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Abstract: We provide a local version of an approach to proving asymptotic stability based on Halanay's inequality. Our result is amenable to nonlinear systems that contain input and state delays. It provides robustness estimates for systems containing actuator uncertainty, in the sense of input-to-state stability. Our numerical example illustrates how our method leads to useful bounds on the allowable uncertainties and on the basin of attraction.

Keywords: Stability of delay systems, input-to-state stability, time-delay systems, nonlinear time-delay systems, robustness analysis

1. INTRODUCTION

Hanalay's inequality was introduced in the celebrated work Halanay (1966), and has since been generalized in multiple ways that have played important roles in proofs of stability properties for significant classes of nonlinear systems that contain input and state delays. Some notable contributions in this directions have included Grifa and Pepe (2021), Mazenc et al. (2022), and Pepe (2022), which cover both continuous-time and discrete-time systems. In its most basic form (as presented, e.g., in (Fridman, 2014, Lemma 4.2, p. 138)), Halanay's inequality requires finding nonnegative valued differentiable functions v and constants $a > 0$, $b \in (0, a)$, and $T > 0$ that satisfy

$$\dot{v}(t) \leq -av(t) + b \sup_{\ell \in [t-T,t]} v(\ell) \tag{1}$$

for all $t \geq T$, and then concludes that $v(t)$ exponentially converges to 0 as $t \to +\infty$. Generalizations include Grifa and Pepe (2021); Ruan et al. (2020), where instead of assuming that a and b are constants, the a and b are allowed to depend on the time variable t, including situations where $b(t) > a(t)$ for some choices of t, and these works provide advantages of using Halanay's inequality approaches in stability analyses instead of using standard Lyapunov function approaches. However, the preceding works are only directly applicable to globally exponentially stable systems. This produces an obstacle to stability analysis, because nonlinear systems are often only locally exponentially stable, and in such situations, one cannot use global Halanay's results to prove global stability results. In addition, to the best of the authors' knowledge, there is no general result in the literature that ensures that a nonlinear delayed system whose linear approximation at the origin is exponentially stabilizable enjoys a locally exponentially stability property. More generally, local stability or stabilization of delayed systems is an under-studied topic, which strongly motivates the present work

We prove a local version of the Halanay's inequality based stability result that applies to functions satisfying a nonlinear differential inequality, in a local sense that is amenable to significant applications. We apply it to systems containing small bounded disturbances that can represent actuator uncertainty, leading to our proof of input-to-state-stability (or ISS) inequalities. The results are amenable to nonlinear systems that contain disturbances and delays, and provide estimates of corresponding basins of attraction. We begin by stating and proving our local Halanay's inequality result in Section 2, which we use to prove a local exponential stabilization theorem for systems with state feedback in Section 3. For greater generality, we cover nonlinear systems with time-varying and distributed delays, and we illustrate our findings in Section 4 using van der Pol's equation, whose structure precludes using earlier reported methods to prove global stabilization results, but which are covered by our local Halanay's inequality approach. This provides significantly new estimates for basins of attractions, and sufficient conditions on bounds for the uncertainties for our local stabilization estimates to hold.

We use standard notation which is simplified when no confusion would arise given the context. The dimensions of our Euclidean spaces are arbitrary, unless we indicate otherwise, and $|\cdot|$ is the usual Euclidean vector norm and also denotes the corresponding matrix operator norm. Given matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$, the notation $A \leq B$ means that $B - A$ is nonnegative definite, and I is the identity matrix. When r is a time variable, we use the usual notation $g_r(\ell) = g(r + \ell)$ for functions g and all $\ell \leq 0$ and $r \geq 0$ for which $r + \ell$ is in the domain of g. We

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also use the usual family of functions $\mathcal{K}_\infty$ and the usual definitions of ISS Khalil (2002); Sontag (2001).

2. LOCAL ISS HANALAY'S RESULTS

Let $\alpha : [0, +\infty) \to [0, +\infty)$ be a continuous, nonnegative valued and nondecreasing function such that there are two constants $v_\star > 0$ and $a > 0$ such that

$$\alpha(v_\star) = a. \tag{2}$$

Let $t_\star \geq 0$ and $b > 0$ be two constants. Let $\zeta : [t_\star, +\infty) \to [0, +\infty)$ be a nondecreasing function such that

$$\zeta(t) < bv_\star \tag{3}$$

holds for all $t \geq t_\star$. Let $\mathcal{L}_0 \geq 0$ and $\tau \in (t_\star, +\infty)$ or $\tau = +\infty$. Throughout this section, we let $V : [t_\star - \mathcal{L}_0, \tau) \to [0, +\infty)$ be a C^1 function such that

$$\sup_{m\in[t_\star-\mathcal{L}_0,t_\star]} V(m) < v_\star \tag{4}$$

and such that the inequality

$$\begin{aligned} \dot{V}(t) \leq& -(a+b)V(t) \\ &+ \alpha\left(\sup_{m\in[t-\mathcal{L}_0,t]} V(m)\right) \sup_{m\in[t-\mathcal{L}_0,t]} V(m) + \zeta(t) \end{aligned} \tag{5}$$

holds for all $t \in [t_\star, \tau)$. We use this technical result:

Lemma 1. The inequality $V(t) < v_\star$ holds for all $t \in [t_\star - \mathcal{L}_0, \tau)$.

Proof. We prove this result by contradiction. Suppose that there were a $t_c \in [t_\star - \mathcal{L}_0, \tau)$ such that $V(t_c) = v_\star$ and $V(t) < v_\star$ for all $t \in [t_\star - \mathcal{L}_0, t_c)$. Then (4) gives $t_c > t_\star$, and (2), (3), and (5) give

$$\begin{aligned} \dot{V}(t_c) &< -(a+b)v_\star + \alpha(v_\star)\, v_\star + bv_\star \\ &= -av_\star + av_\star = 0. \end{aligned} \tag{6}$$

From the inequality $\dot{V}(t_c) < 0$ and the continuity of V, we can find a $t_d \in (t_\star, t_c)$ such that $V(t_d) > v_\star$. This contradicts the definition of t_c, so the lemma holds. □

Since $a > 0$ and $b > 0$, there is a unique $\lambda > 0$ such that

$$\lambda = a + b - ae^{\lambda\mathcal{L}_0}. \tag{7}$$

Using this λ, we next state and prove:

Theorem 1. Let V satisfy the requirements of Lemma 1 in the case where $\tau = +\infty$. Then

$$V(t) \leq \sup_{m\in[s-\mathcal{L}_0,s]} V(m)e^{-\lambda(t-s)} + \int_s^t e^{b(m-t)}\zeta(m)\mathrm{d}m \tag{8}$$

holds for all $t \geq s$ and for all $s \geq t_\star$.

Proof. Since α is nondecreasing, Lemma 1 and (5) give

$$\dot{V}(t) \leq -(a+b)V(t) + \alpha(v_\star) \sup_{m\in[t-\mathcal{L}_0,t]} V(m) + \zeta(t) \tag{9}$$

for all $t \geq t_\star$. From (2), we deduce that

$$\dot{V}(t) \leq -(a+b)V(t) + a \sup_{m\in[t-\mathcal{L}_0,t]} V(m) + \zeta(t) \tag{10}$$

holds for all $t \geq t_\star$. Since $a > 0$ and $b > 0$, we can then apply Lemma 3 in Appendix A.1 to conclude. □

Remark 1. The inequality (8) gives an ISS inequality, by upper bounding its right side integral term by $(1/b)\sup_{\ell\in[s,t]} |\zeta(\ell)|$ for all $t \geq s$ and all $s \geq t_\star$.

3. LOCAL EXPONENTIAL STABILIZATION RESULT

We use Theorem 1 to solve a local stabilization problem for a class of nonlinear systems.

3.1 Studied system and preliminary result

Let $h : [0, +\infty) \to [0, +\infty)$ be a piecewise continuous function for which there is a constant $\bar{h} > 0$ such that $0 \leq h(t) \leq \bar{h}$ for all $t \geq 0$. Let $\delta : [0, +\infty) \to \mathbb{R}^n$ be a continuous function that admits a constant $\bar{\Delta}$ such that

$$|\delta(t)| \leq \bar{\Delta} \tag{11}$$

for all $t \geq 0$. Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t - h(t)) + \mathcal{F}(t, x_t) + \delta(t) \tag{12}$$

where x is valued in $\mathbb{R}^n$, the input u is valued in $\mathbb{R}^p$, and $\mathcal{F}$ is a locally Lipschitz continuous function. Let $t_0 \geq 0$. We consider initial functions $x_0 : [t_0 - \bar{h}, t_0] \to \mathbb{R}^n$, and we introduce three assumptions:

Assumption 1. The pair (A, B) is controllable.

Assumption 2. There is a continuous nondecreasing function $\rho : [0, +\infty) \to [0, +\infty)$ that is not identically equal to zero such that

$$|\mathcal{F}(t, \phi)| \leq \sup_{m\in[-\bar{h},0]} |\phi(m)|^2 \rho(|\phi(m)|) \tag{13}$$

holds for all functions $\phi : [-\bar{h}, 0] \to \mathbb{R}^n$ and all $t \geq 0$.

It is well known that Assumption 1 provides a matrix $K \in \mathbb{R}^{p\times n}$ such that the matrix $H = A + BK$ is Hurwitz, and so also a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ and constants $c > 0$ and $\overline{p} > 0$ such that

$$PH + H^\top P \leq -cP \text{ and } I \leq P \leq \overline{p}I, \tag{14}$$

e.g., by using the Pole-Shifting Theorem to find K, then solving the Riccati equation $PH + H^\top P = -I$ for P, then choosing $c > 0$ small enough such that $cP \leq I$ Sontag (1998), then scaling P by a big enough positive constant if needed to satisfy our additional requirement that $P \geq I$. We fix K, P, and $\bar{p}$ satisfying the preceding requirements, and assume that $BK \neq 0$. Our last assumption is the following smallness condition on $\bar{h}$:

Assumption 3. There is an $s_\star > 0$ such that

$$\omega_0 = \left(2|A| + 2|BK| + 1 + 2\rho\left(\sqrt{s_\star}\right)\right)\overline{p} \tag{15}$$

is such that the inequality

$$\left(e^{2.1\omega_0\bar{h}} - 1\right)\frac{\overline{p}^2\bar{\Delta}^2}{\omega_0} < s_\star \tag{16}$$

is satisfied.

In terms of the preceding notation and the function

$$W(x) = x^\top Px, \tag{17}$$

we start with a technical lemma, where Assumption 3 ensures that the (18) is satisfied when the initial function is valued in a small enough neighborhood of the origin:

Lemma 2. Let (12) satisfy Assumptions 1-3. Consider (12) in closed-loop with the feedback $u(t-h(t)) = Kx(t-h(t))$. Let x be a solution of this system such that

$$\sup_{m\in[t_0-\bar{h},t_0]} W(x(m))e^{2.1\omega_0\bar{h}} + \left(e^{2.1\omega_0\bar{h}} - 1\right)\frac{\overline{p}^2\bar{\Delta}^2}{\omega_0} < s_\star. \quad (18)$$

Then x is defined over $[t_0 - \bar{h}, t_0 + 2\bar{h}]$ and

$$\begin{aligned}\sup_{m\in[t_0-\bar{h},t_0+2\bar{h}]} W(x(m)) \le & \sup_{m\in[t_0-\bar{h},t_0]} W(x(m))e^{2.1\omega_0\bar{h}} \\ & + \left(e^{2.1\omega_0\bar{h}} - 1\right)\frac{\overline{p}^2\bar{\Delta}^2}{\omega_0}\end{aligned} \quad (19)$$

is satisfied.

Proof. Consider a solution $x(t)$ of this closed-loop system from the lemma such that (18) holds. Let $[t_0 - \bar{h}, t_0 + t_\infty)$ be the domain of definition of $x(t)$. Then $0 < t_\infty < +\infty$ or $t_\infty = +\infty$. The time derivative of (17) along $x(t)$ satisfies

$$\begin{aligned}\dot{W}(t) =& \; 2x(t)^\top P[Ax(t) + BKx(t-h(t)) \\ & + \mathcal{F}(t, x_t) + \delta(t)] \\ \le& \; 2\overline{p}|x(t)| \big[|A||x(t)| + |BK||x(t-h(t))| \\ & + \sup_{m\in[t-\bar{h},t]} |x(m)|^2\rho(|x(m)|)\big] + 2|x(t)|\overline{p}\bar{\Delta}\end{aligned} \quad (20)$$

for all $t \in (t_0, t_0 + t_\infty)$, by (11), (13), and (14). Since ρ from Assumption 2 is nondecreasing, (14) gives

$$\begin{aligned}\dot{W}(t) \le & \; \overline{p}\left(2|A| + 2|BK| + 1\right) \sup_{m\in[t-\bar{h},t]} W(x(m)) \\ & + 2\overline{p}\sup_{m\in[t-\bar{h},t]} W(x(m)) \\ & \times\rho\left(\sqrt{\sup_{m\in[t-\bar{h},t]} W(x(m))}\right) + \overline{p}\bar{\Delta}^2\end{aligned} \quad (21)$$

by using the triangle inequality to upper bound the last right term in (20), to get $2|x(t)|\overline{p}\bar{\delta} \le |x(t)|^2 + \overline{p}^2\bar{\Delta}^2$. Setting $\bar{\omega}(s) = \overline{p}\left(2|A| + 2|BK| + 1\right)s + 2s\overline{p}\rho\left(\sqrt{s}\right)$, we then have

$$\dot{W}(t) \le \bar{\omega}\left(\sup_{m\in[t-\bar{h},t]} W(x(m))\right) + \overline{p}\bar{\Delta}^2 \quad (22)$$

for all $t \in (t_0, t_0 + t_\infty)$ and, according to the definition of ω_0 in (15), we have $\bar{\omega}(s) \le \omega_0 s$ for all $s \in [0, s_\star]$. We now apply Lemma 4 in Appendix A.2 with $W(x(t))$, $\bar{\omega}$, ω_0, $\overline{p}^2\bar{\Delta}^2$, t_0, t_∞, $2.1\bar{h}$, $\sup_{m\in[t_0-\bar{h},t_0]} W(x(m))$, $s_\star$, and $\bar{h}$ as the choice of $Z(t)$, Ψ, Ψ_0, $\bar{\Delta}$, t_a, τ, q, $\bar{Z}$, ω and T in the lemma, respectively. Assumption 3 ensures that Assumption A.1 from Appendix A.2 (which is needed to apply Lemma 4) holds. Also, (A.19) holds, since $W(x(t)) \le \sup_{m\in[t_0-\bar{h},t_0]} W(x(m))$ for all $t \in [t_0 - \bar{h}, t_0]$. Then (18) ensures that (A.20) holds. Hence, Lemma 4 implies that for all $t \in [t_0 - \bar{h}, t_0 + \min\{t_\infty, 2.1\bar{h}\})$, we have

$$\begin{aligned}W(x(t)) \le & \sup_{m\in[t_0-\bar{h},t_0]} W(x(m))e^{2.1\omega_0\bar{h}} \\ & + \left(e^{2.1\omega_0\bar{h}} - 1\right)\frac{\overline{p}^2\bar{\Delta}^2}{\omega_0}.\end{aligned} \quad (23)$$

Therefore, the finite escape time phenomenon does not occur over $[t_0 - \bar{h}, t_0 + 2\bar{h}]$, so $t_\infty > 2\bar{h}$. □

3.2 ISS result

Using the notation from Section 3.1, we use the function

$$\begin{aligned}\beta(m) =& \; 2\bar{h}|PBK|\left(|A| + |BK|\right)m \\ & + 2\left(\bar{h}|PBK| + |P|\right)m^{3/2}\rho\left(\sqrt{m}\right).\end{aligned} \quad (24)$$

Since ρ is nonnegative valued, nondecreasing and continuous, and since we assumed that $BK \neq 0$, the function β is strictly increasing over $[0, +\infty)$ and $\beta(0) = 0$. It follows that there is a $w_\star > 0$ such that

$$\beta(w_\star) = \tfrac{c}{4} \quad (25)$$

and we fix a $w_\star$ satisfying the preceding requirement in the rest of this subsection. We also assume:

Assumption 4. The inequality

$$\tfrac{4}{c}\left(|PBK|^2\bar{h}^2 + |P|^2\right)\bar{\Delta}^2 \le \tfrac{cw_\star}{4} \quad (26)$$

holds.

Assumption 4 can be viewed as a smallness condition on $\bar{\Delta}$. Let $\gamma > 0$ be the constant such that

$$\gamma = \tfrac{c}{2} - \tfrac{c}{4}e^{2\gamma\bar{h}}. \quad (27)$$

We are ready to state and prove the following result:

Theorem 2. Let (12) satisfy Assumptions 1-4. Then, with the notation from the preceding subsection, consider (12) in closed-loop with $u(t - h(t)) = Kx(t - h(t))$. Consider any solution $x(t)$ of the closed-loop system such that

$$\begin{aligned}& \sup_{m\in[t_0-\bar{h},t_0]} W(x(m))e^{2.1\omega_0\bar{h}} + \left(e^{2.1\omega_0\bar{h}} - 1\right)\frac{\overline{p}^2\bar{\Delta}^2}{\omega_0} \\ & < \min\{s_\star, w_\star\}\end{aligned} \quad (28)$$

holds. Then, for each $s \ge t_0 + \bar{h}$, and with the choice

$$\mathcal{G}(m) = \tfrac{4|PBK|^2\bar{h}}{c}\int_{s-\bar{h}}^m |\delta(r)|^2\,\mathrm{d}r + \tfrac{4|P|^2}{c}\sup_{\ell\in[0,m]}|\delta(\ell)|^2, \quad (29)$$

the inequality

$$\begin{aligned}& |x(t)| \le \\ & \sqrt{\overline{p}\sup_{m\in[s-2\bar{h},s]}|x(m)|^2e^{-\gamma(t-s)} + \int_s^t e^{\frac{c}{4}(m-t)}\mathcal{G}(m)\mathrm{d}m}\end{aligned} \quad (30)$$

holds for all $t \ge s$.

Proof. We consider a trajectory $x(t)$ of the closed-loop system satisfying the conditions of Theorem 2. Let $[t_0 - \bar{h}, t_\infty)$ be the largest domain of definition of x. By Lemma 2, $x(t)$ is defined over $[t_0 - \bar{h}, t_0 + 2\bar{h}]$, and (19) holds for all $t \in [t_0 - \bar{h}, t_0 + 2\bar{h}]$. Then necessarily, $t_\infty > t_0 + 2\bar{h}$. From the definition of $H = A + BK$, we deduce that

$$\dot{x}(t) = Hx(t) - BK\int_{t-h(t)}^t \dot{x}(m)\mathrm{d}m + \mathcal{F}(t, x_t) + \delta(t) \quad (31)$$

for all $t \in [t_0 + \bar{h}, t_\infty)$, since the integral in (31) is $x(t) - x(t - h(t))$. According to (14), the time derivative of W along (31) satisfies the following for all $t \in [t_0 + \bar{h}, t_\infty)$:

$$\begin{aligned}\dot{W}(t) \le & -cW(x(t)) - 2x(t)^\top PBK\int_{t-h(t)}^t \dot{x}(m)\mathrm{d}m \\ & + 2x(t)^\top P\mathcal{F}(t, x_t) + 2x(t)^\top P\delta(t)\end{aligned} \quad (32)$$

Since $t_\infty > 2\bar{h}$, it follows that with the choice $\mathcal{H}(m) = |Ax(m) + BKx(m - h(m)) + \mathcal{F}(m, x_m) + \delta(m)|$, we have

$$\begin{aligned}\dot{W}(t) \le & -cW(x(t)) + 2|x(t)||PBK|\int_{t-h(t)}^t \mathcal{H}(m)\mathrm{d}m \\ & + 2x(t)^\top P\mathcal{F}(t, x_t) + 2x(t)^\top P\delta(t)\end{aligned} \quad (33)$$

for all $t \in [t_0+\bar{h}, t_\infty)$. Consequently,

$$\begin{aligned}\dot{W}(t) \leq & -cW(x(t)) + 2|PBK||x(t)|\int_{t-h(t)}^{t}|Ax(m)|\,\mathrm{d}m\\ & +2|PBK||x(t)|\int_{t-h(t)}^{t}|BKx(m-h(m))|\,\mathrm{d}m\\ & +2|PBK||x(t)|\int_{t-h(t)}^{t}|\mathcal{F}(m,x_m)|\,\mathrm{d}m\\ & +2x(t)^\top P\mathcal{F}(t,x_t)\\ & +2|PBK||x(t)|\int_{t-h(t)}^{t}|\delta(m)|\,\mathrm{d}m + 2x(t)^\top P\delta(t)\end{aligned} \tag{34}$$

for all $t \in [t_0+\bar{h}, t_\infty)$. From our upper bound $\bar{h}$ on h, Assumption 2, and (14), we deduce that

$$\begin{aligned}\dot{W}(t) \leq & -cW(x(t)) + 2\bar{h}|PBK||A|\sup_{m\in[t-\bar{h},t]}W(x(m))\\ & +2\bar{h}|PBK||BK|\sup_{m\in[t-2\bar{h},t]}W(x(m))\\ & +2|PBK||x(t)|\int_{t-h(t)}^{t}\sup_{r\in[m-\bar{h},m]}|x(r)|^2\rho(|x(r)|)\mathrm{d}m\\ & +2|x(t)||P|\sup_{m\in[t-\bar{h},t]}|x(m)|^2\rho(|x(m)|)\\ & +2|PBK||x(t)|\int_{t-h(t)}^{t}|\delta(m)|\,\mathrm{d}m+2|P||x(t)||\delta(t)|\end{aligned} \tag{35}$$

for all $t \in [t_0+\bar{h}, t_\infty)$. Since ρ is nondecreasing, we get

$$\begin{aligned}\dot{W}(t) \leq & \ 2\bar{h}|PBK|\left(|A|+|BK|\right)\sup_{m\in[t-2\bar{h},t]}W(x(m))\\ & +2\bar{h}|PBK|\rho\left(\sup_{m\in[t-2\bar{h},t]}\sqrt{W(x(m))}\right)\mathrm{d}m\\ & \times\sup_{m\in[t-2\bar{h},t]}W(x(m))^{3/2}+2|P||x(t)||\delta(t)|\\ & +2|P|\sup_{m\in[t-2\bar{h},t]}W(x(m))^{3/2}\\ & \times\rho\left(\sup_{m\in[t-2\bar{h},t]}\sqrt{W(x(m))}\right)\\ & +2|PBK||x(t)|\int_{t-h(t)}^{t}|\delta(m)|\,\mathrm{d}m - cW(x(t))\end{aligned} \tag{36}$$

for all $t \in [t_0+\bar{h}, t_\infty)$. Using the triangle inequality, Jensen's inequality, and (14) to get

$$\begin{aligned}& 2|PBK||x(t)|\int_{t-h(t)}^{t}|\delta(m)|\mathrm{d}m \leq \tfrac{c}{4}W(x(t))\\ & +\tfrac{4}{c}|PBK|^2\bar{h}\int_{t-h(t)}^{t}|\delta(m)|^2\mathrm{d}m \ \text{ and}\\ & 2|P||x(t)||\delta(t)| \leq \tfrac{c}{4}W(x(t))+\tfrac{4}{c}|P|^2|\delta(t)|^2,\end{aligned} \tag{37}$$

it follows that

$$\begin{aligned}\dot{W}(t) \leq & -\tfrac{c}{2}W(x(t))+\tfrac{4}{c}|PBK|^2\bar{h}\int_{t-h(t)}^{t}|\delta(m)|^2\,\mathrm{d}m\\ & +2\bar{h}|PBK|\left(|A|+|BK|\right)\sup_{m\in[t-2\bar{h},t]}W(x(m))\\ & +2\left(\bar{h}|PBK|+|P|\right)\sup_{m\in[t-2\bar{h},t]}W(x(m))^{3/2}\\ & \times\rho\left(\sup_{m\in[t-2\bar{h},t]}\sqrt{W(x(m))}\right)+\tfrac{4}{c}|P|^2|\delta(t)|^2\end{aligned} \tag{38}$$

holds for all $t \in [t_0+\bar{h}, t_\infty)$. Therefore, we have

$$\dot{W}(t) \leq -\tfrac{c}{2}W(x(t))+\beta\left(\sup_{m\in[t-2\bar{h},t]}W(x(m))\right)+\delta_\sharp(t) \tag{39}$$

for all $t \in [t_0+\bar{h}, t_\infty)$, where β was defined in (24) and

$$\delta_\sharp(t)=\tfrac{4}{c}|PBK|^2\bar{h}\int_{t-h(t)}^{t}|\delta(m)|^2\,\mathrm{d}m+\tfrac{4}{c}|P|^2|\delta(t)|^2. \tag{40}$$

Note that (11) gives $|\delta_\sharp(t)| \leq (4/c)\left(|PBK|^2\bar{h}^2+|P|^2\right)\bar{\Delta}^2$ for all $t \geq \bar{h}$. Hence, (26) gives

$$|\delta_\sharp(t)| \leq \tfrac{cw_\star}{4} \tag{41}$$

for all $t \geq \bar{h}$. Then let us recall that (19) holds. Consequently (28) ensures that

$$\sup_{m\in[t_0-\bar{h},t_0+2\bar{h}]}W(x(m)) < w_\star \tag{42}$$

We can now apply Theorem 1 with $\lambda=\gamma$, and with

$$\begin{aligned}& V(t)=W(x(t)),\ a=b=c/4,\ \alpha=\beta,\\ & \zeta(t)=\sup_{\ell\in[0,t]}|\delta_\sharp(\ell)|,\ \mathcal{L}_0=2\bar{h},\ v_\star=w_\star,\ \tau=t_\infty,\end{aligned} \tag{43}$$

and $t_\star = t_0+\bar{h}$. Then (25) ensures that (2) is satisfied. Then (41)-(42) imply (3)-(4). Using Lemma 1, we can prove that the finite escape time phenomenon does not occur, so $t_\infty=+\infty$, and Theorem 1 gives

$$\begin{aligned}W(x(t)) \leq & \sup_{m\in[s-2\bar{h},s]}W(x(m))e^{-\gamma(t-s)}\\ & +\int_s^t e^{\frac{c}{4}(m-t)}\sup_{\ell\in[0,m]}|\delta_\sharp(\ell)|\mathrm{d}m\end{aligned} \tag{44}$$

for all $t \geq s \geq t_0+\bar{h}$ where γ is the constant defined in (27), and where the sup was needed in (43) and in the integrand in (44) because Theorem 1 requires its function ζ to be nondecreasing. Hence, (14) gives

$$\begin{aligned}|x(t)|^2 \leq & \ \overline{p}\sup_{m\in[s-2\bar{h},s]}|x(m)|^2e^{-\gamma(t-s)}\\ & +\int_s^t e^{\frac{c}{4}(m-t)}\sup_{\ell\in[0,m]}|\delta_\sharp(\ell)|\mathrm{d}m\end{aligned} \tag{45}$$

for all $t \geq s \geq t_0+\bar{h}$. This allows us to conclude. □

4. ILLUSTRATION OF THEOREM 2

Consider the controlled van der Pol equation

$$\begin{cases}\dot{x}_1(t) = x_2(t)\\ \dot{x}_2(t) = -x_1(t)+\epsilon(1-x_1^2)x_2+u(t-h(t))\end{cases} \tag{46}$$

for constants $\epsilon > 0$ and a piecewise continuous choice $h(t)$ of the delay; see, e.g., (Khalil, 2002, Section 13.2) for simpler cases with no delays. The dynamics are used to represent oscillations in vacuum tube circuits, and provide a fundamental equation in nonlinear oscillation theory. The system has the form (12) with the choices

$$\begin{aligned}& A=\begin{bmatrix}0 & 1\\ -1 & \epsilon\end{bmatrix},\ B=\begin{bmatrix}0\\ 1\end{bmatrix},\\ & \text{and } \mathcal{F}(t,x_t)=\begin{bmatrix}0\\ -\epsilon x_1^2(t)x_2(t)\end{bmatrix}\end{aligned} \tag{47}$$

with $\delta=0$. Using the Mathematica computer program, we can check that Assumption 1-2 hold with $\bar{\Delta}=0$, $K=[-1,-2]$, $\rho(s)=\epsilon s$,

$$P=\begin{bmatrix}3.85664 & 0.7625\\ 0.7625 & 1.20395\end{bmatrix} \tag{48}$$

when $\epsilon=0.1$, where P was found by first solving for a positive definite symmetric matrix $P_1 \in \mathbb{R}^{2\times 2}$ such that

$P_1 H + H^\top P_1 = -I$ holds with $H = A + BK$, then choosing $c = 0.75$ in order to satisfy $cP_1 \le I$, and then multiplying P_1 by 3.05 to satisfy the requirement that $P \ge I$ with the choice $P = 3.05P_1$. Also, since $\bar{\Delta} = 0$, Assumption 3-4 hold for any $s_* > 0$. We can then also use Mathematica to compute the basin of attraction from Theorem 2. For instance, when the delay h is the zero function, we can check that we can satisfy the requirements of Theorem 2 with $w_\star = s_* = 0.48052$ and all initial functions that are bounded by 0.344019. If we instead use the delay bound $\bar{h} = 0.025$ and keep all other parameter values the same as before, then the basin of attraction consists of all initial functions that are bounded by 0.111809. Finally, if one instead uses $\bar{h} = 0.035$ with all other parameters the same as before, then the basin of attraction consists of all initial functions that are bounded by 0.0729156. This illustrates the trade-off that increasing the bound $\bar{h}$ on the allowable input delays $h(t)$ can reduce the basin of attraction.

5. CONCLUSION

We provided a local version of Halanay's inequality to prove local asymptotic stability for nonlinear systems that contain state or input delays and uncertainties. Our new results are significant, because of the well-known benefits of using global versions of Halanay's inequality to prove global asymptotic stability for systems with unknown delays, and because many significant systems are only locally asymptotically stable and so are beyond the scope of global versions of Halanay's inequality. Another significant benefit of our work is that we allow the dynamics to contain unknown nonlinearities that violate the standard linear growth conditions and that can contain distributed delays. We illustrated how our methods provide new estimates for basins of attraction for a controlled Van der Pol equation.

Appendix A.1. ISS INEQUALITY

We prove a key lemma that we used in our proof of Theorem 1. First let $t_\star \ge 0$ and $\mathcal{L}_0 \ge 0$ be given constants. Consider a C^1 function $V : [t_\star - \mathcal{L}_0, +\infty) \to [0, +\infty)$, a nonnegative valued nondecreasing continuous function ζ, and constants $a > 0$ and $b > 0$ such that

$$\dot{V}(t) \le -(a+b)V(t) + a \sup_{m\in[t-\mathcal{L}_0,t]} V(m) + \zeta(t) \quad \text{(A.1)}$$

for all $t \ge t_\star$. Let $\lambda > 0$ be the constant in (7). We prove:

Lemma 3. The inequality

$$V(t) \le \sup_{m\in[s-\mathcal{L}_0,s]} V(m)e^{-\lambda(t-s)} + \int_s^t e^{b(m-t)}\zeta(m)\mathrm{d}m \quad \text{(A.2)}$$

holds for all $t \ge s$ and all $s \ge t_\star$.

Proof. Let $s \ge t_\star$ be a constant. We use the function

$$\theta(t) = \int_s^{\max\{t,s\}} e^{b(m-t)}\zeta(m)\mathrm{d}m. \quad \text{(A.3)}$$

Notice that $\dot{\theta}(t) = -b\theta(t) + \zeta(t)$ for all $t > s$. Using the nondecreasing property of ζ to get

$$\theta(t) \le \int_s^t e^{b(m-t)}\mathrm{d}m\zeta(t) \le \tfrac{1}{b}\zeta(t) \quad \text{(A.4)}$$

for all $t \ge s$, it follows that θ is nondecreasing over $[s, +\infty)$. Moreover, $\theta(t) = 0$ for all $t \in [s - \mathcal{L}_0, s]$. Hence,

$$\dot{\theta}(t) = -(a+b)\theta(t) + a \sup_{m\in[t-\mathcal{L}_0,t]} \theta(m) + \zeta(t) \quad \text{(A.5)}$$

holds for all $t > s$. We next choose $\chi_\epsilon(t) = \theta(t) + v_l e^{-\lambda(t-s)} + \epsilon$ and $v_l = \sup_{m\in[s-\mathcal{L}_0,s]} V(m)$, where $\epsilon > 0$ is a constant. Then

$$\chi_\epsilon(t) > \theta(t) + v_l e^{-\lambda(t-s)} \ge v_l = \sup_{m\in[s-\mathcal{L}_0,s]} V(m) \quad \text{(A.6)}$$

holds for all $t \in [s - \mathcal{L}_0, s]$. Therefore,

$$\chi_\epsilon(t) > V(t) \text{ for all } t \in [s - \mathcal{L}_0, s]. \quad \text{(A.7)}$$

We next prove that $\chi_\epsilon(t) > V(t)$ holds for all $t \ge s - \mathcal{L}_0$. Let us start to prove this by observing that

$$\begin{aligned}\dot{\chi}_\epsilon(t) \;=\; & -(a+b)\theta(t) + a \sup_{m\in[t-\mathcal{L}_0,t]} \theta(m) + \zeta(t) \\ & - \lambda v_l e^{-\lambda(t-s)}\end{aligned} \quad \text{(A.8)}$$

for all $t > s$. From (7), we deduce that

$$\begin{aligned}\dot{\chi}_\epsilon(t) \;=\; & -(a+b)\theta(t) + a \sup_{m\in[t-\mathcal{L}_0,t]} \theta(m) + \zeta(t) \\ & - (a + b - ae^{\lambda\mathcal{L}_0})v_l e^{-\lambda(t-s)} \\ =\; & -(a+b)\left(\theta(t) + v_l e^{-\lambda(t-s)}\right) + \zeta(t) \\ & + a\left(\sup_{m\in[t-\mathcal{L}_0,t]} \theta(m) + v_l e^{-\lambda(t-s-\mathcal{L}_0)}\right).\end{aligned} \quad \text{(A.9)}$$

Now, observe that

$$\begin{aligned}&\sup_{m\in[t-\mathcal{L}_0,t]} \left(\theta(m) + v_l e^{-\lambda(m-s)}\right) \\ &\le \sup_{m\in[t-\mathcal{L}_0,t]} \theta(m) + v_l e^{-\lambda(t-s-\mathcal{L}_0)}.\end{aligned} \quad \text{(A.10)}$$

As an immediate consequence,

$$\begin{aligned}\dot{\chi}_\epsilon(t) \;\ge\; & -(a+b)\left(\theta(t) + v_l e^{-\lambda(t-s)}\right) \\ & + a \sup_{m\in[t-\mathcal{L}_0,t]} \left(\theta(m) v_l e^{-\lambda(m-s)}\right) + \zeta(t)\end{aligned} \quad \text{(A.11)}$$

for all $t > s$. Using the definition of χ_ϵ, we obtain

$$\begin{aligned}\dot{\chi}_\epsilon(t) \ge\; & -(a+b)\left(\chi_\epsilon(t) - \epsilon\right) \\ & + a \sup_{m\in[t-\mathcal{L}_0,t]} \left(\chi_\epsilon(m) - \epsilon\right) + \zeta(t) \\ =\; & -(a+b)\chi_\epsilon(t) + a \sup_{m\in[t-\mathcal{L}_0,t]} \chi_\epsilon(m) + \zeta(t) + b\epsilon.\end{aligned} \quad \text{(A.12)}$$

We next proceed by contradiction. Bearing in mind (A.7), suppose that there were a $t_c > s$ such that $\chi_\epsilon(t) > V(t)$ for all $t \in [s - \mathcal{L}_0, t_c)$ and $\chi_\epsilon(t_c) = V(t_c)$. By (A.12),

$$\begin{aligned}\dot{\chi}_\epsilon(t_c) \;\ge\; & -(a+b)V(t_c) + a \sup_{m\in[t_c-\mathcal{L}_0,t_c]} \chi_\epsilon(m) \\ & + \zeta(t_c) + b\epsilon.\end{aligned} \quad \text{(A.13)}$$

On the other hand (A.1) gives

$$-\dot{V}(t_c) \ge (a+b)V(t_c) + a \sup_{m\in[t_c-\mathcal{L}_0,t_c]} V(m) - \zeta(t_c). \quad \text{(A.14)}$$

By adding (A.13) and (A.14), we obtain

$$\begin{aligned}&\dot{\chi}_\epsilon(t_c) - \dot{V}(t_c) \ge \\ &a\left[\sup_{m\in[t_c-\mathcal{L}_0,t_c]} \chi_\epsilon(m) - \sup_{m\in[t_c-\mathcal{L}_0,t_c]} V(m)\right] + b\epsilon.\end{aligned} \quad \text{(A.15)}$$

The definition of t_c ensures that $\sup_{m\in[t_c-\mathcal{L}_0,t_c]} \chi_\epsilon(m) \ge V(t)$ for all $t \in [t_c - \mathcal{L}_0, t_c]$. It follows that

$$\sup_{m\in[t_c-\mathcal{L}_0,t_c]} \chi_\epsilon(m) - \sup_{m\in[t_c-\mathcal{L}_0,t_c]} V(m) \ge 0.$$

We deduce from (A.15) that $\dot{\chi}_\epsilon(t_c) - \dot{V}(t_c) > 0$. Since $\chi_\epsilon(t_c) - V(t_c) = 0$, we deduce that there is $t_d \in (s, t_c)$ such that $\chi_\epsilon(t_d) - V(t_d) < 0$. This contradicts the definition of t_c. Hence, $\chi_\epsilon(t) > V(t)$ holds for all $t \geq s - \mathcal{L}_0$. Since $\epsilon > 0$ is arbitrary, we deduce that $\theta(t) + v_l e^{-\lambda(t-s)} \geq V(t)$ for all $t \geq s - \mathcal{L}_0$. Therefore, the conclusion of the lemma follows.

Appendix A.2. TECHNICAL RESULT

We prove the key lemma that we used in the proof of Lemma 2. We use constants $T > 0$, $q > 0$, $\Psi_0 > 0$, $\omega > 0$, $\tau > 0$, $\bar{\Delta} \geq 0$ and $t_a \geq 0$ and a continuous, nondecreasing function $\Psi : [0, +\infty) \to [0, +\infty)$ such that

$$\Psi(\ell) \leq \Psi_0 \ell \tag{A.16}$$

for all $\ell \in [0, \omega]$. Let $Z : [t_a - T, t_a + \tau) \to [0, +\infty)$ be a nonnegative valued function of class C^1 such that

$$\dot{Z}(t) \leq \Psi\left(\sup_{\ell \in [t-T,t]} Z(\ell)\right) + \bar{\Delta} \tag{A.17}$$

for all $t \in [t_a, t_a + \tau)$. We use the following assumption:

Assumption A.1. The inequality

$$\left(e^{\Psi_0 q} - 1\right) \tfrac{\bar{\Delta}}{\Psi_0} < \omega \tag{A.18}$$

is satisfied.

In the following lemma, the existence of values $\bar{Z} > 0$ such that (A.20) is satisfied follows from (A.18):

Lemma 4. Let Assumption A.1 hold. Let Z be such that

$$Z(\ell) \leq \bar{Z} \text{ for all } \ell \in [t_a - T, t_a] \tag{A.19}$$

where $\bar{Z} \in \mathbb{R}$ is such that

$$\bar{Z} e^{\Psi_0 q} + \left(e^{\Psi_0 q} - 1\right) \tfrac{\bar{\Delta}}{\Psi_0} < \omega. \tag{A.20}$$

Then

$$Z(t) \leq \bar{Z} e^{\Psi_0 q} + \left(e^{\Psi_0 q} - 1\right) \tfrac{\bar{\Delta}}{\Psi_0} \tag{A.21}$$

holds for all $t \in [t_a - T, t_a + \min\{\tau, q\})$.

Proof. We first prove that $Z(t) < \omega$ for all $t \in [t_a - T, t_a + \min\{\tau, q\})$. To prove this, we proceed by contradiction. Note that (A.19)-(A.20) imply that $Z(t) < \omega$ for all $t \in [t_a - T, t_a]$, and let us suppose that there were a $t_c \in [t_a, t_a + \min\{\tau, q\})$ such that $Z(t) < \omega$ for all $t \in [t_a - T, t_a + t_c)$ and $Z(t_c) = \omega$. Then

$$\dot{Z}(t) \leq \Psi_0 \sup_{\ell \in [t-T,t]} Z(\ell) + \bar{\Delta} \tag{A.22}$$

for all $t \in [t_a, t_a + t_c]$, by the bound (A.16). Now set

$$\begin{aligned} \xi_\epsilon(t) = &\ (\bar{Z} + \epsilon) e^{(\Psi_0 + \epsilon)(t - t_a)} \\ &+ \left[e^{(\Psi_0 + \epsilon)(t - t_a)} - 1\right] \tfrac{\bar{\Delta}}{\Psi_0 + \epsilon} \end{aligned} \tag{A.23}$$

with $\epsilon > 0$. Note that $Z(t_a) \leq \bar{Z} < \bar{Z} + \epsilon \leq \xi_\epsilon(t_a)$. We next show that $Z(t) < \xi_\epsilon(t)$ for all $t \in [t_a, t_c]$. We argue by contradiction. Suppose there were a $t_f \in [t_a, t_c]$ such that

$$Z(t) < \xi_\epsilon(t) \text{ for all } t \in [t_a, t_f) \text{ and } Z(t_f) = \xi_\epsilon(t_f). \tag{A.24}$$

Simple algebraic calculations that combine terms give

$$\dot{\xi}_\epsilon(t) = (\Psi_0 + \epsilon) \xi_\epsilon(t) + \bar{\Delta} \tag{A.25}$$

for all $t > t_a$. On the other hand, (A.22) and (A.24) give $\dot{Z}(t_f) \leq \Psi_0 \xi_\epsilon(t_f) + \bar{\Delta}$. Consequently, (A.25) gives $\dot{Z}(t_f) < \dot{\xi}_\epsilon(t_f)$, which we can combine with (A.24) to deduce that there is a $t_g \in [t_a, t_f)$ such that $\xi_\epsilon(t_g) < Z(t_g)$. This contradicts the definition of t_f. Hence,

$$Z(t) < \xi_\epsilon(t) \text{ for all } t \in [t_a, t_c]. \tag{A.26}$$

Since ϵ is an arbitrary positive number, we deduce that

$$Z(t) \leq \bar{Z} e^{\Psi_0 (t - t_a)} + \left(e^{\Psi_0 (t - t_a)} - 1\right) \tfrac{\bar{\Delta}}{\Psi_0} \tag{A.27}$$

for all $t \in [t_a, t_c]$. Since $t_c \in [t_a - T, t_a + q]$, it follows that

$$Z(t_c) \leq \bar{Z} e^{\Psi_0 q} + \left(e^{\Psi_0 q} - 1\right) \tfrac{\bar{\Delta}}{\Psi_0}. \tag{A.28}$$

Since $Z(t_c) = \omega$, we obtain

$$\omega \leq \bar{Z} e^{\Psi_0 q} + \left(e^{\Psi_0 q} - 1\right) \tfrac{\bar{\Delta}}{\Psi_0}. \tag{A.29}$$

This contradicts (A.20). Hence, $Z(t) < \omega$ for all $t \in [t_a - T, t_a + \min\{\tau, q\})$. It follows from (A.16) that

$$\dot{Z}(t) \leq \Psi_0 \sup_{\ell \in [t-T,t]} Z(\ell) + \bar{\Delta} \tag{A.30}$$

for all $t \in [t_a, \min\{\tau, q\})$. Arguing as in (A.22)-(A.26) except with t_c replaced by $\min\{\tau, q\}$, we obtain $Z(t) \leq \xi_\epsilon(t)$ for all $t \in [t_a, \min\{\tau, q\})$ which implies that

$$Z(t) \leq \bar{Z} e^{\Psi_0 (t - t_a)} + \left(e^{\Psi_0 (t - t_a)} - 1\right) \tfrac{\bar{\Delta}}{\Psi_0} \tag{A.31}$$

for all $t \in [t_a, \min\{\tau, q\})$. Since

$$Z(t) \leq \bar{Z} \leq \bar{Z} e^{\Psi_0 q} + \left(e^{\Psi_0 q} - 1\right) \tfrac{\bar{\Delta}}{\Psi_0} \tag{A.32}$$

for all $t \in [t_a - T, t_a]$, we can conclude. □

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