Event-Triggered Control under Unknown Input and Unknown Measurement Delays using Interval Observers

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***Abstract*— We provide a new input-to-state stabilizing event-triggered feedback design for linear systems with unknown input delays, unknown measurement delays, and unknown additive disturbances. Our trigger times are computed using only the matrices defining the system and time-lagged sampled state values. We use the theory of positive systems, interval observers, and a vector version of Halanay's inequality. We illustrate our method using a marine robotic model.**

***Index Terms*— Event-triggered, delay**

I. Introduction

This paper continues our development (begun in [7], [8], [9], [10]) of event-triggered control methods for continuous time systems using interval observers (as defined in [2], [12]), positive systems, and the use of matrices of absolute values instead of Euclidean norms. Although our prior event-triggered works allowed input delays (as well as time-varying coefficients in linear systems and additive uncertainty [10]), they assume that the input delays are known in advance. Here we study more realistic systems having unknown delays in the input. We allow time-varying delays, and unknown disturbances. Although neither this work nor our prior event-triggered controlled works assume that our systems to be controlled are positive, our analysis uses new synergies of the theory of positive systems, which has been shown in our prior works to lead to less triggering than traditional approaches, and the vector Halanay's inequality approach that was presented in [6] in the absence of event triggering.

We prove input-to-state stability (or ISS), and we rule out Zeno's phenomenon. Allowing unknown input delays (as compared with our prior chain predictor based work [7] that required exact knowledge of the input delays to design the control and constantness of the delays), and trigger rules that only require sampled delayed state measurements with unknown measurement delays, make this work novel and significant. These innovations are made possible by our new event trigger rule and new Hurwitzness condition which have never appeared before, and which (as we illustrate below) let us achieve ISS under realistic bounds on the unknown time-varying input and measurement delays. This also contrasts with other notable event-triggered works for nonlinear systems, such as [1], [4], [11] (where the delays are known) and [14], [15] (where the measurement delays were only allowed to capture the effects of sampling). Since event-triggered controls only change values when it is essential to change them [3], they are useful for underwater robotics, owing to constraints in underwater communication. Hence, another contribution of this work is our application to a model of the BlueROV2 underwater vehicle, which is used to study corals, under uncertain input and measurement delays.

We use standard notation. The dimensions of our Euclidean spaces are arbitrary, unless we indicate otherwise. We set $\mathbb{Z}_0 = \{0, 1, 2, ...\}$. For a matrix $G = [g_{ij}] \in \mathbb{R}^{r\times s}$, we set $|G| = [|g_{ij}|]$, so the entries of $|G|$ are the absolute values of the corresponding entries g_{ij} of G. By G^+, we denote the matrix whose entries are $\max\{0, g_{ij}\}$ in the ith row and jth column for each i and j, and $G^- = G^+ - G$. Note for later use that $|G| = G^+ + G^-$. For matrices $D = [d_{ij}]$ and $E = [e_{ij}]$ of the same size, we write $D < E$ (resp., $D \le E$) provided $d_{ij} < e_{ij}$ (resp., $d_{ij} \le e_{ij}$) for all i and j, 0 denotes the zero matrix, and I denotes the identity matrix. A matrix $M \in \mathbb{R}^{n\times n}$ is called Metzler provided all of its off-diagonal elements are nonnegative. Given $M \in \mathbb{R}^{n\times n}$, let D_M denote the diagonal matrix such that all main diagonal entries of $M - D_M$ are equal to zero, $R_M = D_M + (M - D_M)^+$, and $N_M = (M - D_M)^-$. For matrix valued functions $M(r) = [m_{ij}(r)]$, the supremum is entrywise, i.e., $\sup_{r\in S} M(r)$ has the (i, j) entry $\sup_{r\in S} m_{ij}(r)$ for each subset S of the domain of M and all i and j when these suprema are finite. We also use the standard definitions of ISS (which we also use to denote input-to-state stable) and comparison functions $\mathcal{KL}$ and $\mathcal{K}_\infty$, e.g., from [13]. We use $||\cdot||$ to denote the usual Euclidean norm and corresponding matrix norm, and $||\cdot||_J$ denotes the supremum in this norm over any interval J.

II. Studied System

We first consider the general class of linear systems

$$\dot{x}(t) = Ax(t) + Bu(t) + \delta(t) \tag{1}$$

with x valued in $\mathbb{R}^n$ and the input u (which will contain an unknown nonzero time-varying delay Δ in the sequel) valued in $\mathbb{R}^m$, and where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are known matrices, and δ is an unknown locally bounded piecewise continuous function (but see Section IV for local analogs for nonlinear systems). Our control values will have the form $u(t) = Kx(t_i - \Delta(t_i))$ for all $t \in [t_i, t_{i+1})$ and $i \in \mathbb{Z}_0$ for constant matrices $K \in \mathbb{R}^{m\times n}$ (which arises from zero-order hold and input delays on the controller side) where the t_i's admit positive constants $\underline{T}$ and $\overline{T}$ such that $\underline{T} \le t_{i+1} - t_i \le \overline{T}$ for all $i \in \mathbb{Z}_0$ where $t_0 = 0$. The t_i's will be event trigger

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times, and we will add an assumption on K. We assume that the initial functions for (1) are constant. The t_i's will be found using sampled values of the time-lagged measurements $y(t) = x(t - \lambda(t))$ of the state of (1) at triggering times t_i, whose delay λ is also unknown. Our first assumption is:

Assumption 1: There are known constants $\overline{\Delta} \geq 0$ and $\overline{\lambda} \geq 0$ and a known function $\overline{\delta} : \mathbb{R} \to [0, +\infty)^n$ such that

$$0 \leq \Delta(t) \leq \overline{\Delta}, \quad 0 \leq \lambda(t) \leq \overline{\lambda}, \quad \text{and} \quad |\delta(t)| \leq \overline{\delta}(t) \tag{2}$$

hold for all $t \geq 0$.

Fixing A, B, and K as above, we use the function

$$\Omega(r) = e^{Ar} + \int_0^r e^{A\ell} \mathrm{d}\ell BK. \tag{3}$$

Since $\Omega(0) = I$, there is a constant $\nu > 0$ such that the matrix $\Omega(r)$ is nonsingular for each $r \in [0, \nu]$. Fixing $\overline{\Delta}$, $\overline{\lambda}$, $\overline{\delta}$, and ν satisfying the preceding requirements and setting

$$\begin{aligned} &\Gamma_1 = \sup_{s\in[0,\nu]} \left|I - \Omega(s)^{-1}\right|, \\ &\Gamma_2 = (\Gamma_1 + I) \sup_{L\in[0,T]} \left|\int_0^L e^{A\ell} BK \mathrm{d}\ell\right|, \\ &\Gamma_3 = \sup_{\ell\in[0,T]} |e^{A\ell} A|, \quad \text{and} \quad \Gamma_4 = \sup_{\ell\in[0,T]} |e^{A\ell} BK| \end{aligned} \tag{4}$$

for any constant $T \geq \nu + \overline{\Delta}$, we also assume:

Assumption 2: With the above notation and the choice $H = A + BK$, the matrix $K \in \mathbb{R}^{m\times n}$ is such that the matrix

$$\begin{aligned} M_* = {} & R_H + N_H + |BK|\Gamma_1 + \max\left\{\overline{\Delta}, \overline{\lambda}\right\} \big[|BKA| \\ & + |(BK)^2| + |BK|\Gamma_2\big(|A| + |BK|\big)\big] \\ & + \overline{\lambda}|BK|\big[(I + \Gamma_1)(\Gamma_3 + \Gamma_4) + |A| + |BK|\big] \end{aligned} \tag{5}$$

is Hurwitz.

Our novel Hurwitzness condition from Assumption 2 can always be satisfied after a change of coordinates if (A, B) is controllable and if $\overline{\Delta}$, $\overline{\lambda}$, and ν are small enough. This is done by choosing K so that $A + BK$ has distinct negative eigenvalues, then using a diagonalizing change of coordinates as in [10], so A, B, and K and x are replaced by $P^{-1}AP$, $P^{-1}B$, KP, and $P^{-1}x$ respectively for an invertible matrix P. After this coordinate change, $H = R_H + N_H$ is diagonal and Hurwitz. Then, by continuity of eigenvalues as functions of the matrix entries, Assumption 2 holds when $\overline{\Delta}$, $\overline{\lambda}$, and ν are small enough; see Section V for an illustration of how Assumption 2 can allow realistic delay values.

III. Main Result

A. Control and Statement of Result

In terms of the control u that is defined by

$$u(t) = Kx(t_i - \Delta(t_i)) \text{ for all } t \in [t_i, t_{i+1}) \text{ and } i \in \mathbb{Z}_0 \tag{6}$$

and the notation from the previous section, the closed-loop system and triggering times t_i we consider are defined by:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + BKx(t_i - \Delta(t_i)) + \delta(t) \\ &\quad \text{for all } t \in [t_i, t_{i+1}) \text{ and } i \in \mathbb{Z}_0, \\ \dot{h}_i(t) &= Ah_i(t) + BKx(t_i - \lambda(t_i)) \\ &\quad \text{for all } t \geq t_i \text{ and } i \in \mathbb{Z}_0, \\ h_i(t_i) &= x(t_i - \lambda(t_i)) \text{ for all } i \in \mathbb{Z}_0, \text{ and} \\ t_{i+1} &= \sup\{b \in [t_i, t_i + T] : |h_i(b) - h_i(t_i)| \\ &\quad \leq \Gamma_1|h_i(b)|\} \text{ for all } i \in \mathbb{Z}_0, \end{aligned} \tag{7}$$

where $t_0 = 0$. Then t_1 is the supremum of all $b \geq t_0 = 0$ such that the inequality in the supremum in (7) holds with $i = 0$ or $t_1 = t_0 + T = T$ (whichever is smaller), and then the other trigger times t_i for $i \geq 2$ are inductively defined in the same way. Our supremum in (7) ensures that $t_{i+1} - t_i \leq T$ for all $i \geq 0$, so there are infinitely many times t_i. Also, only the discrete state measurements $h_i(t_i)$ are needed to find the t_i's, which are found using solutions $h_i(t)$ of the initial value problem for the h_i dynamics from the second and third equalities in (7), and T can be arbitrarily large. Hence, the triggering rule does not need to continuously monitor $x(t)$ to determine future triggering times, which is a significantly novel feature. We prove the following, whose last conclusion rules out Zeno's phenomenon:

Theorem 1: Let the system (1) satisfy Assumptions 1-2. Then (1) in closed loop with the control from (6)-(7) is ISS to the origin on $\mathbb{R}^n$. Also, $t_{i+1} - t_i \geq \nu$ for all $i \geq 0$.

Remark 1: Our theorem is new, even in the special case where there are no measurement delays, in which case we can choose $\overline{\lambda} = 0$ and then Assumption 2 is the requirement that $R_H + N_H + |BK|\Gamma_1 + \overline{\Delta}\big[|BK|\Gamma_2\big(|A| + |BK|\big) + |BKA| + |(BK)^2|\big]$ is Hurwitz. The proof of Theorem 1 builds comparison functions in the final ISS estimate, namely, $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that all solutions $x(t)$ of (7) for all initial states $x(0) \in \mathbb{R}^n$ and δ's satisfy

$$||x(t)|| \leq \beta(||x(0)||, t) + \gamma\big(||\delta||_{[0,t]}\big) \tag{8}$$

for all $t \geq 0$, with positive constants c_a and c_b such that these requirements hold with $\beta(r, t) = c_a r e^{-c_b t}$; see also Remark 2 below. Hence, all solutions $x(t)$ converge to $0 \in \mathbb{R}^n$ exponentially at rate c_b when δ is zero, with an overshoot (given by the second right side term in (8)) for nonzero δ.

B. Proof of Theorem 1

1) Ruling Out Zeno's phenomenon: We prove that $t_{i+1} - t_i \geq \nu$ for all $i \in \mathbb{Z}_0$. We proceed by contradiction.

Suppose that there were an $i \in \mathbb{Z}_0$ such that $t_{i+1} - t_i < \nu$. By applying variation of parameters to the h_i dynamics from (7), we deduce from our choice of Ω in (3) that $h_i(t) = \Omega(t - t_i)h_i(t_i)$ for all $t \geq t_i$. Since $\Omega(r)$ is invertible for each $r \in [0, \nu]$ and $t_{i+1} - t_i < \nu$, this provides a constant $\varepsilon \in (0, \nu + t_i - t_{i+1})$ such that $h_i(t_i) = \Omega(t - t_i)^{-1}h_i(t)$ for all $t \in [t_i, t_{i+1} + \varepsilon)$, because $0 \leq t_{i+1} + \varepsilon - t_i < \nu$. Then $|h_i(t_i) - h_i(t)| \leq |I - \Omega(t - t_i)^{-1}||h_i(t)| \leq \Gamma_1|h_i(t)|$ for all $t \in [t_i, t_{i+1} + \varepsilon)$. This contradicts the definition of t_{i+1}. Hence, $t_{i+1} - t_i \geq \nu$ holds for all $i \in \mathbb{Z}_0$.

2) Stability Analysis: The analysis has three parts. First, we derive a vector valued bound for $|x(t) - x(t_i)|$ at times $t \in [t_i, t_{i+1})$ for all $i \in \mathbb{Z}_0$. In the second part, we use the bound on $|x(t) - x(t_i)|$ from the first part to find a vector valued upper bound on $\dot{s}(t)$ where $s(t) = \overline{x}(t) - \underline{x}(t)$ is the difference between upper and lower vectors in an interval observer $(\overline{x}, \underline{x})$ for x. In the final part, we use the bound on $\dot{s}(t)$ from the second part and a variant of the ISS vector Halanay's inequality from [6] to obtain our ISS estimate (8).

First Part. Defining the function σ by $\sigma(t) = t_i$ when $t \in [t_i, t_{i+1})$ and $i \geq 0$, and setting $z(t) = Ax(t) + BKx(\sigma(t) -$

$\Delta(t_i)$), we observe that our choice $H = A + BK$ gives

$$\begin{aligned} \dot{x}(t) &= Hx(t) + BK[x(t_i) - x(t)] \\ &\quad + BK[x(t_i - \Delta(t_i)) - x(t_i)] + \delta(t) \\ &= Hx(t) + BK[x(t_i) - x(t)] \\ &\quad - BK \int_{(t_i - \Delta(t_i))^+}^{t_i} (z(\ell) + \delta(\ell)) \mathrm{d}\ell + \delta(t) \end{aligned} \tag{9}$$

for all $t \in [t_i, t_{i+1})$ and $i \in \mathbb{Z}_0$, where we used the constantness of the initial functions for the x dynamics. Substituting our formula for z into (9) gives

$$\begin{aligned} \dot{x}(t) &= Hx(t) + BK[x(\sigma(t)) - x(t)] \\ &\quad + R_1 \int_{(\sigma(t) - \Delta(\sigma(t)))^+}^{\sigma(t)} x(\ell) \mathrm{d}\ell \\ &\quad + R_2 \int_{(\sigma(t) - \Delta(\sigma(t)))^+}^{\sigma(t)} x(\sigma(\ell) - \Delta(\sigma(\ell))) \mathrm{d}\ell \\ &\quad + R_3 \int_{(\sigma(t) - \Delta(\sigma(t)))^+}^{\sigma(t)} \delta(\ell) \mathrm{d}\ell + \delta(t) \end{aligned} \tag{10}$$

for all $t \geq 0$ with $R_1 = -BKA$, $R_2 = -(BK)^2$ and $R_3 = -BK$. Next note that our choice of t_{i+1} from (7) gives

$$|h_i(t) - h_i(t_i)| \leq \Gamma_1 |h_i(t)| \tag{11}$$

for all $t \in [t_i, t_{i+1})$ and all $i \in \mathbb{Z}_0$. On the other hand, (7) and the Fundamental Theorem of Calculus give

$$\begin{aligned} \dot{x}(t) - \dot{h}_i(t) &= A(x(t) - h_i(t)) \\ &\quad + BK[x(t_i - \Delta(t_i)) - x(t_i - \lambda(t_i))] \\ &\quad + \delta(t) \text{ for all } t \in [t_i, t_{i+1}) \\ x(t_i) - h_i(t_i) &= \int_{(t_i - \lambda(t_i))^+}^{t_i} [Ax(\ell) + \delta(\ell)] \mathrm{d}\ell \\ &\quad + BK \int_{(t_i - \lambda(t_i))^+}^{t_i} x(\sigma(\ell) - \Delta(\sigma(\ell))) \mathrm{d}\ell \end{aligned} \tag{12}$$

for all $i \in \mathbb{Z}_0$. Since $|z(t)| \leq |Ax(t)| + |BKx(\sigma(t) - \Delta(\sigma(t)))|$, we can use the second equality of (12) to get

$$\begin{aligned} & |x(t_i - \Delta(t_i)) - x(t_i - \lambda(t_i))| \\ & \leq \int_{(t_i - \overline{\Delta}^\sharp)^+}^{t_i} [|A||x(\ell)| + |\delta(\ell)|] \mathrm{d}\ell \\ & + |BK| \int_{(t_i - \overline{\Delta}^\sharp)^+}^{t_i} |x(\sigma^\flat(\ell))| \mathrm{d}\ell, \end{aligned} \tag{13}$$

$$\begin{aligned} & |e^{(t-t_i)A}(x(t_i) - h_i(t_i))| \leq \Gamma_3 \int_{(t_i - \overline{\lambda})^+}^{t_i} |x(\ell)| \mathrm{d}\ell \\ & + \Gamma_4 \int_{(t_i - \overline{\lambda})^+}^{t_i} |x(\sigma^\flat(\ell))| \mathrm{d}\ell + \Gamma_5 \int_{(t_i - \overline{\lambda})^+}^{t_i} |\delta(\ell)| \mathrm{d}\ell, \end{aligned} \tag{14}$$

and
$$\begin{aligned} & |x(t_i) - h_i(t_i)| \leq \int_{(t_i - \overline{\lambda})^+}^{t_i} [|A||x(\ell)| + |\delta(\ell)|] \mathrm{d}\ell \\ & + |BK| \int_{(t_i - \overline{\lambda})^+}^{t_i} |x(\sigma^\flat(\ell)))| \mathrm{d}\ell \end{aligned} \tag{15}$$

for all $t \in [t_i, t_{i+1})$ and $i \in \mathbb{Z}_0$, where the function $\sigma^\flat$ is defined by $\sigma^\flat(\ell) = \sigma(\ell) - \Delta(\sigma(\ell))$ for all $\ell \geq 0$, $\overline{\Delta}^\sharp = \max\{\overline{\Delta}, \overline{\lambda}\}$, Γ_3 and Γ_4 are defined in (4), and $\Gamma_5 = \sup_{\ell \in [0,T]} |e^{A\ell}|$.

Letting $\mathcal{G}_j(t_i)$ for $j = 1, 2$ denote the right sides of (13) and (14) respectively, and applying the method of variation of parameters to the dynamics for $x - h_i$ from (12) on $[t_i, t]$, it now follows from (12), (13), and (14) that

$$\begin{aligned} & |x(t) - h_i(t)| \leq \left| \int_{t_i}^t e^{A(t-\ell)} \delta(\ell) \mathrm{d}\ell \right| \\ & + \left| \int_{t_i}^t e^{A(t-\ell)} BK \mathrm{d}\ell \right| \mathcal{G}_1(t_i) + \mathcal{G}_2(t_i), \end{aligned} \tag{16}$$

and so also

$$\begin{aligned} & |h_i(t)| \leq |x(t)| + \left| \int_{t_i}^t e^{A(t-\ell)} \delta(\ell) \mathrm{d}\ell \right| \\ & + \left| \int_{t_i}^t e^{A(t-\ell)} BK \mathrm{d}\ell \right| \mathcal{G}_1(t_i) + \mathcal{G}_2(t_i) \end{aligned} \tag{17}$$

for all $t \in [t_i, t_{i+1})$ and all $i \in \mathbb{Z}_0$, since $|h_i(t)| - |x(t)| \leq |x(t) - h_i(t)|$. Using (11) and (16) and then (17), and letting $\mathcal{G}_3(t_i)$ denote the right side of (15), we can bound each of the three right side terms in $|x(t) - x(t_i)| \leq |x(t) - h_i(t)| + |h_i(t) - h_i(t_i)| + |h_i(t_i) - x(t_i)|$ (which is obtained by applying the triangle inequality twice) separately to get

$$\begin{aligned} |x(t) - x(t_i)| &\leq \Gamma_1 |h_i(t)| + \int_{t_i}^t e^{|A|(t-\ell)} \overline{\delta}(\ell) \mathrm{d}\ell \\ &\quad + \left| \int_{t_i}^t e^{A(t-\ell)} BK \mathrm{d}\ell \right| \mathcal{G}_1(t_i) \\ &\quad + \mathcal{G}_2(t_i) + \mathcal{G}_3(t_i) \\ &\leq \Gamma_1 |x(t)| + \Gamma_1 \int_{t_i}^t e^{|A|(t-\ell)} \overline{\delta}(\ell) \mathrm{d}\ell \\ &\quad + \Gamma_2 \mathcal{G}_1(t_i) + \int_{t_i}^t e^{|A|(t-\ell)} \overline{\delta}(\ell) \mathrm{d}\ell \\ &\quad + (I + \Gamma_1) \mathcal{G}_2(t_i) + \mathcal{G}_3(t_i) \end{aligned} \tag{18}$$

for all $t \in [t_i, t_{i+1})$ and $i \in \mathbb{Z}_0$, so (14)-(15) give

$$\begin{aligned} |x(t) - x(t_i)| &\leq \Gamma_1 |x(t)| + \Gamma_2 |A| \int_{(t_i - \overline{\Delta}^\sharp)^+}^{t_i} |x(\ell)| \mathrm{d}\ell \\ &\quad + \Gamma_2 |BK| \int_{(t_i - \overline{\Delta}^\sharp)^+}^{t_i} |x(\sigma^\flat(\ell))| \mathrm{d}\ell \\ &\quad + \Gamma_2 \int_{(t_i - \overline{\Delta}^\sharp)^+}^{t} \overline{\delta}(\ell) \mathrm{d}\ell \\ &\quad + (\Gamma_1 + I) \int_{t_i}^t e^{|A|(t-\ell)} \overline{\delta}(\ell) \mathrm{d}\ell \\ &\quad + (\Gamma_3^\sharp + |A|) \int_{(t_i - \overline{\lambda})^+}^{t_i} |x(\ell)| \mathrm{d}\ell \\ &\quad + (\Gamma_4^\sharp + |BK|) \int_{(t_i - \overline{\lambda})^+}^{t_i} |x(\sigma^\flat(\ell))| \mathrm{d}\ell \\ &\quad + (I + \Gamma_5^\sharp) \int_{(t_i - \overline{\lambda})^+}^{t} \overline{\delta}(\ell) \mathrm{d}\ell, \end{aligned} \tag{19}$$

where $\Gamma_j^\sharp = (I + \Gamma_1)\Gamma_j$ for $j = 3, 4, 5$. Setting $\Gamma_6 = \Gamma_2 |BK|$ and $\Gamma_7 = (\Gamma_1 + I)e^{T|A|} + \Gamma_2$, this provides the bounds

$$\begin{aligned} |x(t) - x(t_i)| &\leq \Gamma_1 |x(t)| + \Gamma_2 |A| \int_{(t_i - \overline{\Delta}^\sharp)^+}^{t_i} |x(\ell)| \mathrm{d}\ell \\ &\quad + \Gamma_6 \int_{(t_i - \overline{\Delta}^\sharp)^+}^{t_i} |x(\sigma^\flat(\ell))| \mathrm{d}\ell \\ &\quad + \Gamma_7 \int_{(t_i - \overline{\Delta}^\sharp)^+}^{t} \overline{\delta}(\ell) \mathrm{d}\ell \\ &\quad + (\Gamma_3^\sharp + |A|) \int_{(t_i - \overline{\lambda})^+}^{t_i} |x(\ell)| \mathrm{d}\ell \\ &\quad + (\Gamma_4^\sharp + |BK|) \int_{(t_i - \overline{\lambda})^+}^{t_i} |x(\sigma^\flat(\ell))| \mathrm{d}\ell \\ &\quad + (I + \Gamma_5^\sharp) \int_{(t_i - \overline{\lambda})^+}^{t} \overline{\delta}(\ell) \mathrm{d}\ell. \end{aligned} \tag{20}$$

Second Part. We next introduce the interval observer

$$\left\{ \begin{aligned} \dot{\overline{x}}(t) &= R_H \overline{x}(t) - N_H \underline{x}(t) \\ &\quad + (BK[x(\sigma(t)) - x(t)])^+ \\ &\quad + \int_{(\sigma^\flat(t))^+}^{\sigma(t)} (R_1 x(\ell))^+ \mathrm{d}\ell \\ &\quad + \int_{(\sigma^\flat(t))^+}^{\sigma(t)} (R_2 x(\sigma^\flat(\ell)))^+ \mathrm{d}\ell \\ &\quad + \int_{(\sigma^\flat(t))^+}^{\sigma(t)} (R_3 \delta(\ell))^+ \mathrm{d}\ell + \delta(t)^+ \\ \dot{\underline{x}}(t) &= R_H \underline{x}(t) - N_H \overline{x}(t) \\ &\quad - (BK[x(\sigma(t)) - x(t)])^- \\ &\quad - \int_{(\sigma^\flat(t))^+}^{\sigma(t)} (R_1 x(\ell))^- \mathrm{d}\ell \\ &\quad - \int_{(\sigma^\flat(t))^+}^{\sigma(t)} (R_2 x(\sigma^\flat(\ell)))^- \mathrm{d}\ell \\ &\quad - \int_{(\sigma^\flat(t))^+}^{\sigma(t)} (R_3 \delta(\ell))^- \mathrm{d}\ell - \delta(t)^- \end{aligned} \right. \tag{21}$$

which is defined for all $t \geq 0$. Since the matrix

$$\begin{bmatrix} R_H & N_H \\ N_H & R_H \end{bmatrix} \in \mathbb{R}^{2n \times 2n} \tag{22}$$

is Metzler, formula (10) for the x dynamics, (21), and the formula $Hx(t) = R_H x(t) - N_H x(t)$ imply that if

$$\underline{x}(0) \le x(0) \le \overline{x}(0), \ \ \underline{x}(0) \le 0, \ \text{ and } \ 0 \le \overline{x}(0), \tag{23}$$

then

$$\underline{x}(t) \le x(t) \le \overline{x}(t), \ \ \underline{x}(t) \le 0, \ \text{ and } \ 0 \le \overline{x}(t) \tag{24}$$

hold for all $t \ge 0$, e.g., by applying [5, Lemma 1] to the dynamics for $(\overline{x}, -\underline{x})$, and the dynamics for $(\overline{x} - x, x - \underline{x})$.

Now, let us consider initial conditions such that (23) are satisfied and let $s(t) = \overline{x}(t) - \underline{x}(t)$. Then (21) gives

$$\begin{aligned} \dot{s}(t) &= (R_H + N_H)s(t) + |BK[x(\sigma(t)) - x(t)]| \\ &\quad + \int_{(\sigma^\flat(t))^+}^{\sigma(t)} |R_1 x(\ell)|\mathrm{d}\ell \\ &\quad + \int_{(\sigma^\flat(t))^+}^{\sigma(t)} |R_2 x(\sigma^\flat(\ell))|\mathrm{d}\ell \\ &\quad + \int_{(\sigma^\flat(t))^+}^{\sigma(t)} |R_3\delta(\ell)|\mathrm{d}\ell + |\delta(t)| \\ &\le (R_H + N_H)s(t) + |BK[x(\sigma(t)) - x(t)]| \\ &\quad + |R_1| \int_{(\sigma(t)-\overline{\Delta})^+}^{\sigma(t)} |x(\ell)|\mathrm{d}\ell \\ &\quad + |R_2| \int_{(\sigma(t)-\overline{\Delta})^+}^{\sigma(t)} |x(\sigma^\flat(\ell))|\mathrm{d}\ell \\ &\quad + \int_{(\sigma(t)-\overline{\Delta})^+}^{\sigma(t)} |R_3\delta(\ell)|\mathrm{d}\ell + |\delta(t)| \end{aligned} \tag{25}$$

for all $t \ge 0$. By using (20) to bound the second term in the upper bounds in (25), we can use (25) to obtain

$$\begin{aligned} \dot{s}(t) &\le (R_H + N_H)s(t) + |BK|\Gamma_1|x(t)| \\ &\quad + |BK|\Gamma_2|A| \int_{(\sigma(t)-\overline{\Delta}^\sharp)^+}^{\sigma(t)} |x(\ell)|\mathrm{d}\ell \\ &\quad + |BK|\Gamma_6 \int_{(\sigma(t)-\overline{\Delta}^\sharp)^+}^{\sigma(t)} |x(\sigma^\flat(\ell))|\mathrm{d}\ell \\ &\quad + \int_{(\sigma(t)-\overline{\Delta})^+}^{\sigma(t)} |R_3\delta(\ell)|\mathrm{d}\ell + |\delta(t)| \\ &\quad + |BK|\Gamma_7 \int_{(\sigma(t)-\overline{\Delta}^\sharp)^+}^{t} \overline{\delta}(\ell)\mathrm{d}\ell \\ &\quad + |R_1| \int_{(\sigma(t)-\overline{\Delta})^+}^{\sigma(t)} |x(\ell)|\mathrm{d}\ell \\ &\quad + |R_2| \int_{(\sigma(t)-\overline{\Delta})^+}^{\sigma(t)} |x(\sigma^\flat(\ell))|\mathrm{d}\ell \\ &\quad + |BK|(\Gamma_3^\sharp + |A|) \int_{(\sigma(t)-\overline{\lambda})^+}^{\sigma(t)} |x(\ell)|\mathrm{d}\ell \\ &\quad + \Gamma_* \int_{(\sigma(t)-\overline{\lambda})^+}^{\sigma(t)} |x(\sigma^\flat(\ell))|\mathrm{d}\ell \\ &\quad + |BK|(I + \Gamma_5^\sharp) \int_{(\sigma(t)-\overline{\lambda})^+}^{t} \overline{\delta}(\ell)\mathrm{d}\ell, \end{aligned} \tag{26}$$

where $\Gamma_* = |BK|(\Gamma_4^\sharp + |BK|)$.

Since (24) gives $\underline{x}(t) - \overline{x}(t) \le x(t) \le \overline{x}(t) - \underline{x}(t)$ and so also $|x(t)| \le s(t)$ for all $t \ge 0$, we can upper bound each right side term in (26) that involves x using s to get

$$\begin{aligned} \dot{s}(t) &\le (R_H + N_H + |BK|\Gamma_1)s(t) \\ &\quad + |BK|\Gamma_2|A| \int_{(\sigma(t)-\overline{\Delta}^\sharp)^+}^{\sigma(t)} s(\ell)\mathrm{d}\ell \\ &\quad + |BK|\Gamma_6 \int_{(\sigma(t)-\overline{\Delta}^\sharp)^+}^{\sigma(t)} s(\sigma^\flat(\ell))\mathrm{d}\ell \\ &\quad + \overline{\delta}(t) + |R_1| \int_{(\sigma(t)-\overline{\Delta})^+}^{\sigma(t)} s(\ell)\mathrm{d}\ell \\ &\quad + |R_2| \int_{(\sigma(t)-\overline{\Delta}^\sharp)^+}^{\sigma(t)} s(\sigma^\flat(\ell))\mathrm{d}\ell \\ &\quad + (|BK|\Gamma_7 + |R_3|) \int_{(\sigma(t)-\overline{\Delta}^\sharp)^+}^{t} \overline{\delta}(\ell)\mathrm{d}\ell \\ &\quad + |BK|(\Gamma_3^\sharp + |A|) \int_{(\sigma(t)-\overline{\lambda})^+}^{\sigma(t)} |s(\ell)|\mathrm{d}\ell \\ &\quad + \Gamma_* \int_{(\sigma(t)-\overline{\lambda})^+}^{\sigma(t)} |s(\sigma^\flat(\ell))|\mathrm{d}\ell \\ &\quad + |BK|(I + \Gamma_5^\sharp) \int_{(\sigma(t)-\overline{\lambda})^+}^{t} \overline{\delta}(\ell)\mathrm{d}\ell \end{aligned} \tag{27}$$

for all $t \ge 0$, by also assuming that the s dynamics has constant initial functions. Collecting terms in (27) gives

$$\begin{aligned} &\dot{s}(t) \le \\ &(R_H + N_H + |BK|\Gamma_1)s(t) + \Gamma_8 \sup_{\ell \in [(t-2T-2\overline{\Delta}^\sharp)^+, t]} s(\ell) \\ &+ (|BK|(\Gamma_5^\sharp + \Gamma_7 + I) + |R_3|) \int_{(\sigma(t)-\overline{\Delta}^\sharp)^+}^{t} \overline{\delta}(\ell)\mathrm{d}\ell + \overline{\delta}(t) \end{aligned} \tag{28}$$

for all $t \ge 0$, with $\Gamma_8 = \overline{\Delta}^\sharp|BK|[\Gamma_2|A| + \Gamma_6] + \overline{\Delta}^\sharp(|R_1| + |R_2|) + \overline{\lambda}|BK|(\Gamma_3^\sharp + \Gamma_4^\sharp + |A| + |BK|)$.

Third Part. We apply [6, Theorem 1] to the function $w(t) = s(t)$ and $\tau = 2(T + \overline{\Delta}^\sharp)$, after extending s to the domain $[-\tau, +\infty)$ by defining s to be constant on $[-\tau, 0]$, so we use a piecewise C^1 function w instead of assuming that w is C^1 as was assumed in [6]; this is valid because the proof of [6, Theorem 1] remains valid if the C^1 requirement on w in [6] is relaxed to only requiring it to be piecewise C^1. This ensures that s satisfies an ISS property when

$$R_H + N_H + |BK|\Gamma_1 + \Gamma_8 \tag{29}$$

is Hurwitz; this follows by applying [6, Theorem 1] with $P = D + R_H + N_H + |BK|\Gamma_1 + \Gamma_8$ for a diagonal D whose main diagonal entries are large enough positive constants, by enlarging P as needed to ensure that $P > 0$ without relabelling. By our choices of the Γ_i's, it follows from Assumption 2 that (29) is Hurwitz. Hence, choosing any constant $h > 0$ in [6, Theorem 1] and [6, Remark 1] provides $\beta_0 \in \mathcal{KL}$ and $\gamma_0 \in \mathcal{K}_\infty$ such that

$$||s(t)|| \le \beta_0(||s||_{[0,h]}, t) + \gamma_0(||\delta||_{[0,t]}) \tag{30}$$

for all $t \ge 0$ for all solutions of the $s(t)$ dynamics from (25). The remaining part of the proof converts the estimate (30) to an ISS estimate for the state variable $x(t)$, as follows.

Note that (7) provides the constant $\bar{c}_1 = \max\{1, ||A|| + ||BK||\}$ such that $||\dot{x}(t)|| \le \bar{c}_1(||x||_{[t-T-\overline{\Delta},t]} + ||\delta||_{[0,t]})$ for all $t \ge 0$. Hence, the fundamental theorem of calculus gives

$$||x(t)|| \le ||x(0)|| + \bar{c}_1 \int_0^t ||x||_{[\ell-T-\overline{\Delta},\ell]}\mathrm{d}\ell + \bar{c}_1 t||\delta||_{[0,t]} \tag{31}$$

for all $t \ge 0$. Recalling our assumption that the initial functions are constant, it follows from (31) that the function $\mathcal{S}(t) = ||x||_{[t-T-\overline{\Delta},t]}$ satisfies

$$\mathcal{S}(t) \le ||x(0)|| + \bar{c}_1 \int_0^t \mathcal{S}(\ell)\mathrm{d}\ell + \bar{c}_1 t||\delta||_{[0,t]} \tag{32}$$

for all $t \ge 0$. Hence, we can apply Gronwall's inequality to $\mathcal{S}$ to check that the constant $\bar{c}_2 = \max\{1, \bar{c}_1 h\}e^{\bar{c}_1 h}$ is such that $||x(t)|| \le \mathcal{S}(t) \le \bar{c}_2(||x(0)|| + ||\delta||_{[0,t]})$ for all $t \in [0, h]$. This allows us to use the s dynamics from the equality in (25) and to then apply Gronwall's inequality to the function $||s(t)||$ to get the constant $\bar{c}_3 = \max\{1, h\}(2\bar{c}_2||BK|| + \bar{c}_2(||R_1|| + ||R_2||)\overline{\Delta} + \overline{\Delta}||R_3|| + 1)e^{||R_H+N_H||h}$ such that $||s(t)|| \le \bar{c}_3(||s(0)|| + ||x(0)|| + ||\delta||_{[0,t]})$ for all $t \in [0, h]$. Since we can assume that $\bar{x}(0) \le (1 + h)|x(0)|$ and $\underline{x}(0) \ge -(1 + h)|x(0)|$, it follows that we can then use the formula $s = \overline{x} - \underline{x}$ to get $||s(0)|| \le 2(1 + h)||x(0)||$, and so also $||s(t)|| \le \bar{c}_3\big[(3 + 2h)||x(0)|| + ||\delta||_{[0,t]}\big]$ for all $t \in [0, h]$. Hence,

$$||s||_{[0,h]} \le \bar{c}_3\big[(3 + 2h)||x(0)|| + ||\delta||_{[0,h]}\big]. \tag{33}$$

Since $|x(t)| \leq s(t)$ holds for all $t \geq 0$, and since the values $x(t)$ do not depend on $\delta(\ell)$ values for times $\ell \geq t$, we can substitute (33) into (30) and then use the fact that $\beta_0(a+b,t) \leq \beta_0(2a,t) + \beta_0(2b,0)$ holds for all $a \geq 0$, $b \geq 0$ and $t \geq 0$, to satisfy the ISS requirement with $\beta(r,t) = \beta_0(2(3+2h)\bar{c}_3 r, t)$ and $\gamma(s) = \gamma_0(s) + \beta_0(2\bar{c}_3 s, 0)$.

Remark 2: In terms of diagonal matrices D whose main diagonal entries are all positive, matrices $P > 0$, constants $h > 0$ and $\tau > 0$, and a constant $c_0 \in (0,1)$ and $U_0 \in [1,+\infty)^n$ such that $M_0 U_0 = c_0 U_0$ where $M_0 = e^{-Dh} + \int_{-h}^{0} e^{D\ell} \mathrm{d}\ell P$, the β_0 from [6, Theorem 1] and [6, Remark 1] is $\beta_0(r,t) = n||U_0|| e^{(t-h)\ln(c_0)/(\tau+h)} r$. The existence of c_0 and U_0 follows because $M_0 > 0$ is Schur stable [6]. This and the last part of the proof of Theorem 1 ensure the exponential stability condition from Remark 1. We next use the preceding β_0 to obtain local analogs for nonlinear systems.

IV. Local Result for Nonlinear Systems

We provide local exponential stability analogs for

$$\dot{x}(t) = Ax(t) + Bu(t) + \Phi(x(t)) \tag{34}$$

where Φ satisfies the following (but similar arguments allow additive uncertainty $\delta(t)$ in (34) with suitable bounds on $\bar{\delta}$):

Assumption 3: Each component Φ_i of Φ is continuous and satisfies $\lim_{z\to 0} \Phi_i(z)/||z|| = 0$ for $i = 1, \ldots, n$.

The preceding assumption is very general, because we can rewrite any C^1 nonlinear function $G : \mathbb{R}^n \to \mathbb{R}^n$ that satisfies $G(0) = 0$ in the form $G(x) = G_0 x + \Phi(x)$, where Φ is the remainder term in the first order Maclaurin approximation of G (which will be of second order at the origin and so satisfy Assumption 3), and then combine the constant matrices A and G_0 to get one constant coefficient of $x(t)$ when the given system has the form $\dot{x}(t) = Ax(t) + Bu(t) + G(x(t))$ for such a function G. Assumption 3 covers cases where each component Φ_i grows quadratically near the origin, and we assume that (34) satisfies the standard uniqueness of solutions property, as in Section II. We continue our other notation from the preceding sections, where $\overline{\Delta}^\sharp = \max\{\overline{\Delta}, \overline{\lambda}\}$ and $\Gamma_5 = \sup_{\ell\in[0,T]} |e^{A\ell}|$, and we use the following matrix:

$$B_* = I + (T + \overline{\Delta}^\sharp)|BK|\left[(1+\Gamma_1)(\Gamma_5 + e^{T|A|}) + \Gamma_2 + 2I\right] \tag{35}$$

Also, like in Theorem 1, our closed loop system will use (6), whose trigger times t_i are defined by $t_0 = 0$,

$$\begin{array}{rcl} \dot{h}_i(t) & = & Ah_i(t) + BKx(t_i - \lambda(t_i)) \\ & & \text{for all } t \geq t_i \text{ and } i \in \mathbb{Z}_0, \\ h_i(t_i) & = & x(t_i - \lambda(t_i)) \text{ for all } i \in \mathbb{Z}_0, \text{ and} \\ t_{i+1} & = & \sup\{b \in [t_i, t_i + T] : |h_i(b) - h_i(t_i)| \\ & & \leq \Gamma_1 |h_i(b)|\} \text{ for all } i \in \mathbb{Z}_0, \end{array} \tag{36}$$

where T and ν satisfy the requirements from Section II. With $H = A + BK$ where K satisfies Assumption 2, and letting $\mathcal{D}_c$ denote the diagonal matrix whose main diagonal entries are all c and $\mathcal{B}_c$ denote the closed 2-norm ball of radius c centered at 0 for any constant $c > 0$, and $\mathbf{1} \in \mathbb{R}^{n\times n}$ be the matrix of all 1's, we prove the following, where the existence of the ϵ such that $M_* + \epsilon\mathbf{1}$ is Hurwitz follows from continuity of eigenvalues of a matrix as functions of the matrix entries:

Theorem 2: Let H, K, M_*, and Φ satisfy Assumptions 2-3, where $\overline{\Delta}$ and $\overline{\lambda}$ are constants such that $0 \leq \Delta(t) \leq \overline{\Delta}$ and $0 \leq \lambda(t) \leq \overline{\lambda}$ hold for all $t \geq 0$, and let $d_{\max}$ be the largest main diagonal element of H. Let $\overline{d}_0$, ϵ, and $\overline{x}$ be positive values such that $||\Phi(x)|| \leq \overline{d}_0||x||$ for all $x \in \mathcal{B}_{\overline{x}}$, $\overline{d}_0 B_* \mathbf{1} \leq \epsilon\mathbf{1}$, and $M_* + \epsilon\mathbf{1}$ is Hurwitz. Choose any constants $h > 0$ and $d > \max\{0, d_{\max}\}$, and then choose the function β_0 from Remark 2 with $P = M_* + \epsilon\mathbf{1} + \mathcal{D}_d$, $D = \mathcal{D}_d$, and $\tau = 2(T + \overline{\Delta}^\sharp)$. Choose the constants

$$\begin{aligned} \overline{d}_1 &= ||A|| + ||BK|| + \overline{d}_0, \quad \overline{d}_2 = e^{\overline{d}_1 h}, \text{ and} \\ \overline{d}_3 &= \overline{d}_2\big[2||BK|| + \overline{\Delta}(||BKA|| + ||(BK)^2|| \\ &\quad + \overline{d}_0||BK||) + \overline{d}_0\big] \max\{1,h\} e^{||R_H + N_H||h}. \end{aligned} \tag{37}$$

Then, we can find positive constants c_* and r_* such that for each initial state $x(0)$ in

$$\begin{aligned} &\mathcal{S}_0 = \\ &\left\{x_0 \in \mathcal{B}_{\overline{x}} : \max\left\{\overline{d}_2||x_0||, \beta_0\big([3+2h]\overline{d}_3||x_0||, h\big)\right\} < \overline{x}\right\}, \end{aligned}$$

the solution $x : [0,+\infty) \to \mathbb{R}^n$ of (34), in closed loop with (6) and (36), satisfies $||x(t)|| \leq c_* e^{-r_* t}||x(0)||$ for all $t \geq 0$.

Proof: The proof has two parts. In the first part, we find constants $c_* > 0$ and $r_* > 0$ such that the local exponential stability estimate holds for all $t \geq 0$ and all solutions $x : [0,+\infty) \to \mathbb{R}^n$ of the closed loop system such that

$$||x||_{[0,+\infty)} \leq \overline{x} \tag{38}$$

holds. In the second part, we show that all solutions $x(t)$ of the closed loop system such that $x(0) \in \mathcal{S}_0$ satisfy (38).

First Part. We indicate the changes that are needed in the proof of Theorem 1. The argument up through (26) remains the same, except with $\delta(t)$ and $\bar{\delta}(t)$ replaced by $\Phi(x(t))$ and $|\Phi(x(t))|$, respectively. With these replacements, we collect the terms involving Φ and use the bounds $B_*|\Phi(x)| \leq \overline{d}_0 B_* \mathbf{1}|x| \leq \epsilon\mathbf{1}|x|$ (which follow from our assumptions, since $||x|| \leq |x_1| + \ldots + |x_n|$), and then the bound $|x(t)| \leq s(t)$ in the result to replace the terms involving $|x|$'s by terms that use s, to get (28) except with Γ_8 replaced by $\Gamma_8 + \epsilon\mathbf{1}$ and $\bar{\delta}$ replaced by 0, if (38) holds. Since $||\Phi(x)|| \leq \overline{d}_0||x||$ for all $x \in \mathcal{B}_{\overline{x}}$, we can then use the reasoning from the third part of the proof of Theorem 1 except with $\bar{c}_i$ replaced by $\overline{d}_i$ for $i = 1, 2, 3$ and $\delta = 0$, to get the local exponential stability estimate for those solutions $x(t)$ such that (38) holds.

Second Part. We show that (38) holds along all solutions $x(t)$ of the closed loop system such that $x(0) \in \mathcal{S}_0$. First, we show that $||x||_{[0,h]} \leq \overline{x}$. To this end, note that the first part of the proof gives $||x||_{[0,t]} \leq \overline{d}_2||x(0)||$ for all initial states $x(0)$ and $t \in [0,h]$ for which the corresponding solution $x(t)$ of the closed loop system satisfies $||x||_{[0,t]} \leq \overline{x}$. Now we argue by contradiction. If $x(0) \in \mathcal{S}_0$ were such that $x(t)$ satisfied $||x||_{[0,h]} > \overline{x}$, then choose the largest $t_* \in [0,h)$ such that $||x||_{[0,t_*]} \leq \overline{x}$. By the continuity of $||x||_{[0,t]}$ as a function of t, this gives $||x||_{[0,t_*]} = \overline{x}$. On the other hand, the preceding argument and our choice of $\mathcal{S}_0$ give $||x||_{[0,t_*]} \leq \overline{d}_2||x(0)|| < \overline{x}$, which is a contradiction. Hence, $||x||_{[0,h]} \leq \overline{x}$.

Therefore, we can apply the first part of the proof of this theorem to get $||x(t)|| \leq \beta(||x(0)||, t)$ for all initial states

$x(0) \in \mathcal{B}_{\overline{x}}$ and all $t \geq 0$ for which $||x||_{[0,t]} \leq \overline{x}$, where $\beta(r,t) = \beta_0([3+2h]\overline{d}_3 r, t)$ is from the proof of Theorem 1 except with the $\overline{c}_i$'s replaced by the $\overline{d}_i$'s and where we can use $3+2h$ instead of $2[3+2h]$ in the β formula because $\delta = 0$. Also, our choices of β and $\mathcal{S}_0$ give $\beta(||x_0||, h) = \beta_0\left([3+2h]\overline{d}_3||x_0||, h\right) < \overline{x}$ for all $x_0 \in \mathcal{S}_0$.

We next use the preceding observations to check that $||x||_{[h,+\infty)} \leq \overline{x}$. We again argue by contradiction. If $x(0) \in \mathcal{S}_0$ were such that the corresponding solution $x(t)$ satisfied $||x||_{[h,+\infty)} > \overline{x}$, then choose the largest $t_{**} \geq h$ such that $||x||_{[h,t_{**}]} \leq \overline{x}$. Again using the continuity of $||x||_{[0,t]}$, this gives $||x||_{[h,t_{**}]} = \overline{x}$. On the other hand, the preceding paragraph implies that $||x||_{[h,t_{**}]} \leq \beta(||x(0)||, h) < \overline{x}$, which is a contradiction. Hence, (38) holds for all solutions of the closed loop system having initial states $x(0) \in \mathcal{S}_0$. ■

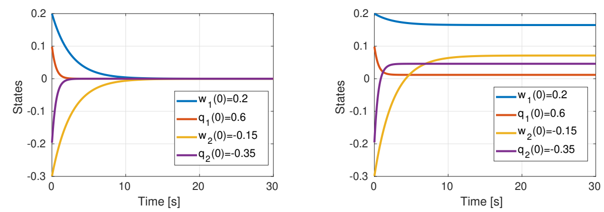
V. Illustration

Consider a dynamics for the depth and pitch degrees-of-freedom of an autonomous underwater vehicle, e.g., the BlueROV2 which is used for environmental surveys such as the study of corals. The vehicle has a Doppler Velocity Logger (or DVL) that estimates its velocity. When close to the sea floor, the DVL experiences bottom lock, which makes it impractical to continuously change the control values. Hence, we use Theorem 1 to design a control system for the depth plane, under input and measurement delays, where unlike in [7], the delays are time-varying and unknown. This is an innovative significant departure from event-triggered works like [7], which did not allow measurement delays, and where the chain predictors required exact knowledge of a constant input delay. Following [9], we linearize the dynamics to get (1) with the diagonal matrix $A = \text{diag}\{-0.387, -1.8\}$ and $B = [0.038, 1.5]^\top$. To choose K, we searched for a K that yielded the largest allowable $\overline{\Delta}$ and $\overline{\lambda}$ subject to the constraint that Assumption 2 holds with $T = 1$.

This produced $K = [-0.0085, -0.054]$, which enabled us to satisfy Assumptions 1-2 with the lower bound $\nu = 0.16$ on the intersample times, the bound $\overline{\lambda} = 2.86$ on λ, and the bound $\overline{\Delta} = 0.64$ on Δ. To validate the efficacy of our method, we used MATLAB to simulate the closed loop system from Theorem 1 with the preceding parameter values, and with random delays that took their values on the allowable intervals $\lambda(t) \in [0,1]$ and $\Delta(t) \in [0, 0.64]$. We report the results in Fig. 1, where (w,q) represent the depth and pitch components of the state respectively, which have subscripts 1 and 2 to indicate two initial state vectors $(0.2, 0.6)$ and $(-0.15, -0.35)$ in our two simulations in each panel. In Fig. 1(b), the uncertainty had the constant components $\delta_i(t) = 0.1$ for $i = 1, 2$. Since our simulations exhibit ISS under realistic delays, they illustrate our result.

VI. Conclusion

Our new global input-to-state stability theorem for event-triggered linear systems allowed uncertain input delays and uncertain measurement delays, and used new synergies of interval observers, positive systems, and an ISS vector Halanay's inequality approach. Our trigger times are computed



(a) (b)

Fig. 1: Simulations (a) without and (b) with uncertainty

from time-lagged sampled state measurements with unknown measurement delays. We also provided a basin of attraction estimate for a local analog that allows nonlinearities. We aim to develop methods to tune the control parameters to maximize the convergence rate, minimize overshoots, and maximize the basin of attraction in the nonlinear case. We also aim to use marine robotic experiments in our LSU lab to further compare the performance of our event-triggered controls with other possible solutions, such as adjustable time-delay processes based on time-delay tools that solve the event-triggered control problem for systems with delays.

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