assumption that Γ is Schur stable, this implies exponential stability of (43), hence the desired ISS property of (34), by the boundedness of δ_1 (which can be shown, e.g., by showing that the strict Lyapunov function for (43) that can be constructed from [2, Theorem 5.13] is an ISS Lyapunov function for (34)).

Third Step: ISS Conclusion of Theorem 1. We can find a constant $\bar{c} > 0$ such that all solutions of the closed loop system from the statement of Theorem 1 satisfy

$$||X(t)|| \leq \bar{c}\,(||X(\sigma(t))|| + \sup\{||\delta_2(\ell)|| : 0 \leq \ell \leq t\}) \tag{56}$$

for all $t \geq 0$. We can express $\bar{c}$ in terms of the bounds $\bar{\delta}$, $\bar{\delta}_B$, and ν from Assumption 1, as follows. Using the fact that $||\Phi_M^\top(\sigma(t), t)|| \leq e^{||M||_\infty \nu}$ for all $t \geq 0$ (e.g., using properties of transition matrices from [18, Appendix C]), we can use the structure of the system (2) to get

$$\begin{aligned} &||\dot{X}(t)|| \leq ||M - \delta_1||_\infty ||||X(t)|| + ||\delta_2(t)|| \\ &+||B+\delta_B||_\infty ||B||_\infty e^{||M||_\infty \nu} \sup_{(r,s)\in\mathcal{S}} ||\chi^{-1}(r, s)||\, ||X(\sigma(t)|| \end{aligned}$$

for all $t \geq 0$. Then we can apply the Fundamental Theorem of Calculus to the solution X on the interval $[\sigma(t), t]$ to get

$$\begin{aligned} &||X(t)|| \leq \\ &||M - \delta_1||_\infty \textstyle\int_{\sigma(t)}^t ||X(\ell)|| \mathrm{d}\ell + \nu \displaystyle\sup_{\ell \in [0,t]} ||\delta_2(\ell)|| + \Big(1+ \\ &\nu ||B+\delta_B||_\infty ||B||_\infty e^{||M||_\infty \nu} \sup_{(r,s)\in\mathcal{S}} ||\chi^{-1}(r, s)||\Big) ||X(\sigma(t)||. \end{aligned}$$

Then applying Gronwall's inequality to the preceding inequality on the interval $[\sigma(t), t]$ and recalling our bounds $\bar{\delta}$

and $\bar{\delta}_B$ on δ_1 and δ_B from (3) allows us to choose

$$\bar{c} = e^{(||M||_\infty + ||\bar{\delta}||_\infty)\nu} \max\Big\{\nu, \Big(1 + \nu(||B||_\infty + ||\bar{\delta}_B||_\infty)||B||_\infty e^{||M||_\infty \nu} \sup_{(r,s)\in\mathcal{S}} ||\chi^{-1}(r,s)||\Big)\Big\}.$$

We can combine (56) with the ISS conclusion from the second step to obtain the ISS property of the closed loop system from the theorem. This proves Theorem 1.

Remark 5 *The last part of the preceding proof shows how bigger values of $\bar{\delta}$ and $\bar{\delta}_B$ can lead to bigger $\bar{c}$ values, and therefore to larger bounds on the right side of the final ISS estimate. Since the formula for $\bar{c}$ also depends on the upper bound ν on the sample intervals, this can also provide guidance on how to choose sample times that lead to a suitable upper bound ν on the sample periods, by choosing ν so that $\bar{c}$ lies in a suitable range to ensure satisfactory performance in terms of a small enough upper bound on the right side of the bound for the state.*

6 Discrete Time Analogs

6.1 Stability Theorem

Using discrete time analogs of Theorems 1 and 2, it is possible to prove asymptotic stability conditions for systems of the form

$$X_{i+1} = (M - \Delta_i)X_i \tag{57}$$

with $M \geq 0$, and with the matrices Δ_i admitting a matrix $\overline{\Delta} \geq 0$ such that $0 \leq \Delta_i \leq \overline{\Delta}$ for all i. Although

$$M - \overline{\Delta} \leq M - \Delta_i \leq M \tag{58}$$

holds for all i, the inequalities (58) cannot be used to study the stability of (57) when some entries of $M - \bar{\Delta}$ are negative. On the other hand, we have the following theorem, whose proof in Appendix A4 uses discrete time analogs of our continuous time theorems from the preceding sections:

Theorem 4 *Let*

$$\overline{S}_j = \frac{(M+\overline{\Delta})^j + (M-\overline{\Delta})^j}{2} \text{ and } \underline{S}_j = M^j + \frac{(M-\overline{\Delta})^j - (M+\overline{\Delta})^j}{2} \tag{59}$$

for all $i \geq 0$. Assume that

$$\lim_{j\to+\infty} \overline{S}_j = 0 \text{ and } \lim_{j\to+\infty} \underline{S}_j = 0. \tag{60}$$

Then the system (57) is globally asymptotically stable to the origin on $\mathbb{R}^n$. □

6.2 Illustration

To illustrate the preceding theorem let

$$M = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \text{ and } \Delta_i = \begin{bmatrix} \delta_{1,i} & 0 \\ 0 & \delta_{2,i} \end{bmatrix}, \tag{61}$$

and assume that there is a constant $\bar{\delta} > 0$ such that $0 \leq \delta_{k,i} \leq \bar{\delta}$ for $k = 1, 2$ and all $i \geq 0$. Then Lemma A.1 from Appendix A1 below with $n = 1$ gives

$$M + \overline{\Delta} = \mathcal{P}^{-1} \begin{bmatrix} \frac{1}{2} + \bar{\delta} & 0 \\ 0 & \bar{\delta} - \frac{1}{2} \end{bmatrix} \mathcal{P}, \tag{62}$$

where $\bar{\Delta} = \bar{\delta}I$,

$$\mathcal{P} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \text{ and} \tag{63}$$

$$M - \overline{\Delta} = \mathcal{P}^{-1} \begin{bmatrix} \frac{1}{2} - \bar{\delta} & 0 \\ 0 & -\bar{\delta} - \frac{1}{2} \end{bmatrix} \mathcal{P}. \tag{64}$$

Hence, with the choices

$$\bar{\delta}_j^\sharp = \frac{(\frac{1}{2}+\bar{\delta})^j}{2} \text{ and } \bar{\mu}_j^\sharp = \frac{(\bar{\delta}-\frac{1}{2})^j}{2}, \tag{65}$$

the functions from (59) are

$$\overline{S}_j = \mathcal{P}^{-1} \begin{bmatrix} \bar{\delta}_j^\sharp + (-1)^j \bar{\mu}_j^\sharp & 0 \\ 0 & \bar{\mu}_j^\sharp + (-1)^j \bar{\delta}_j^\sharp \end{bmatrix} \mathcal{P} \tag{66}$$

and

$$\begin{aligned} \underline{S}_j &= M^j + \mathcal{P}^{-1} \begin{bmatrix} (-1)^j \bar{\mu}_j^\sharp - \bar{\delta}_j^\sharp & 0 \\ 0 & (-1)^j \bar{\delta}_j^\sharp - \bar{\mu}_j^\sharp \end{bmatrix} \mathcal{P} \\ &= \mathcal{P}^{-1} \begin{bmatrix} (\frac{1}{2})^j + (-1)^j \bar{\mu}_j^\sharp - \bar{\delta}_j^\sharp & 0 \\ 0 & \left(-\frac{1}{2}\right)^j + (-1)^j \bar{\delta}_j^\sharp - \bar{\mu}_j^\sharp \end{bmatrix} \mathcal{P}, \end{aligned} \tag{67}$$

by Lemma A.1 from Appendix A1 below with $\alpha = 0$ and $\beta = 1/2$. Hence, if $\bar{\delta} < 1/2$, then (60) will hold, which will ensure global asymptotic stability of the system.

7 Observer Designs

We next illustrate the usefulness of our continuous time transition matrix estimation and Gramian approaches, in the context of new observer designs. First we provide a finite time observer, based on our Gramian approach but not using our estimation methods. This finite time observer has the advantage of being a fixed time one, meaning, the convergence of the observer to the true state value occurs at a time that is independent of the initial state. Then we provide a new asymptotic observer for more general systems, using a convergence proof that is based on both transition matrix estimation and the Gramian. The proof of the convergence property for this asymptotic observer makes essential use of the arguments from the proof of Theorem 1 and the result from Theorem 2. Therefore, through its uses of the Gramian in our first observer design (which is an essential ingredient from Theorem 1) and the results from Theorems 1-2, this section is strongly connected to, and helps illustrate the value of, the earlier sections.

For simplicity, we assume in this section that the coefficient matrices are constant and that the sample intervals $t_{i+1} - t_i$ are of a constant positive length ν, but analogous arguments produce results for unevenly spaced sample times t_i as in the previous sections and for time-varying coefficient matrices, by replacing matrix exponentials by transition matrices in the relevant places. In both of our observer designs, we will use the matrix valued functions

$$G(S) = \left(\int_0^\nu e^{S^\top \ell} C^\top C e^{S\ell} \mathrm{d}\ell\right)^{-1} \tag{68}$$

for suitable functions S, where the existence of the inverse will follow from the observability of (S, C) for matrices S that we specify later (again by [18, Theorem 5, p.109]).

7.1 Finite Time Observer

We consider the system

$$\begin{cases} \dot{x}(t) = Ax(t) + f(y(t), u(t)) \\ y(t) = Cx(t) \end{cases} \quad (69)$$

where x is valued in $\mathbb{R}^n$, f is locally Lipschitz, u is valued in $\mathbb{R}^p$, and y is valued in $\mathbb{R}^q$. We assume that this system is forward complete, and we introduce a sequence $t_i = i\nu$ with $\nu > 0$ being a given constant. See, e.g., [8], [13], and [16] (which did not use Gramian approaches and also did not use estimators for transition matrices) for the importance of systems with nonlinearities of the form we have in (69) for modeling the vibration of elastic membranes, pendulums, and single-link robotic manipulators. Our objective is to construct a finite time observer for this system, under this standard assumption:

Assumption 3 *The pair (A, C) is observable.* □

We also use the matrix valued function $R : \mathbb{R} \to \mathbb{R}^{n\times q}$ defined by

$$R(\ell) = e^{A\ell}G(A)e^{A^\top \ell}C^\top \quad (70)$$

for all $\ell \in \mathbb{R}$, where G was defined in (68).

Let us propose the observer that is defined by

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + f(y(t), u(t)) \\ &\quad + R(t - t_i)\left[y(t) - Ce^{A(t-t_i)}\hat{x}(t_i) - C\hat{z}(t)\right] \quad (71) \\ \dot{\hat{z}}(t) &= A\hat{z}(t) + f(y(t), u(t)),\ \hat{z}(t_i) = 0 \end{aligned}$$

for all $t \in [t_i, t_{i+1})$ and $i \geq 0$, where the solutions of the $\hat{x}$ dynamics are defined by solving the first equation of (71) on $[t_0, t_1) = [0, \nu)$ with the initial state $\hat{x}(0) = 0$, then solving it on $[t_1, t_2)$ with the initial state $\hat{x}(t_1) = \hat{x}(t_1^-)$, and then repeating this process on subsequent intervals $[t_i, t_{i+1})$ for $i \geq 2$. We prove:

Theorem 5 *Let Assumption 3 hold. Then for each solution $x(t)$ of (69), the corresponding solution $\hat{x}(t)$ of the observer (71) is such that $\hat{x}(t) = x(t)$ for all $t \geq \nu$.* □

Proof. By applying variation of paramters to the dynamics in (69), we deduce that

$$y(t) = Ce^{A(t-t_i)}x(t_i) + C\int_{t_i}^{t} e^{A(t-\ell)}f(y(\ell), u(\ell))\mathrm{d}\ell. \quad (72)$$

Since $\hat{z}(t_i) = 0$, it follows that $y(t) = Ce^{A(t-t_i)}x(t_i) + C\hat{z}(t)$, so

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + f(y(t), u(t)) \\ &\quad + R(t - t_i)Ce^{A(t-t_i)}[x(t_i) - \hat{x}(t_i)]. \end{aligned} \quad (73)$$

Here and in the sequel, all equalities and inequalities are for all $t \in [t_i, t_{i+1})$ and $i \geq 0$, unless otherwise indicated. Let $\tilde{x}(t) = x(t) - \hat{x}(t)$. Then

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) - R(t - t_i)Ce^{A(t-t_i)}\tilde{x}(t_i). \quad (74)$$

By integrating this system, we deduce that

$$\begin{aligned} &\tilde{x}(t) = \\ &\left[e^{A(t-t_i)} - \int_{t_i}^{t} e^{A(t-\ell)}R(\ell - t_i)Ce^{A(\ell-t_i)}\mathrm{d}\ell\right]\tilde{x}(t_i). \end{aligned} \quad (75)$$

Thus, in particular, the equality

$$\begin{aligned} &\tilde{x}(t_{i+1}) = \\ &\left[e^{A\nu} - \int_{t_i}^{t_i+\nu} e^{A(t_{i+1}-\ell)}R(\ell - t_i)Ce^{A(\ell-t_i)}\mathrm{d}\ell\right]\tilde{x}(t_i) \end{aligned} \quad (76)$$

holds. Consequently,

$$\begin{aligned} \tilde{x}(t_{i+1}) &= e^{A\nu}\left[I - \int_0^{\nu} e^{-A\ell}R(\ell)Ce^{A\ell}\mathrm{d}\ell\right]\tilde{x}(t_i) \\ &= e^{A\nu}\left[I - \int_0^{\nu} e^{-A\ell}e^{A\ell}G(A)e^{A^\top\ell}C^\top Ce^{A\ell}\mathrm{d}\ell\right]\tilde{x}(t_i), \end{aligned} \quad (77)$$

where the last inequality is a consequence of the definition (70) of R. We deduce that

$$\begin{aligned} \tilde{x}(t_{i+1}) &= e^{A\nu}\left[I - \int_0^{\nu} G(A)e^{A^\top\ell}C^\top Ce^{A\ell}\mathrm{d}\ell\right]\tilde{x}(t_i) \\ &= 0 \end{aligned} \quad (78)$$

Thus, in particular, $\tilde{x}(\nu) = 0$. From (74), we deduce that $\tilde{x}(t) = 0$ for all $t \geq \nu$. This concludes the proof. □

7.2 Observer for Time-Varying Systems

We next consider the more general class of systems

$$\begin{cases} \dot{\xi}(t) = [A - \Delta(t)]\xi(t) + f(Y(t), u(t)) \\ Y(t) = C\xi(t) \end{cases} \quad (79)$$

where ξ is valued in $\mathbb{R}^n$, u is valued in $\mathbb{R}^p$, and Y is valued in $\mathbb{R}^q$. We assume that f is locally Lipschitz, that this system is forward complete, and that Assumption 3 is satisfied, where A, Δ, and f are assumed to be known.

Before presenting our observer design, we introduce the following dynamic extension whose purpose is to remove the term $f(Y(t), u(t))$ in (79):

$$\begin{aligned} \dot{\xi}_\star(t) &= [A - \Delta(t)]\xi_\star(t) + f(Y(t), u(t)) \\ &\quad + L[Y(t) - C\xi_\star(t)]. \end{aligned} \quad (80)$$

The term $L[Y(t) - C\xi_\star(t)]$ is here only to introduce a degree of freedom. Then, with the choice, $x(t) = \xi_\star(t) - \xi(t)$, we obtain

$$\begin{cases} \dot{x}(t) = [M - \Delta(t)]x(t) \\ y(t) = Cx(t) \end{cases} \quad (81)$$

with $M = A - LC$. In terms of our notation (68), we assume the following, where the existence of the inverse in (83) follows from the observability of (A, C) as before:

Assumption 4 *The pair (A, C) is observable, the origin of (81) is globally uniformly asymptotically stable, and M is Metzler. Also, there is a matrix $\overline{\Delta}$ such that*

$$0 \leq \Delta(t) \leq \overline{\Delta} \quad (82)$$

for all $t \geq 0$. Finally, with the choices

$$\begin{aligned} &\overline{\beta}(s) = \tfrac{e^{(M-\overline{\Delta})s} + e^{(M+\overline{\Delta})s} - 2e^{Ms}}{2}, \\ &\underline{\beta}(s) = \tfrac{e^{(M-\overline{\Delta})s} - e^{(M+\overline{\Delta})s}}{2},\ \ H(\ell) = e^{M\ell}G(M)e^{M^\top\ell}C^\top, \end{aligned} \quad (83)$$

and

$$\begin{aligned}\overline{\mathcal{G}}_c(r) &= \overline{\beta}(r) - \int_0^r \Big[\underline{\beta}(\nu-\ell)\left(H(\ell)Ce^{M\ell}\right)^+ \\ &\quad -\overline{\beta}(\nu-\ell)\left(H(\ell)Ce^{M\ell}\right)^-\Big]\,\mathrm{d}\ell \\ \underline{\mathcal{G}}_c(r) &= \underline{\beta}(r) - \int_0^r \Big[\overline{\beta}(\nu-\ell)\left(H(\ell)Ce^{M\ell}\right)^+ \\ &\quad -\underline{\beta}(\nu-\ell)\left(H(\ell)Ce^{M\ell}\right)^-\Big]\,\mathrm{d}\ell,\end{aligned} \tag{84}$$

the matrix

$$\kappa = \begin{bmatrix} \overline{\mathcal{G}}_c(\nu)^+ & \underline{\mathcal{G}}_c(\nu)^- \\ \underline{\mathcal{G}}_c(\nu)^- & \overline{\mathcal{G}}_c(\nu)^+ \end{bmatrix} \tag{85}$$

is Schur stable. □

Our condition on (81) in Assumption 4 holds if $\overline{\Delta}$ is sufficiently small and M is Hurwitz and Metzler. In this case, bounds $\bar{\Delta}$ on the allowable functions $\Delta(t)$ can be found using quadratic Lyapunov functions. For instance, we can solve the Riccati equation $P_*M + M^\top P_* = -I$ for a positive definite matrix P_*, and then the time derivative of the quadratic Lyapunov function $v(x) = x^\top P_* x$ along solutions of $\dot{x}(t) = [M - \Delta(t)]x(t)$ satisfies $\dot{v} \le -||x(t)||^2 - 2x^\top(t)P_*\Delta(t)x(t)$, and then can choose $\overline{\Delta}$ so that all of its entries are strictly less than $1/(2n||P_*||)$, because that will ensure that $||P_*||||\Delta(t)|| < \frac{1}{2}$ when Δ satisfies (82). Here the n in the denominator of the upper bound on the entries of $\bar{\Delta}$ arises from the need to convert a condition on Euclidean 2-norms of $\Delta(t)$ into a condition of the form (82) on the entries of $\Delta(t)$. We define our observer by

$$\begin{aligned}\dot{\hat{x}}(t) &= [M - \Delta(t)]\hat{x}(t) \\ &\quad + H(t-t_i)[y(t) - Ce^{M(t-t_i)}\hat{x}(t_i)]\end{aligned} \tag{86}$$

for all $t \in [t_i, t_{i+1})$ and $i \ge 0$, where H is from (83), and the solutions of (86) are defined analogously to those of the $\hat{x}$ dynamics in (71). We then have the following theorem:

Theorem 6 *Let Assumption 4 hold. Then for each solution $x(t)$ of (81) and the corresponding solution $\hat{x}(t)$ of the observer (86), we have* $\lim_{t\to+\infty}(x(t) - \hat{x}(t)) = 0$. □

Proof. Since $x(t) = \Phi_{M-\Delta}(t, t_i)x(t_i)$ for all $t \ge t_i$, we have

$$\begin{aligned}&\dot{\hat{x}}(t) = [M - \Delta(t)]\hat{x}(t) \\ &+ H(t-t_i)[C\Phi_{M-\Delta}(t,t_i)x(t_i) - Ce^{M(t-t_i)}\hat{x}(t_i)].\end{aligned} \tag{87}$$

Here and in the sequel, all equalities and inequalities are for all $t \in [t_i, t_{i+1})$ and $i \ge 0$, unless otherwise indicated. Thus

$$\begin{aligned}\dot{\hat{x}}(t) &= [M - \Delta(t)]\hat{x}(t) \\ &\quad + H(t-t_i)Ce^{M(t-t_i)}[x(t_i) - \hat{x}(t_i)] \\ &\quad + H(t-t_i)C\left[\Phi_{M-\Delta}(t,t_i) - e^{M(t-t_i)}\right]x(t_i).\end{aligned} \tag{88}$$

As an immediate consequence of (81) and (88), the variable $\tilde{x}(t) = x(t) - \hat{x}(t)$ satisfies the following:

$$\begin{aligned}\dot{\tilde{x}}(t) &= [M - \Delta(t)]\tilde{x}(t) - H(t-t_i)Ce^{M(t-t_i)}\tilde{x}(t_i) \\ &\quad - H(t-t_i)C\left[\Phi_{M-\Delta}(t,t_i) - e^{M(t-t_i)}\right]x(t_i).\end{aligned} \tag{89}$$

Applying variation of parameters to (89) gives

$$\begin{aligned}\tilde{x}(t) &= \Phi_{M-\Delta}(t,t_i)\tilde{x}(t_i) \\ &\quad + \int_{t_i}^t \Phi_{M-\Delta}(t,\ell)\left[-H(\ell-t_i)Ce^{M(\ell-t_i)}\tilde{x}(t_i)\right. \\ &\quad \left. - H(\ell-t_i)C\mathcal{E}(\ell,t_i)x(t_i)\right]\mathrm{d}\ell \\ &= \Big[\Phi_{M-\Delta}(t,t_i) \\ &\quad - \int_{t_i}^t \Phi_{M-\Delta}(t,\ell)H(\ell-t_i)Ce^{M(\ell-t_i)}\mathrm{d}\ell\Big]\tilde{x}(t_i) \\ &\quad - \int_{t_i}^t \Phi_{M-\Delta}(t,\ell)H(\ell-t_i)C\mathcal{E}(\ell,t_i)\mathrm{d}\ell\, x(t_i)\end{aligned} \tag{90}$$

and so also

$$\begin{aligned}\tilde{x}(t) &= \Big[\Phi_M(t,t_i) \\ &\quad - \int_{t_i}^t \Phi_M(t,\ell)H(\ell-t_i)Ce^{M(\ell-t_i)}\mathrm{d}\ell\Big]\tilde{x}(t_i) \\ &\quad + \Big[\mathcal{E}(t,t_i) \\ &\quad - \int_{t_i}^t \mathcal{E}(t,\ell)H(\ell-t_i)Ce^{M(\ell-t_i)}\mathrm{d}\ell\Big]\tilde{x}(t_i) \\ &\quad - \int_{t_i}^t \Phi_{M-\Delta}(t,\ell)H(\ell-t_i)C\mathcal{E}(\ell,t_i)\mathrm{d}\ell\, x(t_i),\end{aligned} \tag{91}$$

where $\mathcal{E}(r,s) = \Phi_{M-\Delta}(r,s) - e^{M(r-s)}$ for all $r \ge 0$ and $s \ge 0$. Thus, in particular,

$$\begin{aligned}&\tilde{x}(t_{i+1}) = \Big[\Phi_{M-\Delta}(t_{i+1},t_i) - e^{M\nu} \\ &- \int_{t_i}^{t_{i+1}} \mathcal{E}(t_{i+1},\ell)H(\ell-t_i)Ce^{M(\ell-t_i)}\mathrm{d}\ell\Big]\tilde{x}(t_i) \\ &- \int_{t_i}^{t_{i+1}} \Phi_{M-\Delta}(t_{i+1},\ell)H(\ell-t_i)C\mathcal{E}(\ell,t_i)\mathrm{d}\ell\, x(t_i) \\ &+ e^{M\nu}\Big[I - \int_{t_i}^{t_{i+1}} e^{M(t_i-\ell)}H(\ell-t_i)Ce^{M(\ell-t_i)}\mathrm{d}\ell\Big]\tilde{x}(t_i).\end{aligned} \tag{92}$$

From the definition of H in (83), it follows that

$$\begin{aligned}&\tilde{x}(t_{i+1}) = e^{M\nu}\Big\{I \\ &- \int_{t_i}^{t_{i+1}} G(M)e^{M^\top(\ell-t_i)}C^\top Ce^{M(\ell-t_i)}\mathrm{d}\ell\Big\}\tilde{x}(t_i) \\ &+ \Big[\Phi_{M-\Delta}(t_{i+1},t_i) - e^{M\nu} \\ &- \int_{t_i}^{t_{i+1}} \mathcal{E}(t_{i+1},\ell)H(\ell-t_i)Ce^{M(\ell-t_i)}\mathrm{d}\ell\Big]\tilde{x}(t_i) \\ &- \int_{t_i}^{t_{i+1}} \Phi_{M-\Delta}(t_{i+1},\ell)H(\ell-t_i)C\mathcal{E}(\ell,t_i)\mathrm{d}\ell\, x(t_i).\end{aligned} \tag{93}$$

From the definition of G in (68), we obtain

$$\tilde{x}(t_{i+1}) = \mathcal{G}(t_i)\tilde{x}(t_i) + \mathcal{H}(t_i)x(t_i) \tag{94}$$

with the choices

$$\begin{aligned}\mathcal{G}(t_i) &= \mathcal{E}(t_{i+1},t_i) \\ &\quad - \int_{t_i}^{t_{i+1}} \mathcal{E}(t_{i+1},\ell)H(\ell-t_i)Ce^{M(\ell-t_i)}\mathrm{d}\ell \\ \mathcal{H}(t_i) &= -\int_{t_i}^{t_{i+1}} \Phi_{M-\Delta}(t_{i+1},\ell)H(\ell-t_i)C\mathcal{E}(\ell,t_i)\mathrm{d}\ell,\end{aligned} \tag{95}$$

since the quantity in curly braces in (93) is zero.

We next study the discrete time system

$$\varsigma(t_{i+1}) = \mathcal{G}(t_i)\varsigma(t_i). \tag{96}$$

In terms of our notation from Assumption 4, we then have

$$\underline{\beta}(t_{i+1}-\ell) \le \mathcal{E}(t_{i+1},\ell) \le \overline{\beta}(t_{i+1}-\ell), \tag{97}$$

by Theorem 2. This gives

$$\begin{aligned}&\underline{\beta}(t_{i+1}-\ell)\left(H(\ell-t_i)Ce^{M(\ell-t_i)}\right)^+ \\ &\le \mathcal{E}(t_{i+1},\ell)\left(H(\ell-t_i)Ce^{M(\ell-t_i)}\right)^+ \\ &\le \overline{\beta}(t_{i+1}-\ell)\left(H(\ell-t_i)Ce^{M(\ell-t_i)}\right)^+\end{aligned} \tag{98}$$

and

$$\begin{aligned}&\underline{\beta}(t_{i+1}-\ell)\left(H(\ell-t_i)Ce^{M(\ell-t_i)}\right)^-\\&\leq \mathcal{E}(t_{i+1},\ell)\left(H(\ell-t_i)Ce^{M(\ell-t_i)}\right)^-\\&\leq \overline{\beta}(t_{i+1}-\ell)\left(H(\ell-t_i)Ce^{M(\ell-t_i)}\right)^-.\end{aligned}\tag{99}$$

Therefore

$$\begin{aligned}&\underline{\beta}(t_{i+1}-\ell)\left(H(\ell-t_i)Ce^{M(\ell-t_i)}\right)^+\\&-\overline{\beta}(t_{i+1}-\ell)\left(H(\ell-t_i)Ce^{M(\ell-t_i)}\right)^-\\&\leq \mathcal{E}(t_{i+1},\ell)H(\ell-t_i)Ce^{M(\ell-t_i)}\\&\leq \overline{\beta}(t_{i+1}-\ell)\left(H(\ell-t_i)Ce^{M(\ell-t_i)}\right)^+\\&-\underline{\beta}(t_{i+1}-\ell)\left(H(\ell-t_i)Ce^{M(\ell-t_i)}\right)^-.\end{aligned}\tag{100}$$

Hence, with our choice of $\mathcal{G}$ from (95), and with our choices of $\overline{\mathcal{G}}_c$ and $\underline{\mathcal{G}}_c$ from Assumption 4, we have

$$\underline{\mathcal{G}}_c(\nu)\leq \mathcal{G}(t_i)\leq \overline{\mathcal{G}}_c(\nu).\tag{101}$$

Therefore, the last part of our proof of Theorem 1 implies that the discrete time system (96) is exponentially stable if the matrix κ we defined in (85) is Schur stable. Since the origin of (81) is globally uniformly asymptotically stable, we also deduce from the proof of Theorem 1 that the solutions of (94) exponentially converge to the origin, which gives the desired result, by the reasoning from the third step of the proof of Theorem 1. □

8 Application

In the important special case where δ_2 and δ_B in (2) are 0, the proof of Theorem 1 implies that its closed loop system with the control (8) is uniformly globally exponentially stable to 0 on $\mathbb{R}^n$ when (9) is Schur stable. On the other hand, we show in this section how to use (17) to get an alternative sufficient condition that ensures this exponential stability. We show that our Schur stability requirement from Theorem 1 is less restrictive than the condition that would come from only using (17), by proving that the Schur condition is satisfied under larger bounds $\bar{\delta}$ on δ_1 than the largest bound that we could allow if we instead only used (17). This will illustrate an advantage of using our Schur stability condition. Although we assume in this section that $\delta_B=0$ and $\delta_2=0$ and that $t_i=i\nu$ for all $i\geq 0$ and that δ_1 is a constant matrix $\bar{\delta}$, similar reasoning applies for time varying δ_1's, as well as for ISS cases with nonzero δ_2's.

Consider the dynamics (2) with

$$M=B=I\in\mathbb{R}^{2\times 2}\ \text{ and }\ \bar{\delta}=\begin{bmatrix}0&\delta_*\\\delta_*&0\end{bmatrix}\in\mathbb{R}^{2\times 2}\tag{102}$$

for a constant $\delta_*\in(0,2)$. This gives $S=I$, and the function χ from (5) is

$$\chi(r,s)=\int_0^{s-r}e^{-2\ell}\mathrm{d}\ell I=\tfrac{1}{2}(1-e^{-2(s-r)})I.\tag{103}$$

Also, since $e^{(M\pm\bar{\delta})\ell}=e^{M\ell}e^{\pm\bar{\delta}\ell}$, we have

$$e^{(M+\bar{\delta})\ell}=\begin{bmatrix}\frac{e^{(1+\delta_*)\ell}+e^{(1-\delta_*)\ell}}{2}&\frac{e^{(1+\delta_*)\ell}-e^{(1-\delta_*)\ell}}{2}\\\frac{e^{(1+\delta_*)\ell}-e^{(1-\delta_*)\ell}}{2}&\frac{e^{(1+\delta_*)\ell}+e^{(1-\delta_*)\ell}}{2}\end{bmatrix}\tag{104}$$

and

$$e^{(M-\bar{\delta})\ell}=\begin{bmatrix}\frac{e^{(1-\delta_*)\ell}+e^{(1+\delta_*)\ell}}{2}&\frac{e^{(1-\delta_*)\ell}-e^{(1+\delta_*)\ell}}{2}\\\frac{e^{(1-\delta_*)\ell}-e^{(1+\delta_*)\ell}}{2}&\frac{e^{(1-\delta_*)\ell}+e^{(1+\delta_*)\ell}}{2}\end{bmatrix}.\tag{105}$$

Consequently, $\overline{\xi}$ and $\underline{\xi}$ from (5) are

$$\begin{aligned}&\overline{\xi}(r,s)=e^{r-s}(\cosh(\delta_*(r-s))-1)I\\&\text{and }\ \underline{\xi}(r,s)=-e^{r-s}\sinh(\delta_*(r-s))\begin{bmatrix}0&1\\1&0\end{bmatrix},\end{aligned}\tag{106}$$

and the bounding functions from (7) are

$$\begin{aligned}&\overline{\mathcal{F}}_c(r,s)=e^{L_*}(\cosh(\delta_*L_*)-1)\,I\\&+\tfrac{2}{1-e^{-2L_*}}\int_0^{L_*}e^{L_*-2\ell}\sinh(\delta_*(L_*-\ell))\mathrm{d}\ell\begin{bmatrix}0&1\\1&0\end{bmatrix}\\&\text{and }\ \underline{\mathcal{F}}_c(r,s)=-e^{L_*}\sinh(\delta_*L_*)\begin{bmatrix}0&1\\1&0\end{bmatrix}\\&-\tfrac{2}{1-e^{-2L_*}}\int_0^{L_*}e^{L_*-2\ell}[\cosh(\delta_*(L_*-\ell))-1]\mathrm{d}\ell\, I,\end{aligned}\tag{107}$$

where $L_*=s-r$, because in this case, $(Se^{-M^\top\ell}\chi^{-1})^-$ is the zero matrix. Since $\cosh(s)\geq 1$ and $\sinh(s)\geq 0$ for all $s\geq 0$, we conclude that in this example, we have $\bar{\mathcal{F}}_c^+=\bar{\mathcal{F}}_c$ and $\underline{\mathcal{F}}_c^-=-\underline{\mathcal{F}}_c$ at all pairs (r,s) where $L_*>0$. Then with the preceding $\bar{\mathcal{F}}_c^+$ and $\underline{\mathcal{F}}_c^-$, the closed loop system from Theorem 1 will be globally exponentially stable to 0 on $\mathbb{R}^n$ provided that the Γ from (9) is Schur stable.

Next, observe that in this case, the functions (35) are

$$\begin{aligned}W(i)&=-\int_{t_i}^{t_{i+1}}\Phi_D(t_{i+1},\ell)e^{-\ell+\sigma(\ell)}\mathrm{d}\ell\tfrac{2}{1-e^{-2\nu}}I\\&\quad+\Phi_D(t_{i+1},t_i),\end{aligned}\tag{108}$$

where $\Phi_D(t,\ell)=\Phi_{M-\delta_1}(t,\ell)-\Phi_{\underline{M}}(t,\ell)$ as before. Then, in this particular case where $\delta_1=\bar{\delta}$ and $\delta_B=0$, we obtain

$$\begin{aligned}&W(i)=e^{(t_{i+1}-t_i)(I-\bar{\delta})}-e^{(t_{i+1}-t_i)I}\\&-\tfrac{2}{1-e^{-2\nu}}\int_{t_i}^{t_{i+1}}\left[e^{(t_{i+1}-\ell)(I-\bar{\delta})}-e^{(t_{i+1}-\ell)I}\right]e^{t_i-\ell}\mathrm{d}\ell\end{aligned}\tag{109}$$

and therefore also

$$\begin{aligned}&W(i)=e^{\nu(I-\bar{\delta})}-e^{\nu I}\\&-\tfrac{2}{1-e^{-2\nu}}\int_0^{\nu}\left[e^{(\nu-\ell)(I-\bar{\delta})}-e^{(\nu-\ell)I}\right]e^{-\ell}\mathrm{d}\ell I.\end{aligned}\tag{110}$$

Therefore, we deduce from (17) that

$$\begin{aligned}||W(i)||&\leq\nu^\sharp\int_0^{\nu}(\nu-\ell)||\bar{\delta}||e^{(\nu-\ell)(1+||\bar{\delta}||)}e^{-\ell}\mathrm{d}\ell\\&\quad+\nu||\bar{\delta}||e^{\nu(1+||\bar{\delta}||)}\\&=\nu^\sharp||\bar{\delta}||\int_0^{\nu}(\nu-\ell)e^{\nu(1+||\bar{\delta}||)-\ell(2+||\bar{\delta}||)}\mathrm{d}\ell\\&\quad+\nu||\bar{\delta}||e^{\nu(1+||\bar{\delta}||)}\\&=\nu^\sharp||\bar{\delta}||e^{\nu(1+||\bar{\delta}||)}\left[\nu\int_0^{\nu}e^{-\ell(2+||\bar{\delta}||)}\mathrm{d}\ell\right.\\&\quad\left.-\int_0^{\nu}\ell e^{-\ell(2+||\bar{\delta}||)}\mathrm{d}\ell\right]+\nu||\bar{\delta}||e^{\nu(1+||\bar{\delta}||)}\end{aligned}$$

and so also

$$\begin{aligned}||W(i)|| &= \nu^\sharp ||\bar{\delta}|| e^{\nu(1+||\bar{\delta}||)} \Big[\nu \tfrac{1-e^{-\nu(2+||\bar{\delta}||)}}{2+||\bar{\delta}||} \\ &\quad + \tfrac{e^{-\nu(2+||\bar{\delta}||)}\nu}{2+||\bar{\delta}||} \\ &\quad + \tfrac{-1+e^{-\nu(2+||\bar{\delta}||)}}{(2+||\bar{\delta}||)^2}\Big] + \nu||\bar{\delta}|| e^{\nu(1+||\bar{\delta}||)} \\ &= \nu^\sharp ||\bar{\delta}|| e^{\nu(1+||\bar{\delta}||)} \Big(\tfrac{\nu}{2+||\bar{\delta}||} \\ &\quad + \tfrac{-1+e^{-\nu(2+||\bar{\delta}||)}}{(2+||\bar{\delta}||)^2}\Big) + \nu||\bar{\delta}|| e^{\nu(1+||\bar{\delta}||)},\end{aligned}$$

where $\nu^\sharp = \frac{2}{1-e^{-2\nu}}$. This provides the condition

$$\begin{aligned}||W(i)|| \le&\ \tfrac{2\delta_* e^{\nu(1+\delta_*)}}{(1-e^{-2\nu})(2+\delta_*)}\Big(\nu + \tfrac{e^{-\nu(2+\delta_*)}-1}{2+\delta_*}\Big) \\ &+ \nu\delta_* e^{\nu(1+\delta_*)}.\end{aligned} \tag{111}$$

Hence, by using (17), we conclude that the closed loop system from Theorem 1 satisfies the desired exponential stability property if the right side of (111) is in $(0, 1)$, by the second part of our proof of Theorem 1. This is more restrictive than our Schur stability condition from Theorem 1, because for instance, if $\delta_* = 0.36$ and $\nu = 0.72$ in the preceding example, then simple Mathematica calculations imply that the eigenvalues of the matrix Γ in (9) are $\{0.969621, -0.767802, 0.246164, -0.170381\}$ (which means that the Schur stability requirement is satisfied) but that the right side of (111) is 1.0879. If instead we have $\delta_* = 1$ and $\nu = 0.36$, then Γ has eigenvalues $\{0.95009, -0.688578, 0.292117, -0.178081\}$ but the right side of (111) is 1.11283. It follows that Theorem 1 gives a better stability condition than the one that we would have obtained by only applying (17).

In Figs. 1-2, we graphically illustrate the preceding result, using plots that we generated using the Mathematica program. In Fig. 1, we show the spectral radius (i.e., the largest of the absolute values of the four eigenvalues) of the matrix Γ from Theorem 1, as a function of the disturbance parameter δ_* and sample rate ν, for the choices from the preceding example. It illustrates a range of δ_* and ν values for which the eigenvalues of Γ remain in $(-1, 1)$ to satisfy our Schur stability requirement from Theorem 1. In Fig. 2, we plot the norms of closed loop solutions obtained from applying Theorem 1 to the previous example with $\delta_* = 0.36$ and $\nu = 0.72$. Fig. 2 shows rapid convergence of the solutions to zero, and it illustrates the effects of the sampling in our control (which produces cusps in the plots of the norms at the sample times), and so also illustrates Theorem 1.

9 Conclusions

We provided new results on feedback stabilization for dynamics with sampling and uncertainty. We allowed additive uncertainty on the right side of the systems, in addition to uncertainty in both vector fields defining our time-varying linear systems. Our methods combined positive systems approaches with new matrix inequalities that estimate fundamental solutions for time-varying linear systems that contain uncertain coefficient matrices. Our matrix inequality estimations are of independent interest, because of the difficulty in estimating fundamental matrix

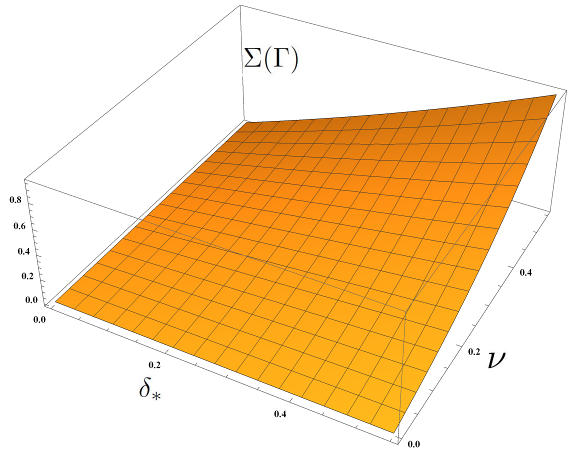

Fig. 1. Spectral Radius $\Sigma(\Gamma)$ of Γ as Function of Disturbance Parameter δ_* and Sample Rate ν, Illustrating Range of δ_* and ν Values for which our Schur Condition on Γ is Satisfied

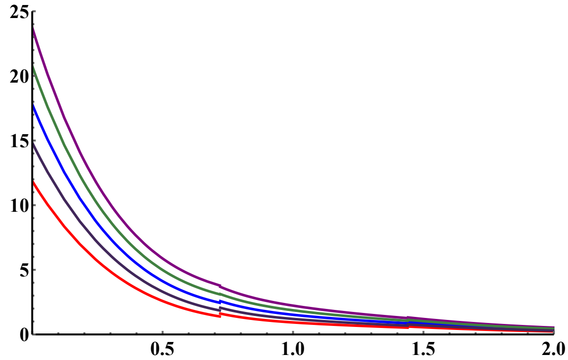

Fig. 2. Norm of State of Closed Loop System from Theorem 1 for Example from Section 8, $\delta_* = 0.36$, $\nu = 0.72$, Control (8), and Initial States $(1, 1)$ (Red), $(1.5, 1)$ (Black), $(2, 1)$ (Blue), $(2.5, 1)$ (Green), and $(3, 1)$ (Purple), Showing Rapid Convergence to 0

solutions for time-vayring linear systems that contain uncertainties in their coefficients. Our example illustrated the potential advantages of our new stabilization approach. In our future work, we aim to extend our work to nonlinear input delayed systems having right sides that include Metzler matrices.

Appendix A1: Key Lemmas

We first provide two lemmas that we used in our proofs of our transition matrix estimation theorems. The first one will use the fact that for the matrix

$$\mathcal{P} = \begin{bmatrix} I & I \\ -I & I \end{bmatrix} \in \mathbb{R}^{(2n)\times(2n)}, \tag{A.1}$$

we have

$$\mathcal{P}^{-1} = \begin{bmatrix} \frac{I}{2} & -\frac{I}{2} \\ \frac{I}{2} & \frac{I}{2} \end{bmatrix}. \tag{A.2}$$

The following then follows from simple calculations:

Lemma A.1 *For all matrices $\alpha \in \mathbb{R}^{n\times n}$ and $\beta \in \mathbb{R}^{n\times n}$,* *we have*

$$\mathcal{P}\begin{bmatrix}\alpha & \beta\\ \beta & \alpha\end{bmatrix}\mathcal{P}^{-1}=\begin{bmatrix}\alpha+\beta & 0\\ 0 & \alpha-\beta\end{bmatrix}. \quad (A.3)$$

Also,

$$\mathcal{P}^{-1}\begin{bmatrix}\gamma & 0\\ 0 & \rho\end{bmatrix}\mathcal{P}=\begin{bmatrix}\frac{\gamma+\rho}{2} & \frac{\gamma-\rho}{2}\\ \frac{\gamma-\rho}{2} & \frac{\gamma+\rho}{2}\end{bmatrix} \quad (A.4)$$

holds for all matrices $\gamma \in \mathbb{R}^{n\times n}$ and $\rho \in \mathbb{R}^{n\times n}$. □

We also use the following consequence of the uniqueness of solutions property for (1) with $\mathcal{F}=\mathcal{R}$, which is shown by checking that $\mathcal{H}(t,t_0)=\mathcal{L}\Phi_{\mathcal{A}}(t,t_0)\mathcal{L}^{-1}$ satisfies $\mathcal{H}(t_0,t_0)=I$ and $\frac{\partial}{\partial t}H(t,t_0)=\mathcal{R}\mathcal{H}(t,t_0)$ for all $t\geq 0$ and $t_0\geq 0$:

Lemma A.2 *Let $\mathcal{A}:[0,+\infty)\to\mathbb{R}^{n\times n}$. Let $\mathcal{L}\in\mathbb{R}^{n\times n}$ be an invertible matrix and let $\mathcal{R}=\mathcal{L}\mathcal{A}\mathcal{L}^{-1}$. Then*

$$\Phi_{\mathcal{R}}(t,t_0)=\mathcal{L}\Phi_{\mathcal{A}}(t,t_0)\mathcal{L}^{-1} \quad (A.5)$$

for all $t_0\in[0,+\infty)$ and $t\in[0,+\infty)$. □

Appendix A2: Proof of Theorem 2

We consider $\mathcal{M}(t)$ and $\Delta(t)$ such that $\mathcal{M}(t)$ is Metzler for all $t\geq 0$ and such that there is a $\bar{\Delta}$ satisfying the requirements of Theorem 2. Let

$$\Omega(t)=\begin{bmatrix}\mathcal{M}(t) & \Delta(t)\\ \Delta(t) & \mathcal{M}(t)\end{bmatrix},\ \overline{\Omega}(t)=\begin{bmatrix}\mathcal{M}(t) & \overline{\Delta}(t)\\ \overline{\Delta}(t) & \mathcal{M}(t)\end{bmatrix} \quad (A.6)$$

$$\text{and }\ \underline{\Omega}(t)=\begin{bmatrix}\mathcal{M}(t) & 0\\ 0 & \mathcal{M}(t)\end{bmatrix}. \quad (A.7)$$

Then $\underline{\Omega}(t)\leq\Omega(t)\leq\overline{\Omega}(t)$. Here and in the sequel, inequalities and equalities should be understood to hold for all $t\geq 0$ unless otherwise indicated. It follows from the comparison lemma that

$$\Phi_{\underline{\Omega}}(t,t_0)\leq\Phi_{\Omega}(t,t_0)\leq\Phi_{\overline{\Omega}}(t,t_0). \quad (A.8)$$

According to Lemma A.1,

$$\Omega(t)=\mathcal{P}^{-1}\begin{bmatrix}\mathcal{M}(t)+\Delta(t) & 0\\ 0 & \mathcal{M}(t)-\Delta(t)\end{bmatrix}\mathcal{P} \quad (A.9)$$

and

$$\overline{\Omega}(t)=\mathcal{P}^{-1}\begin{bmatrix}\mathcal{M}(t)+\overline{\Delta}(t) & 0\\ 0 & \mathcal{M}(t)-\overline{\Delta}(t)\end{bmatrix}\mathcal{P}. \quad (A.10)$$

Then, from (A.8) and Lemma A.2, we have

$$\begin{aligned}\Phi_{\underline{\Omega}}(t,t_0) &\leq \mathcal{P}^{-1}\begin{bmatrix}\Phi_{\mathcal{M}+\Delta}(t,t_0) & 0\\ 0 & \Phi_{\mathcal{M}-\Delta}(t,t_0)\end{bmatrix}\mathcal{P}\\ &\leq \mathcal{P}^{-1}\begin{bmatrix}\Phi_{\mathcal{M}+\overline{\Delta}}(t,t_0) & 0\\ 0 & \Phi_{\mathcal{M}-\overline{\Delta}}(t,t_0)\end{bmatrix}\mathcal{P}.\end{aligned} \quad (A.11)$$

Hence, Lemma A.1 also gives

$$\begin{aligned}\Phi_{\underline{\Omega}}(t,t_0) &= \begin{bmatrix}\Phi_{\mathcal{M}}(t,t_0) & 0\\ 0 & \Phi_{\mathcal{M}}(t,t_0)\end{bmatrix}\\ &\leq \begin{bmatrix}\frac{\Phi_{\mathcal{M}+\Delta}(t,t_0)+\Phi_{\mathcal{M}-\Delta}(t,t_0)}{2} & \frac{\Phi_{\mathcal{M}+\Delta}(t,t_0)-\Phi_{\mathcal{M}-\Delta}(t,t_0)}{2}\\ \frac{\Phi_{\mathcal{M}+\Delta}(t,t_0)-\Phi_{\mathcal{M}-\Delta}(t,t_0)}{2} & \frac{\Phi_{\mathcal{M}+\Delta}(t,t_0)+\Phi_{\mathcal{M}-\Delta}(t,t_0)}{2}\end{bmatrix} \quad (A.12)\\ &\leq \mathcal{G}_{\mathcal{M},\overline{\Delta}}(t,t_0),\end{aligned}$$

where

$$\mathcal{G}_{\mathcal{M},\overline{\Delta}}(t,t_0)=\begin{bmatrix}\frac{\Phi_{\mathcal{M}+\overline{\Delta}}(t,t_0)+\Phi_{\mathcal{M}-\overline{\Delta}}(t,t_0)}{2} & \frac{\Phi_{\mathcal{M}+\overline{\Delta}}(t,t_0)-\Phi_{\mathcal{M}-\overline{\Delta}}(t,t_0)}{2}\\ \frac{\Phi_{\mathcal{M}+\overline{\Delta}}(t,t_0)-\Phi_{\mathcal{M}-\overline{\Delta}}(t,t_0)}{2} & \frac{\Phi_{\mathcal{M}+\overline{\Delta}}(t,t_0)+\Phi_{\mathcal{M}-\overline{\Delta}}(t,t_0)}{2}\end{bmatrix}. \quad (A.13)$$

We straightforwardly deduce from considering the upper left and then the upper right submatrices in (A.12) that

$$\begin{aligned}\Phi_{\mathcal{M}}(t,t_0) &\leq \frac{\Phi_{\mathcal{M}+\Delta}(t,t_0)+\Phi_{\mathcal{M}-\Delta}(t,t_0)}{2}\\ &\leq \frac{\Phi_{\mathcal{M}+\overline{\Delta}}(t,t_0)+\Phi_{\mathcal{M}-\overline{\Delta}}(t,t_0)}{2}\\ &\text{and } \frac{-\Phi_{\mathcal{M}+\overline{\Delta}}(t,t_0)+\Phi_{\mathcal{M}-\overline{\Delta}}(t,t_0)}{2}\\ &\leq \frac{-\Phi_{\mathcal{M}+\Delta}(t,t_0)+\Phi_{\mathcal{M}-\Delta}(t,t_0)}{2}\leq 0\end{aligned} \quad (A.14)$$

are satisfied. Adding the corresponding left, middle, and right sides of the inequalities in (A.14) gives the desired conclusion of the theorem.

Appendix A3: Proof of Theorem 3

From (21), it follows that $P^+\underline{\rho}\leq P^+\rho\leq P^+\overline{\rho}$ and $-P^-\overline{\rho}\leq -P^-\rho\leq -P^-\underline{\rho}$. We deduce from the formula $P\rho=(P\rho)^+-(P\rho)^-$ that $P^+\underline{\rho}-P^-\overline{\rho}\leq P\rho\leq P^+\overline{\rho}-P^-\underline{\rho}$ is satisfied. Therefore, we get

$$\begin{aligned}\left(P^+\underline{\rho}-P^-\overline{\rho}\right)(P^{-1})^+ &\leq P\rho(P^{-1})^+\\ &\leq \left(P^+\overline{\rho}-P^-\underline{\rho}\right)(P^{-1})^+\end{aligned} \quad (A.15)$$

$$\begin{aligned}&\text{and } -\left(P^+\overline{\rho}-P^-\underline{\rho}\right)(P^{-1})^-\\ &\leq -P\rho(P^{-1})^- \leq -\left(P^+\underline{\rho}-P^-\overline{\rho}\right)(P^{-1})^-.\end{aligned} \quad (A.16)$$

By adding the inequalities in (A.15)-(A.16), we obtain

$$\underline{\varsigma}\leq P\rho P^{-1}\leq\overline{\varsigma}, \quad (A.17)$$

where $\overline{\varsigma}$ and $\underline{\varsigma}$ are defined as follows:

$$\begin{aligned}\overline{\varsigma} &= \left(P^+\overline{\rho}-P^-\underline{\rho}\right)(P^{-1})^+ - \left(P^+\underline{\rho}-P^-\overline{\rho}\right)(P^{-1})^- \text{ and}\\ \underline{\varsigma} &= \left(P^+\underline{\rho}-P^-\overline{\rho}\right)(P^{-1})^+ - \left(P^+\overline{\rho}-P^-\underline{\rho}\right)(P^{-1})^-.\end{aligned}$$

Then, since the formula (25) gives $M=F+\overline{\varsigma}^+$, we get

$$P(A+\rho)P^{-1}=F+P\rho P^{-1}=M-\Delta \quad (A.18)$$

where M is from (25) and $\Delta=\overline{\varsigma}^+-P\rho P^{-1}$. Then $M(t)$ is Metzler for all $t\geq 0$, and (A.17) gives $0\leq\Delta\leq\overline{\varsigma}^+-\underline{\varsigma}$.

Next, Theorem 2 ensures that

$$\begin{aligned}&\Phi_M(t,t_0)+\frac{\Phi_{M-\overline{\varsigma}^++\underline{\varsigma}}(t,t_0)-\Phi_{M+\overline{\varsigma}^+-\underline{\varsigma}}(t,t_0)}{2}\\ &\leq\Phi_{M-\Delta}(t,t_0)\leq\frac{\Phi_{M-\overline{\varsigma}^++\underline{\varsigma}}(t,t_0)+\Phi_{M+\overline{\varsigma}^+-\underline{\varsigma}}(t,t_0)}{2}\end{aligned} \quad (A.19)$$

for all $t \geq t_0 \geq 0$. According to Lemma A.2, these inequalities in combination with (A.18) give

$$\underline{\mu}(t,t_0) \leq P\Phi_{A+\rho}(t,t_0)P^{-1} = \Phi_{M-\Delta}(t,t_0) \leq \overline{\mu}(t,t_0)$$

for all $t \geq t_0 \geq 0$, where

$$\begin{array}{l} \overline{\mu}(r,s) = \frac{\Phi_{M-\overline{\zeta}^+ + \underline{\zeta}}(r,s) + \Phi_{M+\overline{\zeta}^+ - \underline{\zeta}}(r,s)}{2} \text{ and} \\ \underline{\mu}(r,s) = \Phi_M(r,s) + \frac{\Phi_{M-\overline{\zeta}^+ + \underline{\zeta}}(r,s) - \Phi_{M+\overline{\zeta}^+ - \underline{\zeta}}(r,s)}{2}. \end{array} \tag{A.20}$$

Consequently,

$$\begin{array}{rcl} (P^{-1})^+\underline{\mu}(t,t_0) & \leq & (P^{-1})^+P\Phi_{A+\rho}(t,t_0)P^{-1} \\ & \leq & (P^{-1})^+\overline{\mu}(t,t_0) \end{array} \tag{A.21}$$

and

$$\begin{array}{rcl} -(P^{-1})^-\overline{\mu}(t,t_0) & \leq & -(P^{-1})^-P\Phi_{A+\rho}(t,t_0)P^{-1} \\ & \leq & -(P^{-1})^-\underline{\mu}(t,t_0). \end{array} \tag{A.22}$$

Adding (A.21)-(A.22) and recalling that $R = P^{-1}$, we get

$$\begin{array}{l} (P^{-1})^+\underline{\mu}(t,t_0) - (P^{-1})^-\overline{\mu}(t,t_0) \leq \Phi_{A+\rho}(t,t_0)P^{-1} \\ \leq (P^{-1})^+\overline{\mu}(t,t_0) - (P^{-1})^-\underline{\mu}(t,t_0). \end{array}$$

From these inequalities, we deduce that

$$\begin{array}{l} \left[(P^{-1})^+\underline{\mu}(t,t_0) - (P^{-1})^-\overline{\mu}(t,t_0)\right] P^+ \\ \leq \Phi_{A+\rho}(t,t_0)P^{-1}P^+ \\ \leq \left[(P^{-1})^+\overline{\mu}(t,t_0) - (P^{-1})^-\underline{\mu}(t,t_0)\right] P^+ \end{array} \tag{A.23}$$

and

$$\begin{array}{l} -\left[(P^{-1})^+\overline{\mu}(t,t_0) - (P^{-1})^-\underline{\mu}(t,t_0)\right] P^- \\ \leq -\Phi_{A+\rho}(t,t_0)P^{-1}P^- \\ \leq -\left[(P^{-1})^+\underline{\mu}(t,t_0) - (P^{-1})^-\overline{\mu}(t,t_0)\right] P^-. \end{array} \tag{A.24}$$

We conclude by adding (A.23)-(A.24), since $\tilde{\zeta} = \overline{\zeta}^+ - \underline{\zeta}$.

Appendix A4: Proof of Theorem 4

This appendix provides a proof of Theorem 4. The proof is based on discrete time analogs of our continuous time transition matrix estimation theorems that are of independent interest, and which we prove first. Consider matrices $M_i \in \mathbb{R}^{n\times n}$, $\Delta_i \in \mathbb{R}^{n\times n}$, and $\bar{\Delta}_i \in \mathbb{R}^{n\times n}$ such that

$$M_i \geq 0 \text{ and } 0 \leq \Delta_i \leq \overline{\Delta}_i \tag{A.25}$$

for all $i \in \mathbb{Z}_{\geq 0}$. Let us find an upper and a lower bound for

$$\begin{array}{l} E_{i,j} = \\ (M_{j-1+i} - \Delta_{j-1+i}).....(M_{i+1} - \Delta_{i+1})(M_i - \Delta_i), \end{array} \tag{A.26}$$

which is the state transition matrix for $X_{i+1} = (M_i - \Delta_i)X_i$, meaning $X_{j+i} = E_{i,j}X_i$ when $j \geq i$ with $j \geq 1$. We use the matrices

$$\Omega_i = \begin{bmatrix} M_i & \Delta_i \\ \Delta_i & M_i \end{bmatrix}, \ \overline{\Omega}_i = \begin{bmatrix} M_i & \overline{\Delta}_i \\ \overline{\Delta}_i & M_i \end{bmatrix}, \tag{A.27}$$

and $\kappa_{j,i} = \Omega_{j-1+i}.....\Omega_{i+1}\Omega_i$,and $\overline{\kappa}_{j,i} = \overline{\Omega}_{j-1+i}.....\overline{\Omega}_{i+1}\overline{\Omega}_i$. According to Lemma A.1 from Appendix A1,

$$\mathcal{P}\kappa_{j,i}\mathcal{P}^{-1} = \begin{bmatrix} D_{i,j} & 0 \\ 0 & E_{i,j} \end{bmatrix} \tag{A.28}$$

holds for all i and j, where $\mathcal{P}$ is the $(2n) \times (2n)$ block matrix with each block being $I \in \mathbb{R}^{n\times n}$, $D_{i,j} = (M_{j-1+i} + \Delta_{j-1+i}).....(M_{i+1} + \Delta_{i+1})(M_i + \Delta_i)$ and $E_{i,j}$ defined in (A.26) and

$$\mathcal{P}\overline{\kappa}_{j,i}\mathcal{P}^{-1} = \begin{bmatrix} \overline{D}_{i,j} & 0 \\ 0 & \overline{E}_{i,j} \end{bmatrix}, \tag{A.29}$$

$$\begin{array}{l} \text{where } \overline{D}_{i,j} = \\ (M_{j-1+i} + \overline{\Delta}_{j-1+i}).....(M_{i+1} + \overline{\Delta}_{i+1})(M_i + \overline{\Delta}_i) \\ \text{and } \overline{E}_{i,j} = \\ (M_{j-1+i} - \overline{\Delta}_{j-1+i}).....(M_{i+1} - \overline{\Delta}_{i+1})(M_i - \overline{\Delta}_i). \end{array} \tag{A.30}$$

Notice that since $\Delta_i \geq 0$ for all i, the inequalities

$$\underline{\kappa}_{j,i} \leq \kappa_{j,i} \leq \overline{\Omega}_{j-1+i}.....\overline{\Omega}_{i+1}\overline{\Omega}_i \tag{A.31}$$

are satisfied with

$$\underline{\kappa}_{j,i} = \begin{bmatrix} M_{j-1+i}...M_{i+1}M_i & 0 \\ 0 & M_{j-1+i}...M_{i+1}M_i \end{bmatrix}. \tag{A.32}$$

As an immediate consequence of (A.28), (A.29), and (A.31), the inequalities

$$\underline{\kappa}_{j,i} \leq \mathcal{P}^{-1} \begin{bmatrix} D_{i,j} & 0 \\ 0 & E_{i,j} \end{bmatrix} \mathcal{P} \leq \mathcal{P}^{-1} \begin{bmatrix} \overline{D}_{i,j} & 0 \\ 0 & \overline{E}_{i,j} \end{bmatrix} \mathcal{P} \tag{A.33}$$

holds. We deduce from Lemma A.1 that

$$\underline{\kappa}_{j,i} \leq \begin{bmatrix} \frac{D_{i,j}+E_{i,j}}{2} & \frac{D_{i,j}-E_{i,j}}{2} \\ \frac{D_{i,j}-E_{i,j}}{2} & \frac{D_{i,j}+E_{i,j}}{2} \end{bmatrix} \leq \begin{bmatrix} \frac{\overline{D}_{i,j}+\overline{E}_{i,j}}{2} & \frac{\overline{D}_{i,j}-\overline{E}_{i,j}}{2} \\ \frac{\overline{D}_{i,j}-\overline{E}_{i,j}}{2} & \frac{\overline{D}_{i,j}+\overline{E}_{i,j}}{2} \end{bmatrix},$$

which we can combine with (A.25) to get

$$\begin{array}{l} M_{j-1+i}...M_{i+1}M_i \leq \frac{D_{i,j}+E_{i,j}}{2} \leq \frac{\overline{D}_{i,j}+\overline{E}_{i,j}}{2} \\ \text{and } \frac{\overline{E}_{i,j}-\overline{D}_{i,j}}{2} \leq \frac{E_{i,j}-D_{i,j}}{2} \leq 0 \end{array} \tag{A.34}$$

for all i and j. By adding these inequalities, we obtain the desired estimate

$$M_{j-1+i}...M_{i+1}M_i + \frac{\overline{E}_{i,j}-\overline{D}_{i,j}}{2} \leq E_{i,j} \leq \frac{\overline{D}_{i,j}+\overline{E}_{i,j}}{2}. \tag{A.35}$$

We now use the preceding observations to prove Theorem 4. To this end, notice that the preceding analysis and the choices (59) of $\underline{S}_j$ and $\bar{S}_j$ give

$$\underline{S}_j \leq (M - \Delta_{j-1+i})......(M - \Delta_i) \leq \overline{S}_j \tag{A.36}$$

for all $j \geq 1$. Hence, the theorem follows by the order preservation property of limits.

References

[1] K. Astrom and B. Wittenmark. *Computer-Controlled Systems. Theory and Design. Third Edition.* Dover Publications, Mineola, NY, 1997.

[2] N. Bof, R. Carli, and L. Schenato. Lyapunov theory for discrete time systems. *arXiv Preprint 1809.05289*, 2018, https://arxiv.org/abs/1809.05289.

[3] A. Borri, F. Cacace, A. De Gaetano, A. Germani, C. Manes, P. Palumbo, S. Panunzi, and P. Pepe. Luenberger-like observers for nonlinear time-delay systems with application to the artificial pancreas: The attainment of good performance. *IEEE Control Systems Magazine*, 37(4):33-49, 2017.

[4] A. Borri and P. Pepe. Event-triggered control of nonlinear systems with time-varying state delays. *IEEE Transactions on Automatic Control*, to appear, doi: 10.1109/TAC.2020.3009173.

[5] F. Cacace, A. Germani, and C. Manes. A new approach to design interval observers for linear systems. *IEEE Transactions on Automatic Control*, 60(6):1665-1670, 2014.

[6] F. Cacace, A. Germani, and C. Manes. An enhanced observer for nonlinear systems with time-varying measurement delays. *IEEE Transactions on Automatic Control*, to appear, doi: 10.1109/TAC.2020.3046723.

[7] J.-L. Gouzé, A. Rapaport, and Z. Hadj-Sadok. Interval observers for uncertain biological systems. *Ecological Modelling*, 133(1-2):45-56, 2000.

[8] T. Insperger an G. Stepan. Stability chart for the delayed Mathieu equation. *Proceedings of the Royal Society Series A: Mathematical, Physical, and Engineering Sciences*, 458(2024):1989-1998, 2002.

[9] J. Klamka. *Controllability and Minimum Energy Control*. Springer, New York, 2019.

[10] M. Malisoff. Tracking and parameter identification for model reference adaptive control. *International Journal of Robust and Nonlinear Control*, 30(4), 1582-1606, 2020.

[11] M. Mattioni, S. Monaco, and D. Normand-Cyrot. Immersion and invariance stabilization of strict-feedback dynamics under sampling. *Automatica*, 76:78-86, 2017.

[12] M. Mattioni, S. Monaco, and D. Normand-Cyrot. Feedforwarding under sampling. *IEEE Transactions on Automatic Control*, 64(11):4668-4675, 2019.

[13] F. Mazenc, S. Ahmed, and M. Malisoff. Finite time estimation through a continuous-discrete observer. *International Journal of Robust and Nonlinear Control*, 28(16):4831-4849, 2018.

[14] F. Mazenc, V. Andrieu, and M. Malisoff. Design of continuous-discrete observers for time-varying nonlinear systems. *Automatica*, 57(7):135-144, 2015.

[15] F. Mazenc and M. Malisoff. Feedback stabilization with discrete measurements using bounds on fundamental matrices. In *Proceedings of the IEEE Conference on Decision and Control (Austin, TX, 13-15 December 2021)*, to appear.

[16] F. Mazenc, M. Malisoff, and Z.-P. Jiang. Reduced-order fast converging observers for systems with discrete measurements and measurement error. *Systems and Control Letters*, 150:104892, 2021.

[17] T. Raissi, D. Efimov, and A. Zolghadri. Interval state estimation for a class of nonlinear systems. *IEEE Transactions on Automatic Control*, 57(1):260-265, 2012

[18] E. Sontag. *Mathematical Control Theory, Second Edition*. Springer, New York, NY, 1998.

[19] E. Sontag. Input to state stability: basic concepts and results. In *Nonlinear and Optimal Control Theory*, P. Nistri and G. Stefani, Eds. Springer, Berlin Germany, 2008, pp. 163-220.

[20] V. Tanasa, S. Monaco, and D. Normand-Cyrot. Backstepping control under multi-rate sampling. *IEEE Transactions on Automatic Control*, 61(5):1208-1222, 2016.