Interval Observation and Control for Continuous-Time Persidskii Systems

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***Abstract*—The problems of interval estimation and stabilization are studied for a class of generalized Persidskii systems. For this class of models with nonlinearities satisfying the incremental passivity property, the nonnegativity conditions are proposed and a nonlinear interval observer is synthesized. A nonlinear feedback is designed that uses the interval estimates. The conditions of stability of estimation and regulation errors are formulated using linear matrix inequalities. The efficiency of the proposed interval observer and controller is demonstrated for a recurrent neural network and a mechanical system with cubic nonlinearity.**

***Index Terms*—Interval observers, Persidskii systems, Nonlinear systems, Robust control, Estimation, LMIs**

I. Introduction

In many application areas, various estimation or control problems have to be solved in the presence of significant nonlinearity and uncertainty of the process models. For example, identification of parameters and modeling are very complicated in systems biology and neuroscience [1], while the presence of several agents (including humans) introduces a lot of uncertainty in collaborative robotics [2]. In these scenarios, the presence of incertitude becomes non-negligible and has to be incorporated in the design modifying also the control/estimation goals. Moreover, in the nonlinear framework, synthesis of a general control or estimation algorithm is difficult, then a common way to overpass this issue consists in utilization of canonical models: linear parameter-varying systems [3], Lur'e models [4], [5], homogeneous dynamics [6].

Conventionally, a state observer contains a copy of the process dynamics with output injection, then in the presence of uncertain parameters and/or external disturbances, such an estimator design, which must follow the ideal value of the state, cannot be implemented (auxiliary restrictive hypotheses have to be introduced) [7]. However, an observation by interval remains still possible since this type of estimators calculates a set of admissible values (interval) for the state at each instant of time using input-output information and the bounds on uncertainty [8], [9]. The length of the interval is minimized via a parametric adjustment of the observer gains, and it becomes proportional to the size of the model uncertainty. It should be emphasized that the interval estimation is not a simplification of the original observation problem, in fact it is an extension since the mean value of the interval can be used as a pointwise estimate of the state, while the bounds of the interval provide an assessment of the accuracy of the observation for the assumed model uncertainty. There are many approaches for designing interval/set-membership estimators [10], [11], [12], and this work focuses on an interval observer design based on the non-negative systems theory [13], [14], [15], [16]. In this way, the main technical difficulty consists in simultaneously ensuring the stability and non-negativity of the interval estimation error dynamics by an appropriate design of the observer structure.

The goal of this work is to propose an interval observer for generalized Persidskii systems. This class of nonlinear models was first introduced for stability analysis in [17], where a linear combination of the integrals of the nonlinearities was used as a Lyapunov function. Next, that result was extended in [18] by augmenting the Lyapunov function through a combination of the absolute values of the states. Furthermore, Persidskii systems were studied in the context of diagonal stability [19], [20], sliding mode control [21], [22], [23] and Lur'e systems [4], with applications to opinion dynamics [24], neural networks [25], [26], [27] and filters [28]. Following the foundational results [17], [18], availability of a canonical form of Lyapunov function constitutes the main advantage of Persidskii dynamics. Hence, to analyze stability and robustness of the suggested nonlinear interval observer, the Lyapunov function method will be used, whose application can be reduced to verification of linear matrix inequalities (LMIs). Moreover, to design an interval observer, the conditions of nonnegativity for the Persidskii systems are revisited. Next, a nonlinear control is designed based on the interval estimates, which stabilizes the uncertain system in a vicinity of the origin. Finally, the utility of the developed observer is demonstrated on the problems of interval estimation for a recurrent neural network and a mechanical system with cubic velocity friction.

With respect to its preliminary version [29], this work contains the proofs, more detailed explanations, different examples and the control design part.

The paper is organized as follows. The preliminaries on the interval estimation and Persidskii systems are given in Section II. The problem statement is introduced in Section III. Nonnegativity conditions are investigated in Section IV. An interval observer is presented in Section V. The use of these interval estimates for output stabilization of uncertain Persidskii systems is considered in Section VI. Applications of the designed estimator to the problems of interval estimation and control are shown in Section VII.

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Notation

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, where $\mathbb{R}$ is the set of real numbers.
- $|\cdot|$ denotes the absolute value in $\mathbb{R}$, $\|\cdot\|$ is used for the Euclidean norm on $\mathbb{R}^n$.
- For a (Lebesgue) measurable function $d : \mathbb{R}_+ \to \mathbb{R}^m$ and $[t_0, t_1) \subset \mathbb{R}_+$ define the norm $\|d\|_{[t_0,t_1)} = \text{ess sup}_{t\in[t_0,t_1)}\|d(t)\|$, then $\|d\|_\infty = \|d\|_{[0,+\infty)}$ and the set of d with the property $\|d\|_\infty < +\infty$ we further denote as $\mathcal{L}_\infty^m$ (the set of essentially bounded measurable functions).
- A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if $\alpha(0) = 0$ and the function is strictly increasing. The function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is increasing to infinity. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed $t \in \mathbb{R}_+$ and $\beta(s, \cdot)$ is decreasing to zero for each fixed $s > 0$.
- A finite series of integers $1, 2, ..., n$ is denoted by $\overline{1,n}$, and $\{\overline{1,n}\} = \{1, 2, ..., n\}$.
- Denote the identity matrix of dimension $n \times n$ by I_n, the vector of dimension n or the matrix of dimension $n \times m$ with all elements equal 1 by $\mathtt{1}_n$ and $\mathtt{1}_{n\times m}$, respectively. The canonical basis vectors in $\mathbb{R}^n$ are denoted as $\mathfrak{e}_i = [0 \dots 0\ 1\ \dots 0]^\top$ for $i = \overline{1,n}$, where 1 appears in the i^{th} position.
- $\text{diag}\{g\} \in \mathbb{D}_+^n$ represents a diagonal matrix of dimension $n \times n$ with a vector $g \in \mathbb{R}_+^n$ on the main diagonal, where $\mathbb{D}_+^n \subset \mathbb{R}^{n\times n}$ is the set of nonnegative diagonal matrices.
- For a matrix $A \in \mathbb{R}^{n\times n}$ the vector of its eigenvalues is denoted by $\lambda(A)$, its i^{th} row and column by $A^{(i)}$ and $A^{[i]}$, respectively, for $i = \overline{1,n}$; $\|A\|_{\max} = \max_{i,j=\overline{1,n}} |A_{i,j}|$ (the elementwise maximum norm, it is not sub-multiplicative) and $\|A\|_2 = \sqrt{\max_{i=\overline{1,n}} \lambda_i(A^\top A)}$ (the induced matrix norm), the relations $\|A\|_{\max} \leq \|A\|_2 \leq n\|A\|_{\max}$ are satisfied between these norms.
- The relation $P \prec 0$ ($P \preceq 0$) means that a symmetric matrix $P \in \mathbb{R}^{n\times n}$ is negative (semi-)definite.

II. PRELIMINARIES

The used standard stability notions and their definitions can be found in [30].

In this paper, it is assumed that if the upper limit of a summation or a sequence is smaller than the lower one, then the corresponding terms (conditions) have to be omitted.

A. Interval arithmetic

For two vectors $x_1, x_2 \in \mathbb{R}^n$ or matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$, the relations $x_1 \leq x_2$ and $A_1 \leq A_2$ are understood elementwise. Given a matrix $A \in \mathbb{R}^{m\times n}$, define $A^+ = \max\{0, A\}$, $A^- = A^+ - A$ (similarly for vectors) and denote the matrix of absolute values of all elements by $|A| = A^+ + A^-$.

Lemma 1. [31] *Let $x \in \mathbb{R}^n$ be a vector variable, $\underline{x} \leq x \leq \overline{x}$ for some $\underline{x}, \overline{x} \in \mathbb{R}^n$. If $A \in \mathbb{R}^{m\times n}$ is a constant matrix, then*

$$A^+\underline{x} - A^-\overline{x} \leq Ax \leq A^+\overline{x} - A^-\underline{x}. \tag{1}$$

B. Nonnegative systems

A matrix $A \in \mathbb{R}^{n\times n}$ is called Hurwitz if all its eigenvalues have negative real parts, it is called Metzler if all its elements outside the main diagonal are nonnegative, and it is called nonnegative if $A \geq 0$.

Any solution of the linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\omega(t),\ t \geq 0, \\ y(t) &= Cx(t) + D\omega(t), \end{aligned} \tag{2}$$

with $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$ and a Metzler matrix $A \in \mathbb{R}^{n\times n}$, is elementwise nonnegative for all $t \geq 0$ provided that $x(0) \geq 0$, $\omega : \mathbb{R}_+ \to \mathbb{R}_+^q$ and $B \in \mathbb{R}_+^{n\times q}$ [32], [33]. The output solution $y(t)$ is nonnegative if $C \in \mathbb{R}_+^{p\times n}$ and $D \in \mathbb{R}_+^{p\times q}$. The dynamical systems are called nonnegative if for initial conditions in $\mathbb{R}_+^n$ the nonnegativity of the solutions is guaranteed [33].

Lemma 2. [15] *Given the matrices $A \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$. If there is a matrix $L \in \mathbb{R}^{n\times p}$ such that the matrices $A - LC$ and Y have the same eigenvalues, then there is a matrix $S \in \mathbb{R}^{n\times n}$ such that $Y = S(A - LC)S^{-1}$ provided that the pairs $(A - LC, \chi_1)$ and (Y, χ_2) are observable for some $\chi_1 \in \mathbb{R}^{1\times n}$, $\chi_2 \in \mathbb{R}^{1\times n}$.*

This result allows to represent the system (2) in its nonnegative form via a similarity transformation of coordinates.

Lemma 3. [34] *Let $D \in \Xi \subset \mathbb{R}^{n\times n}$ be a matrix variable satisfying the interval constraints $\Xi = \{D \in \mathbb{R}^{n\times n} : D_a - \Delta \leq D \leq D_a + \Delta\}$ for some $D_a^\top = D_a \in \mathbb{R}^{n\times n}$ and $\Delta \in \mathbb{R}_+^{n\times n}$. If for some constant $\mu \in \mathbb{R}_+$ and $\Upsilon \in \mathbb{D}_+^n$ the Metzler matrix $Y = \mu \mathtt{1}_{n\times n} - \Upsilon$ has the same eigenvalues as the matrix D_a, then there is an orthogonal matrix $S \in \mathbb{R}^{n\times n}$ such that the matrix $S^\top DS$ is Metzler for any $D \in \Xi$ provided that $\mu > n\|\Delta\|_{\max}$.*

In the last lemma, the existence of similarity transformation is established for an interval of matrices.

Lemma (Finsler's lemma)**.** [35] *Let $x \in \mathbb{R}^n \setminus \{0\}$ and $P, R \in \mathbb{R}^{n\times n}$ are symmetric, then $x^\top Px \prec 0$ whenever $x^\top Rx = 0$ if and only if there exists $\rho \in \mathbb{R}$ such that $P - \rho R \prec 0$.*

C. Persidskii systems

Consider the following class of systems [36], [37]:

$$\dot{x}(t) = A_0x(t) + \sum_{j=1}^{M} A_jf^j(H_jx(t)) + Bu(t) + d(t), \tag{3}$$

$$y(t) = \begin{bmatrix} C_0x(t) \\ C_1f^1(H_1x(t)) \\ \vdots \\ C_Mf^M(H_Mx(t)) \end{bmatrix} + v(t),\ t \geq 0,$$

where $x(t) = [x_1(t) \dots x_n(t)]^\top \in \mathbb{R}^n$ is the state vector, $x(0) \in \mathbb{R}^n$; $u(t) \in \mathbb{R}^m$ is the known input, $u \in \mathcal{L}_\infty^m$; $y(t) \in \mathbb{R}^p$ is the output signal, $p = \sum_{g=0}^{M} p_g$ and $C_g \in \mathbb{R}^{p_g\times k_g}$ for $g = \overline{0,M}$, which is available for measurements and containing the perturbation $v(t) \in \mathbb{R}^p$, $v \in \mathcal{L}_\infty^p$; and $d(t) \in \mathbb{R}^n$

is the external disturbance, $d \in \mathcal{L}_{\infty}^{n}$; $f^j : \mathbb{R}^{k_j} \to \mathbb{R}^{k_j}$ with diagonal structure $f^j(s) = [f_1^j(s_1) \dots f_{k_j}^j(s_{k_j})]^\top$, $j = \overline{1,M}$ are the continuous functions ensuring existence of solutions of the system (3) in the forward time, the matrices $A_g \in \mathbb{R}^{n \times k_g}$ and $H_g \in \mathbb{R}^{k_g \times n}$ for $g = \overline{0,M}$, where for brevity we use the convention $k_0 = n$ and $H_0 = I_n$ with $f^0(x) = x$.

The model (3) belongs to the class of Persidskii system [18], [19] under the following passivity (or sector) condition imposed on the nonlinearities: for any $j = \overline{1,M}$ and $i = \overline{1,k_j}$:

$$s f_i^j(s) > 0 \quad \forall s \in \mathbb{R} \setminus \{0\}.$$

In the property above, it is stated that all nonlinearities belong to a sector and may take zero values at zero only. If $H_1 = I_n$ and $A_r = 0$ for all $r = \overline{2,M}$, then we recover the system studied by Persidskii in the conventional framework [18]. In the case of $M = 1$, (3) belongs also to the class of Lur'e systems widely investigated in the absolute stability theory [5]. In this work we will need a more strong incremental passivity (or monotonicity) property ensured by the following assumption:

Assumption 1. *For any $j = \overline{1,M}$ and $i = \overline{1,k_j}$:*

$$(s_1 - s_2)\left(f_i^j(s_1) - f_i^j(s_2)\right) > 0 \quad \forall s_1, s_2 \in \mathbb{R},\ s_1 \neq s_2.$$

Taking $s_2 = 0$, and since $f_i^j(0) = 0$, Assumption 1 implies the sector condition, while additionally asking certain monotonicity of the nonlinearities (it is also called incremental passivity of f_i^j).

After a proper re-indexing and decomposition of f^j, there exists $\nu \in \{\overline{0,M}\}$ such that for all $z = \overline{1,\nu}$ and $i = \overline{1,k_z}$:

$$\lim_{s \to \pm\infty} f_i^z(s) = \pm\infty;$$

and due to Assumption 1 for all $j = \overline{1,M}$ and $i = \overline{1,k_j}$:

$$\lim_{s \to \pm\infty} \int_0^s f_i^j(\sigma) d\sigma = +\infty.$$

Thus, some of the nonlinearities are radially unbounded, and $\nu = 0$ corresponds to the case when all nonlinearities are bounded (at least for negative or positive argument).

III. Problem statement

Consider the system (3). The following assumptions will be used in this work.

Assumption 2. *Let $x(0) \in [\underline{x}_0, \overline{x}_0]$ for some known $\underline{x}_0, \overline{x}_0 \in \mathbb{R}^n$.*

Assumption 3. *There exist known signals $\underline{d}, \overline{d} \in \mathcal{L}_{\infty}^{n}$ and a constant $V > 0$ such that $\underline{d}(t) \leq d(t) \leq \overline{d}(t)$ and $-V1_p \leq v(t) \leq V1_p$ for all $t \geq 0$.*

Assumptions 2 and 3 represent standard hypotheses for the interval estimation [8], [9].

The objective of this work is to design an interval observer for the system (3), which is a dynamical system that utilizes the properties stated in assumptions 1–3 (i.e., taking the information on the initial conditions $[\underline{x}_0, \overline{x}_0]$, the admissible bounds on the values of the exogenous input $[\underline{d}(t), \overline{d}(t)]$, the upper bound on the measurement noise V, and the features of the nonlinearities), the information about the system matrices A_g, H_g, B and C_g for $g = \overline{0,M}$, and generates interval estimates $\underline{x}(t), \overline{x}(t) \in \mathbb{R}^n$ such that $\overline{x} - \underline{x} \in \mathcal{L}_{\infty}^{n}$ and

$$\underline{x}(t) \leq x(t) \leq \overline{x}(t) \quad \forall t \geq 0. \tag{4}$$

For an interval observer, it is also desired that for $M = 0$ (there is no nonlinearity), with $\underline{d}(t) = d(t) = \overline{d}(t)$ for all $t \geq 0$ and $V = 0$ (there is no uncertainty), $\limsup_{t \to +\infty} \|\underline{x}(t) - x(t)\| + \|\overline{x}(t) - x(t)\| = 0$. Next, it is required to design a control feedback dependent on $\underline{x}(t), \overline{x}(t)$ and other known quantities that forces $x(t)$ to a neighborhood of the origin, whose size is proportional to $\max\{\|\overline{d}\|_\infty, \|\underline{d}\|_\infty, V\}$.

IV. Nonnegativity in Persidskii systems

In the simplest disturbance-free case, the system (3) with $M = 1$ and $H_1 = I_n$ takes the form of the conventional Persidskii model:

$$\dot{x}(t) = A_0 x(t) + A_1 f^1(x(t)) + Bu(t).$$

In such a case the nonnegativity conditions are similar to the ones for (2) [18], [38]: the matrices A_0 and A_1 are Metzler and $Bu(t) \in \mathbb{R}_+^m$ for all $t \geq 0$. However, for $H_1 \neq I_n$ these conditions become more sophisticated:

Lemma 4. *For any $x(0) \in \mathbb{R}_+^n$, the special case of the system* (3),

$$\dot{x}(t) = A_1 f^1(H_1 x(t)),$$

satisfying Assumption 1, has solution $x(t) \geq 0$ for all $t \geq 0$, provided that for all $i = \overline{1,k_1}$ one of the following properties is verified:

i) $H_1^{(i)} \geq 0$ and $A_1^{[i]} \geq 0$,
ii) $H_1^{(i)} \leq 0$ and $A_1^{[i]} \leq 0$.

Moreover, if $H_1^{(i)} = \kappa \mathfrak{e}_r^\top$ for some $\kappa \in \mathbb{R}$, $i \in \{\overline{1,k_1}\}$ and $r \in \{\overline{1,n}\}$, then $(A_1)_{r,i}$ can be arbitrary.

Proof. Assume that $x(t) \geq 0$ for a given $t \in \mathbb{R}_+$, then we need to show that $\dot{x}(t) \geq 0$ under the imposed conditions, which prevents the variable $x(t)$ to change the sign. For the cases i) and ii), we get $H_1^{(i)} x(t) \geq 0$ or $H_1^{(i)} x(t) \leq 0$ while $A_1^{[i]} f_i^1(H_1^{(i)} x(t)) \geq 0$ for all $i = \overline{1,k_1}$ (due to $s f_i^1(s) > 0$ for all $s \in \mathbb{R} \setminus \{0\}$), which ensures $\dot{x}(t) \geq 0$, that is needed for nonnegativity. If $H_1^{(i)} = \kappa \mathfrak{e}_r^\top$ for some $\kappa \in \mathbb{R}$, $i \in \{\overline{1,k_1}\}$ and $r \in \{\overline{1,n}\}$ (the remaining case), then for $x_r(t) = 0$ the constraint $\dot{x}_r(t) \geq 0$ is kept under any gain $(A_1)_{r,i}$ of $f_i^1(H_1^{(i)} x(t)) = f_i^1(\kappa x_r(t))$, since $f_i^1(0) = 0$. □

The conditions given in Lemma 4 generalize the previous ones for the conventional case: it is easy to check that if $H_1 = I_n$, then A_1 can be chosen Metzler according to Lemma 4.

Since the proposed nonnegativity conditions depend on the pair of matrices A_1 and H_1, then we further will use the notation

$$(A_1, H_1) \in \mathcal{N}$$

if the restrictions formulated in Lemma 4 are verified for these A_1 and H_1 (hence, the property $(A_0, H_0) = (A_0, I_n) \in \mathcal{N}$

implies that A_0 is Metzler). Then we can present the nonnegativity conditions for (3):

Corollary 1. *Under Assumption 1, the system* (3) *is nonnegative if* $(A_g, H_g) \in \mathcal{N}$ *for all* $g = \overline{0, M}$ *and* $Bu(t) + d(t) \geq 0$ *for all* $t \geq 0$.

Similarly the conditions for nonnegativity of discrepancy between two solutions of (3) with different initial conditions can be formulated:

Corollary 2. *Under Assumption 1, consider a copy of the system* (3)*:*

$$\dot{\xi}(t) = A_0\xi(t) + \sum_{j=1}^{M} A_j f^j(H_j\xi(t)) + Bu(t) + d(t),$$

with the same inputs u, d *and* $\xi(0) \in \mathbb{R}^n$ *verifying* $x(0) \geq \xi(0)$*, then* $x(t) \geq \xi(t)$ *for all* $t \geq 0$ *provided that* $(A_g, H_g) \in \mathcal{N}$ *for all* $g = \overline{0, M}$.

Proof. Consider the error $e(t) = x(t) - \xi(t)$, then $e(0) \geq 0$ and it is enough to show that $\dot{e}(t) \geq 0$, where

$$\dot{e}(t) = A_0 e(t) + \sum_{j=1}^{M} A_j \left(f^j(H_j x(t)) - f^j(H_j\xi(t))\right)$$
$$= \sum_{g=0}^{M} A_g \left(f^g(H_g x(t)) - f^g(H_g\xi(t))\right).$$

It is straightforward to check that if $e(t) \geq 0$ for a given $t \in \mathbb{R}_+$, then due to $(A_g, H_g) \in \mathcal{N}$ for all $g = \overline{0, M}$, we have $H_g^{(i)} e(t) \geq 0$ or $H_g^{(i)} e(t) \leq 0$ implying $A_g^{[i]} f_i^g(H_g^{(i)} e(t)) \geq 0$ for all $i = \overline{1, k_g}$. Under Assumption 1, in such a case $A_g^{[i]} \left(f_i^g(H_g^{(i)} x(t)) - f_i^g(H_g^{(i)} \xi(t))\right) \geq 0$ for all $g = \overline{0, M}$ and $i = \overline{1, k_g}$, which ensures $\dot{e}(t) = \sum_{g=0}^{M} \sum_{i=1}^{k_g} A_g^{[i]} \left(f_i^g(H_g^{(i)} x(t)) - f_i^g(H_g^{(i)} \xi(t))\right) \geq 0$ implying nonnegativity. The special case $H_g^{(i)} = \kappa \mathbf{e}_r^\top$ for some $\kappa \in \mathbb{R}$, $i \in \{\overline{1, k_g}\}$ and $r \in \{\overline{1, n}\}$ can be analyzed similarly. □

The conditions of corollaries 1 and 2 will be used later in the design of interval observer.

Remark 1. As in the case of linear system (2), if $(A_g, H_g) \notin \mathcal{N}$, then a transformation of coordinates can be considered. Indeed, the transformation $z = Sx$ with an invertible matrix $S \in \mathbb{R}^{n \times n}$ preserves the Persidskii form of the system (3) in the new coordinates:

$$\dot{z}(t) = \tilde{A}_0 z(t) + \sum_{j=1}^{M} \tilde{A}_j f^j(\tilde{H}_j z(t)) + SBu(t) + Sd(t),$$

$$y(t) = \begin{bmatrix} \tilde{C}_0 z(t) \\ C_1 f^1(\tilde{H}_1 z(t)) \\ \vdots \\ C_M f^M(\tilde{H}_M z(t)) \end{bmatrix} + v(t),$$

where

$$\tilde{A}_0 = SA_0S^{-1}, \ \tilde{C}_0 = C_0S^{-1};$$
$$\tilde{A}_j = SA_j, \ \tilde{H}_j = H_jS^{-1}, \ j = \overline{1, M},$$

and Lemma 2 can be used to derive S. However, to find such a S providing Metzler property to the matrix $\tilde{A}_0$ and simultaneously $(A_j, H_j) \in \mathcal{N}$ for all $j = \overline{1, M}$ can be difficult. If all matrices A_j are sufficiently close to each other, then the result of Lemma 3 can be adapted to derive S.

However, if such a transformation S cannot be constructed, then interval inclusion of the nonlinearities can be utilized for the design of interval observers. Such an inclusion is well developed in the theory of the bounding decomposition functions for mixed monotone mappings [39], which in our case has a simple realization:

Lemma 5. *Let* $s, \underline{s}, \overline{s} \in \mathbb{R}^{k_1}$ *and* $f^1 : \mathbb{R}^{k_1} \to \mathbb{R}^{k_1}$ *verify Assumption 1, then*

$$\underline{s} \leq s \leq \overline{s} \Leftrightarrow f^1(\underline{s}) \leq f^1(s) \leq f^1(\overline{s}).$$

Proof. First, let $\underline{s} \leq s \leq \overline{s}$. By contradiction, assume that $f_i^1(\underline{s}_i) > f_i^1(s_i)$ for some $i \in \{\overline{1, k_1}\}$ (the strict inequality implies that $s_i \neq \underline{s}_i$), then $s_i - \underline{s}_i > 0$ by the assumed relations, and we get that $(s_i - \underline{s}_i)\left(f_i^1(\underline{s}_i) - f_i^1(s_i)\right) > 0$, which violates the restrictions introduced in Assumption 1, hence, $f^1(\underline{s}) \leq f^1(s)$. The relation $f^1(s) \leq f^1(\overline{s})$ can be proven by the same arguments. Second, let $f^1(\underline{s}) \leq f^1(s) \leq f^1(\overline{s})$, and in a similar way, $s_i - \underline{s}_i < 0$ for some $i \in \{\overline{1, k_1}\}$ violating the left-hand side of the relation formulated in this lemma. In such a case $f_i^1(\underline{s}_i) \leq f_i^1(s_i)$, and multiplying these inequalities we get that $(s_i - \underline{s}_i)\left(f_i^1(\underline{s}_i) - f_i^1(s_i)\right) \geq 0$ that contradicts Assumption 1, hence $\underline{s} \leq s$. □

If $\underline{x} \leq x \leq \overline{x}$ for some $x, \underline{x}, \overline{x} \in \mathbb{R}^n$, $H_1 \in \mathbb{R}^{k_1 \times n}$ and $f^1 : \mathbb{R}^{k_1} \to \mathbb{R}^{k_1}$ verifies Assumption 1, then $\underline{s} = H_1^+\underline{x} - H_1^-\overline{x} \leq s = H_1 x \leq H_1^+\overline{x} - H_1^-\underline{x} = \overline{s}$ due to (1), and using Lemma 5 we obtain:

$$f^1(H_1^+\underline{x} - H_1^-\overline{x}) \leq f^1(H_1 x) \leq f^1(H_1^+\overline{x} - H_1^-\underline{x}). \quad (5)$$

V. Interval observer design

The system (3) can be equivalently presented as follows:

$$\begin{aligned} \dot{x}(t) \ &= \ (A_0 - L_0C_0)x(t) + \sum_{j=1}^{M}(A_j - L_jC_j)f^j(H_j x(t)) \\ &\quad + L\left(y(t) - v(t)\right) + Bu(t) + d(t), \end{aligned}$$

where $L = [L_0 \ L_1 \dots L_M] \in \mathbb{R}^{n \times p}$, with $L_g \in \mathbb{R}^{n \times p_g}$ for $g = \overline{0, M}$, is an observer gain to be properly chosen next. As usual [8], [9], such a gain L has to ensure nonnegativity and stability of the dynamics of the interval estimation errors. Let us focus first on the former property, then following corollaries 1 and 2 we may ask for existence of L providing $(A_g - L_gC_g, H_g) \in \mathcal{N}$ for all $g = \overline{0, M}$, which is however a restrictive requirement. For brevity of presentation, we will not introduce a transformation of coordinates discussed in Remark 1, which can relax the conservatism in some cases (synthesis of the transformation matrix S is not an easy task). There is another approach presented in [40] that mixes the advantages of a (partial) transformation of coordinates with simplicity of formulation in the original variables, whose utilization requires introduction of the following additional conditions:

Assumption 4. *There exist $\ell \in \{\overline{0,M}\}$, $L \in \mathbb{R}^{n\times p}$ and $N \in \mathbb{R}^{n\times p_0}$ such that $(D_r, H_r) \in \mathcal{N}$ for all $r = \overline{0,\ell}$, where we denote $D_g = TA_g - L_gC_g$ for all $g = \overline{0,M}$ and $T = I_n - NC_0$.*

Obviously, if $T = I_n$ or $N = 0$, then we have just the initial case without any transformation of coordinates. Note that in Remark 1 the transformation S acts on A_j with $j = \overline{1,M}$ in a similar way yielding SA_j for the dynamics of the new variables. The variable ℓ determines the number of terms in (3) that can be well processed by L and T, the choice $\ell \geq 0$ (that the linear part $D_0 = TA_0 - L_0C_0$ is always Metzler) is not obligatory and accepted to streamline the presentation.

Under Assumption 4, it is straightforward to check that

$$\begin{aligned} x(t) &= z(t) + N(y_0(t) - v_0(t)),\ z(t) = Tx(t), \\ \dot{z}(t) &= \sum_{g=0}^{M} D_g f^g(H_g x(t)) + TBu(t) \\ &\quad + L(y(t) - v(t)) + Td(t), \end{aligned}$$

then the interval observer for (3) can be selected as follows:

$$\begin{aligned} \underline{x}(t) &= \underline{z}(t) + Ny_0(t) - |N|\mathbb{1}_{p_0}V, \\ \overline{x}(t) &= \overline{z}(t) + Ny_0(t) + |N|\mathbb{1}_{p_0}V, \\ \dot{\underline{z}}(t) &= \sum_{r=0}^{\ell} \left(D_r f^r(H_r\underline{x}(t)) - \underline{\kappa}_r(\underline{x}(t))\right) + TBu(t) \\ &\quad + \sum_{k=\ell+1}^{M} D_k^+ f^k(H_k^+\underline{x}(t) - H_k^-\overline{x}(t)) \\ &\quad - D_k^- f^k(H_k^+\overline{x}(t) - H_k^-\underline{x}(t)) \\ &\quad + Ly(t) - |L|\mathbb{1}_p V + T^+\underline{d}(t) - T^-\overline{d}(t), \qquad (6) \\ \dot{\overline{z}}(t) &= \sum_{r=0}^{\ell} \left(D_r f^r(H_r\overline{x}(t)) + \overline{\kappa}_r(\overline{x}(t))\right) + TBu(t) \\ &\quad + \sum_{k=\ell+1}^{M} D_k^+ f^k(H_k^+\overline{x}(t) - H_k^-\underline{x}(t)) \\ &\quad - D_k^- f^k(H_k^+\underline{x}(t) - H_k^-\overline{x}(t)) \\ &\quad + Ly(t) + |L|\mathbb{1}_p V + T^+\overline{d}(t) - T^-\underline{d}(t), \\ \underline{z}(0) &= T^+\underline{x}_0 - T^-\overline{x}_0,\ \overline{z}(0) = T^+\overline{x}_0 - T^-\underline{x}_0, \end{aligned}$$

where $y_0 = C_0x(t) \in \mathbb{R}^{p_0}$ represents the first p_0 (linear in the state) measured outputs (similarly, $v_0(t) \in \mathbb{R}^{p_0}$ further corresponds to the first p_0 elements of the output perturbation v), $\underline{\kappa}_r, \overline{\kappa}_r : \mathbb{R}^n \to \mathbb{R}^n_+$ are derived using the following rules: if $H_r^{(i)} = \varkappa \mathfrak{e}_g^\top$ for some $\varkappa \in \mathbb{R}$, $i \in \{\overline{1,k_r}\}$ and $g \in \{\overline{1,n}\}$ with $(D_r)_{g,i}\varkappa < 0$, then

$$\begin{aligned} \underline{\kappa}_{r,g}(\underline{x}(t)) &= |(D_r)_{g,i}|[f_i^r(H_r\underline{x}(t) + 2H_r^+|N|\mathbb{1}_{p_0}V) \\ &\quad - f_i^r(H_r\underline{x}(t))], \\ \overline{\kappa}_{r,g}(\overline{x}(t)) &= |(D_r)_{g,i}|[-f_i^r(H_r\overline{x}(t) - 2H_r^+|N|\mathbb{1}_{p_0}V) \\ &\quad + f_i^r(H_r\overline{x}(t))], \end{aligned}$$

and the elements of $\underline{\kappa}_r, \overline{\kappa}_r$ are zeros otherwise. Note that if $T = I_n$ (or $N = 0$), then $\underline{\kappa}_r = \overline{\kappa}_r = 0$ for all $r = \overline{0,\ell}$. The desired interval inclusion of the state is realized by (6) under the introduced hypotheses:

Lemma 6. *Let assumptions 1–4 be satisfied for the system* (3)*, then the interval observer* (6) *ensures the property* (4)*.*

Proof. Define the interval estimation errors:

$$\begin{aligned} \underline{e}(t) &= x(t) - \underline{x}(t) = \underline{\epsilon}(t) + |N|\mathbb{1}_{p_0}V - Nv_0, \\ \overline{e}(t) &= \overline{x}(t) - x(t) = \overline{\epsilon}(t) + |N|\mathbb{1}_{p_0}V + Nv_0, \\ \underline{\epsilon}(t) &= z(t) - \underline{z}(t),\ \overline{\epsilon}(t) = \overline{z}(t) - z(t), \end{aligned}$$

whose nonnegativity it is necessary to substantiate, then under given assumptions their dynamics take the form:

$$\begin{aligned} \dot{\underline{\epsilon}}(t) &= \sum_{r=0}^{\ell} \left(D_r\left(f^r(H_rx(t)) - f^r(H_r\underline{x}(t))\right) + \underline{\kappa}_r(\underline{x}(t))\right) \\ &\quad + \sum_{k=\ell+1}^{M} \underline{\delta}_x^k(t) + \underline{\delta}_v(t) + \underline{\delta}_d(t), \\ \dot{\overline{\epsilon}}(t) &= \sum_{r=0}^{\ell} \left(D_r\left(f^r(H_r\overline{x}(t)) - f^r(H_rx(t))\right) + \overline{\kappa}_r(\overline{x}(t))\right) \\ &\quad + \sum_{k=\ell+1}^{M} \overline{\delta}_x^k(t) + \overline{\delta}_v(t) + \overline{\delta}_d(t), \end{aligned}$$

where

$$\begin{aligned} \underline{\delta}_v(t) &= |L|\mathbb{1}_pV - Lv(t),\ \overline{\delta}_v(t) = |L|\mathbb{1}_pV + Lv(t), \\ \underline{\delta}_d(t) &= Td(t) - T^+\underline{d}(t) + T^-\overline{d}(t), \\ \overline{\delta}_d(t) &= T^+\overline{d}(t) - T^-\underline{d}(t) - Td(t), \\ \underline{\delta}_x^k(t) &= D_kf^k(H_kx(t)) - D_k^+f^k(H_k^+\underline{x}(t) - H_k^-\overline{x}(t)) \\ &\quad + D_k^-f^k(H_k^+\overline{x}(t) - H_k^-\underline{x}(t)), \\ \overline{\delta}_x^k(t) &= D_k^+f^k(H_k^+\overline{x}(t) - H_k^-\underline{x}(t)) - D_kf^k(H_kx(t)) \\ &\quad - D_k^-f^k(H_k^+\underline{x}(t) - H_k^-\overline{x}(t)). \end{aligned}$$

Note that $\underline{\delta}_v(t), \overline{\delta}_v(t) \in \mathbb{R}^p_+$ and $\underline{\delta}_d(t), \overline{\delta}_d(t) \in \mathbb{R}^n_+$ for all $t \geq 0$ by Assumption 3 and (1). Due to assumptions 2 and 3, $\underline{\epsilon}(0), \overline{\epsilon}(0) \in \mathbb{R}^n_+$ and $|N|\mathbb{1}_{p_0}V \geq \pm Nv_0$, hence $\underline{e}(0), \overline{e}(0) \in \mathbb{R}^n_+$. By induction, let $\underline{e}(t), \overline{e}(t) \in \mathbb{R}^n_+$ for some $t > 0$, therefore $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$, then $\underline{\delta}_x^k(t), \overline{\delta}_x^k(t) \in \mathbb{R}^n_+$ for all $k = \overline{\ell+1,M}$ due to (1) and (5). Consequently, it can be shown that $\dot{\underline{\epsilon}}(t), \dot{\overline{\epsilon}}(t) \in \mathbb{R}^n_+$ follows the arguments of Corollary 2 and the properties of $\underline{\kappa}_r, \overline{\kappa}_r$ for $r = \overline{0,\ell}$, which ensure nonnegativity of $D_r\left(f^r(H_rx(t)) - f^r(H_r\underline{x}(t))\right) + \underline{\kappa}_r(\underline{x}(t))$ and $D_r\left(f^r(H_r\overline{x}(t)) - f^r(H_rx(t))\right) + \overline{\kappa}_r(\overline{x}(t))$ when the related elements of $\underline{\epsilon}(t)$ or $\overline{\epsilon}(t)$ become zero. Indeed,

$$\begin{aligned} D_r[f^r(H_rx(t)) - f^r(H_r\underline{x}(t))] &= D_r[f^r(H_r\{\underline{\epsilon}(t) \\ &\quad + |N|\mathbb{1}_{p_0}V - Nv_0(t) + \underline{x}(t)\}) - f^1(H_r\underline{x}(t))], \\ D_r[f^r(H_r\overline{x}(t)) - f^r(H_rx(t))] &= D_r[f^r(H_r\overline{x}(t)) \\ &\quad - f^r(H_r\{\overline{x}(t) - \overline{\epsilon}(t) - |N|\mathbb{1}_{p_0}V - Nv_0(t))\}] \end{aligned}$$

and for $\underline{\epsilon}(t) = 0$ or $\overline{\epsilon}(t) = 0$ these terms may become negative depending on the signs of the respective elements of D_r. Finally, it implies the property $\underline{\epsilon}(t), \overline{\epsilon}(t) \in \mathbb{R}^n_+$ for the next instants of time, leading to $\underline{e}(t), \overline{e}(t) \in \mathbb{R}^n_+$ for all $t \geq 0$. $\square$

To demonstrate that (6) is indeed an interval observer for (3) it is left to prove the boundedness of $\overline{x}(t)-\underline{x}(t)$. To this end denote

$$\Gamma=\left[\begin{array}{cc} I_n & -I_n \end{array}\right];$$

$$\mathcal{D}_r=\left[\begin{array}{cc} D_r & 0\\ 0 & D_r \end{array}\right],\ \mathcal{H}_r=\left[\begin{array}{cc} H_r & 0\\ 0 & H_r \end{array}\right],\ r=\overline{0,\ell};$$

$$\mathcal{D}_k=\left[\begin{array}{cc} D_k^+ & -D_k^-\\ -D_k^- & D_k^+ \end{array}\right],\ \mathcal{H}_k=\left[\begin{array}{cc} H_k^+ & -H_k^-\\ -H_k^- & H_k^+ \end{array}\right],$$
$$k=\overline{\ell+1,M}.$$

Theorem 1. *Let assumptions 1–4 be satisfied for the system* (3)*, the functions* $\underline{\kappa}_r,\overline{\kappa}_r$ *be globally bounded for* $r=\overline{0,\ell}$ *and* $v_0(t)$ *be continuously differentiable or* $N=0$*, then* (6) *is an interval observer provided that there exist* $P_1^\top=P_1\in\mathbb{R}^{2n\times 2n}$*,* $P_2^\top=P_2\in\mathbb{R}^{n\times n}$*,* $\Lambda^j\in\mathbb{D}_+^{2k_j}$ *for* $j=\overline{1,M}$*,* $\Phi^\top=\Phi\in\mathbb{R}^{2n\times 2n}$*,* $\Psi\in\mathbb{R}^{2n\times 2n}$*,* $\Omega_g\in\mathbb{R}^{2n\times 2k_g}$*,* $\Upsilon_{g,g}\in\mathbb{D}_+^{2k_g}$ *(*$\tilde{\Upsilon}_{g,g}=\mathcal{H}_g^\top\Upsilon_{g,g}\mathcal{H}_g$*) for* $g=\overline{0,M}$*,* $\Upsilon_{0,j}\in\mathbb{D}_+^{2k_j}$ *(*$\tilde{\Upsilon}_{0,j}=\mathcal{H}_j^\top\Upsilon_{0,j}$*) for* $j=\overline{1,M}$*,* $\Upsilon_{z,s}\in\mathbb{D}_+^{2n}$ *(*$\tilde{\Upsilon}_{z,s}=\mathcal{H}_z\Upsilon_{z,s}\mathcal{H}_s^\top$*) for* $z=\overline{1,M-1}$ *and* $s=\overline{z+1,M}$ *such that the following LMIs are satisfied:*

$$P_1\succeq 0,\ P_2\succ 0,\ \Phi\succ 0,\ Q\preceq 0,$$
$$\rho_1\left(P+2\sum_{j=1}^{M}\mathcal{H}_j^\top\Lambda^j\mathcal{H}_j\right)\preceq\sum_{r=0}^{\nu}\tilde{\Upsilon}_{r,r}$$
$$+\rho_2\sum_{z=1}^{\nu-1}\sum_{s=z+1}^{\nu}\mathcal{H}_z^\top\tilde{\Upsilon}_{z,s}\mathcal{H}_s+\rho_3\sum_{j=1}^{M}\mathcal{H}_j^\top\Upsilon_{0,j}\mathcal{H}_j,$$

for some $\rho_1,\rho_2,\rho_3>0$*, where* $P=P_1+\Gamma^\top P_2\Gamma$ *and*

$$Q=\left[\begin{array}{cccccc} -\Psi^\top-\Psi & P+\Psi^\top\mathcal{D}_0-\Omega_0 & \mathcal{H}_1^\top\Lambda^1+\Psi^\top\mathcal{D}_1-\Omega_1 & \cdots & \mathcal{H}_M^\top\Lambda^M+\Psi^\top\mathcal{D}_M-\Omega_M & \Psi^\top\\ P+\mathcal{D}_0^\top\Psi-\Omega_0^\top & \Omega_0^\top\mathcal{D}_0+\mathcal{D}_0^\top\Omega_0+\Upsilon_{0,0} & \Omega_0^\top\mathcal{D}_1+\mathcal{D}_0^\top\Omega_1+\tilde{\Upsilon}_{0,1} & \cdots & \Omega_0^\top\mathcal{D}_M+\mathcal{D}_0^\top\Omega_M+\tilde{\Upsilon}_{0,M} & \Omega_0^\top\\ \Lambda^1\mathcal{H}_1+\mathcal{D}_1^\top\Psi-\Omega_1^\top & \mathcal{D}_1^\top\Omega_0+\Omega_1^\top\mathcal{D}_0+\tilde{\Upsilon}_{0,1}^\top & \Omega_1^\top\mathcal{D}_1+\mathcal{D}_1^\top\Omega_1+\Upsilon_{1,1} & \cdots & \Omega_1^\top\mathcal{D}_M+\mathcal{D}_1^\top\Omega_M+\tilde{\Upsilon}_{1,M} & \Omega_1^\top\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \Lambda^M\mathcal{H}_M+\mathcal{D}_M^\top\Psi-\Omega_M^\top & \mathcal{D}_M^\top\Omega_0+\Omega_M^\top\mathcal{D}_0+\tilde{\Upsilon}_{0,M}^\top & \mathcal{D}_M^\top\Omega_1+\Omega_M^\top\mathcal{D}_1+\tilde{\Upsilon}_{1,M} & \cdots & \Omega_M^\top\mathcal{D}_M+\mathcal{D}_M^\top\Omega_M+\Upsilon_{M,M} & \Omega_M^\top\\ \Psi & \Omega_0 & \Omega_1 & \cdots & \Omega_M & -\Phi \end{array}\right].$$

Proof. The relations (4) are satisfied by Lemma 6. It is left to demonstrate that $\overline{x}-\underline{x}\in\mathcal{L}_\infty^n$ in (6). For this purpose, define the extended state vector as $X=[\underline{x}^\top\ \overline{x}^\top]^\top$, whose dynamics admit the differential equation:

$$\dot{X}(t)=\delta(t)+\sum_{r=0}^{\ell}\left[\begin{array}{cc} D_r & 0\\ 0 & D_r \end{array}\right]\left[\begin{array}{c} f^r(H_r\underline{x}(t))\\ f^r(H_r\overline{x}(t)) \end{array}\right]$$
$$+\sum_{k=\ell+1}^{M}\left[\begin{array}{cc} D_k^+ & -D_k^-\\ -D_k^- & D_k^+ \end{array}\right]\left[\begin{array}{c} f^k(H_k^+\underline{x}(t)-H_k^-\overline{x}(t))\\ f^k(H_k^+\overline{x}(t)-H_k^-\underline{x}(t)) \end{array}\right],$$

where

$$\delta(t)=\left[\begin{array}{c} \underline{\delta}(t)\\ \overline{\delta}(t) \end{array}\right],$$
$$\underline{\delta}(t)=TBu(t)+Ly(t)+N\dot{y}_0(t)-\sum_{r=0}^{\ell}\underline{\kappa}_r(\underline{x}(t))$$
$$-|L|1_pV+T^+\underline{d}(t)-T^-\overline{d}(t),$$
$$\overline{\delta}(t)=TBu(t)+Ly(t)+N\dot{y}_0(t)+\sum_{r=0}^{\ell}\overline{\kappa}_r(\overline{x}(t))$$
$$+|L|1_pV+T^+\overline{d}(t)-T^-\underline{d}(t)$$

is a new input, whose boundedness is independent of $X(t)$ since $\underline{\kappa}_r,\overline{\kappa}_r$ are bounded for $r=\overline{0,\ell}$ (to simplify the notation we write just $\delta(t)$ omitting dependence on other variables). Denote

$$F^k(s)=\left[\begin{array}{c} f^k(s)\\ f^k(s) \end{array}\right],\ k=\overline{0,M},$$

then we obtain a new generalized Persidskii system:

$$\dot{X}(t)=\mathcal{D}_0X(t)+\sum_{j=1}^{M}\mathcal{D}_jF^j(\mathcal{H}_jX(t))+\delta(t),\qquad (7)$$
$$Y(t)=\Gamma X(t),$$

where $Y(t)\in\mathbb{R}^n$ is an auxiliary output, and boundedness of $Y(t)$ we would like to analyze in the presence of the input $\delta(t)$. To this end, following [37], [41], in order to investigate input-to-output stability property [42] of (7), consider a candidate Lyapunov function:

$$W(X)=X^\top PX+2\sum_{j=1}^{M}\sum_{i=1}^{2k_j}\Lambda_i^j\int_0^{\mathcal{H}_j^{(i)}X}F_i^j(s)ds,\qquad (8)$$

where $P=P_1+\Gamma^\top P_2\Gamma$ and $\Lambda^j=\text{diag}[\Lambda_1^j\ldots\Lambda_{2k_j}^j]$, which admits the estimates:

$$\alpha_1(\|Y\|)\leq W(X)\leq\alpha_2(\|X\|),\quad\forall X\in\mathbb{R}^{2n},$$

for some $\alpha_1,\alpha_2\in\mathcal{K}_\infty$ due to imposed restrictions ($P_2\succ 0$), and whose time derivative for the system dynamics takes the form:

$$\dot{W}=\dot{X}^\top PX+X^\top P\dot{X}+2\sum_{j=1}^{M}\dot{X}^\top\mathcal{H}_j^\top\Lambda^jF^j(\mathcal{H}_jX)$$
$$-2[\sum_{g=0}^{M}F^g(\mathcal{H}_gX)^\top\Omega_g^\top+\dot{X}^\top\Psi^\top](\dot{X}-\sum_{g=0}^{M}\mathcal{D}_gF^g(\mathcal{H}_gX)-\delta),$$

where the last line is added by applying the descriptor approach [43] (it equals to zero, while helping us in optimization of the resulted matrix inequalities). Opening the brackets, adding and subtracting the terms weighted by the matrices

Φ and $\tilde{\Upsilon}_{g,s}$ with $g=\overline{0,M}$ and $s=\overline{g,M}$, this expression can be rewritten as follows:

$$\dot{W}=\zeta^{\top}Q\zeta-\sum_{g=0}^{M}F^{g}(\mathcal{H}_{g}X)^{\top}\Upsilon_{g,g}F^{g}(\mathcal{H}_{g}X)$$
$$-2\sum_{k=0}^{M-1}\sum_{s=k+1}^{M}F^{k}(\mathcal{H}_{k}X)^{\top}\tilde{\Upsilon}_{k,s}F^{s}(\mathcal{H}_{s}X)+\delta^{\top}\Phi\delta$$
$$\leq-\sum_{g=0}^{M}F^{g}(\mathcal{H}_{g}X)^{\top}\Upsilon_{g,g}F^{g}(\mathcal{H}_{g}X)$$
$$-2\sum_{k=0}^{M-1}\sum_{s=k+1}^{M}F^{k}(\mathcal{H}_{k}X)^{\top}\tilde{\Upsilon}_{k,s}F^{s}(\mathcal{H}_{s}X)+\delta^{\top}\Phi\delta,$$

where

$$\zeta^{\top}=\left[\dot{X}^{\top}\ X^{\top}\ F^{1}(\mathcal{H}_{1}X)^{\top}\dots F^{M}(\mathcal{H}_{M}X)^{\top}\ \delta^{\top}\right]^{\top}.$$

To get the desired input-to-output stability property in (7) we need further to substantiate the upper bound:

$$\dot{W}\leq-\alpha(W)+\delta^{\top}\Phi\delta$$

for some $\alpha\in\mathcal{K}_{\infty}$. Thus, the following property has to be demonstrated:

$$\alpha(W(X))\leq\sum_{g=0}^{M}F^{g}(\mathcal{H}_{g}X)^{\top}\Upsilon_{g,g}F^{g}(\mathcal{H}_{g}X)$$
$$+2\sum_{k=0}^{M-1}\sum_{s=k+1}^{M}F^{k}(\mathcal{H}_{k}X)^{\top}\tilde{\Upsilon}_{k,s}F^{s}(\mathcal{H}_{s}X)$$

for a suitable $\alpha\in\mathcal{K}_{\infty}$. Recall that in W all terms are unbounded (quadratic in X and the integrals of the nonlinearities), while in the right-hand side of the above inequality the unbounded terms are weighted by $\tilde{\Upsilon}_{r,s}$ for $r=\overline{0,\nu}$ with $s=\overline{r,\nu}$ and $\tilde{\Upsilon}_{0,j}$ for $j=\overline{1,M}$. Hence, the fulfillment of the related matrix inequality for $\rho_i>0$, $i=\overline{1,2,3}$ given in the formulation of the theorem guarantees the existence of such a function α [35]. □

Remark 2. While calculating the expression for $\dot{W}$ in the proof of Theorem 1, the terms $\dot{X}^{\top}PX+X^{\top}P\dot{X}$ can be dissolved by substituting the expression for $\dot{X}$, then we can assign $\Omega_0=0$ and the term $\mathcal{D}_0^{\top}P+P\mathcal{D}_0$ will appear in the block $Q_{2,2}$ (with respective terms in other parts). Such a treatment may be beneficial if the matrix $\mathcal{D}_0^{\top}P+P\mathcal{D}_0$ has good properties (see also LMIs in Theorem 2 below, where such a derivation is used). Note also that if the matrix $\mathcal{D}_0$ is Hurwitz and Metzler (the latter property is satisfied by Assumption 4), then existence of $P\in\mathbb{D}_{+}^{2n}$ is an equivalent characterization [32].

Remark 3. A necessary condition for feasibility of LMIs given in the formulation of Theorem 1 is that all matrices on the main diagonal of Q are nonnegative definite, i.e., $\Omega_g^{\top}\mathcal{D}_g+\mathcal{D}_g^{\top}\Omega_g\preceq0$ for $g=\overline{0,M}$ with some $\Omega_g\in\mathbb{R}^{2n\times2k_g}$, which can be easily checked. The result of this theorem also allows the asymptotic gain, for the estimation accuracy $Y=\overline{x}-\underline{x}$ from the uncertainty collected in δ, be evaluated.

VI. Stabilization using interval observer

Following [16], to solve the problem of robust stabilization of (3), one can design a feedback based on the interval estimates $\overline{x}(t),\underline{x}(t)$ providing convergence of them to a vicinity of the origin proportional to the size of uncertainty $\|\overline{d}\|_{\infty}$, $\|\underline{d}\|_{\infty}$ and V and ensuring that $\overline{x}-\underline{x}\in\mathcal{L}_{\infty}^{n}$. In such a case, under the interval inclusion (4), the state $x(t)$ has to approach the same vicinity of the origin and stay there.

The following control is proposed in this paper:

$$u(t)=\sum_{r=0}^{\ell}[\underline{K}_rf^{r}(H_r\underline{x}(t))+\overline{K}_rf^{r}(H_r\overline{x}(t))] \quad (9)$$
$$+\sum_{k=\ell+1}^{M}[R_k^{+}f^{k}(H_k^{+}\underline{x}(t)-H_k^{-}\overline{x}(t))$$
$$+R_k^{-}f^{k}(H_k^{+}\overline{x}(t)-H_k^{-}\underline{x}(t))]$$
$$+Zy(t)+\underline{Q}\underline{d}(t)+\overline{Q}\overline{d}(t),$$

where $\underline{K}_r,\overline{K}_r\in\mathbb{R}^{m\times k_r}$ for $r=\overline{0,\ell}$, $R_r^{+},R_r^{-}\in\mathbb{R}^{m\times k_r}$ for $r=\overline{\ell+1,M}$, $Z\in\mathbb{R}^{m\times p}$ and $\underline{Q},\overline{Q}\in\mathbb{R}^{m\times n}$ are the matrix gains to be calculated. For the extended state vector $X=[\underline{x}^{\top}\ \overline{x}^{\top}]^{\top}$ and the augmented nonlinearities $F^{g}(s)=[f^{g}(s)^{\top}\ f^{g}(s)^{\top}]^{\top}$ with $g=\overline{0,M}$ (as before), and

$$\mathcal{A}_r=\begin{bmatrix}D_r+B\underline{K}_r & B\overline{K}_r\\ B\underline{K}_r & D_r+B\overline{K}_r\end{bmatrix},\ r=\overline{0,\ell};$$
$$\mathcal{A}_k=\begin{bmatrix}D_k^{+}+BR_k^{+} & BR_k^{-}-D_k^{-}\\ BR_k^{+}-D_k^{-} & D_k^{+}+BR_k^{-}\end{bmatrix},\ k=\overline{\ell+1,M};$$
$$\mathcal{Z}=\begin{bmatrix}I_n\\ I_n\end{bmatrix}(L+BZ),\ \mathcal{Q}=\begin{bmatrix}I_n+B\underline{Q} & B\overline{Q}\\ B\underline{Q} & I_n+B\overline{Q}\end{bmatrix},$$

the closed-loop system (6), (9) can be rewritten in the form:

$$\dot{X}(t)=\mathcal{A}_0X(t)+\sum_{j=1}^{M}\mathcal{A}_jF^{j}(\mathcal{H}_jX(t)) \quad (10)$$
$$+\mathcal{Z}\left(y(t)-v(t)\right)+\tilde{\delta}(t),$$

where

$$\tilde{\delta}(t)=V\begin{bmatrix}-I_n\\ I_n\end{bmatrix}|L|1_p+\mathcal{Z}v(t)+\mathcal{Q}\begin{bmatrix}\underline{d}(t)\\ \overline{d}(t)\end{bmatrix}$$

is a new external input. The influence of $\tilde{\delta}(t)$ on (10) can be minimized if the control parameters $Z,\underline{Q},\overline{Q}$ are chosen as follows:

$$(\underline{Q},\overline{Q})=\underset{\underline{Q},\overline{Q}\in\mathbb{R}^{m\times n}}{\arg\min}\left\|\mathcal{Q}\begin{bmatrix}\underline{d}(t)\\ \overline{d}(t)\end{bmatrix}\right\|_{\infty}, \quad (11)$$
$$Z=\underset{Z\in\mathbb{R}^{m\times p}}{\arg\min}\|\mathcal{Z}\|.$$

Thus, preservation of the Persidskii form of the closed-loop system (10) and minimization of the influence of the uncertainty (11) are the main features determining the shape of the control (9).

In this work, to streamline the control design the case with $N=0$ is considered. Consequently, according to Assumption 4, the selection of observer gain ensures that $A_0-L_0C_0$ is Metzler, and all other matrices $A_j-L_jC_j$ for $j=\overline{1,M}$ are not restricted. An important property of the system (10) is that

$y(t)-v(t)$ is a functional nonlinearity of the state $X(t)$ since straightforward derivations show:

$$\|y-v\|^2 = x^\top C_0^\top C_0 x + \sum_{j=1}^{M} f^j(H_j x)^\top C_j^\top C_j f^j(H_j x),$$
$$\|x\|^2 \le \|X\|^2,\ \|f^j(H_j x)\|^2 \le \|F^j(\mathcal{H}_j X)\|^2,\ j=\overline{1,M}.$$

Define $\varrho_g > 0$ for $g=\overline{0,M}$ as a solution of the LMI

$$\begin{bmatrix} I_{p_g} & C_g \\ C_g^\top & \varrho_g I_{k_g} \end{bmatrix} \succeq 0,$$

then $C_g^\top C_g \le \varrho_g I_{k_g}$ and, hence,

$$x^\top C_0^\top C_0 x \le \varrho_0 \|X\|^2,$$
$$f^j(H_j x)^\top C_j^\top C_j f^j(H_j x) \le \varrho_j \|F^j(\mathcal{H}_j X)\|^2,\ j=\overline{1,M}.$$

Therefore, the following result can be formulated:

Theorem 2. *Let assumptions 1–4 be satisfied for the system* (3) *with* $N=0$*, the control* (9) *gains* $\underline{K}_r, \overline{K}_r \in \mathbb{R}^{m\times k_r}$ *for* $r=\overline{0,\ell}$*,* $R_r^+, R_r^- \in \mathbb{R}^{m\times k_r}$ *for* $r=\overline{\ell+1,M}$ *be given and* $Z\in\mathbb{R}^{m\times p}$ *and* $\underline{Q}, \overline{Q} \in \mathbb{R}^{m\times n}$ *be chosen from* (11)*, then* (6) *is an interval observer guaranteeing that*

$$\|x(t)\| \le \|X(t)\| \le \beta(\|\underline{x}_0\| + \|\overline{x}_0\|, t) + \gamma(\|\tilde{\delta}\|_\infty),\ t\ge 0$$

for some $\beta\in\mathcal{KL}$ *and* $\gamma\in\mathcal{K}$ *provided that there exist* $P^\top = P \in \mathbb{R}^{2n\times 2n}$*,* $\Lambda^j \in \mathbb{D}_+^{2k_j}$ *for* $j=\overline{1,M}$*,* $\phi_1>0$*,* $\Phi_2^\top = \Phi_2 \in \mathbb{R}^{2n\times 2n}$*,* $\Psi\in\mathbb{R}^{2n\times 2n}$*,* $\Omega_g \in \mathbb{R}^{2n\times 2k_g}$*,* $\Upsilon_{g,g}\in\mathbb{D}_+^{2k_g}$ *(*$\tilde{\Upsilon}_{g,g} = \phi_1 \varrho_g I_{2k_g} + \Upsilon_{g,g}$*) for* $g=\overline{0,M}$*,* $\Upsilon_{0,j}\in\mathbb{D}_+^{2k_j}$ *for* $j=\overline{1,M}$ *(*$\tilde{\Upsilon}_{0,s} = \mathcal{H}_s^\top \Upsilon_{0,s}$*),* $\Upsilon_{z,s}\in\mathbb{D}_+^{2n}$ *(*$\tilde{\Upsilon}_{z,s} = \mathcal{H}_z \Upsilon_{z,s} \mathcal{H}_s^\top$*) for* $z=\overline{1,M-1}$ *and* $s=\overline{z+1,M}$ *such that the following LMIs are satisfied:*

$$P\succeq 0,\ \Phi_2 \succ 0,\ Q' \preceq 0,\ P+\rho_1\sum_{j=1}^{M}\mathcal{H}_j^\top \Lambda^j \mathcal{H}_j \succ 0,$$
$$\sum_{r=0}^{\nu}\mathcal{H}_g^\top \Upsilon_{g,g}\mathcal{H}_g + \rho_2\sum_{j=1}^{M}\mathcal{H}_j^\top \Upsilon_{0,j}\mathcal{H}_j$$
$$+\rho_3\sum_{z=1}^{\nu-1}\sum_{s=z+1}^{\nu}\mathcal{H}_z^\top \tilde{\Upsilon}_{z,s}\mathcal{H}_s + \mathcal{H}_s^\top \tilde{\Upsilon}_{s,z}^\top \mathcal{H}_z \succ 0$$

for some $\rho_1,\rho_2,\rho_3 > 0$*, where*

$$Q' = \begin{bmatrix} -\Psi^\top-\Psi & \Psi^\top\mathcal{A}_0-\Omega_0 & \Psi^\top\mathcal{A}_1-\Omega_1 & \cdots & \Psi^\top\mathcal{A}_M-\Omega_M & \Psi^\top\mathcal{Z} & \Psi^\top \\ \mathcal{A}_0^\top\Psi-\Omega_0^\top & \tilde{\Omega}_0^\top\mathcal{A}_0+\mathcal{A}_0^\top\tilde{\Omega}_0+\tilde{\Upsilon}_{0,0} & \tilde{\Omega}_0^\top\mathcal{A}_1+\mathcal{A}_0^\top\tilde{\Omega}_1+\tilde{\Upsilon}_{0,1} & \cdots & \tilde{\Omega}_0^\top\mathcal{A}_M+\mathcal{A}_0^\top\tilde{\Omega}_M+\tilde{\Upsilon}_{0,M} & \tilde{\Omega}_0^\top\mathcal{Z} & \tilde{\Omega}_0^\top \\ \mathcal{A}_1^\top\Psi-\Omega_1^\top & \mathcal{A}_1^\top\tilde{\Omega}_0+\tilde{\Omega}_1^\top\mathcal{A}_0+\tilde{\Upsilon}_{0,1}^\top & \tilde{\Omega}_1^\top\mathcal{A}_1+\mathcal{A}_1^\top\tilde{\Omega}_1+\tilde{\Upsilon}_{1,1} & \cdots & \tilde{\Omega}_1^\top\mathcal{A}_M+\mathcal{A}_1^\top\tilde{\Omega}_M+\tilde{\Upsilon}_{1,M} & \tilde{\Omega}_1^\top\mathcal{Z} & \tilde{\Omega}_1^\top \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathcal{A}_M^\top\Psi-\Omega_M^\top & \mathcal{A}_M^\top\tilde{\Omega}_0+\tilde{\Omega}_M^\top\mathcal{A}_0+\tilde{\Upsilon}_{0,M}^\top & \mathcal{A}_M^\top\tilde{\Omega}_1+\tilde{\Omega}_M^\top\mathcal{A}_1+\tilde{\Upsilon}_{1,M}^\top & \cdots & \tilde{\Omega}_M^\top\mathcal{A}_M+\mathcal{A}_M^\top\tilde{\Omega}_M+\tilde{\Upsilon}_{M,M} & \tilde{\Omega}_M^\top\mathcal{Z} & \tilde{\Omega}_M^\top \\ \mathcal{Z}^\top\Psi & \mathcal{Z}^\top\tilde{\Omega}_0 & \mathcal{Z}^\top\tilde{\Omega}_1 & \cdots & \mathcal{Z}^\top\tilde{\Omega}_M & -\phi_1 I_p & 0 \\ \Psi & \tilde{\Omega}_0 & \tilde{\Omega}_1 & \cdots & \tilde{\Omega}_M & 0 & -\Phi_2 \end{bmatrix},$$
$$\tilde{\Omega}_0 = P+\Omega_0;\ \tilde{\Omega}_j = \Omega_j + \mathcal{H}_j^\top\Lambda^j,\ j=\overline{1,M}.$$

Proof. Under introduced restrictions the relations (4) are satisfied by Lemma 6. We need to establish an estimate on behavior of $X(t)$. Utilizing the same Lyapunov function (8) for P given in the formulation of the theorem and $\Lambda^j = \mathrm{diag}[\Lambda_1^j \dots \Lambda_{2k_j}^j]$, using the Finsler's lemma [35], the following estimates can be obtained:

$$\alpha_1(\|X\|) \le W(X) \le \alpha_2(\|X\|), \quad \forall X\in\mathbb{R}^{2n},$$

for some $\alpha_1,\alpha_2\in\mathcal{K}_\infty$. The time derivative of W for the closed-loop system (10) takes the form:

$$\dot{W} = \dot{X}^\top P X + X^\top P\dot{X} + 2\sum_{j=1}^{M}\dot{X}^\top\mathcal{H}_j^\top\Lambda^j F^j(\mathcal{H}_j X)$$
$$-2\left(\sum_{g=0}^{M} F^g(\mathcal{H}_g X)^\top\Omega_g^\top + \dot{X}^\top\Psi^\top\right)$$
$$\times\left(\dot{X} - \sum_{g=0}^{M}\mathcal{A}_g F^g(\mathcal{H}_g X) - \mathcal{Z}\,(y-v) - \tilde{\delta}\right),$$

where the matrices $\Lambda^j\in\mathbb{D}_+^{2k_j}$ for $j=\overline{1,M}$, $\Psi\in\mathbb{R}^{2n\times 2n}$ and $\Omega_g\in\mathbb{R}^{2n\times 2k_g}$ for $g=\overline{0,M}$ are given in the formulation of theorem, while the last term is added by applying the descriptor approach [43] (it again equals zero due to (10), and its introduction allows us to get simpler matrix inequalities). Opening the brackets, adding and subtracting the terms weighted by ϕ_1, the matrices Φ_2 and $\tilde{\Upsilon}_{g,s}$ with $g=\overline{0,M}$ and $s=\overline{g,M}$, this expression can be rewritten as follows:

$$\dot{W} \le \zeta^\top Q'\zeta - \sum_{g=0}^{M} F^g(\mathcal{H}_g X)^\top\Upsilon_{g,g}F^g(\mathcal{H}_g X)$$
$$-2\sum_{k=0}^{M-1}\sum_{s=k+1}^{M} F^k(\mathcal{H}_k X)^\top\tilde{\Upsilon}_{k,s}F^s(\mathcal{H}_s X) + \tilde{\delta}^\top\Phi_2\tilde{\delta}$$
$$\le -\sum_{g=0}^{M} F^g(\mathcal{H}_g X)^\top\Upsilon_{g,g}F^g(\mathcal{H}_g X)$$
$$-2\sum_{k=0}^{M-1}\sum_{s=k+1}^{M} F^k(\mathcal{H}_k X)^\top\tilde{\Upsilon}_{k,s}F^s(\mathcal{H}_s X) + \tilde{\delta}^\top\Phi_2\tilde{\delta},$$

where

$$\zeta^\top = \left[\dot{X}^\top\ X^\top\ F^1(\mathcal{H}_1 X)^\top \dots F^M(\mathcal{H}_M X)^\top\ (y-v)^\top\ \tilde{\delta}^\top\right]^\top$$

and the property

$$\|y-v\|^2 \le \varrho_0\|X\|^2 + \sum_{j=1}^{M}\varrho_j\|F^j(\mathcal{H}_j X)\|^2 \qquad (12)$$

established above was utilized. In order to prove input-to-state stability of (10) with respect to the inputs in $\tilde{\delta}$ we need to demonstrate that there exists $\alpha \in \mathcal{K}_\infty$ such that

$$\sum_{g=0}^{M} F^g(\mathcal{H}_g X)^\top \Upsilon_{g,g} F^g(\mathcal{H}_g X)$$
$$+2\sum_{k=0}^{M-1}\sum_{s=k+1}^{M} F^k(\mathcal{H}_k X)^\top \tilde{\Upsilon}_{k,s} F^s(\mathcal{H}_s X) \geq \alpha(\|X\|)$$

for all $X \in \mathbb{R}^{2n}$. Recall that the first ν nonlinearities are unbounded and $X^\top \tilde{\Upsilon}_{0,j} F^j(\mathcal{H}_j X)$ is also unbounded for $j = \overline{1,M}$, and the required property follows from the last LMI via S-procedure [35]. Therefore, $\overline{x}-\underline{x} \in \mathcal{L}_\infty^n$ and (6) is an interval observer for (3). The desired estimate for x follows from (4) (i.e., $\|x\|^2 \leq \|X\|^2$) taking into account the initialization of the observer (6). □

Remark 4. If $C_g \geq 0$ for all $g = \overline{0,M}$ (in this case the interval inclusion $\underline{x} \leq x \leq \overline{x}$ implies that $C_0\underline{x} \leq C_0 x \leq C_0\overline{x}$), then the estimate (12) can be relaxed to

$$\|y-v\|^2 \leq X^\top \mathbb{C}_0 X + \sum_{j=1}^{M} F^j(\mathcal{H}_j X)^\top \mathbb{C}_j F^j(\mathcal{H}_j X),$$
$$\mathbb{C}_g = \begin{bmatrix} C_g^\top C_g & 0 \\ 0 & C_g^\top C_g \end{bmatrix},\ g = \overline{0,M},$$

and the result of Theorem 2 holds for

$$\tilde{\Upsilon}_{g,g} = \phi_1 \mathbb{C}_g + \Upsilon_{g,g},\ g = \overline{0,M}.$$

In the conditions of Theorem 2 it is assumed that the control gains $\underline{K}_r, \overline{K}_r \in \mathbb{R}^{m\times k_r}$ for $r = \overline{0,\ell}$ and $R_r^+, R_r^- \in \mathbb{R}^{m\times k_r}$ for $r = \overline{\ell+1,M}$ are given. Introducing additional restrictions, the LMIs of the theorem can be rewritten in order to have these gains as their solutions:

Corollary 3. *Let $k_j = n$ and $H_j = I_n$ for $j = \overline{1,M}$, assumptions 1–4 be satisfied for the system* (3) *with $N = 0$, the gains $Z \in \mathbb{R}^{m\times p}$ and $\underline{Q}, \overline{Q} \in \mathbb{R}^{m\times n}$ be chosen from* (11)*, then* (6) *with the control* (9) *is an interval observer ensuring that*

$$\|x(t)\| \leq \|X(t)\| \leq \beta(\|\underline{x}_0\| + \|\overline{x}_0\|, t) + \gamma(\|\tilde{\delta}\|_\infty),\ t \geq 0$$

for some $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ provided that there exist $P, \Lambda^j \in \mathbb{D}_+^{2n}$ for $j = \overline{1,M}$, $\phi_1 > 0$, $\Phi_2^\top = \Phi_2 \in \mathbb{R}^{2n\times 2n}$, $\mathfrak{G}_{g,s} \in \mathbb{D}_+^{2n}$ for $g = \overline{0,M}$ and $s = \overline{g,M}$, $\mathfrak{U}_g \in \mathbb{R}^{m\times 2n}$ for $g = \overline{0,M}$, such that the following LMIs are satisfied:

$$P \succ 0,\ \Phi_2 \succ 0,\ \mathfrak{Q} \preceq 0;\ \Lambda^j \succ 0,\ j = \overline{1,M};$$
$$\mathfrak{U}_k \geq 0,\ k = \overline{\ell+1,M};$$
$$\sum_{r=0}^{\nu} \Upsilon_{g,g} + \rho_2 \sum_{j=1}^{M} \Upsilon_{0,j} + \rho_3 \sum_{z=1}^{\nu-1}\sum_{s=z+1}^{\nu} \Upsilon_{z,s} \succ 0$$

for some $\rho_2, \rho_3 > 0$, and by assigning $\begin{bmatrix} \underline{K}_0 & \overline{K}_0 \end{bmatrix} = \mathfrak{U}_0 P^{-1}$, $\begin{bmatrix} \underline{K}_r & \overline{K}_r \end{bmatrix} = \mathfrak{U}_r(\Lambda^r)^{-1}$ for $r = \overline{1,\ell}$ and $\begin{bmatrix} R_k^+ & R_k^- \end{bmatrix} = \mathfrak{U}_k(\Lambda^k)^{-1}$ for $k = \overline{\ell+1,M}$, where

$$\mathfrak{Q} = \begin{bmatrix} \mathfrak{Q}_{0,0} & \mathfrak{Q}_{0,1} & \cdots & \mathfrak{Q}_{0,M} & \mathcal{Z} & I_{2n} \\ \mathfrak{Q}_{0,1}^\top & \mathfrak{D}_1\Lambda^1 + \Lambda^1\mathfrak{D}_1^\top + \mathfrak{B}\mathfrak{U}_1 + \mathfrak{U}_1^\top\mathfrak{B}^\top + \mathfrak{G}_{1,1} & \cdots & \mathfrak{D}_M\Lambda^M + \mathfrak{B}\mathfrak{U}_M + \Lambda^1\mathfrak{D}_1^\top + \mathfrak{U}_1^\top\mathfrak{B}^\top + \mathfrak{G}_{1,M} & \mathcal{Z} & I_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathfrak{Q}_{0,M}^\top & \mathfrak{D}_1\Lambda^1 + \mathfrak{B}\mathfrak{U}_1 + \Lambda^M\mathfrak{D}_M^\top + \mathfrak{U}_M^\top\mathfrak{B}^\top + \mathfrak{G}_{1,M} & \cdots & \mathfrak{D}_M\Lambda^M + \Lambda^M\mathfrak{D}_M^\top + \mathfrak{B}\mathfrak{U}_M + \mathfrak{U}_M^\top\mathfrak{B}^\top + \mathfrak{G}_{M,M} & \mathcal{Z} & I_{2n} \\ \mathcal{Z}^\top & \mathcal{Z}^\top & \cdots & \mathcal{Z}^\top & -\phi_1 I_p & 0 \\ I_{2n} & I_{2n} & \cdots & I_{2n} & 0 & -\Phi_2 \end{bmatrix},$$

$$\mathfrak{Q}_{0,0} = \mathfrak{D}_0 P + P\mathfrak{D}_0^\top + \mathfrak{B}\mathfrak{U}_0 + \mathfrak{U}_0^\top\mathfrak{B}^\top + \mathfrak{G}_{0,0},$$
$$\mathfrak{Q}_{0,j} = \mathfrak{D}_j\Lambda^j + \mathfrak{B}\mathfrak{U}_j + P\mathfrak{D}_0^\top + \mathfrak{U}_0^\top\mathfrak{B}^\top + \mathfrak{G}_{0,j},\ j = \overline{1,M},$$
$$\mathfrak{D}_r = \begin{bmatrix} D_r & 0 \\ 0 & D_r \end{bmatrix},\ r = \overline{0,\ell};\ \mathfrak{D}_k = \begin{bmatrix} D_k^+ & -D_k^- \\ -D_k^- & D_k^+ \end{bmatrix},\ k = \overline{\ell+1,M};\ \mathfrak{B} = \begin{bmatrix} B \\ B \end{bmatrix}.$$

Proof. Let us consider how the restrictions $k_j = n$ and $H_j = I_n$ for $j = \overline{1,M}$ transform the matrix inequalities of Theorem 2. Under these conditions $\mathcal{H}_j = I_{2n}$, hence, $\tilde{\Omega}_j = \Omega_j + \Lambda^j$ and $\tilde{\Upsilon}_{z,s} = \Upsilon_{z,s}$. Moreover, it is easy to check that a variation of Ω_g for $g = \overline{0,M}$ does not influence the feasibility of inequalities, then substitution $\Omega_g = 0$ leads to $\tilde{\Omega}_0 = P$ and $\tilde{\Omega}_j = \tilde{\Omega}_j^\top = \Lambda^j$, and finally, the choice $\Psi = 0$ just simplifies the structure without impacting the resolution of $Q' \preceq 0$. The LMI including the constant ρ_1 is satisfied since $P \succ 0$, the inequality including ρ_2 and ρ_3 is reformulated in the corollary. In order to put the control gains as the variables in LMIs, note that $\mathcal{A}_r = \mathfrak{D}_r + \mathfrak{B}\mathfrak{S}_r$ with $\mathfrak{S}_r = \begin{bmatrix} \underline{K}_r & \overline{K}_r \end{bmatrix}$ for $r = \overline{0,\ell}$ and $\mathcal{A}_k = \mathfrak{D}_k + \mathfrak{B}\mathfrak{S}_k$ with $\mathfrak{S}_k = \begin{bmatrix} R_k^+ & R_k^- \end{bmatrix} \geq 0$ for $k = \overline{\ell+1,M}$. Denote $S = \mathrm{diag}[P^{-1}, (\Lambda^1)^{-1}, \ldots (\Lambda^M)^{-1}, I_p, I_{2n}]$, which is an invertible matrix since P, Λ^j are assumed to be nonsingular in the conditions of the corollary, then the property $Q' \preceq 0$ is equivalent to $\mathfrak{Q} = SQ'S^\top \preceq 0$ given in the corollary using notation: $\mathfrak{U}_0 = \mathfrak{S}_0 P^{-1}$, $\mathfrak{G}_{0,0} = P^{-1}\tilde{\Upsilon}_{0,0}P^{-1}$, $\mathfrak{U}_j = \mathfrak{S}_j(\Lambda^j)^{-1}$ and $\mathfrak{G}_{0,j} = P^{-1}\tilde{\Upsilon}_{0,j}(\Lambda^j)^{-1}$ for $j = \overline{1,M}$, $\mathfrak{G}_{j,s} = (\Lambda^j)^{-1}\tilde{\Upsilon}_{j,s}(\Lambda^s)^{-1}$ for $j = \overline{1,M}$ and $s = \overline{j+1,M}$, under substitution $P \to P^{-1}$ and $\Lambda^j \to (\Lambda^j)^{-1}$ (these manipulations allow the terms $\mathcal{A}_0^\top P + P\mathcal{A}_0$ and $\mathcal{A}_j^\top\Lambda^j + \Lambda^j\mathcal{A}_j$ for $j = \overline{1,M}$, appearing on the main diagonal of Q', to be transformed into $P\mathcal{A}_0^\top + \mathcal{A}_0 P = \mathfrak{D}_0 P + P\mathfrak{D}_0^\top + \mathfrak{B}\mathfrak{U}_0 + \mathfrak{U}_0^\top\mathfrak{B}^\top$ and $\Lambda^j\mathcal{A}_j^\top + \mathcal{A}_j\Lambda^j = \mathfrak{D}_j\Lambda^j + \Lambda^j\mathfrak{D}_j^\top + \mathfrak{B}\mathfrak{U}_j + \mathfrak{U}_j^\top\mathfrak{B}^\top$ for $j = \overline{1,M}$, respectively). It is clear that $\mathfrak{U}_k = \begin{bmatrix} R_k^+ & R_k^- \end{bmatrix} (\Lambda^k)^{-1} \geq 0$ for $k = \overline{\ell+1, M}$ since $\begin{bmatrix} R_k^+ & R_k^- \end{bmatrix} \geq 0$ and $\Lambda^k > 0$, as it is required in the corollary. $\square$

Let us demonstrate applicability of the theoretical results given in sections V and VI.

VII. Examples

In this section two applications of the proposed interval observer are presented. One deals with recurrent neural networks (RNNs) illustrating ability of (6) to evaluate accurately the bounding behavior of complex nonlinear systems, and another with a simple nonlinear mechanical dynamics showing efficiency of the control (9). In [29], the observer (6) was tested for a Lotka-Volterra model.

A. Recurrent neural networks

RNNs are popular tools for classification and prediction of time series (e.g., handwritten symbol recognition) or modeling and estimation of dynamical systems [44], [45], [46]. Their main advantage consists in well developed and implemented procedures for learning the unstructured functional interrelations from data. In this paper we will consider continuous-time RNNs initially proposed in [25], which can be written in the form of (3), where $x(t) \in \mathbb{R}^n$ corresponds to the state of RNN, $u(t) \in \mathbb{R}^m$ is the external input (to be classified), and $d(t) \in \mathbb{R}^n$ will be used to model the deviations of different realizations of external inputs from the mean one. Usually $M = 1$ and f^1 is the activation function in the hidden layer, but recently several works have been published indicating that combination of different activation functions in one RNN improves its learning abilities [47], hence, the case with $M > 1$ is of interest. Note that the incremental passivity condition stated in Assumption 1 is usually satisfied for many types of activation functions: sign (bipolar step), hyperbolic tangent, bipolar sigmoid, saturation (hard hyperbolic tangent), etc. The matrices A_g for $g = \overline{0, M}$ and H_j for $j = \overline{1, M}$ represent the adjustable weights in the output and the hidden layers, respectively, which we suppose to be properly tuned. A typical scenario of utilization of such a RNN is that for a collection of input sequences, the neural network is trained in a way providing state convergence to different levels (or to different attractors) depending on the presented input.

In such a setting, the robustness evaluation problem for a trained RNN consists in quantifying the state behavior in the presence of different inputs that may slightly (or significantly) deviate from ones presented in the database for learning or testing. Such a problem can be solved by deriving analytical estimates using annular short-time stability [48], or by calculating the mean input $u(t)$ of the database together with admissible deviations from it, which can be presented by $d(t)$. To this end, we may assume that everything is measured without noise, i.e., $C_g = I_{k_g}$ for $g = \overline{0, M}$ and $V = 0$, then design of interval observer for (3) under assumptions 2 and 3 provides another solution to the robustness evaluation problem, where the bounds $\underline{d}$ and $\overline{d}$ correspond to the admissible deviations of possible external inputs from the mean/nominal one. Under assumption that the state is measured it is easy to design the gain L satisfying Assumption 4 for $\ell = M$. Therefore, we can construct (6) and the results of Lemma 6 and Theorem 1 hold. Note that since activation functions are usually bounded, the stability of the matrix A_0 is sufficient to establish boundedness of the state x (the same for $A_0 - L_0C_0$ and $\underline{x}, \overline{x}$).

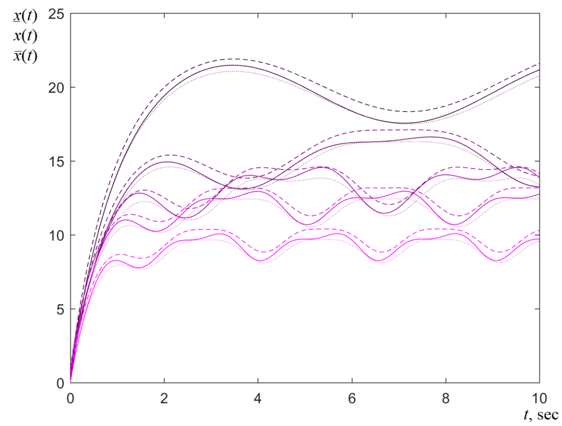

Figure 1. The results of interval estimation by (6) for a RNN: $x(t)$ (solid lines), $\underline{x}(t)$ (dot lines), $\overline{x}(t)$ (dash lines) versus time t [sec]

Example. Take $n = m = 5$ with $B = I_n$; $M = 2$ with sign and hyperbolic tangent activation functions; A_0 to be a Hurwitz diagonal matrix, $k_1 = k_2 = 15$ be the number of neurons in the hidden layer with different activation, and A_1, A_2, H_1, H_2 composed by random numbers; the inputs $u(t)$ and $d(t)$ are harmonic signals of amplitude 5, hence, $\underline{d} = 0$ and $\overline{d} = 5$. The observer gains L_0, L_1, L_2 are chosen to satisfy Assumption 4 such that all conditions of Lemma 6 and Theorem 1 (with all LMIs) are verified. In Fig. 1, the results of simulation are shown (solid lines correspond to the trajectories of the RNN, dot and dash curves represent the lower and upper interval estimates, respectively). As we can see the proposed interval observer generates very reasonable and bounded estimates for a highly nonlinear RNN.

B. Stabilization of a mechanical system

Consider a mechanical system with cubic velocity friction term:

$$\dot{x}_1(t) = x_2(t),\ y(t) = x_1(t) + v(t),$$
$$\dot{x_2}(t) = -a_1x_1(t) - a_2x_2(t) - a_3x_2^3(t) + u(t) + d(t),$$

where $x_1(t), x_2(t) \in \mathbb{R}$ are the system position and velocity, respectively, $u(t) \in \mathbb{R}$ is the control and $d(t) \in \mathbb{R}$ is a bounded disturbance, which contains a bias, $y(t), v(t) \in \mathbb{R}$ are the measured output and the respective noise; a_1, a_2 and a_3 are positive parameters. The goal is stabilization of this system in a vicinity of the origin under assumptions 2 and 3 (the system is input-to-state stable for $u = 0$, and the role of the control law is to improve attenuation of disturbances d). It is easy

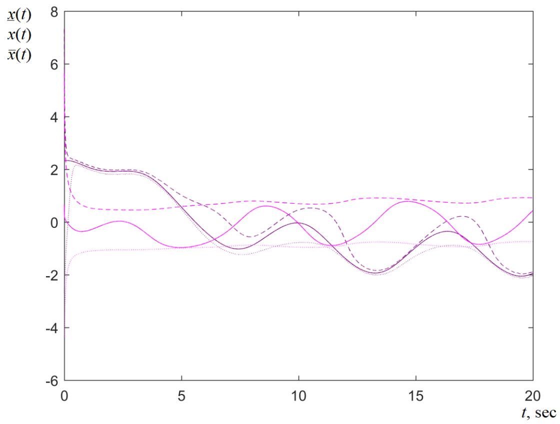

Figure 2. The results of interval estimation by (6) for a mechanical system: $x(t)$ (solid lines), $\underline{x}(t)$ (dot lines), $\overline{x}(t)$ (dash lines) versus time t [sec]

to check that this model can be rewritten in the form (3) for $M = 1$ and $f^1(x) = [x_1^3\ x_2^3]^\top$ satisfying Assumption 1, with

$$A_0 = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -a_3 \end{bmatrix},$$
$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C_0 = C_1 = [1\ 0],\ H_1 = I_2.$$

Taking

$$L_0 = \begin{bmatrix} L_{01} \\ -a_1 \end{bmatrix},\ L_1 = \begin{bmatrix} L_{11} \\ 0 \end{bmatrix},$$

where $L_{01}, L_{11} > 0$ are design parameters, all conditions of Assumption 4 are verified for $\ell = 1$. Fixing the values of all parameters and gains, the control matrices $\underline{K}_r, \overline{K}_r$ for $r = \overline{0,1}$, Z and $\underline{Q}, \overline{Q}$ can be designed, and from (11) we get:

$$Z = [a_1\ 0],\ \underline{Q} = [0\ 1], \overline{Q} = [0\ -1].$$

Example. Let

$$a_1 = 0.1,\ a_2 = 0.5,\ a_3 = 1,\ L_{01} = 1,\ L_{11} = 1,$$

then for

$$\underline{K}_0 = [0\ -2], \overline{K}_0 = [-1\ -2],$$
$$\underline{K}_1 = [0\ -1], \overline{K}_1 = [0\ -1]$$

the LMIs of Theorem 2 are verified. Select $d(t) = 1 + \sin(t)$ with $\underline{d} = 0$ and $\overline{d} = 2$, then in Fig. 2, the results of simulation are shown for the controlled system with $x(0) = [2.2\ 0.4]^\top$ (solid lines correspond to the trajectories of the system, dot and dash curves represent the lower and upper interval estimates, respectively). As we can see, the proposed interval observer (6) generates accurate estimates and the control (9) regulates the system to a vicinity of the origin in spite of the influence of d. Since all admissible trajectories are evaluated by (6), the control (9) can be complemented with a model predictive one as in [49], [50], which can be a direction of future research.

VIII. Conclusion

The paper addressed the issue of design of interval observers for a class of generalized Persidskii systems. The conditions of nonnegativity in this class of models were established. A possible structure of the interval observer was given, together with respective stability conditions. These conditions were formulated in terms of LMIs. A control design was proposed based on interval estimates, whose applicability can also be checked via LMIs. To illustrate the efficiency of the proposed interval observer, it was applied to RNNs. Another application dealt with stabilization of a mechanical system with cubic friction term. Relaxing the imposed assumptions on the nonlinearity can be considered as a direction of future research.

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