# ISS of rapidly time-varying systems via a novel superposition-based presentation

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***Abstract*— We treat the input-to-state stability (ISS) of linear continuous-time systems with multiple time-scales. Such systems contain rapidly-varying, piecewise continuous and almost periodic coefficients with small parameters (time-scales). For systems with a single small parameter, a novel time-delay approach to averaging was recently introduced, whereas a complementary method for systems with multiple independent small parameters has been presented lately by the authors. The latter method relies on a novel system transformation, leading to a new system whose ISS guarantees the ISS of the original one. In this work, we unify this transformation with a new superposition-based system presentation. We employ time-varying Lyapunov functions for ISS analysis, where the novel system presentation plays a crucial role in deriving essentially less conservative compensating upper bounds. The analysis yields LMI conditions for ISS, leading to explicit bounds on the small parameters, decay rate and ISS gains. The LMIs are accompanied by suitable feasibility guarantees. Numerical examples demonstrate the efficacy of the proposed approach in comparison to existing methods.**

***Index Terms*— stability, averaging, time-varying systems, ISS.**

## I. Introduction

Systems with almost periodic signals and/or excitations are central to physics and engineering. Applications of such systems include vibrational control [5], power systems [18] and time-delay systems [20] (see also references therein). Such systems often involve componenets evolving over multiple time-scales (see e.g. [9] for applications to systems biology). Hence, it is not surprising that perturbation theory played an essential part in the analysis of systems with rapidly time-varying coefficients and have led to important results [2], [11], [12], [19], [16]. However, the majority of such results is qualitative in nature.

The method of averaging is an important perturbation-based technique for the study of stability of systems with oscillatory control inputs [3], [13], [15]. The fundamental idea behind asymptotic averaging is that stability of the first-order averaged system guarantees stability of the original rapidly-varying system for small enough values of the time-scale parameter (see e.g. [17, Chapter 8]). However, it is often the case that asymptotic averaging provides only an existence result, without an efficient and explicit bound on the small parameter under which the stability of the original system is preserved. For singularly perturbed systems, such bounds were derived in, e.g., [12] and [6] via a direct Lyapunov approach.

Recently, several advances in quantitative asymptotic averaging have been made. First, [8] presented a constructive time-delay approach to periodic averaging of a system with a single fast time-scale. The approch relies on backward averaging of the system, which yields a neutral-type system presentation where the delay magnitude is equal to to the time-scale parameter. The stability of the delayed system was shown to guarantee the stability of the original system. Stability of the delayed system was analyzed via a direct Lyapunov method, leading to LMI conditions which yield an efficient upper bound on the small parameter that preserves the stability of the original system. This method is also well suited for averaging of systems with time-varying delays, where the delay magnitude is of equal order to the time-scale parameter. These results were extended to $L_2$-gain analysis for periodic averaging and to stochastic systems in [21]. Second, [10] presented a complementary method for asymptotic averaging for ISS and stability in the presence of constant delays and multiple time-scales. Differently from [8], this method employs a non-delayed system transformation, leading to a new system whose ISS guarantees the ISS of the original system. ISS analysis of the transformed system was performed via a direct Lyapunov method leading to LMIs which provide quantitative estimates on the small parameters, internal decay rate and ISS gains. The LMIs were accompanied by suitable feasibility guarantees. The approach in [10] was further extended to rapidly time-varying systems with constant delay, where the novel transformation decoupled the effects of the delay and time-scale parameter on stability, thereby leading to stability results for non-small delay (relative to the small parameter).

In this work we study ISS of rapidly time-varying systems with multiple time-scales. We employ a novel *superposition-based* presentation of the system, in conjunction with a modified tranformation based on [10]. The superposition-based presentation relies on two key ingredients: first, we write the rapidly-varying system matrices as superpositions of constant matrices with rapidly-varying *scalar* coefficients. Second, we force the latter coefficients to have zero averages. We then adapt the transformation from [10] to a new, superposition-based form, thereby obtaining a transformed system whose ISS guarantees the ISS of the original system. ISS of the transformed system is studied via time-varying Lyapunov functions leading to LMIs which provide quantitative estimates on the small parameters, internal decay

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rate and ISS gains. The LMIs are backed by theoretical feasibility guarantees. Extensive numerical examples show that our approach essentially improves the small parameter bounds for which ISS of the original system is preserved.

The article is organized as follows. Section 2 presents ISS results for rapidly time-varying systems. Numerical examples are given in Section 3. Conclusions are drawn in Section 4.

*Notations:* Throughout the paper $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with the vector norm $|\cdot|$, $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices with the induced matrix norm $\|\cdot\|$. We also denote $\mathbb{Z}_+=\{0,1,2,\dots\}$ and $\mathbb{R}_{\geq 0}=[0,\infty)$. The superscript $\top$ denotes matrix transposition, and the notation $P>0$, for $P\in\mathbb{R}^{n\times n}$ means that $P$ is symmetric and positive definite. The symmetric elements of the symmetric matrix are denoted by $*$. For $0<P\in\mathbb{R}^{n\times n}$ and $x\in\mathbb{R}^n$, we write $|x|_P^2=x^\top Px$. $\otimes$ denotes the Kronecker product. The standard lexicographic order on $\mathbb{R}^n$ is denoted by $\leq_{\text{lex}}$.

## II. ISS-LIKE ESTIMATES OF RAPIDLY TIME-VARYING SYSTEMS

### A. Problem formulation

The recent work [8] considered the fast-varying system

$$\dot{x}(t)=A\left(\tfrac{t}{\epsilon}\right)x(t)+B\left(\tfrac{t}{\epsilon}\right)d(t),\ t\geq 0 \tag{1}$$

where $x(t)\in\mathbb{R}^n$ for $t\geq 0$, $\epsilon>0$ is a small parameter defining a fast time-scale, $d$ is a piecewise continuous disturbance and $A:\mathbb{R}\to\mathbb{R}^{n\times n}$ and $B:\mathbb{R}\to\mathbb{R}^{n\times n_d}$ are piecewise continuous matrix functions, which are norm-bounded uniformly for $t\in[0,\infty)$. Under the assumption that there exist $0<T$ and $A_{av},B_{av}$, such that

$$\begin{array}{l} T^{-1}\int_t^{t+T}B(\tau)d\tau=B_{av}+\Delta B(t),\\ T^{-1}\int_t^{t+T}A(\tau)d\tau=A_{av}+\Delta A(t),\quad \forall t\in\mathbb{R}\end{array} \tag{2}$$

with $A_{av}$ is Hurwitz and $\Delta A,\Delta B:\mathbb{R}\to\mathbb{R}^{n\times n}$ sufficiently small in norm, [8] proposed a novel time-delay transformation, leading to quantitative estimate on $\epsilon$ for which ISS of (1) is preserved.

In this work we consider a more general system in the following *superposition-based presentation*:

$$\begin{aligned}\dot{x}(t)\ &=\left[A_{av}+\textstyle\sum_{i=1}^{N}a_i\left(\tfrac{t}{\epsilon_i}\right)A_i\right]x(t)\\ &+\left[B_{av}+\textstyle\sum_{i=1}^{N_d}b_i\left(\tfrac{t}{\epsilon_{d,i}}\right)B_i\right]d(t),\quad t\geq 0\end{aligned} \tag{3}$$

where $x(t)\in\mathbb{R}^n$ for $t\geq 0$, $N,N_d\in\mathbb{N}$, $\{\epsilon_i\}_{i=1}^N$ and $\{\epsilon_{d,i}\}_{i=1}^{N_d}$ are positive small parameters, $\{A_i\}_{i=1}^N\subseteq\mathbb{R}^{n\times n}$, $\{B_i\}_{i=1}^{N_d}\subseteq\mathbb{R}^{n\times n_d}$ are constant matrices, and $\{a_i\}_{i=1}^N$, $\{b_i\}_{i=1}^{N_d}$ are piecewise continuous scalar functions which are uniformly bounded on $[0,\infty)$. We allow the arguments of the scalar functions to depend on *different and independent* time-scales.

For simplicity of presentation we will proceed with the case $N=N_d=2$. The general case follows the same arguments (see also Remark 8). We make the following assumption:

**Assumption 1:** $\{a_i\}_{i=1}^2$ and $\{b_j\}_{j=1}^2$ are almost periodic and there exist positive $\{T_i\}_{i=1}^2$, $\{T_{d,j}\}_{j=1}^2$ such that

$$\begin{array}{l} T_i^{-1}\int_t^{t+T_i}a_i(\tau)d\tau=:\Delta a_i(t),\\ T_{d,j}^{-1}\int_t^{t+T_{d,j}}b_j(\tau)d\tau=:\Delta b_j(t),\quad\forall t\in\mathbb{R}\end{array} \tag{4}$$

with $\{\Delta a_i\}_{i=1}^2,\{\Delta b_j\}_{j=1}^2$ satisfying

$$\begin{array}{l}\sup_{\tau\in\mathbb{R}}\|\Delta a_i(\tau)\|^2\leq\Delta_{a_i,M},\\ \sup_{\tau\in\mathbb{R}}\|\Delta b_j(\tau)\|^2\leq\Delta_{b_j,M},\quad 1\leq i,j\leq 2\end{array} \tag{5}$$

for some positive constants $\{\Delta_{a_i,M}\}_{i=1}^2,\{\Delta_{b_j,M}\}_{j=1}^2$. We further assume that $A_{av}$ is Hurwitz.

*Remark 1:* The assumption that the averages of $a_i$ and $b_j$ are zero is an essential component of the proposed superposition-based system presentation and leads to essentially less conservative LMI conditions for ISS. Note that this assumption poses no loss of generality, since we can always substract the averages from the corresponding functions, while retaining $\Delta a_i$ and $\Delta b_j$ on the right-hand side of (5) and modifying the matrices $A_{av}$ and $B_{av}$.

*Remark 2:* The system (1) can be presented as (3) by fixing $\epsilon_i=\epsilon_{d,j}=\epsilon$, $1\leq i\leq N$, $1\leq j\leq N_d$ and presenting $A\left(\frac{t}{\epsilon}\right)$, $B\left(\frac{t}{\epsilon}\right)$ as linear combinations of constant matrices with time-varying coefficients. In this case $N,N_d\leq n^2$.

We aim to derive efficient and constructive conditions which guarantee ISS-like estimates for (3), with respect to $d$ and $\dot{d}$ (see Theorem 1).

### B. System transformation and Lyapunov analysis

Inspired by [14], for $t\geq 0$, $1\leq i,j\leq 2$ we introduce

$$\begin{array}{l}\varrho_{\epsilon,i}(t)=-\frac{1}{\epsilon_iT_i}\int_t^{t+\epsilon_iT_i}g_i(\tau)a_i\left(\frac{\tau}{\epsilon_i}\right)d\tau,\\ \omega_{\epsilon_{d,j}}(t)=-\frac{1}{\epsilon_{d,j}T_{d,j}}\int_t^{t+\epsilon_{d,j}T_{d,j}}g_{d,j}(\tau)b_j\left(\frac{\tau}{\epsilon_{d,j}}\right)d\tau,\\ g_i(\tau)=t+\epsilon_iT_i-\tau,\ g_{d,j}(\tau)=t+\epsilon_{d,j}T_{d,j}-\tau.\end{array} \tag{6}$$

*Remark 3:* Differently from [10], $\varrho_{\epsilon,i}$, $\omega_{\epsilon_{d,i}}$, $a_i$ and $b_i$, $i=1,2$ are scalar (as opposed to matrix-valued) functions. Hence, efficient upper bounds on them can be easily obtained via standard tools from analysis. This fact plays a key role in obtaining the less conservative LMI conditions for ISS-like estimates below.

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*Remark 4:* Differently from [10], $\varrho_{\epsilon,i}$, $\omega_{\epsilon_{d,i}}$, $a_i$ and $b_i$, $i=1,2$ are scalar (as opposed to matrix-valued) functions. Hence, efficient upper bounds on them can be easily obtained via standard tools from analysis. This fact plays a key role in obtaining the less conservative LMI conditions for ISS-like estimates below.

Differentiating (7), we have for $t \geq 0$

$$\begin{aligned} &\dot{\varrho}_{\epsilon,i}(t) = a_i\left(\tfrac{t}{\epsilon_i}\right) - \Delta a_i\left(\tfrac{t}{\epsilon_i}\right), \\ &\dot{\omega}_{\epsilon_{d,j}}(t) = b_j\left(\tfrac{t}{\epsilon_{d,j}}\right) - \Delta b_j\left(\tfrac{t}{\epsilon_{d,j}}\right). \end{aligned} \tag{8}$$

We introduce the following transformation

$$z(t) = x(t) - \sum_{i=1}^{2} \varrho_{\epsilon,i}(t)A_i x(t) - \sum_{i=1}^{2} \omega_{\epsilon_{d,i}}(t)B_i d(t). \tag{9}$$

*Remark 5:* For the case of (1) with a single time-scale, the time-delay transformation employed in [8] has the form

$$\begin{aligned} &z(t) = x(t) - G(t), \\ &G(t) = \tfrac{1}{\epsilon T}\int_{t-\epsilon T}^{t} g(\tau)\left[A(\tau)x(\epsilon\tau) + B(\tau)d(\epsilon\tau)\right]d\tau, \\ &g(\tau) = \tau - t + \epsilon T, \end{aligned}$$

which leads to a neutral-type system. This transformation allows for ISS analysis which employs averaging of $B\left(\frac{t}{\epsilon}\right)$ for measurable functions $d$, whereas (9) allows ISS for non differentiable $d$ without averaging of $B\left(\frac{t}{\epsilon}\right)$ only, which may be restrictive.

Compared to [8], here we consider multiple fast time-scales and unify the transformation in [10] with a novel system presentation. The non-delayed transformation (9) simplifies the Lyaponov-based analysis whereas the new system presentation (3) significantly improves the results in the numerical examples (see Section III).

**Assumption 2:** We assume that $I_n - \sum_{i=1}^{2} \varrho_{\epsilon,i}(t)A_i$ is invertible for all $t \geq 0$ with

$$\sup_{t\geq 0}\left\|\left(I_n - \sum_{i=1}^{2}\varrho_{\epsilon,i}(t)A_i\right)^{-1}\right\| \leq \delta_{1,x} < \infty.$$

Note that a sufficient condition for Assumption 2 to hold is $\sum_{i=1}^{2} \epsilon_i T_i a_{i,M}\|A_i\| < 2$, where $a_{i,M} := \sup_{\tau\in\mathbb{R}} |a_i(\tau)|$. Indeed, in this case we have

$$\sup_{t\geq 0}\left\|\sum_{i=1}^{2}\varrho_{\epsilon,i}(t)A_i\right\| \leq \frac{\sum_{i=1}^{2}\epsilon_i T_i a_{i,M}\|A_i\|}{2} =: \delta_{2,x} < 1. \tag{10}$$

By using a Neumann series, the latter implies we can take

$$\delta_{1,x} = \frac{2}{2 - \sum_{i=1}^{2}\epsilon_i T_i a_{i,M}\|A_i\|} = \frac{1}{1-\delta_{2,x}}. \tag{11}$$

We will further employ the notation

$$\delta_d := \sup_{t\geq 0}\left\|\sum_{i=1}^{n}\omega_{\epsilon_d,i}(t)B_i\right\|. \tag{12}$$

Analogously to (10), we have

$$\delta_d \leq \frac{1}{2}\sum_{i=1}^{2}\epsilon_i T_i b_{i,M}\|B_i\|, \quad b_{i,M} := \sup_{\tau\in\mathbb{R}}|b_i(\tau)|. \tag{13}$$

Note that $d \in C^1([0,\infty))$ implies that $z(t)$ is continuously differentiable. By employing (3) we obtain the following expression for $\dot{z}(t),\ t \geq 0$:

$$\begin{aligned} \dot{z}(t) \quad &= A_{av}\left[z(t) + \textstyle\sum_{i=1}^{2}\varrho_{\epsilon,i}(t)A_i x(t) + \sum_{i=1}^{2}\omega_{\epsilon_{d,i}}(t)\right. \\ &\left. + \times B_i d(t)\right] + B_{av}d(t) - \textstyle\sum_{i=1}^{2}\omega_{\epsilon_{d,i}}(t)B_i\dot{d}(t) \\ &+ \textstyle\sum_{i=1}^{2}\left[\Delta a_i\left(\tfrac{t}{\epsilon_i}\right)A_i x(t) + \Delta b_i\left(\tfrac{t}{\epsilon_{d,i}}\right)B_i d(t)\right] \\ &- \textstyle\sum_{i=1}^{2}\varrho_{\epsilon,i}(t)A_i\left[A_{av} + \sum_{j=1}^{2}a_j\left(\tfrac{t}{\epsilon_j}\right)A_j\right]x(t) \\ &- \textstyle\sum_{i=1}^{2}\varrho_{\epsilon,i}(t)A_i\left[B_{av} + \sum_{j=1}^{2}b_j\left(\tfrac{t}{\epsilon_{d,j}}\right)B_j\right]d(t). \end{aligned} \tag{14}$$

Recall that $\leq_{\rm lex}$ is the lexicographic order on $\mathbb{R}^n$ ($(i,j) \leq_{\rm lex} (k,l)$ iff $i < k$ or $i = k,\ j \leq l$). To vectorize (14), let

$$\begin{aligned} &\Upsilon_\varrho(t) = \mathrm{col}\left\{\varrho_{\epsilon,i}(t)x(t)\right\}_{i=1}^{2}, \\ &\mathcal{Z}_\omega(t) = \mathrm{col}\left\{\omega_{\epsilon_{d,j}}(t)d(t)\right\}_{j=1}^{2}, \\ &\mathcal{Z}_\varrho(t) = \mathrm{col}\left\{\varrho_{\epsilon,i}(t)d(t)\right\}_{i=1}^{2}, \\ &\Xi_\omega(t) = \mathrm{col}\left\{\omega_{\epsilon_{d,j}}(t)\dot{d}(t)\right\}_{j=1}^{2}, \\ &\Upsilon_{\varrho,a}(t) = \mathrm{col}\left\{\varrho_{\epsilon,i}(t)a_k\left(\tfrac{t}{\epsilon_k}\right)x(t)\right\}_{\{(i,k)\}_{\leq_{\rm lex}}}, \\ &\mathcal{Z}_{\varrho,b}(t) = \mathrm{col}\left\{\varrho_{\epsilon,i}(t)b_j\left(\tfrac{t}{\epsilon_{d,j}}\right)d(t)\right\}_{\{(i,j)\}_{\leq_{\rm lex}}}, \\ &\Upsilon_{\Delta a}(t) = \mathrm{col}\left\{\Delta a_i\left(\tfrac{t}{\epsilon_i}\right)x(t)\right\}_{i=1}^{2}, \\ &\mathcal{Z}_{\Delta b}(t) = \mathrm{col}\left\{\Delta b_j\left(\tfrac{t}{\epsilon_{d,j}}\right)d(t)\right\}_{j=1}^{2}, \\ &\mathbb{A} = \begin{bmatrix}A_1 & A_2\end{bmatrix},\ \mathbb{A}_1 = [A_1^2\ A_1A_2\ A_2A_1\ A_2^2], \\ &\mathbb{A}_2 = [A_1B_1\ A_1B_2\ A_2B_1\ A_2B_2],\ \mathbb{B} = \begin{bmatrix}B_1 & B_2\end{bmatrix}, \\ &\mathbb{W} = \begin{bmatrix}W_1 & W_2\end{bmatrix}, \quad W_i = A_{av}A_i - A_iA_{av},\ 1 \leq i \leq 2. \end{aligned} \tag{15}$$

Employing (9) and (14), we obtain the following expression for $\dot{z}(t),\ t \geq 0$:

$$\begin{aligned} \dot{z}(t) \quad &= A_{av}z(t) + B_{av}d(t) + \mathbb{A}\Upsilon_{\Delta a}(t) + \mathbb{B}\mathcal{Z}_{\Delta b}(t) \\ &- \mathbb{A}\left(I_2 \otimes B_{av}\right)\mathcal{Z}_\varrho(t) + \mathbb{W}\Upsilon_\varrho(t) - \mathbb{B}\Xi_\omega(t) \\ &+ A_{av}\mathbb{B}\mathcal{Z}_\omega(t) - \mathbb{A}_1\Upsilon_{\varrho,a}(t) - \mathbb{A}_2\mathcal{Z}_{\varrho,b}(t). \end{aligned} \tag{16}$$

Let

$$\begin{aligned} &H_\varrho = \mathrm{col}\left\{\mathfrak{h}_\varrho^{(i)}\right\}_{i=1}^{2},\ H_\omega = \mathrm{col}\left\{\mathfrak{h}_\omega^{(j)}\right\}_{j=1}^{2}, \\ &H_{\varrho,a} = \mathrm{col}\left\{\mathfrak{h}_{\varrho,a}^{(i,k)}\right\}_{\{(i,k)\}_{\leq_{\rm lex}}}, \\ &H_{\varrho,b} = \mathrm{col}\left\{\mathfrak{h}_{\varrho,b}^{(i,j)}\right\}_{\{(i,j)\}_{\leq_{\rm lex}}} \end{aligned} \tag{17}$$

be vectors with nonnegative entries such that for any $1 \leq i,k,j \leq 2$ and $t \geq 0$ the following conditions hold

$$\begin{aligned} &I) \quad \varrho_{\epsilon,i}^2(t) \leq \mathfrak{h}_\varrho^{(i)}, \quad II) \quad \omega_{\epsilon_{d,j}}^2(t) \leq \mathfrak{h}_\omega^{(j)}, \\ &III) \quad \varrho_{\epsilon,i}^2(t)a_k^2\left(\tfrac{t}{\epsilon_k}\right) \leq \mathfrak{h}_{\varrho,a}^{(i,k)}, \\ &IV) \quad \varrho_{\epsilon,i}^2(t)b_j^2\left(\tfrac{t}{\epsilon_{d,j}}\right) \leq \mathfrak{h}_{\varrho,b}^{(i,j)}. \end{aligned} \tag{18}$$

Note that all the inequalities involve scalar functions. Let $\Lambda_{\Upsilon_\varrho}, \Lambda_{\mathcal{Z}_\varrho}, \Lambda_{\Upsilon_{\Delta a}} \in \mathbb{R}^{2\times 2}$, $\Lambda_{\mathcal{Z}_\omega}, \Lambda_{\Xi_\omega}, \Lambda_{\mathcal{Z}_{\Delta b}} \in \mathbb{R}^{2\times 2}$ and $\Lambda_{\Upsilon_{\varrho,a}}, \Lambda_{\mathcal{Z}_{\varrho,b}} \in \mathbb{R}^{4\times 4}$ be *diagonal* matrices with *positive* diagonal entries and recall (15). By (5) and (18) we have

$$\begin{aligned} &\Upsilon_\varrho^\top(t)\left(\Lambda_{\Upsilon_\varrho} \otimes I_n\right)\Upsilon_\varrho(t) \leq \left|\Lambda_{\Upsilon_\varrho}H_\varrho\right|_1 |x(t)|^2, \\ &\mathcal{Z}_\varrho^\top(t)\left(\Lambda_{\mathcal{Z}_\varrho} \otimes I_n\right)\mathcal{Z}_\varrho(t) \leq \left|\Lambda_{\mathcal{Z}_\varrho}H_\varrho\right|_1 |d(t)|^2, \end{aligned} \tag{19}$$

$$\mathcal{Z}_\omega^\top(t)\left(\Lambda_{\mathcal{Z}_\omega}\otimes I_n\right)\mathcal{Z}_\omega(t)\leq \left|\Lambda_{\mathcal{Z}_\omega}H_\omega\right|_1|d(t)|^2,$$
$$\Xi_\omega^\top(t)\left(\Lambda_{\Xi_\omega}\otimes I_n\right)\Xi_\omega(t)\leq \left|\Lambda_{\Xi_\omega}H_\omega\right|_1\left|\dot d(t)\right|^2,$$
$$\Upsilon_{\varrho,a}^\top(t)\left(\Lambda_{\Upsilon_{\varrho,a}}\otimes I_n\right)\Upsilon_{\varrho,a}(t)\leq \left|\Lambda_{\Upsilon_{\varrho,a}}H_{\varrho,a}\right|_1|x(t)|^2,$$
$$\mathcal{Z}_{\varrho,b}^\top(t)\left(\Lambda_{\mathcal{Z}_{\varrho,b}}\otimes I_n\right)\mathcal{Z}_{\varrho,b}(t)\leq \left|\Lambda_{\mathcal{Z}_{\varrho,b}}H_{\varrho,b}\right|_1|d(t)|^2,$$
$$\Upsilon_{\Delta a}^\top(t)\left(\Lambda_{\Upsilon_{\Delta a}}\otimes I_n\right)\Upsilon_{\Delta a}(t)\leq \left|\Lambda_{\Upsilon_{\Delta a}}\Delta_{a,M}\right|_1|x(t)|^2,$$
$$\mathcal{Z}_{\Delta b}^\top(t)\left(\Lambda_{\mathcal{Z}_{\Delta b}}\otimes I_n\right)\mathcal{Z}_{\Delta b}(t)\leq \left|\Lambda_{\mathcal{Z}_{\Delta b}}\Delta_{b,M}\right|_1|d(t)|^2$$

where

$$\Delta_{a,M}=\operatorname{col}\left\{\Delta_{a_i,M}\right\}_{i=1}^2,\ \Delta_{b,M}=\operatorname{col}\left\{\Delta_{b_j,M}\right\}_{j=1}^2. \quad (20)$$

The upper bounds (19) will be employed to compensate $\Upsilon_\varrho(t),\mathcal{Z}_\varrho(t),\mathcal{Z}_\omega(t),\Xi_\omega(t),\Upsilon_{\varrho,a}(t),\mathcal{Z}_{\varrho,b}(t),\Upsilon_{\Delta a}(t)$ and $\mathcal{Z}_{\Delta b}(t)$ in the derivation of ISS-like estimates below.

*Remark 6:* Recall that we assume the averages of the functions $\{a_i\}_{i=1}^2$ and $\{b_j\}_{j=1}^2$ to be zero (this is an essential part of the proposed superposition-based system presentation). By doing this, $\{\varrho_{\epsilon,i}\}_{i=1}^2$ and $\{\omega_{\epsilon_d,j}\}_{j=1}^2$ obtained in (7) have smaller $L^\infty$ norms, whence the upper bounds in (18) will be of smaller magnitude. This fact plays a key role in achieving the less conservative LMIs (29) in the Lyapunov analysis.

For ISS-like estimates of (3), let $0<\alpha$ and $0<P\in\mathbb{R}^{n\times n}$. We introduce the Lyapunov function

$$V(t)=|z(t)|_P^2 \quad (21)$$

and the notation

$$Q_\alpha:=PA_{av}+A_{av}^\top P+2\alpha P \quad (22)$$

Differentiating $V$ along the solution to (16), we obtain

$$\begin{aligned}&\dot V+2\alpha V=|z(t)|_{Q_\alpha}^2+2z^\top(t)PB_{av}d(t)\\&+2z^\top(t)P\left[\mathbb{A}\Upsilon_{\Delta a}(t)+\mathbb{B}\mathcal{Z}_{\Delta b}(t)+\mathbb{W}\Upsilon_\varrho(t)+A_{av}\mathbb{B}\mathcal{Z}_\omega(t)\right]\\&-2z^\top(t)P\left[\mathbb{A}\left(I_2\otimes B_{av}\right)\mathcal{Z}_\varrho(t)+\mathbb{B}\Xi_\omega(t)\right]\\&-2z^\top(t)P\left[\mathbb{A}_1\Upsilon_{\varrho,a}(t)+\mathbb{A}_2\mathcal{Z}_{\varrho,b}(t)\right].\end{aligned} \quad (23)$$

Substituting (9) and recalling (15), we have

$$\begin{aligned}&|z(t)|_{Q_\alpha}^2=\left|x(t)-\mathbb{A}\Upsilon_\varrho(t)-\mathbb{B}\mathcal{Z}_\omega(t)\right|_{Q_\alpha}^2=|x(t)|_{Q_\alpha}^2\\&+|\Upsilon_\varrho(t)|^2_{\mathbb{A}^\top Q_\alpha\mathbb{A}}+|\mathcal{Z}_\omega(t)|^2_{\mathbb{B}^\top Q_\alpha\mathbb{B}}-2x^\top(t)Q_\alpha\mathbb{A}\Upsilon_\varrho(t)\\&-2x^\top(t)Q_\alpha\mathbb{B}\mathcal{Z}_\omega(t)+\Upsilon_\varrho^\top(t)\mathbb{A}^\top Q_\alpha\mathbb{B}\mathcal{Z}_\omega(t).\end{aligned} \quad (24)$$

Similarly,

$$\begin{aligned}&z^\top(t)PB_{av}d(t)=\left[x(t)-\mathbb{A}\Upsilon_\varrho(t)-\mathbb{B}\mathcal{Z}_\omega(t)\right]^\top PB_{av}d(t),\\&z^\top(t)P\left[\mathbb{A}\Upsilon_{\Delta a}(t)+\mathbb{B}\mathcal{Z}_{\Delta b}(t)+\mathbb{W}\Upsilon_\varrho(t)+A_{av}\mathbb{B}\mathcal{Z}_\omega(t)\right.\\&\left.-\mathbb{A}\left(I_2\otimes B_{av}\right)\mathcal{Z}_\varrho(t)-\mathbb{B}\Xi_\omega(t)-\mathbb{A}_1\Upsilon_{\varrho,a}(t)-\mathbb{A}_2\mathcal{Z}_{\varrho,b}(t)\right]\\&=\left[x(t)-\mathbb{A}\Upsilon_\varrho(t)-\mathbb{B}\mathcal{Z}_\omega(t)\right]^\top P\left[\mathbb{A}\Upsilon_{\Delta a}(t)+\mathbb{B}\mathcal{Z}_{\Delta b}(t)\right.\\&+A_{av}\mathbb{B}\mathcal{Z}_\omega(t)-\mathbb{A}\left(I_2\otimes B_{av}\right)\mathcal{Z}_\varrho(t)-\mathbb{B}\Xi_\omega(t)-\mathbb{A}_1\Upsilon_{\varrho,a}(t)\\&\left.-\mathbb{A}_2\mathcal{Z}_{\varrho,b}(t)+\mathbb{W}\Upsilon_\varrho(t)\right]\end{aligned} \quad (25)$$

Let

$$\begin{aligned}\eta(t)\ &=\operatorname{col}\left\{x(t),d(t),\dot d(t),\Upsilon_\varrho(t),\Upsilon_{\varrho,a}(t),\Upsilon_{\Delta a}(t)\right.\\&\left.\mathcal{Z}_\varrho(t),\mathcal{Z}_{\varrho,b}(t),\mathcal{Z}_{\Delta b}(t),\mathcal{Z}_\omega(t),\Xi_\omega(t)\right\}\end{aligned} \quad (26)$$

Then, (19) implies that

$$\begin{aligned}&0\leq W=\eta^\top(t)\left[\Lambda_0-\Lambda_1\right]\eta(t),\\&\Lambda_0=\operatorname{diag}\left\{\Lambda_0^{(1)},\Lambda_0^{(2)},\Lambda_0^{(3)},0,0,0,0,0,0,0,0\right\},\\&\Lambda_1=\operatorname{diag}\left\{0,0,0,\Lambda_1^{(1)}\right\},\ \Lambda_0^{(3)}=\left|\Lambda_{\Xi_\omega}H_\omega\right|_1I_{n_d},\\&\Lambda_0^{(1)}=\left(\left|\Lambda_{\Upsilon_\varrho}H_\varrho\right|_1+\left|\Lambda_{\Upsilon_{\varrho,a}}H_{\varrho,a}\right|_1+\left|\Lambda_{\Upsilon_{\Delta a}}\Delta_{a,M}\right|_1\right)I_n,\\&\Lambda_0^{(2)}=\left(\left|\Lambda_{\mathcal{Z}_\varrho}H_\varrho\right|_1+\left|\Lambda_{\mathcal{Z}_\omega}H_\omega\right|_1+\left|\Lambda_{\mathcal{Z}_{\varrho,b}}H_{\varrho,b}\right|_1\right.\\&\qquad\left.+\left|\Lambda_{\mathcal{Z}_{\Delta b}}\Delta_{b,M}\right|_1\right)I_{n_d},\\&\Lambda_1^{(1)}=\operatorname{diag}\left\{\Lambda_{\Upsilon_\varrho},\Lambda_{\Upsilon_{\varrho,a}},\Lambda_{\Upsilon_{\Delta a}},\Lambda_{\mathcal{Z}_\varrho},\Lambda_{\mathcal{Z}_{\varrho,b}}\right.\\&\qquad\left.,\Lambda_{\mathcal{Z}_{\Delta b}},\Lambda_{\mathcal{Z}_\omega},\Lambda_{\Xi_\omega}\right\}\otimes I_n\end{aligned} \quad (27)$$

By (19)-(27) and the S-procedure ( [7])

$$\begin{aligned}&\dot V+2\alpha V-\gamma_1^2|d(t)|^2-\gamma_2^2\left|\dot d(t)\right|^2\\&\leq\dot V+2\alpha V-\gamma_1^2|d(t)|^2-\gamma_2^2\left|\dot d(t)\right|^2+W\\&\leq\eta^\top(t)\Psi_{\epsilon,\epsilon_d}\eta(t)\leq 0,\end{aligned} \quad (28)$$

provided

$$\Psi_{\epsilon,\epsilon_d}=\left[\begin{array}{c|c|c|c}\Psi_{\epsilon,\epsilon_d}^{(1)}&\Psi_{\epsilon,\epsilon_d}^{(2)}&\Psi_{\epsilon,\epsilon_d}^{(3)}&\Psi_{\epsilon,\epsilon_d}^{(4)}\\\hline *&\Psi_{\epsilon,\epsilon_d}^{(5)}&\Psi_{\epsilon,\epsilon_d}^{(6)}&\Psi_{\epsilon,\epsilon_d}^{(7)}\\\hline *&*&\Psi_{\epsilon,\epsilon_d}^{(8)}&\Psi_{\epsilon,\epsilon_d}^{(9)}\\\hline *&*&*&\Psi_{\epsilon,\epsilon_d}^{(10)}\end{array}\right]<0 \quad (29)$$

with

$$\Psi_{\epsilon,\epsilon_d}^{(1)}=\begin{bmatrix}Q_\alpha+\Lambda_0^{(1)}&PB_{av}&0\\ *&-\gamma_1^2I_{n_d}+\Lambda_0^{(2)}&0\\ *&*&-\gamma_2^2I_{n_d}+\Lambda_0^{(3)}\end{bmatrix},$$
$$\Psi_{\epsilon,\epsilon_d}^{(2)}=\begin{bmatrix}-Q_\alpha\mathbb{A}+P\mathbb{W}&-P\mathbb{A}_1&P\mathbb{A}\\-B_{av}^\top P\mathbb{A}&0&0\\0&0&0\end{bmatrix},$$
$$\Psi_{\epsilon,\epsilon_d}^{(3)}=\begin{bmatrix}-P\mathbb{A}\left(I_2\otimes B_{av}\right)&-P\mathbb{A}_2&P\mathbb{B}\\0&0&0\\0&0&0\end{bmatrix},$$
$$\Psi_{\epsilon,\epsilon_d}^{(4)}=\begin{bmatrix}-Q_\alpha\mathbb{B}+PA_{av}\mathbb{B}&-P\mathbb{B}\\-B_{av}^\top P\mathbb{B}&0\\0&0\end{bmatrix},$$
$$\Psi_{\epsilon,\epsilon_d}^{(5)}=\begin{bmatrix}\psi_{\epsilon,\epsilon_d}^{(1)}&\mathbb{A}^\top P\mathbb{A}_1&-\mathbb{A}^\top P\mathbb{A}\\ *&-\left(\Lambda_{\Upsilon_{\varrho,a}}\otimes I_n\right)&0\\ *&*&-\left(\Lambda_{\Upsilon_{\Delta a}}\otimes I_n\right)\end{bmatrix},$$
$$\Psi_{\epsilon,\epsilon_d}^{(6)}=\begin{bmatrix}\psi_{\epsilon,\epsilon_d}^{(4)}&\mathbb{A}^\top P\mathbb{A}_2&-\mathbb{A}^\top P\mathbb{B}\\0&0&0\\0&0&0\end{bmatrix},$$
$$\Psi_{\epsilon,\epsilon_d}^{(7)}=\begin{bmatrix}\psi_{\epsilon,\epsilon_d}^{(2)}&\mathbb{A}^\top P\mathbb{B}\\\mathbb{A}_1^\top P\mathbb{B}&0\\-\mathbb{A}^\top P\mathbb{B}&0\end{bmatrix},\ \Psi_{\epsilon,\epsilon_d}^{(9)}=\begin{bmatrix}\psi_{\epsilon,\epsilon_d}^{(3)}&0\\\mathbb{A}_2^\top P\mathbb{B}&0\\-\mathbb{B}^\top P\mathbb{B}&0\end{bmatrix},$$
$$\Psi_{\epsilon,\epsilon_d}^{(8)}=-\operatorname{diag}\left\{\Lambda_{\mathcal{Z}_\varrho},\Lambda_{\mathcal{Z}_{\varrho,b}},\Lambda_{\mathcal{Z}_{\Delta b}}\right\}\otimes I_n,$$
$$\Psi_{\epsilon,\epsilon_d}^{(10)}=\begin{bmatrix}-\left(\Lambda_{\mathcal{Z}_\omega}\otimes I_n\right)+2\alpha\mathbb{B}^\top P\mathbb{B}&\mathbb{B}^\top P\mathbb{B}\\ *&-\left(\Lambda_{\Xi_\omega}\otimes I_n\right)\end{bmatrix},$$
$$\psi_{\epsilon,\epsilon_d}^{(1)}=-\left(\Lambda_{\Upsilon_\varrho}\otimes I_n\right)+\mathbb{A}^\top Q_\alpha\mathbb{A}-\mathbb{A}^\top P\mathbb{W}-\mathbb{W}^\top P\mathbb{A},$$
$$\psi_{\epsilon,\epsilon_d}^{(2)}=\mathbb{A}^\top Q_\alpha\mathbb{B}-\mathbb{W}^\top P\mathbb{B}-\mathbb{A}^\top PA_{av}\mathbb{B},$$
$$\psi_{\epsilon,\epsilon_d}^{(3)}=\left(I_2\otimes B_{av}\right)^\top\mathbb{A}^\top P\mathbb{B},$$
$$\psi_{\epsilon,\epsilon_d}^{(4)}=\mathbb{A}^\top P\mathbb{A}\left(I_2\otimes B_{av}\right).$$

Summarizing, we arrive at

*Theorem 1:* Consider the system (3) subject to Assumptions 1 and 2. Let $H_\varrho,H_\omega,H_{\varrho,a},H_{\varrho,b}$ be given by (17) and satisfying (18). Given matrices $A_{av},B_{av}$ and positive constants $\alpha, \{\epsilon_i\}_{i=1}^2, \{\epsilon_{d,j}\}_{j=1}^2, \{T_i\}_{i=1}^2 \{T_{d,j}\}_{j=1}^2$, $\{\Delta_{a_i,M}\}_{i=1}^2, \{\Delta_{b_j,M}\}_{j=1}^2$ let there exist $0 < P \in \mathbb{R}^{n\times n}$, diagonal matrices $\Lambda_{\Upsilon_\varrho}, \Lambda_{\mathcal{Z}_\varrho}, \Lambda_{\Upsilon_{\Delta a}} \in \mathbb{R}^{2\times 2}$, $\Lambda_{\mathcal{Z}_\omega}, \Lambda_{\Xi_\omega}, \Lambda_{\mathcal{Z}_{\Delta b}} \in \mathbb{R}^{2\times 2}$ and $\Lambda_{\Upsilon_{\varrho,a}}, \Lambda_{\mathcal{Z}_{\varrho,b}} \in \mathbb{R}^{4\times 4}$ with positive diagonal entries, and positive scalars $\gamma_1^2, \gamma_2^2$ such that $\Psi_{\epsilon,\epsilon_d} < 0$, with $\Psi_{\epsilon,\epsilon_d}$ given in (29). Then (3) satisfies the ISS-like estimates

$$\begin{aligned} |x(t)|^2 &\leq \beta_1^2 e^{-2\alpha t} |x(0)|^2 + \beta_2^2 \max_{s\in[0,t]} |d(s)|^2 \\ &+\beta_3^2 \max_{s\in[0,t]} \left|\dot{d}(s)\right|^2, \quad t \geq 0 \end{aligned} \tag{30}$$

for some $\beta_i > 0,\ i = 1,2,3$. The LMI $\Psi_{\epsilon,\epsilon_d} < 0$ is always feasible for small enough $\alpha, \{\epsilon_i\}_{i=1}^2, \{\epsilon_{d,j}\}_{j=1}^2$, $\{\Delta_{a_i,M}\}_{i=1}^2, \{\Delta_{b_j,M}\}_{j=1}^2$ and large $\gamma_i^2,\ i = 1,2$. Moreover, there exist $\alpha, \{\epsilon_i\}_{i=1}^2, \{\epsilon_{d,j}\}_{j=1}^2$ such that $\Psi_{\epsilon',\epsilon'_d} < 0$ is feasible for all $\epsilon'_i < \epsilon_i,\ i = 1,2$ and $\epsilon'_{d,j} < \epsilon_{d,j},\ j = 1,2$, implying that $\beta_i$ in (30) do not depend on $\epsilon'_i < \epsilon_i,\ i = 1,2$ and $\epsilon'_{d,j} < \epsilon_{d,j},\ j = 1,2$.

*Proof:* Fix $\tau > 0$. Feasibility of (29) implies that for all $t \in [0,\tau]$

$$\begin{aligned} &\dot{V} + 2\alpha V - \gamma_1^2 |d(t)|^2 - \gamma_2^2 \left|\omega_{\epsilon_d}(t)\dot{d}(t)\right|^2 \leq 0 \\ &\Rightarrow V(t) \leq e^{-2\alpha t} V(0) \\ &\quad + \int_0^t e^{-2\alpha(t-s)} \left(\gamma_1^2 |d(s)|^2 + \gamma_2^2 \left|\dot{d}(s)\right|^2\right) ds. \end{aligned}$$

Since $\lambda_{min}(P)|z(t)|^2 \leq V(t) \leq \lambda_{max}(P)|z(t)|^2$ for all $t \geq 0$, we have

$$\begin{aligned} &|z(t)|^2 \leq \tfrac{\lambda_{max}(P)}{\lambda_{min}(P)} e^{-2\alpha t} |z(0)|^2 + \tfrac{\gamma_1^2}{2\alpha\lambda_{min}(P)} \\ &\times \max_{s\in[0,\tau]} |d(s)|^2 + \tfrac{\gamma_2^2}{2\alpha\lambda_{min}(P)} \max_{s\in[0,\tau]} \left|\dot{d}(s)\right|^2, \end{aligned} \tag{31}$$

meaning that (16) satsfies ISS-like estimates with respect to $d$ and $\dot{d}$ . To obtain ISS-like estimates for (3), we employ the transformation (9). By Assumption 2, (10), (12), Young's inequality and the triangle inequality

$$\begin{aligned} |z(0)|^2 &\leq 2\delta_{2,x}^2 |x(0)|^2 + 2\delta_d^2 \max_{s\in[0,\tau]} |d(s)|^2 \\ |x(t)|^2 &\leq \delta_{1,x}^2 \left|z(t) + \textstyle\sum_{i=1}^2 \omega_{\epsilon_{d,i}}(t) B_i d(t)\right|^2 \\ &\leq 2\delta_{1,x}^2 |z(t)|^2 + 2\delta_{1,x}^2 \delta_d^2 \max_{s\in[0,\tau]} |d(s)|^2 . \end{aligned}$$

By combining the latter with (31), we obtain (30) with

$$\begin{aligned} &\beta_1^2 = \tfrac{4\delta_{1,x}^2 \delta_{2,x}^2 \lambda_{max}(P)}{\lambda_{min}(P)}, \quad \beta_3^2 = \tfrac{2\delta_{1,x}^2 \gamma_2^2}{2\alpha\lambda_{min}(P)}, \\ &\beta_2^2 = 2\delta_{1,x}^2 \left[\delta_d^2 \tfrac{2\lambda_{max}(P)+\lambda_{min}(P)}{\lambda_{min}(P)} + \tfrac{\gamma_1^2}{2\alpha\lambda_{min}(P)}\right]. \end{aligned}$$

For LMI feasiblity guarantees, it is enough to consider the case when the small parameters satisfy $\epsilon_i = \epsilon_{d,j} = \epsilon,\ i,j = 1,2$. Recall (7) and (18). It can be easily verified that there exists a constant $\mathcal{K} > 0$ large enough, such that (18) holds when all entries of (17) are equal to $\mathcal{K}\epsilon^2$. Next, choose $\Lambda_{\Upsilon_\varrho}, \Lambda_{\mathcal{Z}_\varrho}, \Lambda_{\Upsilon_{\Delta a}}, \Lambda_{\mathcal{Z}_\omega}, \Lambda_{\Xi_\omega}, \Lambda_{\mathcal{Z}_{\Delta b}} = \lambda I_2$ and $\Lambda_{\Upsilon_{\varrho,a}}, \Lambda_{\mathcal{Z}_{\varrho,b}} = \lambda I_4$, where $\lambda > 0$. Henceforth, we fix these choices. We begin by choosing $\alpha = 0$, $0 < P \in \mathbb{R}^n$ such that $Q_\alpha < 0$ (see (22)). Fixing $P$ and $\epsilon < 1$ we look at the LMI (29). Considering the bottom-right $3\times 3$ block submatrix (which we will henceforth denote as $\Xi_{\epsilon,\epsilon_d}$) we see that $\Xi_{\epsilon,\epsilon_d} < 0$ for $\lambda > \lambda_*$ with $\lambda_* > 0$ large enough (the diagonal elements are linear and negative in $\lambda$). Next, we apply Schur complement with respect to $\Xi_{\epsilon,\epsilon_d}$, to obtain the equivalent matrix inequality

$$\begin{aligned} &\Psi_{\epsilon,\epsilon_d}^{(1)} - \tfrac{1}{\lambda}\begin{bmatrix}\Psi_{\epsilon,\epsilon_d}^{(2)} & \Psi_{\epsilon,\epsilon_d}^{(3)} & \Psi_{\epsilon,\epsilon_d}^{(4)}\end{bmatrix}\left(\lambda^{-1}\Xi_{\epsilon,\epsilon_d}\right)^{-1} \\ &\qquad \times \begin{bmatrix}\Psi_{\epsilon,\epsilon_d}^{(2)} & \Psi_{\epsilon,\epsilon_d}^{(3)} & \Psi_{\epsilon,\epsilon_d}^{(4)}\end{bmatrix}^\top < 0. \end{aligned} \tag{32}$$

Note that $\left(\lambda^{-1}\Xi_{\epsilon,\epsilon_d}\right)^{-1}$ is bounded as $\lambda \to \infty$ (converges to the identity matrix), whereas $\begin{bmatrix}\Psi_{\epsilon,\epsilon_d}^{(2)} & \Psi_{\epsilon,\epsilon_d}^{(3)} & \Psi_{\epsilon,\epsilon_d}^{(4)}\end{bmatrix}$ is independent of $\lambda$. On the other hand, for any $\lambda > 0$, we can always find $\epsilon > 0$ small enough and $\gamma_i > 0,\ i = 1,2$ large enough so that $\Psi_{\epsilon,\epsilon_d}^{(1)} < 0$. Indeed, by choosing $\gamma_i = \lambda^2,\ i = 1,2,\ \epsilon = \frac{1}{\lambda^2}$, we obtain that (32) holds for $\lambda > 0$ large enough, whence feasibility of (29) follows. The fact that feasibility of (29) for some $\alpha, \{\epsilon_i\}_{i=1}^2, \{\epsilon_{d,j}\}_{j=1}^2$ implies its feasibility for all $\epsilon'_i < \epsilon_i,\ i = 1,2$ and $\epsilon'_{d,j} < \epsilon_{d,j},\ j = 1,2$ and the same $\alpha,\ \gamma_i,\ i = 1,2$ follows from monotonicity of (29) with respect to $\epsilon'_i < \epsilon_i,\ i = 1,2$ and $\epsilon'_{d,j} < \epsilon_{d,j},\ j = 1,2$ (meaning that as the small parameters decrease, the eigenvalues of $\Psi_{\epsilon',\epsilon'_d}$ are non-increasing). ■

*Remark 7:* Recall (17) and (18). In the Lyapunov analysis above we assume the scalar bounds on the right-hand side of (18) are identical for all $t \geq 0$. Assume that there exists a partition of $[0,\infty)$ into intervals such that every interval in the partition belong to one of finitely many classes (types), denoted by $\{\mathcal{I}_j\}_{j=1}^\zeta$. As an example, consider Example 2.3.1 below, where we treat a switched system with two functioning modes (35). In this case $\zeta = 2$ and $\mathcal{I}_1$ corresponds to subintervals where $A(\tau) \equiv A_1$, whereas $\mathcal{I}_2$ corresponds to subintervals where $A(\tau) \equiv A_2$. Assume that for each $1 \leq j \leq \zeta$, there exist vectors $H_{\varrho,j}, H_{\omega,j}, H_{\varrho,a,j}, H_{\varrho,b,j}$ whose entries serve as upper bounds in (18) whenever $t \geq 0$ belongs to an interval of type $\mathcal{I}_j$ (the vectors may vary between classes). In this case our proposed approach can be applied to each of the classes separately and will yield $\zeta$ LMIs of the form (29) (one for each class). Feasibility of the LMIs can then be verified simultaneously with *the same* $P$ and $\gamma_i,\ i = 1,2$. Note that the auxilliary matrices $\Lambda_{\Upsilon_\varrho}, \Lambda_{\mathcal{Z}_\varrho}, \Lambda_{\Upsilon_{\Delta a}}, \Lambda_{\mathcal{Z}_\omega}, \Lambda_{\Xi_\omega}, \Lambda_{\mathcal{Z}_{\Delta b}}, \Lambda_{\Upsilon_{\varrho,a}}, \Lambda_{\mathcal{Z}_{\varrho,b}}$ may differ between LMIs corresponding to different classes. This approach is expected to yield less conservative results than choosing bounds in (18) which hold uniformly for all $t \geq 0$, and verifying feasibility of a single LMI (29).

*Remark 8:* For the case of general $N, N_d \in \mathbb{N}$, the approach presented above requires only minor modifications, which are related to the dimensions of the matrices. In particular, in (15) the dimensions of the vectors require changing, whereas the matrices now having the form

$$\begin{aligned} &\mathbb{A} = \begin{bmatrix}A_1 & \dots & A_N\end{bmatrix},\ \mathbb{B} = \begin{bmatrix}B_1 & \dots & B_{N_d}\end{bmatrix}, \\ &\mathbb{A}_1 = [A_1^2 \dots A_1 A_N \dots A_N A_1 \dots A_N^2], \\ &\mathbb{A}_2 = [A_1 B_1 \dots A_1 B_{N_d} \dots A_N B_1 \dots A_N B_{N_d}], \\ &\mathbb{W} = \begin{bmatrix}W_1 & \dots & W_N\end{bmatrix}, \\ &W_i = A_{av} A_i - A_i A_{av},\ 1 \leq i \leq N. \end{aligned} \tag{33}$$

The system (16) (and the derived LMIs) will have the same form with $I_2$ replaced by $I_{N_d}$. Therefore, the Lyapunov analysis and resulting LMIs of Section 2.3 will be identical, subject to the changes in (33) and $I_2$ replaced by $I_{N_d}$ will guarantee (30) for (3).

*Remark 9:* Note that instead of the ISS-like estimates (30), we are also able to obtain standard ISS bounds (meaning, with respect to $d$ only) for (3). Indeed, consider the system (3). In order to avoid introducing the disturbance derivative one can simply not use averaging for $\left[\sum_{i=1}^{2} b_i\left(\frac{t}{\epsilon_{d,i}}\right)B_i\right]d(t)$. Instead, one can treat this term as a norm bounded time-varying matrix-valued function which multiplies the disturbance. In this case the presentation of this matrix valued function as a linear combination is obviously not needed and (9) will be replaced with $z(t) = x(t) - \sum_{i=1}^{2} \varrho_{\epsilon,i}(t)A_i x(t)$. The norm bound on $\sum_{i=1}^{2} b_i\left(\frac{t}{\epsilon_{d,i}}\right)B_i$ will be employed in a standard ISS analysis. This approach is expected to result in larger estimates on the ISS gains.

## III. Numerical examples

### A. Example 3.1: Stabilization by fast switching I

Consider stabilization by fast switching of a linear system [8, Example 2.2]. Let $\epsilon > 0$ and

$$A_{av} = \begin{bmatrix} -0.038 & 0.024 \\ 0.042 & -0.062 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.1 & 0.3 \\ 0.6 & -0.2 \end{bmatrix}, \ A_2 = \begin{bmatrix} -0.13 & -0.16 \\ -0.33 & 0.03 \end{bmatrix}. \tag{34}$$

Set

$$A(\tau) = \begin{cases} A_1, \ \tau \in [k, k+0.4), \ k \in \mathbb{Z}_+ \\ A_2, \ \tau \in [k+0.4, k+1), \ k \in \mathbb{Z}_+ \end{cases}. \tag{35}$$

Given $\tau \in [k, k+1)$, (35) can be written as

$$A(\tau) = \chi_{[k,k+0.4)}(\tau) A_1 + \left[1 - \chi_{[k+0.4,k+1)}(\tau)\right] A_2,$$

where $\chi_{[k,k+0.4)}$ is the indicator function of $[k, k+0.4)$. Note that $A(\tau)$ is 1-periodic. We further set $B(t) \equiv 0_{2\times 1}$.

Consider (3) with (35), $\epsilon_i = \epsilon, T_i = 1, \ i = 1, 2$ and $B_{av} = B_1 = B_2 = 0$. Here

$$a_1(\tau) = \begin{cases} 0.6, \ \tau \in [k, k+0.4), \ k \in \mathbb{Z} \\ -0.4, \ \tau \in [k+0.4, k+1), \ k \in \mathbb{Z} \end{cases}, \quad a_2(\tau) = -a_1(\tau).$$

Note that the latter functions are 1-periodic, meaning that $\Delta_{a_i,M} = 0, \ i = 1, 2$ in (20). Let $t \in [m\epsilon, (m+1)\epsilon), \ m \in \mathbb{Z}_+$ and denote $w = t - m\epsilon \in [0, \epsilon), \ m \in \mathbb{Z}_+$. An explicit computation of $\varrho_{\epsilon,i}(t), \ i = 1, 2$ in (7) yields the bounds $\varrho^2_{\epsilon,1}(t) \leq 0.0144\epsilon^2$ and $\varrho^2_{\epsilon,2}(t) \leq 0.0144\epsilon^2$. We then use the fact that $a_1(\tau)$, $a_2(\tau)$ are indicator functions to separate the analysis into two cases

$$a_1\left(\tfrac{t}{\epsilon}\right)\varrho_{\epsilon_j}(t) = \begin{cases} 0.6\varrho_{\epsilon,j}(t), & w \in [0, 0.4\epsilon) \\ -0.4\varrho_{\epsilon,j}(t), & w \in [0.4\epsilon, \epsilon) \end{cases}$$
$$a_2\left(\tfrac{t}{\epsilon}\right)\varrho_{\epsilon,j}(t) = -a_1\left(\tfrac{t}{\epsilon}\right)\varrho_{\epsilon_j}(t)$$

and obtain tight upper bounds in (18) for each of the cases. Thus, we separate the analysis into the two subintervals $0 \leq w < 0.4\epsilon$ and $0.4\epsilon \leq w < \epsilon$. For each subinterval (and its corresponding bounds (18)) we obtain an LMI of the form (29) (see Remark 7). We verify feasibility for both LMIs with the same $\alpha$ and $P$.

We consider $\alpha \in \{0, 0.005, 0.01\}$ and verify the LMIs of Theorem 1 to obtain the maximal value $\epsilon^*$ which preserves feasibility of the LMIs. Note that $\epsilon^*$ guarantees internal exponential stability (and thus the ISS-like bounds) of (3). The values of $\epsilon^*$ are given in Table I, where we further compare our results to the bounds in the recent work [21]. It is seen that our results essentially improve the results of [21] with a value of $\epsilon^*$ larger by more than 2.5 times.

| | $\alpha = 0$ | $\alpha = 0.005$ | $\alpha = 0.01$ |
|---|---|---|---|
| Zhang & Fridman | 0.1920 | 0.1306 | Unchecked |
| Thm. 1 | 0.4332 | 0.3013 | 0.1662 |

TABLE I

Switched system I - maximum value $\epsilon^*$ preserving LMI feasibility.

Next, we set $B_{av} = [0\ 1]^\top$ and $B_1 = B_2 = 0_{2\times 1}$ and verify feasibility of (29) in order to guarantee (30). Note that in this case the transformation (9) will not result in terms involving $\dot{d}$. Hence, we obtain classical ISS estimates (i.e., we have $\gamma_2 = 0$ in (28) $\beta_3 = 0$ in (30)). Table II presents several pairs $(\beta_1, \beta_2)$ (see proof of Theorem 30) for different choices of $\alpha$ and $\epsilon$. Note that in this case $\delta_{1,x}$ and $\delta_{2,x}$ were computed using the bounds (10) and (11).

| | $\epsilon = 0.002$ | $\epsilon = 0.16$ |
|---|---|---|
| $\alpha = 0.005$ | $(0.0054, 73.503)$ | $(0.5147, 99.266)$ |
| $\alpha = 0.01$ | $(0.006, 76.48)$ | $(0.7126, 389.89)$ |

TABLE II

Switched system I - ISS gains: $(\beta_1, \beta_2)$.

### B. Example 3.2: Stabilization by fast switching II

We consider stabilization by fast switching of a linear system with three functioning modes ( [1] and [4]) . Let $\epsilon > 0$ and

$$A_{av} = \begin{bmatrix} 0.047 & 0.33 \\ -0.6 & -0.87 \end{bmatrix}, \ A_1 = \begin{bmatrix} 0 & 0.5 \\ 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.1 & 0 \\ -1 & -1 \end{bmatrix}, \ A_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{36}$$

and set

$$A(\tau) = \begin{cases} A_1, \ \tau \in [k, k+0.4), \ k \in \mathbb{Z}_+ \\ A_2, \ \tau \in [k+0.4, k+0.87), \ k \in \mathbb{Z}_+ \\ A_3, \ \tau \in [k+0.87, k+1), \ k \in \mathbb{Z}_+ \end{cases}. \tag{37}$$

Note that $A(\tau)$ is 1-periodic. We further set $B(t) \equiv 0_{2\times 1}$.

Consider (3) with (35), $\epsilon_i = \epsilon, T_i = 1, \ i = 1, 2, 3$ and

$B_{av} = B_1 = B_2 = B_3 = 0$. Here for $k \in \mathbb{Z}_+$

$$a_1(\tau) = \chi_{[k,k+0.4)}(\tau) - 0.4,$$
$$a_2(\tau) = \chi_{[k+0.4,k+0.87)}(\tau) - 0.47,$$
$$a_3(\tau) = \chi_{[k+0.87,k+1)}(\tau) - 0.13.$$

where $\chi_{[a,b)}(\tau)$ denotes the indicator of $[a,b) \subseteq \mathbb{R}$. Note that the latter functions are 1-periodic, meaning that $\Delta_{a_i,M} = 0,\ i = 1,2,3$ in (20). Similarly to Example 3.2.1, an explicit computation of $\varrho_{\epsilon,i}(t),\ i = 1,2$ in (7) yields the bounds $\varrho^2_{\epsilon,1}(t) \leq 0.0144\epsilon^2$, $\varrho^2_{\epsilon,2}(t) \leq 0.0155127\epsilon^2$ and $\varrho^2_{\epsilon,3}(t) \leq 0.0031979\epsilon^2$. We then use the fact that $a_1(\tau)$, $a_2(\tau)$ and $a_3(\tau)$ are indicator functions to separate the analysis into three cases, corresponding to the subintervals in (37). For each subinterval (and corresponding bounds (18)) we obtain an LMI of the form (29) (see Remarks 7, 8). We verify feasibility for both LMIs with the same $\alpha$ and $P$.

We consider $\alpha \in \{0, 0.005, 0.25\}$ and verify the LMIs of Theorem 1 to obtain the maximal value $\epsilon^*$ which preserves feasibility of the LMI. Note that $\epsilon^*$ guarantees internal exponential stability (and thus ISS-like bounds) of (3). The values of $\epsilon^*$ are given in Table III.

| | $\alpha = 0$ | $\alpha = 0.005$ | $\alpha = 0.25$ |
|---|---|---|---|
| Thm. 1 | 0.4341 | 0.4177 | 0.0591 |

TABLE III

SWITCHED SYSTEM II - MAXIMUM VALUE $\epsilon^*$ PRESERVING LMI FEASIBILITY.

*Remark 10:* In examples 3.1 and 3.2, presenting the systems as (3) with $A_{av} = 0$ and

$$\text{Example 2.3.1:}\quad a_1(\tau) = \chi_{[k,k+0.4)}(\tau),\quad a_2(\tau) = \chi_{[k+0.4,k+1)}(\tau),$$
$$\text{Example 2.3.2:}\quad a_1(\tau) = \chi_{[k,k+0.4)}(\tau),\quad a_2(\tau) = \chi_{[k+0.4,k+0.87)}(\tau),\quad a_3(\tau) = \chi_{[k+0.87,k+1)}(\tau).$$

leads to essentially smaller $\epsilon^*$. For example, for $\alpha = 0$ we find $\epsilon^* = 0.1566$ (compared to 0.4332) in example 2.3.1 and $\epsilon^* = 0.141$ (compared to 0.4341) in example 2.3.2. The reason for the significantly improved results is that $\varrho_{\epsilon,i}$ become essentially smaller when the averages $a_{av,i}$ are zero (see Remark 1), thereby decreasing the bounds required on the right-hand side of (18).

### C. Example 3.3: Control of a pendulum

We consider a suspended pendulum with the suspension point that is subject to vertical vibrations of small amplitude and high frequency (see [11, Example 10.10] and [8, Example 2.1]). Let $\epsilon > 0$ and

$$A(\tau) = \begin{bmatrix} \cos(\tau) & 1 \\ 0.04 - \cos^2(\tau) & -0.2 - \cos(\tau) \end{bmatrix},\ B(\tau) \equiv 0_{2\times 1}. \tag{38}$$

Note that $A(\tau)$ is $2\pi$ periodic.

Consider (3) with $\epsilon_i = \epsilon, T_i = 2\pi,\ i = 1,2,\ B_0 = B_1 = B_2 = 0_{2\times 1}$ and

$$A_{av} = \begin{bmatrix} 0 & 1 \\ -0.46 & -0.2 \end{bmatrix},\ A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0 & 0 \\ -0.5 & 0 \end{bmatrix},\ a_1(\tau) = \cos(\tau),\ a_2(\tau) = \cos(2\tau).$$

Note that $a_i,\ i = 1,2$ are $2\pi$-periodic, whence $\Delta_{a_i,M} = 0,\ i = 1,2$ in (20). An explicit computation of $\varrho_{\epsilon,i}(t),\ i = 1,2$ in (7) yields

$$\varrho_{\epsilon,1}(t) = \epsilon \sin(\tau),\ a_2(\tau)\,\varrho_{\epsilon,2}(t) = \tfrac{\epsilon}{4}\sin(4\tau),$$
$$\varrho_{\epsilon,2}(t) = a_1(\tau)\,\varrho_{\epsilon,1}(t) = \epsilon\cos(\tau)\sin(\tau),$$
$$a_2(\tau)\,\varrho_{\epsilon,1}(t) = (2\cos^2(\tau) - 1)\varrho_{\epsilon,1}(t),$$
$$a_1(\tau)\,\varrho_{\epsilon,2}(t) = \cos^2(\tau)\,\varrho_{\epsilon,1}(t),\ \tau = \tfrac{t}{\epsilon}$$

which are used to derive the upper bounds in (18). Differently from the previous examples, here we obtain only one LMI of the form (29).

We consider $\alpha \in \left\{0, \frac{1}{10\pi}\right\}$ and verify the LMIs of Theorem 1 to obtain the maximal value $\epsilon^*$ which preserves feasibility of the LMI. Note that $\epsilon^*$ guarantees internal exponential stability (and thus the ISS-like bounds) of (3). The values of $\epsilon^*$ are given in Table IV, where we further compare our results to the bounds in the recent work [21].

| | $\alpha = 0$ | $\alpha = (10\pi)^{-1}$ |
|---|---|---|
| Zhang & Fridman | 0.0074 | 0.005 |
| Thm. 1 | 0.0457 | 0.0321 |

TABLE IV

PENDULUM - MAXIMUM VALUE $\epsilon^*$ PRESERVING LMI FEASIBILITY.

Finally, we consider this example subject to uncertainty. For that purpose, we replace $a_2(\tau) = \cos(2\tau)$ with $a_2(\tau) = \cos(2\tau) + 0.4g(\tau)$, where $\|g\|_\infty \leq 0.1$. In this case we obtain a nonzero $\Delta a_2(t)$ in (4), satisfying $\|\Delta a_2\|_\infty \leq 0.04 =: \Delta_{a_2,M}$. We consider $\alpha \in \left\{0, \frac{1}{10\pi}\right\}$ and verify the LMIs of Theorem 1 to obtain the maximal value $\epsilon^*$ which preserves feasibility of the LMI. We further compare our results with [21, Example 4.1]. The results are given in Table V. Our results are essentially better than the results of [21].

| | $\alpha = 0$ | $\alpha = (10\pi)^{-1}$ |
|---|---|---|
| Fridman & Zhang | 0.0058 | 0.0034 |
| Thm. 1 | 0.0204 | 0.0146 |

TABLE V

PENDULUM WITH UNCERTAINTY - MAXIMUM VALUE $\epsilon^*$ PRESERVING LMI FEASIBILITY.

## IV. CONCLUSION

We introduced a novel quantitative methodology for deriving ISS-like estimates for linear continuous-time systems. The presented methodology relies on a superposition-based system presentation, in conjunction with a system transformation. Differently from the time-delay approach to averaging, the presented method is efficient for ISS of systems with fast-varying almost periodic coefficients subject to multiple independent time-scale parameters, and achieves essentially less conservative LMI conditions for ISS-like estimates. Future work may include extension of the method to systems with delays and applications to control problems that employ averaging.

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