Local Observer for Parameters and for State Variables of Nonlinear Systems

Frederic Mazenc, Michael Malisoff

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Abstract— **We provide new observer designs to simultaneously identify parameters and states of systems whose nonlinearities have order two near the origin, which include cubic terms arising in the study of jump phenomena, process control, and bistable models of aerospace systems. This yields local exponential convergence of the state estimation error to zero, basin of attraction estimates, and fixed time parameter identification. We illustrate our result using Duffing's equation, whose cubic term puts it outside the scope of prior methods.**

I. Introduction

We continue our search for observers for unknown states and unknown model parameters that we began in [1], [2]. Whereas [1], [2] were confined to systems where the state dependent nonlinearity satisfies a linear growth condition and provided global exponential stability, here we replace this linear growth condition by an order 2 condition near the origin (which covers cubic terms that are important in the study of jump phenomena, process control, and bistable models in aerospace engineering), but we instead prove local exponential stability for the state observer with basin of attraction estimates, and fixed time parameter identification, where by fixed time, we mean finite time convergence where the convergence time can be chosen to be independent of the initial state. This contrasts with our prior works (e.g., [3], [4], [5], [6], [7], [8], [9]) for finite-time observers for nonlinear systems, which provided arbitrarily fast convergence of state observers for perturbed systems (and fixed time convergence when the uncertainties are zero and when the system was affine in the unmeasured state), but did not identify model parameters. These studies are motivated by the fact that observers are useful, e.g., to solve feedback control problems; see, e.g., [10], [11], [12], [13], [14], [15], [16], [17], [18].

As in [1], [2], a key motivation is artificial neural network expansions that represent unknown time-varying functions as linear combinations of known basis functions whose constant weights must be identified. While adaptive control can estimate parameters, it is usually not amenable for fixed time parameter identification, and does not estimate states. We address these challenges when there is a linear output that can have measurement delays. By allowing nonlinearities that violate uniform global Lipschitzness conditions on the state-dependent nonlinearity, we cover types of jump dynamics that we study in Section IV below. Our results are novel and of interest even if no unknown parameters or measurement delays are present, because of the order 2 nonlinearities.

We use standard notation. The dimensions of our Euclidean spaces are arbitrary, unless we indicate otherwise, and $|\cdot|$ is the standard Euclidean and corresponding matrix norm. We set $g_t(s) = g(t+s)$ for functions g and all $s \leq 0$ and $t \geq 0$ such that $t+s$ is in the domain of g, $|f|_J$ is the usual supremum over any interval J in the domain of functions f, $|f|_\infty$ is the supremum of a function f, $\lambda_{\max}(M)$ (resp., $\lambda_{\min}(M)$) denote largest (resp., smallest) eigenvalues of square matrices M whose eigenvalues are all real, 0 is the zero matrix, and I denotes the identity matrix.

II. Main Result

A. *Studied System and Statement of Main Result*

Consider the system

$$\begin{cases} \dot{x}(t) = Ax(t) + B\left[\sum_{i=1}^{q} \epsilon_i \gamma_i(t) + u(t)\right] + \Delta(x(t), t) \\ y(t) = Cx(t-d) \end{cases} \quad (1)$$

whose state x is valued in $\mathbb{R}^n$, where the constants ϵ_j in the parameter vector $\epsilon = [\epsilon_1, \ldots, \epsilon_q]^\top$ are unknown, the control u will be specified, the measurement y is real valued, the matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{1\times n}$ are known, $\gamma(t) = [\gamma_1(t), \ldots, \gamma_q(t)]$ is a matrix of known C^1 bounded functions where $\gamma_i : \mathbb{R} \to \mathbb{R}^m$ for each i, $d \geq 0$ is a known constant, and Δ is locally Lipschitz in x uniformly in t and has order 2 at the origin, where the order 2 condition is the requirement that there is a nondecreasing positive definite continuous function $\mathcal{L} : [0, +\infty) \to [0, +\infty)$ such that

$$|\Delta(x, t)| \leq \mathcal{L}(|x|)|x|^2 \quad (2)$$

for all $x \in \mathbb{R}^n$ and $t \geq 0$. The sum in (1) (which can be written as $\gamma(t)\epsilon$) can represent an uncertainty that is parameterized by unknown weights ϵ_j on known basis functions γ_i. Condition (2) often holds, e.g., for systems with time invariant C^2 nonlinearities $G(x)$ such that $G(0) = 0$, since we can use a Maclaurin expansion to write $G(x) = G_1 x + G_2(x)$ for a constant matrix G_1 such that the choice $\Delta(x,t) = G_2(x)$ satisfies (2), and then we can add G_1 to A to get one constant coefficient of x, when the given system has the form $\dot{x}(t) = Ax(t) + B[\gamma(t)\epsilon + u(t)] + G(x(t))$.

Later we specify u such that the closed loop system satisfies standard forward completeness conditions for all initial states $x(0)$ in the basin of attraction of our locally exponentially stabilized system. Our first assumption is:

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Assumption 1: The pair (A, B) is controllable, and the pair (A, C) is observable.

Assumption 1 implies that the matrix

$$E = \int_{-h}^{0} e^{A^\top s} C^\top C e^{As} \mathrm{d}s \tag{3}$$

is invertible for each constant $h > 0$. We fix the constant $h > 0$ in what follows. In terms of the real valued functions

$$\begin{aligned} &\eta_i(t) = \\ &CE^{-1}\int_{t-h}^{t} e^{A^\top (s-t)} C^\top C \int_{s-d}^{t-d} e^{A(s-m-d)} B\gamma_i(m)\mathrm{d}m\mathrm{d}s \end{aligned} \tag{4}$$

for $i = 1, \ldots, q$ and the function $\eta(t) = [\eta_1(t), \ldots, \eta_q(t)]$, our second assumption is the following, which agrees with our main invertibility assumption in our works [1], [2] on systems without order 2 state dependent nonlinearities:

Assumption 2: There are known nonnegative constants $\overline{\gamma}$ and τ_j for $j = 1, \ldots, q$ such that $\tau_1 < \tau_2 < \ldots < \tau_q$ and such that the $q \times q$ matrix

$$W(t) = [\eta^\top(t - \tau_1), \ldots, \eta^\top(t - \tau_q)] \tag{5}$$

is nonsingular for each $t \geq \tau_q + h + d$, and such that $0 < |B\gamma(W^{-1})^\top|_{[\tau_q + h + d, +\infty)} \leq \overline{\gamma}$.

Assumption 1 also provides matrices L and K such that

$$H = A + LC \text{ and } M = A + BK \tag{6}$$

are Hurwitz matrices. We fix L and K that satisfy the preceding requirements in the sequel, and corresponding H and M as defined by (6). Then the Hurwitzness of H and M provides positive definite matrices Q_1 and Q_2 and constants $c_1 > 0$, $c_2 > 0$, and $\overline{d} > 0$ such that the time derivative of

$$V(z_t) = V_0(z(t)) + c_2 \int_{t-2d}^{t} \int_{s}^{t} |z(\ell)|^2 \mathrm{d}\ell \mathrm{d}s \tag{7}$$

along all solutions $z = (z_1, z_2) : [0, +\infty) \to \mathbb{R}^{2n}$ of the $2n$-dimensional system

$$\left\{ \begin{array}{rcl} \dot{z}_1(t) &=& Mz_1(t) + LCz_2(t) \\ && + BK[z_1(t-d) - z_1(t)] \\ \dot{z}_2(t) &=& Hz_2(t) \end{array} \right. \tag{8}$$

with constant initial functions satisfies $\frac{d}{dt}V(z_t) \leq -c_1 V(z_t)$ for all $t \geq 0$ and all $d \in [0, \overline{d}]$, where $V_0(z) = z_1^\top Q_1 z_1 + z_2^\top Q_2 z_2$. We assume that the initial functions for all of our delayed systems are constant at the initial time $t = 0$.

The preceding constants and V can be found using the Lyapunov-Krasovskii methods from [19]; see the appendix below. We assume that $d \in [0, \overline{d}]$ and we use the constants

$$\begin{aligned} \overline{S} = h \sup \big\{ \big| CE^{-1} e^{A^\top s} C^\top C e^{A(s-m)} \big| : \\ s \in [-h, 0],\ m \in [s, 0] \big\} \text{ and} \\ m_0 = \min\{\lambda_{\min}(Q_1), \lambda_{\min}(Q_2)\}, \end{aligned} \tag{9}$$

and the functions

$$\begin{aligned} &\mu_1(s) = 2\mathcal{L}\left(\sqrt{\tfrac{s}{\lambda_{\min}(Q_1)}} + \sqrt{\tfrac{s}{\lambda_{\min}(Q_2)}}\right)\tfrac{1}{m_0}, \\ &\Gamma_1(s) = 2|Q_2|\sqrt{\tfrac{s}{\lambda_{\min}(Q_2)}}\mu_1(s),\ \Gamma_2(s) = \tfrac{1}{2}\mu_1(s), \text{ and} \\ &\Gamma_3(s) = 4\sqrt{\tfrac{s}{\lambda_{\min}(Q_2)}}|Q_2 B\gamma W^{-1}|_{[h+\tau_q+2d,\infty)}\overline{S}\sqrt{q} \end{aligned} \tag{10}$$

where $\mathcal{L}$ is from the order 2 condition (2) on the nonlinearity Δ. We also use the functionals

$$\begin{aligned} &\mathcal{G}^\sharp(x_t) = [\mathcal{G}(x_{t-\tau_1}), \ldots, \mathcal{G}(x_{t-\tau_q})], \\ &\theta^\sharp(y_t, u_t) = [\theta(y_{t-\tau_1}, u_{t-\tau_1}), \ldots, \theta(y_{t-\tau_q}, u_{t-\tau_q})], \\ &\text{ and } \quad \Omega(t, y_t, u_t) = \gamma(t)[\theta^\sharp(y_t, u_t)W^{-1}(t)]^\top, \end{aligned} \tag{11}$$

where

$$\begin{aligned} &\mathcal{G}(x_t) = \\ &-CE^{-1}\int_{t-h}^{t} e^{A^\top(s-t)} C_T \int_{s-d}^{t-d} e^{A(s-d-m)} \Delta_x(m)\mathrm{d}m\mathrm{d}s \\ &\text{and } \theta(y_t, u_t) = y(t) - CE^{-1}\int_{t-h}^{t} e^{A^\top(s-t)} C^\top y(s)\mathrm{d}s \\ &-CE^{-1}\int_{t-h}^{t} e^{A^\top(s-t)} C_T \int_{s-d}^{t-d} e^{A(s-d-m)} Bu(m)\mathrm{d}m\mathrm{d}s \end{aligned} \tag{12}$$

and $\Delta_x(m) = \Delta(x(m), m)$ and $C_T = C^\top C$. We then use the dynamic controller

$$u(t) = \left\{ \begin{array}{ll} K\hat{x}(t), & t \in [0, h + \tau_q + 2d) \\ K\hat{x}(t) - \Omega(t, y_t, u_t), & t \geq h + \tau_q + 2d \end{array} \right. \tag{13}$$

where the estimator $\hat{x}$ of x has the dynamics

$$\begin{array}{rcl} \dot{\hat{x}}(t) &=& M\hat{x}(t) + BK[\hat{x}(t-d) - \hat{x}(t)] \\ && + L[C\hat{x}(t) - y(t)] \end{array} \tag{14}$$

with $\hat{x}(0) = 0$. Letting $v_* > 0$ be a constant such that

$$\Gamma_1(v_*) + (h + \tau_q + 2d)\Gamma_2(v_*)\Gamma_3(v_*) < c_1 \tag{15}$$

where c_1 is the decay rate on V given above, and setting

$$\overline{\mathcal{E}}(t) = \sup\{|Q_2 B\gamma(t)\lambda| : \lambda \in \mathcal{E}\} \tag{16}$$

for any bounded set $\mathcal{E} \subseteq \mathbb{R}^q$ that is known to contain ϵ, our last assumption is this smallness condition on $\overline{\mathcal{E}}$, which can be checked in practice because the right side of (17) is independent of ϵ and bounds on ϵ are typically known:

Assumption 3: With the preceding choices, the inequality

$$|\overline{\mathcal{E}}|_{[0, h+\tau_q+2d]} < \tfrac{1}{2}\sqrt{v_* \lambda_{\min}(Q_2)}\big(c_1 - \Gamma_1(v_*)\big) \tag{17}$$

is satisfied. Also, $d \in [0, \overline{d}]$.

Using the variable $\tilde{x}(t) = \hat{x}(t) - x(t - d)$ and constants

$$\begin{aligned} &R^* = \max\{\lambda_{\max}(Q_1), \lambda_{\max}(Q_2)\}(1 + 2d^2 c_2)|c_d|^2, \\ &\text{where } c_d = (1 + c_{d2}, c_{d2}) \text{ and} \\ &c_{d2} = \int_0^d |e^{(M+LC-BK)\ell} LC|\mathrm{d}\ell \end{aligned} \tag{18}$$

and

$$\Delta_{\mathcal{E}} = \tfrac{4}{\lambda_{\min}(Q_2)}\left(\tfrac{|\overline{\mathcal{E}}|_{[0, h+\tau_q+d]}}{c_1 - \Gamma_1(v_*)}\right)^2, \tag{19}$$

our main result is as follows (but see Remark 2.1 and Section III below for ways to express u in terms of a dynamical extension without distributed terms, and a way to express ϵ in terms of the estimator $\hat{x}$ instead of the state x):

Theorem 1: Let Assumptions 1-3 hold. Consider (1) in closed loop with (13), where the dynamics of $\hat{x}$ is defined by (14), with $\hat{x}(0) = 0$. Then for each initial state $x(0)$ in

$$\mathcal{S}_0 = \big\{ z \in \mathbb{R}^n : |z|^2 < \tfrac{1}{R^*}(v_* - \Delta_{\mathcal{E}}) \big\}, \tag{20}$$

the pair $(\hat{x}(t), \tilde{x}(t))$ satisfies $\lim_{t \to +\infty}(\hat{x}(t), \tilde{x}(t)) = 0$, and

$$\epsilon^\top = [\theta^\sharp(y_t, u_t) + \mathcal{G}^\sharp(x_t)]W^{-1}(t) \tag{21}$$

holds for all $t \geq h + \tau_q + d$.

Remark 2.1: The proof of our theorem allows us to construct positive constants r_i for $i = 1, 2$ such that

$$|(\hat{x}(t), \tilde{x}(t))| \le r_2 e^{-r_1 t}(|x(0)| + (h + \tau_q + 2d)B_*) \quad (22)$$

for all $t \ge \tau_q + h + 2d$ and initial states $x(0) \in \mathcal{S}_0$, where $B_* = \sup\{|B\gamma\lambda|_\infty : \lambda \in \mathcal{E}\}$ and $\mathcal{E}$ is the set we used in (16). This will be done by transforming the functional V from (7) into a new Lyapunov-Krasovskii functional $V^\sharp$, with the choice $(z_1, z_2) = (\hat{x}, \tilde{x})$, and then building a uniform global exponential stability estimate for $V^\sharp$ that is valid for all $t \ge \tau_q + h + 2d$, and then applying a variation of parameters to the dynamics for $(\hat{x}, \tilde{x})$ on $[0, \tau_q + h + 2d]$, after first showing that all solutions $(\hat{x}(t), \tilde{x}(t))$ for all initial states $x(0) \in \mathcal{S}_0$ stay in a bounded set in which the dynamics satisfy a linear growth condition. This will allow us to construct a nondecreasing function $\mathcal{L}^\sharp : [0, +\infty) \to [0, +\infty)$ such that

$$\epsilon^\top W(t) = \theta^\sharp(y_t, u_t) + \mathcal{G}^\sharp(\hat{x}_t) + \mathcal{L}^\sharp\big(J_t\big)\,(J_t)^2 \quad (23)$$

for all $t \ge 2(h + \tau_q + d)$, where $J_t = e^{-r_1 t}(|x(0)| + (h + \tau_q)B_*)$, and where $\mathcal{G}^\sharp(\hat{x}_t)$ is defined as in (11)-(12) except with x in the preceding formulas replaced by $\hat{x}$. The construction of $\mathcal{L}^\sharp$ follows because our growth condition (2) on Δ gives $|\Delta(x(\ell), r)| = |\Delta(\hat{x}(\ell + d) - \tilde{x}(\ell + d), r)| \le 2\mathcal{L}(|\hat{x}|_{[t-h-\tau_q-2d,t+d]} + |\tilde{x}|_{[t-h-\tau_q-2d,t+d]})z(t)$ for all $\ell \in [t - h - \tau_q - 2d, t]$, $r \ge 0$, and $t \ge h + \tau_q + 2d$, where $z(t) = (|\hat{x}|_{[t-h-\tau_q-2d,t+d]} + |\tilde{x}|_{[t-h-\tau_q-2d,t+d]})^2$

Hence, ϵ is expressible in terms of the measured quantitites y_t, $\hat{x}_t$, and u_t at each time t, with an additional error term that exponentially converges to 0 as $t \to +\infty$ if W^{-1} is bounded. See also Section III for ways to express u and W in terms of solutions of ordinary differential equations.

B. Proof of Theorem 1

The proof has three parts. First, we prove (21), and we derive the dynamics for $(\hat{x}, \tilde{x})$, whose linearization at 0 agrees with (8) and so admits the Lyapunov functional (7). In the second part, we show that $\lim_{t\to+\infty} V(\hat{x}_t, \tilde{x}_t) = 0$ from all initial states $x(0) \in \mathcal{S}_0$, provided $\sup_{\ell\in[0,h+\tau_q+2d]} V(\hat{x}_\ell, \tilde{x}_\ell) < v_*$, where v_* is from (15). In the last part, we prove that our choice of S_0 implies that $\sup_{\ell\in[0,h+\tau_q+2d]} V(\hat{x}_\ell, \tilde{x}_\ell) < v_*$ if $x(0) \in \mathcal{S}_0$.

First Part: Dynamics for $(\hat{x}, \tilde{x})$. Let $\rho(t) = Bu(t) + \Delta(x(t), t)$. Then, for each time $t \ge h + d$ when the solution of (1) is defined, we can prove that

$$\begin{aligned} y(t) &= CE^{-1}\textstyle\int_{t-h}^t e^{A^\top(s-t)}C^\top y(s)\mathrm{d}s + \eta(t)\epsilon \\ &+ CE^{-1}\textstyle\int_{t-h}^t e^{A^\top(s-t)}C_T\int_{s-d}^{t-d} e^{A(s-d-m)}\rho(m)\mathrm{d}m\mathrm{d}s. \end{aligned} \quad (24)$$

where $C_T = C^\top C$ and $\eta(t) = [\eta_1(t), \ldots, \eta_q(t)]$ with each η_i defined in (4). To see why (24) holds, first note that we can apply the method of variation of parameters to (1) to get

$$\begin{aligned} e^{A(s-t)}x(t-d) &= x(s-d) + \textstyle\int_{s-d}^{t-d} e^{A(s-d-\ell)}B(\gamma(\ell)\epsilon \\ &+ u(\ell))\mathrm{d}\ell + \textstyle\int_{s-d}^{t-d} e^{A(s-d-\ell)}\Delta(x(\ell), \ell)\mathrm{d}\ell \end{aligned} \quad (25)$$

for all $t \ge s$ and $s \ge d$. Then (24) follows by left multiplying (25) through by $e^{A^\top(s-t)}C_T$, then integrating the result over $s \in [t - h, t]$, and then left multiplying the result by CE^{-1}. Thus, (24) and our choice of θ in (12) gives

$$\begin{aligned} \epsilon^\top\eta^\top(t) &= y(t) - CE^{-1}\textstyle\int_{t-h}^t e^{A^\top(s-t)}C^\top y(s)\mathrm{d}s \\ &- CE^{-1}\textstyle\int_{t-h}^t e^{A^\top(s-t)}C_T\int_{s-d}^{t-d} e^{A(s-d-m)}\rho(m)\mathrm{d}m\mathrm{d}s \end{aligned} \quad (26)$$

and so also

$$\begin{aligned} \epsilon^\top\eta^\top(t) &= \theta(y_t, u_t) \\ &- CE^{-1}\textstyle\int_{t-h}^t e^{A^\top(s-t)}C_T\int_{s-d}^{t-d} e^{A(s-d-m)}\Delta(x(m), m)\mathrm{d}m\mathrm{d}s \end{aligned}$$

for all $t \ge h + d$. Therefore, our choice of $\mathcal{G}$ in (12) and our formula for W in Assumption 2 give

$$\begin{aligned} \epsilon^\top W(t) = [&\theta(y_{t-\tau_1}, u_{t-\tau_1}) + \mathcal{G}(x_{t-\tau_1}), \ldots, \\ &\theta(y_{t-\tau_q}, u_{t-\tau_q}) + \mathcal{G}(x_{t-\tau_q})] \end{aligned} \quad (27)$$

for all $t \ge h + \tau_q + d$. Since Assumption 2 ensures that $W(t)$ is nonsingular for each $t \ge \tau_q + h + d$, (21) follows from our formulas for $\mathcal{G}^\sharp$ and $\theta^\sharp$ in (11). Hence,

$$\begin{aligned} \sum_{i=1}^q \epsilon_i\gamma_i(t) &= \gamma(t)\epsilon = \gamma(t)[\theta^\sharp(y_t, u_t)W^{-1}(t)]^\top \\ &\quad + \gamma(t)[\mathcal{G}^\sharp(x_t)W^{-1}(t)]^\top \end{aligned} \quad (28)$$

for all $t \ge h + \tau_q + d$. Inserting (28) into the x dynamics (1) and recalling our formula for Ω from (11) now gives

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \Delta(x(t), t) + B\,[\Omega(t, y_t, u_t) \\ &\quad + \gamma(t)[\mathcal{G}^\sharp(x_t)W^{-1}(t)]^\top + u(t)]\,. \end{aligned} \quad (29)$$

This motivates our choice

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + B\,[\Omega(t-d, y_{t-d}, u_{t-d}) + u(t-d)] \\ &\quad + L[C\hat{x}(t) - y(t)] \end{aligned} \quad (30)$$

of the observer. Combining (29)-(30) gives the dynamics

$$\begin{aligned} \dot{\tilde{x}}(t) &= H\tilde{x}(t) - B\gamma(t-d)[\mathcal{G}^\sharp(x_{t-d})W^{-1}(t-d)]^\top \\ &\quad - \Delta(x(t-d), t-d) \end{aligned} \quad (31)$$

for the error $\tilde{x}(t) = \hat{x}(t) - x(t - d)$ for all $t \ge h + \tau_q + 2d$. Hence, substituting (13) into (30) gives

$$\begin{cases} \dot{\hat{x}}(t) = M\hat{x}(t) + LC\tilde{x}(t) + BK[\hat{x}(t-d) - \hat{x}(t)] \\ \dot{\tilde{x}}(t) = H\tilde{x}(t) - B\gamma(t-d)[\mathcal{G}^\sharp(x_{t-d})W^{-1}(t-d)]^\top \\ \qquad\quad - \Delta(x(t-d), t-d) \end{cases} \quad (32)$$

when $t \ge h + \tau_q + 2d$. The dynamics (32) have the linear approximation (8) with $(z_1, z_2) = (\hat{x}, \tilde{x})$ which admits the Lyapunov functional V from (7). Hence, along all solutions of (32), our V from (7) satisfies $\dot{V}(t) \le -c_1 V(\hat{x}_t, \tilde{x}_t)$ for all $t \ge h + \tau_q + 2d$ when Δ is not present.

Second Part: Proof that $\lim_{t\to+\infty} V(\hat{x}_t, \tilde{x}_t) = 0$ *if* $\sup_{\ell\in[0,h+\tau_q+2d]} V(\hat{x}_\ell, \tilde{x}_\ell) < v_*$. Using the first part of the proof and recalling the lower bounds $V_0(\hat{x}, \tilde{x}) \ge \max\{\lambda_{\min}(Q_1)|\hat{x}|^2, \lambda_{\min}(Q_2)|\tilde{x}|^2\}$ and $V_0(\hat{x}, \tilde{x}) \ge \min\{\lambda_{\min}(Q_1), \lambda_{\min}(Q_2)\}|(\hat{x}, \tilde{x})|^2$, it follows from our choices of the Γ_i's in (10) that when Δ is present,

$$\begin{aligned} \dot{V}(t) &\le -c_1 V(\hat{x}_t, \tilde{x}_t) + \Gamma_1(V(\hat{x}_t, \tilde{x}_t))V(\hat{x}_t, \tilde{x}_t) \\ &+ \textstyle\int_{t-h-\tau_q-2d}^t \Gamma_2(V(\hat{x}_m, \tilde{x}_m))V(\hat{x}_m, \tilde{x}_m)\mathrm{d}m\Gamma_3^\sharp(t) \end{aligned} \quad (33)$$

along all solutions of (32) for all $t \ge h + \tau_q + 2d$, where $\Gamma_3^\sharp(t) = \Gamma_3(V(\hat{x}_t, \tilde{x}_t))$, and where (in terms of the m_0 we defined in (9)) the term $\Gamma_1(V(\hat{x}(t),\tilde{x}(t)))V(\hat{x}(t),\tilde{x}(t))$ in (33) came from the bound

$$\begin{aligned}&|2\tilde{x}(t)^\top Q_2\Delta(\hat{x}(t)-\tilde{x}(t),t-d)|\\ &\le 4|Q_2||\tilde{x}(t)|\mathcal{L}(|\tilde{x}(t)|+|\hat{x}(t)|)(|\hat{x}(t)|^2+|\tilde{x}(t)|^2) \quad (34)\\ &\le 4|Q_2|\sqrt{\tfrac{V_0(\hat{x}(t),\tilde{x}(t))}{\lambda_{\min}(Q_2)}}\mathcal{L}(V_r(\hat{x}(t),\tilde{x}(t)))\tfrac{V_0(\hat{x}(t),\tilde{x}(t))}{m_0},\end{aligned}$$

and where the first inequality in (34) used (2) and the bound $(|\hat{x}(t)|+|\tilde{x}(t)|)^2 \le 2(|\hat{x}(t)|^2+|\tilde{x}(t)|^2)$ and where

$$V_r(\hat{x}(t),\tilde{x}(t)) = \sqrt{\tfrac{V_0(\hat{x}(t),\tilde{x}(t))}{\lambda_{\min}(Q_1)}}+\sqrt{\tfrac{V_0(\hat{x}(t),\tilde{x}(t))}{\lambda_{\min}(Q_2)}}, \quad (35)$$

and where the third right side term in (33) came from

$$\begin{aligned}&2|\tilde{x}(t)||Q_2B\gamma(t-d)[W^{-1}(t-d)]^\top[\mathcal{G}^\sharp(x_{t-d})]^\top|\\ &\le Q^\sharp(t)\int_{t-h-\tau_q-2d}^t\mathcal{L}\left(V_r(\hat{x}(m),\tilde{x}(m))\right)\tfrac{V(\hat{x}_m,\tilde{x}_m)}{m_0}\mathrm{d}m,\end{aligned} \quad (36)$$

where

$$Q^\sharp(t) = 4\sqrt{\tfrac{V(\hat{x}(t),\tilde{x}(t))}{\lambda_{\min}(Q_2)}}|Q_2B\gamma W^{-1}|_{[h+\tau_q+d,\infty)}\overline{S}\sqrt{q}, \quad (37)$$

using the formula $x(t-d)=\hat{x}(t)-\tilde{x}(t)$ and (2).

Then, one can prove that if $\sup_{\ell\in[0,h+\tau_q+2d]}V(\hat{x}_\ell,\tilde{x}_\ell) < v_*$ (with $v_*>0$ satisfying (15)), then for all $t\ge h+\tau_q+2d$, $V(\hat{x}_t,\tilde{x}_t) < v_\star$, as follows. If the assertion were not true, choose the smallest $t^a > h+\tau_q+2d$ such that $V(\hat{x}_{t^a},\tilde{x}_{t^a}) = v_*$, which exists by the continuity of $V(\hat{x}_t,\tilde{x}_t)$ as a function of t. Then, since the Γ_i's are nondecreasing, (15) implies that the right side of (33) at $t=t^a$ is bounded above by $-c_1v_*+\Gamma_1(v_*)v_*+(h+\tau_q+2d)\Gamma_2(v_*)\Gamma_3(v_*)v_*<0$, contradicting the fact that $V(\hat{x}_{t^a},\tilde{x}_{t^a}) > V(\hat{x}_t,\tilde{x}_t)$ for all $t\in[h+\tau_q,t^a)$. Since the Γ_i's are nondecreasing, this and (33) give

$$\begin{aligned}\dot{V}(t) \;\le\; &-c_1V(\hat{x}_t,\tilde{x}_t)+\Gamma_1(v_*)V(\hat{x}_t,\tilde{x}_t)\\ &+\Gamma_2(v_*)\Gamma_3(v_*)\int_{t-h-\tau_q-2d}^t V(\hat{x}_m,\tilde{x}_m)\mathrm{d}m\end{aligned} \quad (38)$$

for all $t\ge h+\tau_q+2d$. Choosing any constant $p>1$ that is close enough to 1 so that

$$\Gamma_1(v_*)+p(h+\tau_q+2d)\Gamma_2(v_*)\Gamma_3(v_*) < c_1 \quad (39)$$

(which exists by (15)) implies that

$$\begin{aligned}&V^\sharp(\hat{x}_t,\tilde{x}_t) = V(\hat{x}_t,\tilde{x}_t)\\ &+\tfrac{p+1}{2}\Gamma_2(v_*)\Gamma_3(v_*)\int_{t-h-\tau_q-2d}^t\int_\ell^t V(\hat{x}_m,\tilde{x}_m)\mathrm{d}m\mathrm{d}\ell\end{aligned} \quad (40)$$

is such that

$$\begin{aligned}&\dot{V}^\sharp(t)\\ &\le -\delta_1V(\hat{x}_t,\tilde{x}_t)-\delta_2\int_{t-h-\tau_q-2d}^t\int_\ell^t V(\hat{x}_m,\tilde{x}_m)\mathrm{d}m\mathrm{d}\ell \quad (41)\\ &\le -\delta_3V^\sharp(\hat{x}_t,\tilde{x}_t)\end{aligned}$$

along all solutions of (32) for all $t\ge\tau_q+h+2d$, where

$$\begin{aligned}&\delta_1 = c_1-\Gamma_1(v_*)-\tfrac{p+1}{2}(h+\tau_q+2d)\Gamma_2(V_*)\Gamma_3(v_*),\\ &\delta_2 = \left(\tfrac{p+1}{2}-1\right)\tfrac{\Gamma_2(V_*)\Gamma_3(v_*)}{h+\tau_q+2d},\\ &\text{and } \delta_3 = \min\left\{\delta_1,\tfrac{2\delta_2}{(p+1)\Gamma_2(v_*)\Gamma_3(v_*)}\right\}.\end{aligned} \quad (42)$$

Since δ_3 is a positive constant exponential decay rate on $V^\sharp$, we get $\lim_{t\to+\infty}V(\hat{x}_t,\tilde{x}_t)\le\lim_{t\to+\infty}V^\sharp(\hat{x}_t,\tilde{x}_t)=0$.

Third Part: Proof that $\sup_{\ell\in[0,h+\tau_q+2d]}V(\hat{x}_\ell,\tilde{x}_\ell) < v_*$ *when* $x(0)\in\mathcal{S}_0$. We first prove that $V(\hat{x}_\ell,\tilde{x}_\ell)\le R^*|x(0)|^2$ for all $\ell\in[0,d]$, where R^* is the constant from (18). To this end, note that our choice (14) of our observer dynamics and our constant initial function 0 for $\hat{x}$ imply that for all $t\in[0,d]$, we have

$$\dot{\hat{x}}(t) = (M+LC-BK)\hat{x}(t)-LCx(0), \quad (43)$$

because our constantness condition on the initial functions gives $x(t-d)=x(0)$ for all $t\in[0,d]$, so we can apply the method of variation of parameters to (43) to get

$$\begin{aligned}&|\hat{x}(t)|\le c_{d2}|x(0)| \;\text{ and}\\ &|\tilde{x}(t)| = |\hat{x}(t)-x(0)|\le(1+c_{d2})|x(0)|\end{aligned} \quad (44)$$

for all $t\in[0,d]$, where c_{d2} was defined in (18). Hence, for all $t\in[0,d]$, our choice of V and our definitions of R^* and c_d from (18) give

$$\begin{aligned}&V(\hat{x}_t,\tilde{x}_t)\\ &\le\max\{\lambda_{\max}(Q_1),\lambda_{\max}(Q_2)\}(1+2d^2c_2)|(\hat{x},\tilde{x})|^2_{[0,d]}\\ &\le\max\{\lambda_{\max}(Q_1),\lambda_{\max}(Q_2)\}(1+2d^2c_2)|c_d|^2|x(0)|^2\\ &= R^*|x(0)|^2.\end{aligned} \quad (45)$$

Hence, (20) gives $V(\hat{x}_\ell,\tilde{x}_\ell) < v_*$ for all $\ell\in[0,d]$.

We now use the preceding result to prove that $\sup_{\ell\in[d,h+\tau_q+2d]}V(\hat{x}_\ell,\tilde{x}_\ell) < v_*$ when $x(0)\in\mathcal{S}_0$. First note that for values $t\ge d$, the reasoning that led to (33) (except with $\gamma\epsilon$ not replaced by the right side of (28) as it was replaced in the second part of the proof, so on this interval, we do not have Ω in the u formula in (13)) gives

$$\begin{aligned}\dot{V}(t) \;\le\; &-c_1V(\hat{x}_t,\tilde{x}_t)+\Gamma_1(V(\hat{x}_t,\tilde{x}_t))V(\hat{x}_t,\tilde{x}_t)\\ &+2|\tilde{x}^\top(t)Q_2B\gamma(t-d)\epsilon|\end{aligned} \quad (46)$$

for all $t\in[d,h+\tau_q+2d]$. Let t^b be the supremum of all times $t\in[d,h+\tau_q+2d]$ such that $V(\hat{x}_\ell,\tilde{x}_\ell) < v_*$ for all $\ell\in[d,t]$. By the preceding argument and the continuity of $V(\hat{x}_t,\tilde{x}_t)$ as a function of t, we have $t^b>d$. Suppose, for the sake of obtaining a contradiction, that $t^b < h+\tau_q+2d$. Then $V(\hat{x}_{t^b},\tilde{x}_{t^b})=v_*$, again by the continuity of $V(\hat{x}_t,\tilde{x}_t)$ as a function of t. Setting $c_1^\flat = c_1-\Gamma_1(v_*)$ (which is positive, because of (15)), we can then apply Young's inequality to the last term in (46) to check that for all $t\in[d,t^b]$, we have

$$\begin{aligned}\dot{V}(t) &\le -c_1^\flat V(\hat{x}_t,\tilde{x}_t)+2\sqrt{\tfrac{V(\hat{x}_t,\tilde{x}_t)}{\lambda_{\min}(Q_2)}}|Q_2B\gamma\epsilon|_{[0,h+\tau_q+d]}\\ &\le -\tfrac{c_1^\flat}{2}V(\hat{x}_t,\tilde{x}_t)+\tfrac{2}{c_1^\flat\lambda_{\min}(Q_2)}|Q_2B\gamma\epsilon|^2_{[0,h+\tau_q+d]}.\end{aligned} \quad (47)$$

Hence, we can multiply (47) through by the integrating factor $e^{c_1^\flat t/2}$ and then integrate the result on $[d,t]$, to get

$$\begin{aligned}V(\hat{x}_t,\tilde{x}_t) \;&\le\; e^{-c_1^\flat(t-d)/2}V(\hat{x}_d,\tilde{x}_d)\\ &\quad+\tfrac{4}{(c_1^\flat)^2\lambda_{\min}(Q_2)}|Q_2B\gamma\epsilon|^2_{[0,h+\tau_q+d]}\\ &\le\; R^*|x(0)|^2+\tfrac{4|Q_2B\gamma\epsilon|^2_{[0,h+\tau_q+d]}}{(c_1^\flat)^2\lambda_{\min}(Q_2)}\\ &<\; v_*\end{aligned} \quad (48)$$

for all $t\in[d,t^b]$, where the last inequality used our definition (20) of the set $\mathcal{S}_0$. Evaluating (48) at $t=t^b$ then gives the contradiction $V(\hat{x}_{t^b}, \tilde{x}_{t^b}) < v_*$. This proves the theorem.

III. Implementing the Control

We can express our control (13) in terms of dynamic extensions that eliminate integrals from the control expression. This makes our method more amenable to implementations. To see how, first note that for all $t \geq h$, our formula for θ in (12) gives

$$\begin{aligned}&\theta(y_t, u_t) = y(t) - CE^{-1}[L_1(t) - e^{-Ah}L_1(t-h)]\\&-CE^{-1}\int_{t-h}^{t}\mathcal{A}(s,t)\int_{s-d}^{t-d}e^{A(t-d-m)}Bu(m)\mathrm{d}m\mathrm{d}s\end{aligned} \tag{49}$$

and so also

$$\begin{aligned}&\theta(y_t, u_t) = y(t) - CE^{-1}[L_1(t) - e^{-Ah}L_1(t-h)]\\&-CE^{-1}\int_{t-h}^{t}\mathcal{A}(s,t)[L_2(t-d) - e^{A(t-s)}L_2(s-d)]\mathrm{d}s\end{aligned} \tag{50}$$

hence

$$\begin{aligned}&\theta(y_t, u_t) = y(t) - CE^{-1}[L_1(t) - e^{-Ah}L_1(t-h)]\\&-CL_2(t-d)+CE^{-1}\int_{t-h}^{t}e^{A^\top(s-t)}C_T L_2(s-d)]\mathrm{d}s\end{aligned} \tag{51}$$

which gives

$$\begin{aligned}&\theta(y_t, u_t) = y(t) - CE^{-1}[L_1(t) - e^{-Ah}L_1(t-h)]\\&-CL_2(t-d) + CE^{-1}[L_3(t) - e^{A^\top h}L_3(t-h)],\end{aligned} \tag{52}$$

where $\mathcal{A}(s,t) = e^{A^\top(s-t)}C_T e^{A(s-t)}$ and $C_T = C^\top C$ as before, and where L_1, L_2, and L_3 are the states of

$$\begin{cases}\dot{L}_1(t) &= -A^\top L_1(t) + C^\top y(t)\\ \dot{L}_2(t) &= AL_2(t) + Bu(t)\\ \dot{L}_3(t) &= -A^\top L_3(t) + C_T L_2(t-d)\end{cases} \tag{53}$$

with the initial states $L_1(s) = L_2(s) = L_3(s) = 0$ for all $s \leq 0$, and where the (49), (50), and (52) followed from applying variations of parameters to the L_1, L_2, and L_3 dynamics, respectively. This allows us to express the control values as

$$\begin{aligned}&u(t) = K\hat{x}(t) - \gamma(t)[W^{-1}(t)]^\top[\theta(y_{t-\tau_1}, u_{t-\tau_1}),\\&\theta(y_{t-\tau_2}, u_{t-\tau_2}), \ldots, \theta(y_{t-\tau_q}, u_{t-\tau_q})]^\top\end{aligned} \tag{54}$$

for times $t \geq \tau_q + h + 2d$ in terms of the y, L_1, L_2, and L_3 values. This expression for u in terms of y, L_1, L_2, and L_3 can then be substituted into (53) to get a dynamics for L_1, L_2, and L_3 that can be solved to produce L_1, L_2, and L_3 values that allow us to write u without using integral terms.

Also, although our functions η_i from (4) (and therefore also $W(t)$ from (5)) are expressed in terms of integrals, the method from [1, Section III] makes it possible to write each η_i in terms of delayed dynamical extensions. This follows because the integral from the η_i formula can be rewritten as

$$\begin{aligned}&\int_{t-h}^{t}C^\sharp(r-t)\left[\int_{r-d}^{t-d}e^{A(t-d-m)}B\gamma_i(m)\mathrm{d}m\right]\mathrm{d}r\\&=\int_{t-h}^{t}C^\sharp(r-t)\left[L_{4i}(t-d)-e^{(t-r)A}L_{4i}(r-d)\right]\mathrm{d}r\\&=EL_{4i}(t-d) - \int_{t-h}^{t}e^{-A^\top(t-r)}C^\top CL_{4i}(r-d)\mathrm{d}r\\&=EL_{4i}(t-d) - (L_{5i}(t) - e^{-A^\top h}L_{5i}(t-h))\end{aligned} \tag{55}$$

for each i and $t \geq 0$, where $C^\sharp(r) = e^{A^\top r}C^\top Ce^{Ar}$, and where L_{4i} and L_{5i} are the states of the dynamic extension

$$\begin{cases}\dot{L}_{4i}(t) &= AL_{4i}(t) + B\gamma_i(t)\\ \dot{L}_{5i}(t) &= -A^\top L_{5i}(t) + C^\top CL_{4i}(t-d)\end{cases} \tag{56}$$

with $L_{4i}(s) = L_{5i}(s) = 0$ for all $s \leq 0$, where the first equality in (55) followed from applying variation of parameters to the L_{4i} dynamics, and the last equality in (55) followed by applying variation of parameters to the L_{5i} dynamics. This eliminates the need to compute integrals to check Assumption 2, which can then be checked by finding a positive lower bound on the absolute value of the determinant of $W(t)$ over all $t \geq \tau_q + h + d$.

IV. Illustration

Consider Duffing's equation, which contains a type of cubic nonlinearity that plays an important role in the study of jump phenomena, process control, and bistable models in aerospace engineering [20], [21]. It is the second order dynamics

$$\ddot{z} + c\dot{z} + \alpha z + \beta z^3 = u + \delta(t), \tag{57}$$

where c, α, and β are real constants, and where the locally bounded piecewise continuous added uncertainty $\delta(t)$ represents unmodelled terms or errors due to the parameter uncertainties. The special case where $u(t) = \omega_1\cos(\omega_2 t)$ for constants ω_1 and ω_2 and where δ is the zero function is called Duffing's oscillator, and can be viewed as a forced oscillator with a spring whose restoring force $F = -\alpha z - \beta z^3$ when $\alpha > 0$ [22]. Assuming that the additive uncertainty δ on the control that we must identify has the form $\delta(t) = \epsilon_1\gamma_1(t) + \ldots \epsilon_q\gamma_q(t)$, the system (57) can be written as (1) with

$$A = \begin{bmatrix}0 & 1\\ -\alpha & -c\end{bmatrix},\ B = \begin{bmatrix}0\\1\end{bmatrix},\ \Delta(x,t) = \begin{bmatrix}0\\ -\beta x_1^3\end{bmatrix}, \tag{58}$$

and $C = [0,\ 1]$. Then we can satisfy our requirements with $\mathcal{L}(s) = \beta s$, for many choices of the parameters and basis functions, which allows us to implement our control using the strategy from the preceding section. For instance, with $\alpha = -1$, $\beta = 0.01$, $c = 3$, $d = 0$, $q = h = 1$, and $\gamma_1 = 1$, and with K and L such that H and M both have the eigenvalues -2 and -2.02 (where K and L can be found, e.g., using the `StateFeedbackGains` command in Mathematica), then simple Mathematica calculations show that we can satisfy our requirements with the positive definite quadratic matrices

$$\begin{aligned}&Q_1 = \begin{bmatrix}58.0535 & 8.29372\\ 8.29372 & 25.0462\end{bmatrix} \text{ and}\\&Q_2 = \begin{bmatrix}1.12469 & 0.124378\\ 0.124378 & 0.155705\end{bmatrix}\end{aligned} \tag{59}$$

and $c_1 = 0.438442$. This differs significantly from prior treatments of (57), which did not make it possible to identify the state or unknown additive uncertainties on u.

V. Conclusion

We provided a new observer design for systems whose state-dependent nonlinearities have order 2 at 0, which produced fixed time identification of model parameters and exponential convergence of the observation error for the state to 0. This is a significant departure from prior works that did not provide fixed time parameter convergence, or did not allow order 2 nonlinearities. Our method covers examples such as Duffing's equation that have cubic terms, which play important roles in the study of jump phenomena. We also allowed measurement delays. Due to the nonlinearity having order 2, this work provides a local result (instead of the global result from [1], which was confined to systems where the state dependent nonlinearity is uniformly globally Lipschitz in the state uniformly in time), including estimates for basins of attraction. We aim to optimize the parameters in the observer to maximize the convergence rate of the observation error to zero and maximize the basin of attraction.

Appendix: Deriving V from (7)

We derive the Lyapunov-Krasovskii functional (7). We first solve the Riccati equations $P_1M + M^\top P_1 = -I$ and $P_2H + H^\top P_2 = -I$ for positive definite matrices P_1 and P_2 with the M and H in (6), and set $V_a(z_1) = z_1^\top P_1 z_1$. Along solutions of (8), Young's inequality gives $2|z_1^\top(t)P_1LCz_2(t)| \le \frac{1}{2}|z_1(t)|^2 + 2|P_1LC|^2|z_2(t)|^2$, hence

$$\begin{aligned}\dot V_a &\le -|z_1(t)|^2 + 2|z_1^\top(t)P_1LCz_2(t)| \\ &\quad +2|z_1^\top(t)P_1BK|z_1(t-d)-z_1(t)| \\ &\le -\tfrac{1}{2}|z_1(t)|^2 + 2|P_1LC|^2|z_2(t)|^2 \\ &\quad +2|z_1^\top(t)P_1BK|z_1(t-d)-z_1(t)|\end{aligned} \tag{60}$$

for all $t \ge 0$. Hence, with the choices $Q_1 = P_1$ and $Q_2 = (1+2|P_1LC|^2)P_2$, we can use the bound $|z_1(t)-z_1(t-d)| \le \int_{t-d}^t |\dot z_1(\ell)|\mathrm{d}\ell$ and the structure of the z_1 subsystem of (8) to check that our function $V_0(z) = z_1^\top Q_1 z_1 + z_2^\top Q_2 z_2$ satisfies

$$\begin{aligned}\dot V_0(t) &\le -\tfrac{1}{2}|z|^2 + 2|z_1^\top(t)P_1BK|z_1(t-d)-z_1(t)| \\ &\le -\tfrac{1}{2}|z|^2 + \{|z_1(t)|\}\left\{\mathcal{M}_*\int_{t-2d}^t |z(\ell)|\mathrm{d}\ell\right\} \\ &\le -\tfrac{1}{4}|z|^2 + dc_a\int_{t-2d}^t |z(\ell)|^2\mathrm{d}\ell\end{aligned} \tag{61}$$

along all solutions of (8) for all $t \ge 0$, where $\mathcal{M}_* = 2\sqrt{2}|P_1BK|\max\{|M| + |BK|, |LC|\}$ and where $c_a = 16|P_1BK|^2(\max\{|M| + |BK|, |LC|\})^2$ is assumed to be positive, by the relation $|z_1(\ell)| + |z_2(\ell)| \le \sqrt{2}|z(\ell)|$, and where the last inequality in (61) followed from applying Young's inequality $ab \le \frac{1}{4}a^2 + b^2$ with a and b being the terms in curly braces in (61) and then using Jensen's inequality. We now pick any constant $J \in (0,1)$ and set

$$\overline{d} = \frac{1-J}{2\sqrt{2c_a(1+J)}} \text{ and } c_2 = (1+J)\overline{d}c_a, \tag{62}$$

Then $1/4 - 2(1+J)\overline{d}^2c_a > 0$, and the formula (7) gives

$$\begin{aligned}\dot V(t) &\le -\left(\tfrac{1}{4} - 2(1+J)\overline{d}^2c_a\right)|z(t)|^2 \\ &\quad - J\overline{d}c_a\int_{t-d}^t |z(\ell)|^2\mathrm{d}\ell \\ &\le -\left\{\frac{\frac{1}{4}-2(1+J)\overline{d}^2c_a}{\max\{\lambda_{\max}(Q_1),\lambda_{\max}(Q_2)\}}\right\}V_0(z(t)) \\ &\quad -\left\{\frac{J}{2\overline{d}(1+J)}\right\}(1+J)\overline{d}c_a\int_{t-2d}^t\int_s^t |z(\ell)|^2\mathrm{d}\ell\mathrm{d}s\end{aligned} \tag{63}$$

along all solutions of (8) for all $t \ge 0$ when $d \in (0, \overline{d}]$, so we can take c_1 to be the minimum of the positive quantities in curly braces in (63).

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