# Observer design and output feedback stabilization of time varying systems

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# OBSERVER DESIGN AND OUTPUT FEEDBACK STABILIZATION OF TIME VARYING SYSTEMS

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Observer Design and Output Feedback Stabilization of Time Varying Systems
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We certify that we have read this dissertation and that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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# ABSTRACT

## OBSERVER DESIGN AND OUTPUT FEEDBACK STABILIZATION OF TIME VARYING SYSTEMS

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We study observer design and output feedback stabilization of switched and nonlinear time varying systems. For stabilization of delayed switched systems via an observer, we develop a new extension of a recently proposed trajectory based approach which is fundamentally different from classical Lyapunov function based methods. This new extension of trajectory based approach can be applied to a wide range of systems with time varying delays and it tackles the issue of finding appropriate Lypunov functions to establish the stability of delayed switched systems. Our stabilization methodology does not require stabilizability and detectability of all of the subsystems of the switched system and we do not impose any constraint on the derivative of the time varying delay. For nonlinear time varying systems, we build a new type of finite-time smooth observer in the case where a state dependent disturbance affects the linear approximation. We combine this finite time observer design and a switched systems approach to develop stabilizing feedbacks for nonlinear time varying systems whose outputs are only available on some finite time intervals. Again, we use an extension of the trajectory based approach to establish the stability of the closed-loop system. Motivated by the fact that the measured components of the state do not need to be estimated, we also construct reduced order finite time observers for a broad class of nonlinear time-varying systems. We show how these reduced order finite time observers can be used to solve dynamic output feedback stabilization problem for MIMO nonlinear time varying systems. Finally, we design a finite time observer to estimate the exact state of a continuous-time linear time varying system from sampled output in the presence of a piecewise continuous disturbance.

*Keywords:* Observer, Output feedback, Stabilization, Switched system, Nonlinear system, Delay, Time varying.

# ÖZET

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Bu tezde anahtarlı, doğrusal olmayan ve zamanla değişen sistemlerin gözlemleyici tasarımı ve çıktı geribeslemesinin kararlılaştırılması üzerine çalışılmıştır. Gecikmeli ve anahtarlı sistemlerin gözlemleyici ile kararlılaştırılması için son zamanlarda bulunan ve klasik Lyapunov fonksiyonu tabanlı yöntemlerden temelde farklı olan gezinge tabanlı yaklaşımın yeni bir uzantısı geliştirilmiştir. Bu yeni gezinge tabanlı uzantı pek çok,gecikmesi zamanla değişen sisteme uygulanabilmektedir ve gecikmeli, anahtarlı sistemlerin kararlılığını sağlayan uygun Lyapunov fonksiyonu bulma probleminin üstesinden gelmektedir. Kararlılaştırma yöntemimiz, anahtarlı bir sistemin tüm alt sistemlerinin kararlılaştırılabilirlik ve tespit edilebilirliğine gereksinim duymamaktadır ve zamanla değişen gecikmenin türeviyle ilgili hiçbir şart koymamaktadır. Doğrusal olmayan ve zamanla değişen sistemler için, duruma bağlı bozucu etkinin doğrusal yakınsamaya tesir ettiği koşullar altında yeni bir sonlu zamanlı düzgün gözlemleyici geliştirilmiştir. Çıktıları sadece bazı sonlu zaman aralıklarında erişilebilen, doğrusal olmayan ve zamanla değişen sistemlerin kararlılaştırıcı geri beslemesini geliştirmek için bu sonlu zamanlı gözlemleyici tasarımını ve anahtarlı sistem yaklaşımını birleştiriyoruz. Durumun ölçülen bileşenlerinin kestirilmesinin gerekmediği düşüncesinden yola çıkarak, geniş bir doğrusal olmayan ve zamanla değişen sistem sınıfı için azaltılmış dereceli ve sonlu zamanlı gözlemleyiciler geliştirilmiştir. Bu azaltılmış dereceli sonlu zamanlı gözlemleyicilerin çok girdili çok çıktılı, doğrusal olmayan ve zamanla değişen sistemlerin dinamik çıktı geribesleme kararlılaştırılması probleminin çözümü için nasıl kullanılacağını gösteriyoruz. Son olarak, parçalı sürekli bozucu etki altında sürekli, doğrusal ve zamanla değişen sistemin örneklenmiş çıktılarıyla tam durum kestirimi için sonlu zamanlı gözlemleyiciler tasarlıyoruz.

*Anahtar sözcükler*: Gözlemleyici, Çıktı geribeslemesi, Kararlılaştırma, Anahtarlı sistem, Doğrusal olmayan sistem, Gecikme, Zaman değişiyor.

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# Chapter 1

# Introduction

The problem of estimating the value of solutions of a system when some variables cannot be measured is of great relevance, both from a theoretical and an applied point of view. Moreover, when the state of the system is not available for measurement, one aims to design output feedback controls that are computed from output observations. This motivates the problem of designing observers and stabilizing dynamic output feedbacks for switched and nonlinear systems because these systems are ubiquitous in communication networks and congestion control, automotive control, power converters, aircraft and air traffic control, process control, mechanical systems, and many other engineering domains; see, e.g., [1], [2], [3], [4], [5], and [6] for the applications of switched systems, and see [7], [8], [9], [10], [11], and [12] for the applications of nonlinear systems.

We present four new observer designs in this thesis. First, we propose an observer for switched systems with delay in the output. Second, motivated by the fact that finite time convergence of estimation error is often desirable in applications like fault detection and feedback control, we design a finite time converging observer for a class of nonlinear time varying systems. Moreover, since the measured components of the state do not need to be estimated, we will also focus on reduced order finite time observer design for a family of nonlinear time varying systems. Finally, we will present a finite-time converging observer design for linear time varying systems where the measurements are only available at discrete instants.

The stabilization problems we study in this thesis are motivated by the fact that the state of the system is not available for measurement in many engineering applications. Instead, one aims to design output feedback controls that are computed from output observations. We provide solutions to three dynamic output feedback stabilization problems for time varying systems in this thesis. To the best of our knowledge, the stabilization problems that we study in this work have remained unsolved. First, we study output feedback stabilization of switched systems with a delay in the output via an observer. Second, motivated by the fact that the observer values may also be intermittent, meaning there may be intervals during which there are no output measurements, we study the problem of stabilization for a broad class of nonlinear time varying systems with intermittent output measurements via finite time observer, which can be difficult when the dropout periods (when there are no output observations) are long. Finally, we provide a solution for an output feedback stabilization problem for a family of MIMO nonlinear time varying systems using reduced order finite time observer.

## 1.1 Classical Results on Switched Systems

A switched system is a class of hybrid dynamical system consisting of a family of continuous-time subsystems (also called modes) and a rule that governs switching between them [13]. Mathematically, a switched system can be written as

$$\dot{x} = f_\sigma(x) \tag{1.1}$$

where $\{f_\sigma : \sigma \in \Xi\}$ is a family of sufficiently regular (at least locally Lipschitz) functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ that is parametrized on some index set $\Xi$. Typically, $\Xi$ is a compact (often finite) subset of a finite-dimensional linear vector space. In the particular case, when all the individual subsystems are linear i.e.,

$$f_\sigma(x) = A_\sigma x, \quad A_\sigma \in \mathbb{R}^{n\times n},\ \sigma \in \Xi$$

and the index set $\Xi$ is finite i.e., $\Xi = \{1, \ldots, n\}$, we obtain a *switched linear system*.

To define a switched system generated by the above family, we need the notion of a *switching signal*. Given an initial time $t_0$, an initial state $x(t_0) = x_0 \in \mathbb{R}^n$, and a switching sequence $\pi = \{(i_0, t_0), \ldots, (i_k, t_k), \ldots, | i_k \in \Xi,\ k \in \mathbb{Z}_{\geq 0}\}$, the function $\sigma : [0, \infty) \to \Xi = \{1, ..., n\}$ such that $\sigma(t) = i_k$ when $t \in [t_k, t_{k+1})$ is called a *switching signal*. The function $\sigma$ is a piecewise constant function and it has a finite number of discontinuities (switching instants) on every bounded interval of time and it takes a constant value on every interval between two consecutive switching instants. The role of $\sigma$ is to specify the index $\sigma(t) \in \Xi$ of the *active subsystem* at each instant of time $t$. We assume that $\sigma$ is continuous from the right everywhere i.e. $\sigma(t) = \lim_{\theta \to t^+} \sigma(\theta)$ for each $\theta > 0$. We also assume that the state of 1.1 does not jump at the switching instants i.e., the solution $x(\cdot)$ is everywhere continuous. Thus a switched linear system can be described by

$$\dot{x}(t) = A_{\sigma(t)}(x(t)) \ . \tag{1.2}$$

### 1.1.1 Notions of Stability

Stability of switched systems is a significant and challenging problem because switched systems may manifest a complicated dynamical behavior due to their hybrid nature which is highlighted by the following fact.

**Fact 1.1** *([13], [14]). Even if all of the subsystems of the switched system are stable, an unconstrained switching may destabilize it. Conversely, it may be possible to stabilize a switched system through a suitable constrained switching even if all of its subsystems are unstable.*

#### 1.1.1.1 Stability for Arbitrary Switching

If all of the subsystems of a switched system are asymptotically stable, then the existence of a common Lyapunov function implies asymptotically stability of the switched system, uniform over the set of all switching signals. The existence of a common Lyapunov function is a necessary and sufficient condition for the switched system to be asymptotically stable under any arbitrary switching signal. This notion of stability is of great relevance for the case when a system is being controlled by means of switching among a set of stabilizing controllers; see [15] for more discussion on this topic.

#### 1.1.1.2 Stability for Slow Switching

By restricting the class of admissible switching signals, asymptotic stability of the switched system can be achieved provided that all of its subsystems are asymptotically stable. One way to restrict the class of switching signals is to make sure that the intervals between consecutive switching times are large enough. Such slow switching assumption is called *dwell time* approach and it greatly simplifies stability analysis. It is a well known fact that when all of the subsystems of the switched linear system are asymptotically stable, then it is globally asymptotically stable (GAS) if the dwell time is large enough. The required lower bound on dwell time can be obtained from the parameters of individual subsystems; see [16, Lemma 2] for details. Multiple Lyapunov function tools play a vital role in stability analysis of slowly switched systems. The dwell time approach is ubiquitous in switching control literature; see for instance [17], [18], and the references therein.

#### 1.1.1.3 Stabilizing Switching Signals

It is possible to find a switching signal that renders the switched system asymptotically stable. Such a signal may even exist in extreme situations when all the individual subsystems are unstable. For instance, assume that the system (1.2) has two modes. If the matrix pencil of the these modes contains a stable matrix then there exists a piecewise constant switching signal which makes the switched system quadratically stable, see [15, Theorem 11], [19], and [20] for more insight on this notion of stability.

### 1.1.2 Controllability, Reachability and Observability

Consider a switched linear system described by

$$\begin{cases} \dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) &= C_{\sigma(t)}x(t) \end{cases} \tag{1.3}$$

with the state $x$ valued in $\mathbb{R}^{d_x}$, the control input $u$ valued in $\mathbb{R}^{d_u}$, the output $y$ valued in $\mathbb{R}^{d_y}$, and piecewise switching signal $\sigma$ taking value from the finite index set $\Xi = \{1, ..., n\}$. Let $\phi(t; t_0, x_0, u, \sigma)$ denote the state trajectory at time $t$ of switched system (1.3) starting from $x(t_0) = x_0$ with input $u$ and switching signal $\sigma$.

#### 1.1.2.1 Controllability and Reachability

The controllability of switched linear systems is defined as

**Definition 1.1** *([21]). State $x \in \mathbb{R}^{d_x}$ is controllable at time $t_0$, if there exist a time instant $t_f > t_0$, a switching signal $\sigma : [t_0, t_f) \to \Xi$, and input $u : [t_0, t_f) \to \mathbb{R}^{d_u}$, such that $\phi(t_f; t_0, x, u, \sigma) = 0$. The controllable set of system (1.3) at $t_0$ is the set of states which are controllable at $t_0$. The system is said to be (completely) controllable at time $t_0$, if its controllable set at $t_0$ is $\mathbb{R}^{d_x}$.*

Let $\mathscr{V}(A_i, B_i)_\Xi$ denotes the minimum subspace of $\mathbb{R}^{d_x}$, which is invariant under all $A_i,\ i \in \Xi$ and which contains all image spaces of $B_i,\ i \in \Xi$ and let $\sum(A_i, B_i)_\Xi$ denotes the switched linear system (1.3) without output.

**Theorem 1.1** *([21]). For switched linear system $\sum(A_i, B_i)_\Xi$, the controllable set and reachable set are always identical, and they are precisely the subspace $\mathscr{V}(A_i, B_i)_\Xi$.*

**Corollary 1.1.1** *([21]). For switched linear system $\sum(A_i, B_i)_\Xi$, the following statements are equivalent:*

*(i) the system is completely controllable;*

*(ii) the system is completely reachable; and*

*(iii) $\mathscr{V}(A_i, B_i)_\Xi = \mathbb{R}^{d_x}$.*

#### 1.1.2.2 Observability

The observability of switched linear systems is defined as

**Definition 1.2** *([22]). State $x$ is said to be unobservable at $t_0$, if for any switching signal $\sigma$, there is an input $u$, such that $C_\sigma \phi(t; t_0, x, u, \sigma) = C_\sigma \phi(t; t_0, 0, u, \sigma)$ for all $t \geq t_0$. The unobservable set of system (1.3) at $t_0$ is the set of states which are unobservable. The system is said to be (completely) observable at $t_0$, if its unobservable set at $t_0$ is null.*

Let $\sum(C_i, A_i)_\Xi$ denotes the switched linear system (1.3) without input, and let $\mathscr{O}(C_i, A_i)_\Xi$ be the minimal subspace which is invariant under ${A_i}^T, i \in \Xi$ and which contains image spaces of ${C_i}^T, i \in \Xi$. Let

$$\begin{aligned}\mathscr{U}(C_i, A_i)_\Xi &= (\mathscr{O}(C_i, A_i)_\Xi)^\perp \\ &= \{y \in \mathbb{R}^{d_x} : \langle x, y \rangle = 0,\ \forall x \in \mathscr{O}(C_i, A_i)_\Xi\}.\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^n$.

**Theorem 1.2** *([21]). For switched linear system (1.3), the unobservable set is subspace $\mathscr{U}(C_i, A_i)_\Xi$.*

**Corollary 1.2.1** *([22]). For switched linear system (1.3), the following statements are equivalent:*

*(i)* *the system is completely observable;*

*(ii)* *the system* $\sum(A_i{}^T, C_i{}^T)_\Xi$ *is completely controllable and/or reachable; and*

*(iii)* $\mathscr{O}(C_i, A_i)_\Xi = \mathbb{R}^{d_x}$.

### 1.1.3 Feedback Stabilization

It is a well known fact that complete controllability implies state feedback stabilizabilty for a linear time invariant system. This problem has not been fully resolved in literature for general switched systems but some results have been achieved for a special class of switched systems as stated below:

**Theorem 1.3** *([23]). If the summation of the controllable set of all the individual subsystems is the total state space, then the switched system is linear state feedback stabilizable.*

Similarly, an interesting result is provided in ([24]) for the dynamic output feedback stabilization of a class of switched linear systems for the case where all of the subsystems are controllable and observable. This result is stated below:

**Theorem 1.4** *([24]) For continuous-time switched linear system (1.3), suppose that each subsystem is controllable and observable. then, for any given scalar* $\tau > 0$*, there is a dynamic output feedback law such that the closed-loop system is stable under every switching signal with dwell time* $\tau$*.*

In contrast to [24], the dynamic feedback stabilization methodology we state and prove in this dissertation does not require all of the modes of the switched linear system to be stabilizable and detectable. Therefore, our results can be applied to a broad class of switched linear systems wider than those invoked in [24] and this is one of our significant contributions.

## 1.2 Classical Results on Nonlinear Systems

The basic families of nonlinear systems are nonautonomous systems, autonomous systems, and systems with inputs. An *nth* order *nonautonomous* nonlinear system can be described by $n$ first-order one-dimensional differential equations as

$$\begin{cases} \dot{x}_1 &= f_1(t, x_1, x_2, \ldots, x_n) \\ \dot{x}_2 &= f_2(t, x_1, x_2, \ldots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(t, x_1, x_2, \ldots, x_n) \end{cases} \tag{1.4}$$

where $x_1, x_2, \ldots, x_n$ are the state variables, $t$ is the time, and all functions $f_1, f_2, \ldots, f_n$ are nonlinear in all of their arguments. More compactly, the system (1.11) can be written as

$$\dot{x} = f(t, x) \tag{1.5}$$

where the state vector $x = (x_1, x_2, \ldots, x_n)$ is valued in a given open set $\mathcal{X} \subseteq \mathbb{R}^n$. Given a constant $t_0 \geq 0$, $x_0 \in \mathcal{X}$, if $x(t, t_0, x_0)$ can be uniquely defined for all $t \geq t_0$ for all initial conditions $x(t_0, t_0, x_0) = x_0$, then the system (1.11) is called *forward complete*. An *equilibrium point* $x^* = (x_1{}^*, x_2{}^*, \ldots, x_n{}^*)$ of (1.5) is defined to be a vector in $\mathbb{R}^n$ for which $f(t, x^*) = 0$ for all $t \geq 0$. Since $f$ depends on time, the systems defined by (1.5) are also called *time-varying systems*. If there is a constant $T > 0$ such that $f$ satisfies $f(t + T, x) = f(t, x)$ for all $(t, x)$ in its domain, then the time-varing system (1.5) is called *periodic* with a period $T$.

If the right hand side of (1.11) or (1.5) is independent of time variable $t$, the the systems are called *autonomous* or *time-invariant* systems. Compactly, the autonomous systems can be written as

$$\dot{x} = f(x) \tag{1.6}$$

The general nonlinear time-varying *systems with inputs* can be described as

$$\begin{cases} \dot{x}_1 &= f_1(t, x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_p) \\ \dot{x}_2 &= f_2(t, x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_p) \\ &\vdots \\ \dot{x}_n &= f_n(t, x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_p) \end{cases} \tag{1.7}$$

or, more compactly as

$$\dot{x} = f(t, x, u) \tag{1.8}$$

where the variables $u_1, u_2, \ldots, u_p$ are inputs and the input vector $u = (u_1, u_2, \ldots, u_p)$ is valued in a given set $\mathcal{U} \in \mathbb{R}^p$. The input $u$ may represent a *control* or *disturbance*. If the system (1.7) can be written as

$$\dot{x}(t) = \mathcal{F}(t, x) + \mathcal{G}(t, x)u \tag{1.9}$$

for some vector fields $\mathcal{F}$ and $\mathcal{G}$, then the system (1.7) is called *affine in control* or *control affine*.

### 1.2.1 Notions of Stability

Before describing various stability notions for nonlinear systems, we first recall the following classes of comparison functions. We say that a continuous function $\gamma : [0, \infty) \to [0, \infty)$ belongs to class $\mathcal{K}$ and write $\gamma \in \mathcal{K}$ provided it is increasing and $\gamma(0) = 0$. We say that it belongs to class $\mathcal{K}_\infty$ if, in addition, $\gamma(r) \to \infty$ as $r \to \infty$. We say that a continuous function $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ is of class $\mathcal{KL}$ provided for each fixed $s \geq 0$, the function $\beta(\cdot, s)$ belongs to class $\mathcal{K}$, and for each fixed $r \geq 0$, the function $\beta(r, \cdot)$ is non-increasing and $\beta(r, s) \to 0$ as $s \to \infty$. For any constants $p > 0$, $r \in \mathbb{N}$, and $q \in \mathbb{R}^r$, we use the notation $\rho\mathcal{B}_r(q) = \{x \in \mathbb{R}^r : |x - q| \leq \rho\}$, which is denoted by $\rho\mathcal{B}_r$ when $q = 0$. Let $\mathcal{M}(U)$ denote the set of all measurable essentially bounded functions $u : [0.\infty) \to U$; i.e. inputs that are bounded in $|\cdot|_\infty$ where $|\cdot|_\infty$ denote the sup norm of any matrix valued function over its entire domain.

Now we define various notions of stability for nonlinear systems.

**Definition 1.3** *([25]). Assume that the system (1.5) admits the origin 0 as an equilibrium point. The equilibrium* 0 *is* stable *provided for each constant* $\epsilon > 0$*, there exists a constant* $\delta(\epsilon) > 0$ *such that for each initial state* $x_0 \in \mathcal{X} \cap \delta(\epsilon)\mathcal{B}_n$ *and each initial time* $t_0 \geq 0$*, the unique solution* $x(t, t_0, x_0)$ *satisfies* $|x(t, t_0, x_0)| \leq \epsilon$ *for all* $t \geq t_0$*. Otherwise we call the equilibrium* unstable.

**Definition 1.4** *([25]). Assume that the system (1.5) admits the origin 0 as an equilibrium point. The equilibrium* 0 *is* globally uniformly asymptotic stable (GUAS) *if there exists a function* $\beta \in \mathcal{KL}$ *such that for each initial state* $x_0 \in \mathcal{X}$ *and each initial time* $t_0 \geq 0$*, the solution* $x(t, t_0, x_0)$ *for (1.5) satisfies* $|x(t, t_0, x_0)| \leq \beta(|x_0|, t - t_0))$ *for all* $t \geq t_0 \geq 0$*. When the system is autonomous, this property is called* global asymptotic stability (GAS)*. The equilibrium* 0 *is* uniformly asymptotically stable *if there exists a function* $\beta \in \mathcal{KL}$ *and a constant* $\bar{c} > 0$ *independent of* $t_0$ *such that* $|x(t, t_0, x_0)| \leq \beta(|x_0|, t - t_0))$ *for all* $t \geq t_0 \geq 0$ *holds for all initial conditions* $x_0 \in \bar{c}\mathcal{B}_n \cap \mathcal{X}$*. When the system is time-invariant, the preceding property is called* (local) asymptotic stability (LAS)*.*

**Definition 1.5** *([25]). The* basin of attraction *of a LAS equilibrium point of a system is the set of all initial states that generate solutions of the system that converge to the equilibrium point.*

**Definition 1.6** *([25]). Assume that the system (1.5) admits the origin 0 as an equilibrium point. The equilibrium* 0 *is* uniformly exponentially stable *if there exist positive constants* $K_1$*,* $K_2$*, and* $r$ *such that for each initial state* $x_0 \in \mathcal{X} \cap r\mathcal{B}_n$ *and each* $t_0 \geq 0$*, the corresponding solution* $x(t, t_0, x_0)$ *satisfies* $|x(t, t_0, x_0)| \leq K_1 e^{-K_2(t-t_0)}$ *for all* $t \geq t_0$*. When the system is autonomous, this property is called local* exponential stability (LES) *or, it is called* global exponential stability (GES) *if* $r$ *can be taken to be* $+\infty$*. The special case of uniform exponential stability where we can take* $r = +\infty$ *is called* global uniform exponential stability (GUES)*.*

**Definition 1.7** *([25]). If there exist functions* $\beta \in \mathcal{KL}$ *and* $\gamma \in \mathcal{K}$ *such that for each* $u \in \mathcal{M}(U)$ *and each initial condition* $x(t_0) = x_0 \in \mathcal{X}$*, the solution*

*$x(t,t_0,x_0,u)$ of the system (1.8) with input vector $u$ satisfies $|x(t,t_0,x_0,u)| \leq \beta(|x_0|, t-t_0) + \gamma(|u|_{[t_0,t]})$ for all $t \geq t_0$ then the system (1.8) is input-to-state (ISS) stable.*

### 1.2.2 Stabilization

Stabilization is the problem of constructing a control law $u_s(t,x)$ such that the origin of (1.8) is asymptotically stable. If local stabilization is required then techniques based on linear approximation of (1.8) can be used. However, if GUAS is required then nonlinear design techniques like *backstepping* and *forwarding* can be applied. These techniques apply to nonlinear systems with special structure. Backstepping applies to lower triangular systems (feedback systems) of the form

$$\left\{\begin{array}{lcl} \dot{x}_1 & = & f_1(t,x_1,x_2) \\ \dot{x}_2 & = & f_2(t,x_1,x_2,x_3) \\ & \vdots & \\ \dot{x}_n & = & f_n(t,x_1,x_2,\ldots,x_n,u)\ . \end{array}\right. \tag{1.10}$$

Forwarding applies to upper triangular systems (feedforward systems) of the form

$$\left\{\begin{array}{lcl} \dot{x}_1 & = & f_1(t,x_1,\ldots,x_n,u) \\ \dot{x}_2 & = & f_2(t,x_2,\ldots,x_n,u) \\ & \vdots & \\ \dot{x}_n & = & f_n(t,x_n,u)\ . \end{array}\right. \tag{1.11}$$

Recently, it has been shown that Lyapunov based techniques can be used to handle stabilization problem for nonlinear systems; see [25] for details.

## 1.3 Classical Results on Time Delay Systems

Let $C_{in} = C([-\tau,0],\mathbb{R}^n)$ be the set of continuous functions mapping $[-\tau,0]$ to $\mathbb{R}^n$ where $\tau$ is the *maximum time delay* of a system, then a time delay system can be described by the retarded functional differential equation (RFDF) of the form

$$\dot{x}(t) = f(t, x_t) \tag{1.12}$$

where $x(t) \in \mathbb{R}^n$ and $f : \mathbb{R} \times C_{in} \to \mathbb{R}^n$. The equation (1.12) specifies that the derivative of state variable $x$ at time $t$ depends on $t$ and $x(\xi)$ for $t - \tau \leq \xi \leq t$. For future evolution of state, the initial state variable in a time interval of length $\tau$, say, from $t_0 - \tau$ to $t_0$ is specified as $x_{t_0} = \phi$ or $x(t_0 + \theta) = \phi(\theta), -\tau \leq \theta \leq 0$ where $\phi \in C_{in}$ is given.

### 1.3.1 Notions of Stability

In this section, we introduce various notions of stability for time delay systems. Let the usual Euclidean norm of vectors, and the induced norm of matrices are denoted by $|\cdot|$. For a function $\phi \in C([a, b], \mathbb{R}^n)$, define the continuous norm $|\cdot|_c$ by $|\phi|_c = \max_{a \leq \theta \leq b} |\phi(\theta)|$.

**Definition 1.8** *[26]. For the system described by (1.12), the trivial solution $x(t) = 0$ is said to be* stable *if for any $t_0 \in \mathbb{R}$ and any $\epsilon > 0$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that $|x_{t_0}|_c < \delta$ implies $|x(t)| < \epsilon$ for $t \geq t_0$. It is said to be* asymptotically stable *if it is stable, and for any $t_0 \in \mathbb{R}$ and any $\epsilon > 0$, there exists a $\delta_a = \delta_a(t_0, \epsilon) > 0$ such that $|x_{t_0}|_c < \delta_a$ implies $\lim_{t \to \infty} x(t) = 0$. It is said to be* uniformly stable *if it is stable and $\delta(t_0, \epsilon)$ can be chosen independently of $t_0$. It is* uniformly asymptotically stable *if it is uniformly stable and there exists a $\delta_a > 0$ such that for any $\eta > 0$, there exists a $T = T(\delta_a, \eta)$, such that $|x_{t_0}|_c < \delta$ implies $|x(t)| < \eta$ for $t \geq t_0 + T$ and $t_0 \in \mathbb{R}$. It is* globally (uniformly) asymptotically stable *if it is (uniformly) asymptotically stable and $\delta_a$ can be an arbitrarily large, finite number.*

### 1.3.2 Classical Stability Theorems

In this section, we introduce some classical stability theorems for time delay systems.

**Theorem 1.5** *([26]) (Lyapunov-Krasovskii Stability Theorem). Suppose* $f : \mathbb{R} \times C_{in} \to \mathbb{R}^n$ *in (1.12) maps* $\mathbb{R}\times$*(bounded sets in* $C_{in}$*) into bounded sets of* $\mathbb{R}^n$*, and that* $u,\ v,\ w : \mathbb{R}_+ \to \mathbb{R}_+$ *are continuous nondecreasing functions, where additionally* $u(s)$ *and* $v(s)$ *are positive for* $s > 0$*, and* $u(0) = v(0) = 0$*. If there exists a continuous differentiable functional* $V : \mathbb{R} \times C_{in} \to \mathbb{R}$ *such that*

$$u(|\phi(0)|) \le V(t, \phi) \le v(|\phi|_c) \tag{1.13}$$

*and*

$$\dot{V}(t, \phi) \le -w(|\phi(0)|), \tag{1.14}$$

*then the trivial solution of (1.12) is uniformly stable. If* $w(s) > 0$ *for* $s > 0$ *then it is uniformly asymptotically stable. If, in addition,* $\lim_{s\to\infty} u(s) = \infty$*, then it is globally uniformly asymptotically stable.*

**Theorem 1.6** *([26]) (Lyapunov-Razumikhin Stability Theorem). Suppose* $f : \mathbb{R} \times C_{in} \to \mathbb{R}^n$ *takes* $\mathbb{R}\times$*(bounded sets of* $C_{in}$*) into bounded sets of* $\mathbb{R}^n$*, and that* $u,\ v,\ w : \mathbb{R}_+ \to \mathbb{R}_+$ *are continuous nondecreasing functions,* $u(s)$ *and* $v(s)$ *are positive for* $s > 0$*, and* $u(0) = v(0) = 0$*,* $v$ *is strictly increasing. If there exists a continuously differentiable functtion* $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *such that*

$$u(|x|) \le V(t, x) \le v(|x|), \quad for\ t \in \mathbb{R}\ and\ x \in \mathbb{R}^n \tag{1.15}$$

*and the derivative of* $V$ *along the solution* $x(t)$ *of (1.12) satisfies*

$$\dot{V}(t, x(t)) \le -w(|x(t)|) \quad if\ V(t+\theta, x(t+\theta)) \le V(t, x(t)) \tag{1.16}$$

*for* $\theta \in [-\tau, 0]$*, then the system (1.12) is uniformly stable.*

*If, in addition,* $w(s) > 0$ *for* $s > 0$*, and there exists a continuous nondecreasing function* $p(s) > s$ *for* $s > 0$ *such that the condition (1.16) is strengthened to*

$$\dot{V}(t, x(t)) \le -w(|x(t)|) \quad if\ V(t+\theta, x(t+\theta)) \le p(V(t, x(t))) \tag{1.17}$$

*for $\theta \in [-\tau, 0]$, then the system (1.12) is uniformly asymptotically stable.*

*If in addition $\lim_{s\to\infty} u(s) = \infty$, then the system (1.12) is globally uniformly asymptotically stable.*

**Lemma 1.1** *([27]) (Halanay's inequality). If there exists a nonnegative continuous function $f(t)$ on $[t_0 - \tau, t_0]$ such that*

$$\dot{f}(t) \leq -\alpha f(t) + \beta \sup_{s\in[t-\tau,t]} f(s) \tag{1.18}$$

*for $t \geq t_0$ and if $\alpha > \beta > 0$, then there exists $\alpha > 0$ and $k > 0$ such that*

$$f(t) \leq k e^{-\gamma(t-t_0)} \tag{1.19}$$

*for $t \geq t_0$.*

## 1.4 Summary of Contributions

This section provides a cursory glance at the contributions and constitution of the entire dissertation.

*Chapter 2*

We propose a new technique to design observers and stabilizing dynamic output feedbacks for switched linear systems with a time-varying pointwise delay in the output. First, we develop an extension of the trajectory based stability result recently proposed in [28] to establish the stability of the closed-loop switched system. We wish to emphasize that the new extension of the trajectory based approach we state and prove in this chapter is of interest by itself: it can be applied to a wide range of systems, notably to families of systems with time-varying delays wider than those invoked in [28] and [29], and therefore it is one of the important contributions of this chapter. In its simplest version, this new technique entails to verifying that there exist constants $\epsilon \in (0, 1)$ and $T > 0$ such that each trajectory $x$ of a system satisfies an inequality of the form $|x(t)| \leq$

$\epsilon \sup_{\ell \in [t-T,t]} |x(\ell)|$ for all $t \geq T$. Second, the stabilization result we develop in Chapter 2 is, to the best of our knowledge, new.

*Chapter 3*

We have two main objectives this chapter. Our first aim is to construct finite time smooth observers for nonlinear systems. To the best of our knowledge, very few works design finite time smooth observers for nonlinear systems; see, e.g., [30] for a finite time smooth observer for nonlinear systems. The preceding finite time observer design approach was carried out without considering disturbances in the measurements or dynamics. Such disturbances are usually present in practical applications and they affect measurements, and can be state dependent. Motivated by this fact, we provide a finite time state estimation algorithm for nonlinear systems with state dependent disturbances. Nonlinear systems with state dependent disturbances that we consider in this chapter arise in many engineering contexts, e.g., in the modeling of vibrations of elastic membranes; see [31]. Our observer design approach has two main advantages. First, our finite time observers are smooth and they have better robust performance as compared to nonsmooth observers. Second, the convergence time in our method is independent of the initial state and it can be rendered as small as desired by the selection of a parameter (called the artificial delay). Our second result is dynamic output feedback design for a class of nonlinear systems with temporary loss of output measurements. It is motivated by the fact that in many engineering applications, the state is not available for measurement, and the output measurements are only available intermittently, meaning there may be intervals during which there is no output measurement, e.g., due to communication failures in GPS-denied environments. Our strategy has the following steps. We combine our finite time observer design and classical results for switched systems to construct a dynamic output feedback using a continuous-discrete observer. To establish the stability of the closed-loop system, we use an extension of the trajectory based approach proposed in [28]. To the best of our knowledge, the stabilization problem we describe with temporary loss of output measurements has remained unsolved in the literature, even in the case of linear systems, so our proposed tools are of considerable independent interest.

*Chapter 4*

Since systems are frequently time-varying and tracking problems can be recasted into stabilization problems of equilibria of time-varying systems, and since the measured components of the state do not need to be estimated, this chapter is devoted to the construction of finite-time reduced order observers for a family of nonlinear time-varying systems. The observer we will build gives estimates only of the unmeasured variables, as does the asymptotic observer proposed for instance in [32, Chapt. 4, Sec. 4.4.3]. This feature presents the following technical advantages. The design is simpler and in some cases, due to the need to determine fundamental solutions of time-varying systems, considering systems with a smaller dimension than the studied one makes it possible to solve the problem, which would be impossible if we were attempting to construct a full-order observer, due to the difficulty of determining explicit formulas for fundamental solutions of systems of dimension larger than 1. In addition, the reduced order observer we propose yield better performances than full order observers, in some cases. To the best of our knowledge, finite-time reduced order observers are proposed for the first time in the present paper. Additionally, we give a second new result where we show how the finite-time observer we propose can be used to solve a dynamic output feedback stabilization problem for a MIMO nonlinear system.

*Chapter 5*

This chapter has two goals. First, we construct a finite time observer for a linear continuous-time system with sampled output in the presence of a disturbance in the dynamics of the system. The observer is expressed in terms of the fundamental solution of suitable time-varying system. Second, we obtain more explicit formulas for finite time observers that do not contain the fundamental matrix and therefore may be better suited to implementations where the fundamental matrix is not available in explicit closed form.

*Chapter 6*

We summarize the value added by our paper and suggest future research directions in this chapter.

## 1.5 Notation

Throughout the sequel, the notation will be simplified whenever no confusion can arise from the context.

- The dimensions of our Euclidean spaces are arbitrary unless otherwise noted.
- The usual Euclidean norm of vectors, and the induced norm of matrices, are denoted by $|\cdot|$.
- $I$ denotes the identity matrix of any dimension.
- $x^T$ or $A^T$ denotes the transpose of a vector $x$ or a matrix $A$.
- For any constant $\tau > 0$, any continuous function $\phi : [-\tau, +\infty) \to \mathbb{R}^n$ and all $t \geq 0$, we define $\phi_t$ by $\phi_t(m) = \phi(t+m)$ for all $m \in [-\tau, 0]$.
- We let $C_{in}$ denote the set $C([-\tau, 0])$ of all continuous functions $\phi : [-\tau, 0] \to \mathbb{R}^n$.
- A vector or a matrix is nonnegative (resp. positive) if all of its entries are nonnegative (resp. positive).
- We write $M \succ 0$ (resp. $M \preceq 0$) to indicate that $M$ is a symmetric positive definite (resp. negative semi-definite) matrix.
- For two vectors $V = (v_1 ... v_n)^\top$ and $U = (u_1 ... u_n)^\top$, we write $V \leq U$ to indicate that for all $i \in \{1, ..., n\}$, $v_i \leq u_i$.
- A matrix is called Schur stable provided its spectral radius $\sigma$ satisfies $\sigma < 1$.
- Let $|\cdot|_\infty$ denote the sup norm of any matrix valued function over its entire domain.

- Let $\exp(f)$ denotes the real valued function $e^f$ for any real valued function $f$.

- For any continuous function $\Omega : [-\tau, +\infty) \to \mathbb{R}^{n \times n}$, we let $\Phi_\Omega$ denote the function such that
$$\frac{\partial \Phi_\Omega}{\partial t}(t, t_0) = -\Phi_\Omega(t, t_0)\Omega(t)$$
and $\Phi_\Omega(t_0, t_0) = I$ for all $t \in \mathbb{R}$ and $t_0 \in \mathbb{R}$. Then $\mathcal{M}(t, s) = \Phi_\Omega^{-1}(t, s)$ is the fundamental solution associated to $\Omega$ for the system $\dot{x} = \Omega(t)x$; see [33, Lemma C.4.1].

- $t^+$ ($t^-$) denotes time right after (right before) $t$.

# Chapter 2

# Observer based Stabilization of Switched Systems with Delay

*The content of this chapter is based on the publications [34] and [35] which were published by the author and his advisors while the study of this dissertation was in progress.* Many questions related to switched systems such as stability ([36]; [13]; [15]; [37]), controllability ([38]; [21]), observability and reachability ([21]; [39]; [40]; [41]), and synthesis ([42]; [22]) have been extensively studied in various contributions. Stability and stabilization are amongst the most challenging problems pertaining to switched systems due to their hybrid nature [43], and they are the main topic of the present chapter.

Before describing the results we will propose, let us review two basic approaches classically utilized in the literature for establishing the stability of switched systems and some issues related to these techniques:

(i) It is shown in [15] that existence of a common strict Lyapunov function (a Lyapunov function whose derivative along the trajectory of all of the subsystems of the switched system is definite negative) is a necessary and sufficient condition for the switched system to be stable under arbitrary switching. On the other hand, when such a Lyapunov function exists, finding it may be a difficult task because it is an NP-hard problem; see [44].

(ii) It is also shown in [15] that even if a switched system does not possess a common strict Lyapunov function, it may be stable, under requirements such as for instance a condition on the size of the difference between two consecutive switching instants. This fundamental result is termed as dwell-time approach in the literature. When this size is sufficiently large, then a so-called dwell-time result can be established. Typically, the proof uses multiple strict Lyapunov functions. It is worth mentioning that multiple Lyapunov functions may lead to an undesirable attenuation property which can only be mitigated by imposing some strong assumptions; see [45].

Both of the above mentioned approaches are mainly developed for non-delayed systems. But measurement delays are present in many practical applications, such as chemical processes, aerodynamics and communication networks, and they are time-varying (see for instance [46] and [47]). Therefore, the problem of stabilizing switched systems when a time-varying delay is present in the output is strongly motivated. On the other hand, it is a difficult problem because it does not seem possible to directly extend the classical Lyapunov function based approaches mentioned above to the output feedback case considered in this chapter. This comment is not in contradiction with the fact that some control problems for switched systems with delay in the input have been solved. Let us recall some of them, which complement our contribution.

## 2.1 Literature Review

Feedback stabilization of delayed switched linear systems is proposed in [48] using a combination of the multiple Lyapunov functions approach and the merging switching signal technique. An online and offline feedback controller design for delayed switched linear systems in the detection of the switching signal are discussed in [49]. Moreover, [50] and [51] present feedback controller designs for delayed switched systems using a dwell-time based stability analysis approach. Note that

[48], [49], [50], and [51] assume that all of the subsystems of the switched system are controllable. A stabilization problem for a class of delayed switched systems is studied in [52] and [53] under the assumption that the subsystems satisfy a certain Hurwitz convex combination condition. A common Lyapunov function approach is used in [52] and [53] to carry out stability analysis. It is assumed in [52] that the delay is constant and [53] requires the derivative of the delay to be bounded. Also note that [48], [49], [50], [51], [52], and [53] present state feedback designs only and it seems to us that there is no direct way to extend them to the output feedback case considered in this chapter. Moreover, most of these contributions use classical Lyapunov function based approaches to establish stability of the closed-loop switched systems. However, it becomes difficult to search appropriate Lyapunov functions for the case when there is a time varying delay in the output of the switched systems.

## 2.2 Contributions of this chapter

We think that our main result can be regarded as an extension of [48], [49], [50], [51], [52], [53], and [54], offering new advantages that are listed below:

(i) Our main result does not require all of the subsystems to be stabilizable and detectable.

(ii) Our feedback stabilization approach does not assume that all the states are available for feedback.

(iii) We use a new extension of trajectory based approach for stability analysis which circumvents the serious obstacle presented by the search for appropriate Lyapunov functions.

(iv) We allow the delay to be time-varying and piecewise continuous function of time, and we do not impose any constraint on the upper bound of the delay derivative.

(v) We do not assume that the systems have synchronous switching sequences.

(vi) The application of our results is not restricted to the class of delayed switched systems where all the convex combinations of the subsystems in the absence of control must be Hurwitz.

The rest of this chapter is organized as follows. An extension of the trajectory based approach is given in Section 2.3. Section 2.4 is devoted to the main result of the chapter. Section 2.5 discusses computational issues related to the delay bound. The results are illustrated by a numerical example in Section 2.6.

## 2.3 Trajectory based approach and its extension

For stabilization of time varying systems with time varying delays, Lyapunov-Krasovskii functionals or Razhumukin functions [55] are used in much of the literature. However, finding strict Lyapunov functions can be difficult. Instead, it is easier to verify that there exist constants $\epsilon \in (0,1)$ and $T > 0$ such that each trajectory $x$ of a system satisfies an inequality of the form $|x(t)| \leq \epsilon \sup_{\ell \in [t-T,t]} |x(\ell)|$ for all $t \geq T$. This provides motivation for trajectory based approach proposed in [28]. It relies on ISS stability notion and it yields ISS stability with respect to uncertainty. The two main advantages of trajectory based approach are 1) it can be applied to a broad class of time varying systems with time varying delays and 2) it is easier to apply as compared to Lyapunov or small-gain methods. We now provide the following key lemma which forms the basis of trajectory based approach.

**Lemma 2.1** *[28] (Trajectory based approach). Let $T^* > 0$ be a constant. Let a piecewise continuous function $w : [-T^*, +\infty) \to [0, +\infty)$ admits a sequence of real numbers $v_i$ and positive constants $\bar{v}_a$ and $\bar{v}_b$ such that $v_0 = 0$, $v_{i+1} - v_i \in [\bar{v}_a, \bar{v}_b]$ for all $i \geq 0$, $w$ is continuous on each interval $[v_i, v_{i+1})$ for all $i \geq 0$, and $w(v_i{}^-)$ exists and is finite for each $i \in \mathbb{N}$. Let $d : [0, +\infty) \to [0, +\infty)$ be any piecewise continuous function, and assume that there is a constant $\rho \in (0,1)$ such that*

$$w(t) \leq \rho |w|_{[t-T^*,t]} + d(t) \tag{2.1}$$

*holds for all* $t \geq 0$. *Then*

$$w(t) \leq |w|_{[-T^*,0]} e^{\frac{ln(\rho)}{T^*}} + \frac{1}{(1-\rho)^2}|d|_{[0,t]} \tag{2.2}$$

*holds for all* $t \geq 0$.

We now provide with an extension of the trajectory based approach given in [28].

**Lemma 2.2** *Let us consider a constant* $T > 0$ *and* $l$ *functions* $z_g : [-T, +\infty) \to [0, +\infty)$, $g = 1, ..., l$. *Let* $Z(t) = (z_1(t) \ ... \ z_l(t))^\top$ *and, for any* $\theta \geq 0$ *and* $t \geq \theta$, *define* $\mathfrak{V}_\theta(t) = \left( \sup_{s\in[t-\theta,t]} z_1(s) \ ... \ \sup_{s\in[t-\theta,t]} z_l(s) \right)^\top$. *Let* $\Upsilon \in \mathbb{R}^{l\times l}$ *be a nonnegative Schur stable matrix. If for all* $t \geq 0$, *the inequalities*

$$Z(t) \leq \Upsilon \mathfrak{V}_T(t)$$

*are satisfied, then*

$$\lim_{t\to+\infty} z_g(t) = 0$$

*for all* $g = 1, \ldots, l$.

**Proof.** Since $\Upsilon$ is Schur stable, there is an integer $q > 1$ such that

$$|\Upsilon^q|\sqrt{l} < 1 \ . \tag{2.3}$$

From Lemma A.1 of the Appendix A, we deduce that

$$Z(t) \leq \Upsilon^q \mathfrak{V}_{qT}(t) \tag{2.4}$$

for all $t \geq qT$. Consequently,

$$|Z(t)| \leq |\Upsilon^q||\mathfrak{V}_{qT}(t)|.$$

Using

$$\begin{aligned} |\mathfrak{V}_{qT}(t)| &= \sqrt{\sum_{i=1}^{l} \sup_{s\in[t-qT,t]} z_i(s)^2} \\ &\leq \sqrt{l} \sup_{s\in[t-qT,t]} |Z(s)|, \end{aligned} \tag{2.5}$$

we obtain

$$|Z(t)| \leq |\Upsilon^q|\sqrt{l} \sup_{s\in[t-qT,t]} |Z(s)|. \tag{2.6}$$

This inequality, in combination with the inequality (2.3) and Lemma 2.1 allows us to conclude the result. □

## 2.4 Observer and control design

We introduce a range dwell-time condition, i.e. a sequence of real numbers $t_k$ such that there are two positive constants $\underline{\delta}$ and $\overline{\delta}$ such that $t_0 = 0$ and for all $k \in \mathbb{Z}_{\geq 0}$,

$$t_{k+1} - t_k \in [\underline{\delta}, \overline{\delta}] \ . \tag{2.7}$$

**Definition 2.1** *Let $\pi = \{(i_0, t_0), \dots, (i_k, t_k), \dots, |i_k \in \Xi,\ k \in \mathbb{Z}_{\geq 0}\}$ be a switching sequence. The function $\sigma : [0, \infty) \to \Xi = \{1, ..., n\}$ such that $\sigma(t) = i_k$ when $t \in [t_k, t_{k+1})$ is called an associated switching signal.*

We consider the continuous-time switched linear system with output delay:

$$\begin{cases} \dot{x}(t) &=& A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) &=& C_{\sigma(t)}x(t-\tau(t)) \end{cases} \tag{2.8}$$

with $x \in \mathbb{R}^{d_x}$, $u \in \mathbb{R}^{d_u}$, $y \in \mathbb{R}^{d_y}$, for all $t \geq 0$, $\tau(t) \in [0, \overline{\tau}]$ with $\overline{\tau} > 0$ and an initial condition in $C_{\text{in}}$. The delay $\tau(t)$ is supposed to be a piecewise continuous function. For any $i \in \Xi$, $A_i$, $B_i$, and $C_i$ are real and constant matrices of compatible dimensions and $\sigma$ is a switching signal. We introduce an assumption which pertains to the stabilizability and the detectability of the system (2.8), but does not imply that all the pairs $(A_i, B_i)$ are stabilizable and all the pairs $(A_i, C_i)$ are detectable.

**Assumption 2.1** *There are matrices $K_i$ and $L_i$ for all $i \in \Xi$ and constants $T \geq \bar{\tau}$, $a \in [0,1)$, $b \geq 0$, $c \in [0,1)$ and $d \geq 0$ such that the solutions of the system*

$$\dot{\alpha}(t) = M_{\sigma(t)}\alpha(t) + \zeta(t) \tag{2.9}$$

*with $M_i = A_i + B_i K_i$ and $\zeta$ being a piecewise continuous function, satisfy*

$$|\alpha(t)| \leq a|\alpha(t-T)| + b \sup_{\ell \in [t-T,t]} |\zeta(\ell)| \tag{2.10}$$

*for all $t \geq T$. Similarly, the solutions of the system*

$$\dot{\beta}(t) = N_{\sigma(t)}\beta(t) + \eta(t) \tag{2.11}$$

*with $N_i = A_i + L_i C_i$ and $\eta$ being a piecewise continuous function, satisfy the following inequality for all $t \geq T$*

$$|\beta(t)| \leq c|\beta(t-T)| + d \sup_{\ell \in [t-T,t]} |\eta(\ell)| \,. \tag{2.12}$$

**Theorem 2.1** *Let the system (2.8) satisfy Assumption 2.1 and, $s_1$, $s_2$ and $s_3$ be defined by*

$$s_1 = \sup_{i \in \Xi} |B_i K_i| \,, \; s_2 = \sup_{i \in \Xi} |L_i C_i| \,, \; s_3 = \sup_{i \in \Xi} |M_i| \,. \tag{2.13}$$

*If*

$$\tau(t) \leq \bar{\tau} < \bar{\tau}_u \tag{2.14}$$

*for all $t \geq 0$, where*

$$\bar{\tau}_u = \frac{(1-a)(1-c)}{d s_1 s_2 ((1-a) + b s_3)} \,, \tag{2.15}$$

*then the origin of the following feedback system is globally uniformly exponentially stable (GUES):*

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & A_{\sigma(t)}x(t) + B_{\sigma(t)}K_{\sigma(t)}\hat{x}(t) \\ \dot{\hat{x}}(t) & = & A_{\sigma(t)}\hat{x}(t) + B_{\sigma(t)}K_{\sigma(t)}\hat{x}(t) + L_{\sigma(t)}[C_{\sigma(t)}\hat{x}(t) - y(t)] \,. \end{array} \right. \tag{2.16}$$

**Proof.** Let us introduce $\tilde{x}(t) = \hat{x}(t) - x(t)$. Then

$$\dot{\tilde{x}}(t) = A_{\sigma(t)}\tilde{x}(t) + L_{\sigma(t)}[C_{\sigma(t)}\hat{x}(t) - C_{\sigma(t)}x(t-\tau(t))] \,.$$

As an immediate consequence, using the definitions of the matrices $M_i$ and $N_i$, we obtain

$$\begin{cases} \dot{x}(t) &= M_{\sigma(t)}x(t) + B_{\sigma(t)}K_{\sigma(t)}\tilde{x}(t) \\ \dot{\tilde{x}}(t) &= N_{\sigma(t)}\tilde{x}(t) + L_{\sigma(t)}C_{\sigma(t)}[x(t) - x(t-\tau(t))] \ . \end{cases}$$

From Assumption 2.1 and the equality $x(\ell) - x(\ell - \tau(\ell)) = \int_{\ell-\tau(\ell)}^{\ell}[M_{\sigma(m)}x(m) + B_{\sigma(m)}K_{\sigma(m)}\tilde{x}(m)]dm$, it follows that, for all $t \geq T + \bar{\tau}$,

$$|x(t)| \leq a|x(t-T)| + b \sup_{\ell\in[t-T,t]} |B_{\sigma(\ell)}K_{\sigma(\ell)}\tilde{x}(\ell)| \ , \tag{2.17}$$

$$\begin{aligned} |\tilde{x}(t)| \leq c|\tilde{x}(t-T)| + d \sup_{\ell\in[t-T,t]} \Big| L_{\sigma(\ell)}C_{\sigma(\ell)} \\ \times \int_{\ell-\tau(\ell)}^{\ell}[M_{\sigma(m)}x(m) + B_{\sigma(m)}K_{\sigma(m)}\tilde{x}(m)]dm \Big| \ . \end{aligned} \tag{2.18}$$

Using the constants defined in (2.13), we deduce from (2.17) and (2.18) that $(x(t), \tilde{x}(t))$ satisfies:

$$\begin{aligned} |x(t)| &\leq a|x(t-T)| + bs_1 \sup_{\ell\in[t-T-\bar{\tau},t]} |\tilde{x}(\ell)| \ , \\ |\tilde{x}(t)| &\leq ds_2s_3\bar{\tau} \sup_{\ell\in[t-T-\bar{\tau},t]} |x(\ell)| + (c + ds_1s_2\bar{\tau}) \sup_{\ell\in[t-T-\bar{\tau},t]} |\tilde{x}(\ell)| \ . \end{aligned}$$

Lemma 2.2 ensures that the origin of (2.16) is GUES if

$$\begin{bmatrix} a & bs_1 \\ ds_2s_3\bar{\tau} & ds_1s_2\bar{\tau} + c \end{bmatrix}$$

is Schur stable, which is equivalent to

$$\tfrac{a+c+ds_1s_2\bar{\tau}}{2} + \sqrt{\left(\tfrac{a+c+ds_1s_2\bar{\tau}}{2}\right)^2 - ac - ds_1s_2\left(a - bs_3\right)\bar{\tau}} < 1 \ ,$$

from which we derive the simpler condition (2.14). □

## 2.5 Parameters of the delay bound

In this section, we illustrate a method to determine the constants $a$, $b$, $c$, and $d$ appearing in Assumption 2.1.

Consider a continuous-time switched linear system

$$\dot{\xi}(t) = \Omega_{\sigma(t)}\xi(t) + \vartheta(t)\ , \tag{2.19}$$

where $\xi \in \mathbb{R}^{d_\xi}$, the switching signal $\sigma$ is associated to a sequence $t_k$ of the type of those introduced in Section 2.4 and $\vartheta$ is a piecewise continuous function.

**Lemma 2.3** *Let the system (2.19) be such that there are real numbers $d_1 > 0$, $d_2 > 0$, $\mu \geq 1$, $\gamma > 0$ and symmetric positive definite matrices $Q_l$, $l \in \Xi$, such that the LMIs*

$$d_1 I \preceq Q_i \preceq d_2 I\ , \tag{2.20}$$

$$Q_i \preceq \mu Q_j\ , \tag{2.21}$$

$$\Omega_i^\top Q_i + Q_i\Omega_i \preceq -\gamma Q_i \tag{2.22}$$

*are satisfied for all $i,\ j \in \Xi$. Moreover, the constant $\mu_\triangle = \mu e^{-\gamma\underline{\delta}}$ is such that*

$$\mu_\triangle < 1\ . \tag{2.23}$$

*Then, along the trajectory of (2.19), the inequality*

$$|\xi(t)| \leq \sqrt{\frac{d_2}{d_1}\mu\mu_\triangle^\rho e^{\gamma\overline{\delta}}}|\xi(t-T)| + \sqrt{\mu\frac{d_2}{\gamma d_1}T}\sup_{\ell\in[t-T,t]}|\vartheta(\ell)|$$

*holds for all $t \geq T$ where $T > 0$ and $\rho$ is a positive integer depending on the choice of $T$ such that for all $t \in [t_k, t_{k+1})$, we have $t - T \in [t_{k-\rho-1}, t_{k-\rho})$. Moreover, we have $\sqrt{\frac{d_2}{d_1}\mu\mu_\triangle^\rho e^{\gamma\overline{\delta}}} < 1$ when $\rho > \frac{1}{\ln(\mu_\triangle)}\left[\ln\left(\frac{d_1}{d_2\mu}\right) - \gamma\overline{\delta}\right]$.*

**Proof.** Let us define Lyapunov functions:

$$\mathcal{V}_i(\xi) = \xi^\top Q_i\xi,\ \forall i \in \Xi\ .$$

We deduce from (2.22) that when $\sigma(t) = i$, then the derivative of $\mathcal{V}_i$ along the trajectories of (2.19) satisfies

$$\begin{array}{rcl}\dot{\mathcal{V}}_i(\xi(t)) & \leq & -2\gamma\mathcal{V}_i(\xi(t)) + 2\xi(t)^\top Q_i\vartheta(t) \\ & \leq & -\gamma\mathcal{V}_i(\xi(t)) + \frac{1}{\gamma}\vartheta(t)^\top Q_i\vartheta(t)\end{array} \tag{2.24}$$

where the last inequality is deduced from the Young's inequality. Now, let us integrate (2.24) between two instants $s$ and $t$, $t \geq s$, belonging to the same sampling interval where $\sigma(t) = l$. Then

$$e^{\gamma t}\mathcal{V}_l(\xi(t)) \;\; \leq \;\; e^{\gamma s}\mathcal{V}_l(\xi(s)) + \tfrac{1}{\gamma}\int_s^t e^{\gamma m}\vartheta(m)^\top Q_l \vartheta(m)dm.$$

It follows that

$$\begin{aligned} \mathcal{V}_l(\xi(t)) \;\; &\leq \;\; e^{\gamma(s-t)}\mathcal{V}_l(\xi(s)) + \tfrac{1}{\gamma}\int_s^t e^{\gamma m-\gamma t}\vartheta(m)^\top Q_l \vartheta(m)dm \\ &\leq \;\; e^{\gamma(s-t)}\mathcal{V}_l(\xi(s)) + \tfrac{d_2}{\gamma}\int_s^t e^{\gamma m-\gamma t}|\vartheta(m)|^2 dm\ , \end{aligned} \tag{2.25}$$

where the last inequality is a consequence of (2.20). Now, let us consider $T > 0$, $t \geq T$ such that $t \in [t_k, t_{k+1})$ for some $k \in \mathbb{Z}_{\geq 0}$ and let $\rho \in \mathbb{N}$ be such that $t - T \in [t_{k-\rho-1}, t_{k-\rho})$. From (2.25), we deduce that

$$\begin{aligned} \mathcal{V}_{\sigma(t_k)}(\xi(t)) \;\; &\leq \;\; e^{-\gamma(t-t_k)}\mathcal{V}_{\sigma(t_k)}(\xi(t_k)) + \tfrac{d_2}{\gamma}\int_{t_k}^t e^{\gamma m-\gamma t}|\vartheta(m)|^2 dm \\ &\leq \;\; \mu e^{-\gamma(t-t_k)}\mathcal{V}_{\sigma(t_{k-1})}(\xi(t_k)) + \tfrac{d_2}{\gamma}\int_{t_k}^t |\vartheta(m)|^2 dm\ , \end{aligned} \tag{2.26}$$

where the last inequality is a consequence of (2.21). For similar reasons,

$$\begin{aligned} \mathcal{V}_{\sigma(t_{k-1})}(\xi(t_k)) \quad &\leq \;\; \mu_\triangle \mathcal{V}_{\sigma(t_{k-2})}(\xi(t_{k-1}))\tfrac{d_2}{\gamma}\int_{t_{k-1}}^{t_k} |\vartheta(m)|^2 dm \\ &\;\;\vdots \\ \mathcal{V}_{\sigma(t_{k-\rho})}(\xi(t_{k-\rho+1})) \;\; &\leq \;\; \mu_\triangle \mathcal{V}_{\sigma(t_{k-\rho-1})}(\xi(t_{k-\rho})) + \tfrac{d_2}{\gamma}\int_{t_{k-\rho}}^{t_{k-\rho+1}} |\vartheta(m)|^2 dm \end{aligned} \tag{2.27}$$

and

$$\begin{aligned} \mathcal{V}_{\sigma(t_{k-\rho-1})}(\xi(t_{k-\rho})) \;\; \leq \;\; & e^{\gamma(t-T-t_{k-\rho})}\mathcal{V}_{\sigma(t_{k-\rho-1})}(\xi(t-T)) \\ & +\tfrac{d_2}{\gamma}\int_{t-T}^{t_{k-\rho}} |\vartheta(m)|^2 dm\ . \end{aligned} \tag{2.28}$$

We deduce from (2.27) that

$$\mathcal{V}_{\sigma(t_{k-1})}(\xi(t_k)) \;\; \leq \;\; \mu_\triangle^\rho \mathcal{V}_{\sigma(t_{k-\rho-1})}(\xi(t_{k-\rho})) + \tfrac{d_2}{\gamma}\int_{t_{k-\rho}}^{t_k} |\vartheta(m)|^2 dm. \tag{2.29}$$

Combining (2.26), (2.28) and (2.29), we obtain

$$\begin{aligned}
\mathcal{V}_{\sigma(t_k)}(\xi(t)) &\leq \mu e^{-\gamma(t-t_k)}\left\{\mu_\triangle^\rho \mathcal{V}_{\sigma(t_{k-\rho-1})}(\xi(t_{k-\rho})) + \tfrac{d_2}{\gamma}\int_{t_{k-\rho}}^{t_k} |\vartheta(m)|^2 dm\right\} \\
&\quad + \tfrac{d_2}{\gamma}\int_{t_k}^{t} |\vartheta(m)|^2 dm \\
&\leq \mu e^{-\gamma(t-t_k)}\mu_\triangle^\rho e^{\gamma(t-T-t_{k-\rho})}\mathcal{V}_{\sigma(t_{k-\rho-1})}(\xi(t-T)) \\
&\quad + \mu e^{-\gamma(t-t_k)}\mu_\triangle^\rho \tfrac{d_2}{\gamma}\int_{t-T}^{t_{k-\rho}} |\vartheta(m)|^2 dm \\
&\quad + \mu e^{-\gamma(t-t_k)}\tfrac{d_2}{\gamma}\int_{t_{k-\rho}}^{t_k} |\vartheta(m)|^2 dm + \frac{d_2}{\gamma}\int_{t_k}^{t} |\vartheta(m)|^2 dm.
\end{aligned}$$

Then using the definition of range dwell-time condition from (2.7), we get

$$\mathcal{V}_{\sigma(t_k)}(\xi(t)) \leq \mu\ \mu_\triangle^\rho e^{\gamma\bar{\delta}}\mathcal{V}_{\sigma(t_{k-\rho-1})}(\xi(t-T)) + \mu\tfrac{d_2}{\gamma}\int_{t-T}^{t} |\vartheta(m)|^2 dm\ .$$

From (2.20), we deduce that

$$d_1|\xi(t)|^2 \ \leq\ \mu\mu_\triangle^\rho e^{\gamma\bar{\delta}} d_2|\xi(t-T)|^2 + \mu\tfrac{d_2}{\gamma}\int_{t-T}^{t} |\vartheta(m)|^2 dm.$$

Using the inequality $\sqrt{p_1+p_2} \leq \sqrt{p_1}+\sqrt{p_2}$ for all $p_1 \geq 0$, $p_2 \geq 0$, we obtain

$$|\xi(t)| \leq \sqrt{\frac{d_2}{d_1}\mu\mu_\triangle^\rho e^{\gamma\bar{\delta}}}|\xi(t-T)| + \sqrt{\mu\frac{d_2}{\gamma d_1}T}\sup_{\ell\in[t-T,t]}|\vartheta(\ell)|\ .$$

Since (2.23) holds and $T$ is arbitrarily large, one can choose $T$ such that the corresponding $\rho$ is so that $\sqrt{\frac{d_2}{d_1}\mu\mu_\triangle^\rho e^{\gamma\bar{\delta}}} < 1$. This concludes the proof. □

**Remark 2.1**

*1. Note that (2.23) holds if and only if* $\underline{\delta} > \frac{\ln(\mu)}{\gamma}$*, which defines a minimum dwell-time condition.*

*2. Conditions of Lemma 2.3 are always satisfied when the matrices* $\Omega_i$*,* $\forall\ i \in \Xi$*, are Hurwitz; i.e., one can always find symmetric positive definite matrices* $Q_i$*,* $i \in \Xi$*, and real numbers* $d_1 > 0$*,* $d_2 > 0$*,* $\mu \geq 1$*,* $\gamma > 0$ *satisfying the LMIs (2.20), (2.21), and (2.22). In the next section we illustrate an alternative approach for the case where some of* $\Omega_i$*'s are not Hurwitz.*

## 2.6 Illustrative Example

Consider the continuous-time switched linear system (2.8) with $x \in \mathbb{R}^2$, $\tau \in [0, \overline{\tau})$,

$$\sigma(t) = \begin{cases} 1 & \text{if } 4\ell\kappa \leq t < (4\ell+3)\kappa \\ 2 & \text{if } (4\ell+3)\kappa \leq t < 4(\ell+1)\kappa \ , \end{cases} \tag{2.30}$$

where $\kappa > 0$ is to be determined, $\ell = 0, 1, 2, \ldots$, and

$$A_1 = \begin{bmatrix} 0 & -1/2 \\ 2/5 & 0 \end{bmatrix}, \ B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ,$$

$$A_2 = \begin{bmatrix} 0 & -2/5 \\ 1/2 & 0 \end{bmatrix}, \ B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ C_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} .$$

Let us observe that the subsystem $(A_1, B_1, C_1)$ is not stabilizable but it is detectable whereas the subsystem $(A_2, B_2, C_2)$ is stabilizable but not detectable. Moreover, in the absence of control, no convex combination of the $A_1$ and $A_2$ is Hurwitz. Furthermore, the subsystems cannot be stabilized by a static output feedback $u = K_i y$. In this example, we have $\underline{\delta} = \kappa$ and $\overline{\delta} = 3\kappa$ and the switchings are periodic with a period of $4\kappa$. We will determine a set of parameters for the delay bound depending on $\kappa$.

### 2.6.1 Preliminary result

First, we provide a preliminary result which shows how Assumption 2.1 can be satisfied in this particular example where some of the subsystems of the switched systems are not stabilizable and not detectable.

**Lemma 2.4** *Consider the switched linear system*

$$\dot{z}(t) = \Gamma_{\sigma(t)} z(t) + \varrho(t) \tag{2.31}$$

*with $\sigma$ defined by (2.30), and let $\Gamma_1 \in \mathbb{R}^{2\times 2}$, $\Gamma_2 \in \mathbb{R}^{2\times 2}$ and $\kappa > 0$ be such that the matrix $S_\kappa := e^{\Gamma_2 \kappa} e^{3\Gamma_1 \kappa}$ is Schur stable. Let $\Phi_\star$ be the state transition matrix* *of the system (2.31) with* $\varrho = 0$*:*

$$\frac{\partial \Phi_\star}{\partial t}(t,s) = \Gamma_{\sigma(t)}\Phi_\star(t,s)\ ,\ \Phi_\star(s,s) = I\ ,$$

*for all* $t \in \mathbb{R}$ *and* $s \in \mathbb{R}$*. Then, for all* $s \geq 0$*,* $t \geq s$

$$|\Phi_\star(t,s)| \leq p_1 e^{-p_2(t-s)} \tag{2.32}$$

*with* $p_1 = e^{8\kappa \max\{|\Gamma_1|,|\Gamma_2|\}} c_\kappa e^{2d_\kappa}$ *and* $p_2 = d_\kappa/4\kappa$*, where* $c_\kappa > 1$ *and* $d_\kappa > 0$ *are such that for all* $m \in \mathbb{N}$*,*

$$|S_\kappa^m| \leq c_\kappa e^{-d_\kappa m}\ . \tag{2.33}$$

*Moreover, for all* $T > 0$*,*

$$|z(t)| \leq p_1 e^{-p_2 T}|z(t-T)| + \tfrac{p_1\left(1-e^{-p_2 T}\right)}{p_2} \sup_{\ell \in [t-T,t]} |\varrho(\ell)|\ . \tag{2.34}$$

**Proof.** Let us introduce a sequence: $g_\ell = 4\ell\kappa$. Then for all integer $n > 0$,

$$z(g_\ell) = S_\kappa^n z(g_{\ell-n})\ . \tag{2.35}$$

Thus

$$\Phi_\star(g_\ell, g_{\ell-n}) = S_\kappa^n\ . \tag{2.36}$$

Let $t \in \mathbb{R}$ and $s \in \mathbb{R}$ be such that $t > s \geq t - 4\kappa$. Then

$$|\Phi_\star(t,s)| \leq e^{4\kappa \max\{|\Gamma_1|,|\Gamma_2|\}}\ . \tag{2.37}$$

Now, let $t \in \mathbb{R}$ and $s \in \mathbb{R}$ be such that $t + 4\kappa > s$. Then there is $\ell$ such that $t \in [g_\ell, g_{\ell+1})$ and $r \in \mathbb{N}$, $r > 0$ such that $s \in [g_{\ell-r-1}, g_{\ell-r})$. Then

$$|\Phi_\star(t,s)| \leq e^{8\kappa \max\{|\Gamma_1|,|\Gamma_2|\}} |\Phi_\star(g_\ell, g_{\ell-r})|\ . \tag{2.38}$$

It follows that

$$|\Phi_\star(t,s)| \leq e^{8\kappa \max\{|\Gamma_1|,|\Gamma_2|\}} |S_\kappa^r|\ .$$

Since $S_\kappa$ is Schur stable, there are $c_\kappa > 1$ and $d_\kappa > 0$ such that for all $m \in \mathbb{N}$,

$$|S_\kappa^m| \leq c_\kappa e^{-d_\kappa m}. \tag{2.39}$$

Thus

$$|\Phi_\star(t,s)| \leq e^{8\kappa \max\{|\Gamma_1|,|\Gamma_2|\}} c_\kappa e^{-d_\kappa r}. \tag{2.40}$$

Now, notice that $r \geq \frac{t-s}{4\kappa} - 2$. Consequently,

$$|\Phi_\star(t,s)| \leq e^{8\kappa \max\{|\Gamma_1|,|\Gamma_2|\}} c_\kappa e^{2d_\kappa} e^{-d_\kappa \frac{t-s}{4\kappa}}\ . \tag{2.41}$$

From (2.37) and (2.41), we deduce that for all $t \geq s$,

$$|\Phi_\star(t,s)| \leq e^{8\kappa \max\{|\Gamma_1|,|\Gamma_2|\}} c_\kappa e^{2d_\kappa} e^{-d_\kappa \frac{t-s}{4\kappa}}\ . \tag{2.42}$$

This allows us to conclude that (2.32) is satisfied.

Now, by integrating (2.31), we obtain that for all $t \geq T$,

$$\begin{aligned} |z(t)| &= \left|\Phi_\star(t,t-T)z(t-T) + \int_{t-T}^{t} \Phi_\star(t,\ell)\varrho(\ell)d\ell\right| \\ &\leq p_1 e^{-p_2 T}|z(t-T)| + \int_{t-T}^{t} p_1 e^{-p_2(t-\ell)} d\ell \sup_{\ell\in[t-T,t]} |\varrho(\ell)| \end{aligned}$$

where the last inequality is a consequence of (2.32). □

**Remark 2.2** *Since $p_2 > 0$, then $p_1 e^{-p_2 T} < 1$ when $T > \frac{\ln(p_1)}{p_2}$, which determines a lower bound for $T$.*

### 2.6.2 Output feedback stabilization

Let us choose the gain matrices as

$$K_2 = \begin{bmatrix} -1/2 & 0 \\ 0 & -4/7 \end{bmatrix},\ L_1 = \begin{bmatrix} -3/5 & 0 \\ 0 & -4/5 \end{bmatrix}.$$

Setting $\Gamma_1 = M_1 = A_1$ and $\Gamma_2 = M_2 = A_2 + B_2 K_2$, one can easily corroborate that (2.33) is satisfied with the choice of $\kappa = 0.1$, $c_k = 1.01$, and $d_k = 0.001$ for all $m \in \mathbb{N}$. Setting $z = \alpha$, $\Omega_i = \Gamma_i = M_i$ for $i \in \{1,2\}$, and $\varrho = \zeta$, it can be easily verified that (2.32) is satisfied by (2.9) with $p_1 = e^{8\kappa \max\{|\Gamma_1|,|\Gamma_2|\}} c_\kappa e^{2d_\kappa} = 1.7142$ and $p_2 = d_\kappa/4\kappa = 1.0025$. Using Lemma 2.4 with $T = 6$, one can observe that the solutions of system (2.9) satisfy (2.10) with $a = p_1 e^{-p_2 T} = 0.0042$, $b = (p_1/p_2)\left(1 - e^{-p_2 T}\right) = 1.7057$. A similar analysis shows that the solutions of system (2.11) satisfy (2.12) with $c = 0.0052$, $d = 2.1156$ and $T = 6$. Therefore, we conclude that the switched delay system satisfies Assumption 2.1. Finally, application of Theorem 2.1 with $s_1 = 0.5714$, $s_2 = 0.8$, $s_3 = 0.7611$, and with the preceding choices of the parameters yields $\bar{\tau}_u = 0.4465$. Fig. 2.1 shows the simulation of system (2.16) for this particular example for a piecewise continuous sawtooth function $\tau(t)$ of a fundamental frequency of 1 $Hz$ described by $\tau(t) = 0.2(t - \lfloor t \rfloor)$ where the switching signal $\sigma(t)$ is given by (2.30) with $\kappa = 0.1$. The initial conditions are chosen to be $x_1(0) = 7$, $x_2(0) = -5$, $\hat{x}_1(0) = -4$, and $\hat{x}_2(0) = 3$, and the sample rate is 1 $kHz$.

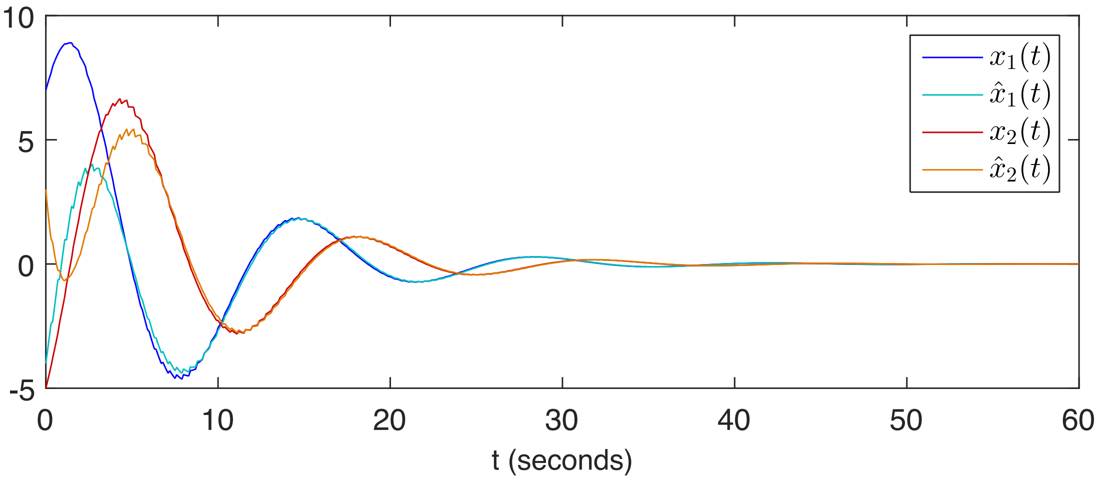


Figure 2.1: Simulation of the system (2.16): Component $x$ and its estimate $\hat{x}$

# Chapter 3

# Finite Time Observer Design and Stabilization of Nonlinear Systems with Intermittent Output Measurements

*The content of this chapter is based on the publication of the author [56].* State estimation of nonlinear systems from output measurements is a basic concern in robotics [57], chemical and biochemical processes [58], biomedical systems [59], communication systems [60], automotive systems [61], networked control systems [62], and many other fields. Due to this strong motivation, various techniques to achieve state observation of nonlinear systems have been discussed in the literature. These techniques have included canonical form observers [63], high-gain observers (as in [64], [65], and [66]), Lyapunov based observers (as in [67] and [68]), and extended Kalman and Luenberger observers (as in [69] and [70]).

The above mentioned observer design techniques have the common disadvantage that they only ensure asymptotic convergence of the estimation error to zero, whereas finite time convergence of estimation errors to zero is often desirable for control and supervision purposes; see [71], [72] and [73], and see [74] for more motivation for finite time control. One can distinguish between two broad families of finite time converging observers, namely, the families composed of nonsmooth observers without delay and the family of the smooth observers with delays. Nonsmooth finite time converging observers have been proposed for instance in [60], [71], [75], [76], and [77]. The main drawback of nonsmooth finite time observers is that their lack of smoothness may generate poor robustness performance, but they have no delay, which is an advantage because the presence of a delay may complicate the implementation of an observer. Another possible drawback is due to the fact that the time of convergence of each trajectory depends on its initial condition. This is not the case for the observers that use artificial delays, for which their instant of convergence of the solutions is independent of the initial conditions and can be rendered as small as desired by the selection of a parameter (called the artificial delay).

## 3.1 Literature Review

Finite time smooth observer designs have been introduced more recently. They were first presented in [78], which was only applicable to linear time invariant (LTI) systems; see also [79] for finite time observers for LTI systems. An extension of the design presented in [78] was carried out in [72] and [73] for linear time varying (LTV) systems and nonlinear systems in observer canonical (normal) form, respectively. A generalized finite time converging observer design technique for nonlinear systems was proposed in [30] and was applicable to noncanonical form nonlinear systems as well. This approach was developed through a Lyapunov based observer from [68].

The preceding finite time observer design approaches were carried out without considering disturbances in the measurements or dynamics. Such disturbances are usually present in practical applications and they affect measurements and can be state dependent. Motivated by this fact, a finite time state estimation algorithm for nonlinear systems with bounded and time-varying disturbances in the dynamics and measurements was recently proposed in [80]. The design approach used in [80] is similar to that of [30]. Other issues pertaining to finite time converging observer design for nonlinear systems have been discussed in [76], [81], and [82].

## 3.2 Contributions of this chapter

The present chapter has two main objectives. Our first aim is to complement [30] and [80], by proposing a finite time converging observer design for Lipschitz nonlinear systems of the form

$$\begin{cases} \dot{x}(t) &= [A + \epsilon(t, x(t))]x(t) + f(t, y(t), u(t)) \\ y(t) &= Cx(t) \end{cases} \tag{3.1}$$

where $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{q\times n}$ are constant matrices, the state $x$ is valued in $\mathbb{R}^n$, the input $u$ is valued in $\mathbb{R}^p$, the output $y$ is valued in $\mathbb{R}^q$, $f$ is a nonlinear function which is assumed to be locally Lipschitz with respect to $y$ and piecewise continuous in its other arguments, the locally Lipschitz function $\epsilon : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ can represent a disturbance, and the dimensions are arbitrary. Systems of the form (3.1) arise in many engineering contexts, e.g., in the modeling of vibrations of elastic membranes; see examples section below.

The key difference between the nonlinear systems in [30] and [80] and (3.1) is the presence of the function $\epsilon$. This disturbance significantly increases the difficulty of constructing a finite time observer, since it makes it impossible to apply [30] or [80] to (3.1) and it seems that there is no direct way to extend them to (3.1). The nonlinear term $\epsilon$ is worth considering because (i) disturbances of this type often affect systems, and (ii) this term will enable us to use a finite time observer to construct dynamic output feedback for a broad family of nonlinear time-varying systems with temporary loss of output measurements. Very few works design finite time observers for Lipschitz nonlinear systems; see, e.g., [77] for a finite time observer for Lipschitz nonlinear systems under homogeneity conditions that are not required here. An advantage of the observer approach adopted here is that the finite convergence time in our method for (3.1) is independent of the initial state. Second, we provide dynamic output feedback design for a class of nonlinear systems with temporary loss of output measurements. We use an assumption that is inspired by [51] and [83] and our finite time observer design to construct a dynamic output feedback through a continuous-discrete observer; see, e.g., [84] for continuous-discrete observers when the output values are available at all times instead of being intermittent. The dynamic output feedback designed here globally exponentially stabilizes the origin of the nonlinear systems with a temporary loss of measurements.

The rest of the chapter is organized as follows. In Section 3.3, we present two lemmas that will be used to prove our main result on finite time observers in Section 3.4. We use observers from Section 3.4 to prove a stabilization result under intermittent observations in Section 3.5. The example in Section 3.6 demonstrates the application of our results to controlled Mathieu equation.

## 3.3 Two Lemmas

The following lemmas will be used to prove the main results of this chapter.

**Lemma 3.1** *Let $\mathcal{A} \in \mathbb{R}^{n\times n}$ be a constant matrix. Consider the system*

$$\dot{\zeta}(t) = [\mathcal{A} + \mathcal{E}(t)]\,\zeta(t) \tag{3.2}$$

*where $\zeta$ is valued in $\mathbb{R}^n$ and $\mathcal{E} : [0, +\infty) \to \mathbb{R}^{n\times n}$ is a bounded locally Lipschitz function. Let $\phi$ denote the fundamental solution of the system (3.2). Then for all $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$ such that $t_1 \geq t_2$, the inequality*

$$\left|\phi(t_1, t_2) - e^{\mathcal{A}(t_1-t_2)}\right| \leq |\mathcal{E}|_\infty (t_1 - t_2) e^{(|\mathcal{A}|+|\mathcal{E}|_\infty)(t_1-t_2)} \tag{3.3}$$

*is satisfied.*

**Proof.** Consider any constants $T_0 \geq 0$ and $T > T_0$. We first prove that for all

$a \in [T_0, T]$ and $b \in [T_0, a]$, the inequality

$$\left|\phi(a,b) - e^{\mathcal{A}(a-b)}\right| \leq |\mathcal{E}|_\infty (a-b) e^{(|\mathcal{A}|+|\mathcal{E}|_\infty)(a-b)} \tag{3.4}$$

holds. Recalling that $\mathcal{E}$ is locally Lipschitz, let $K_\mathcal{E} \geq 0$ be a Lipschitz constant for $\mathcal{E}(t)$ on $[T_0, T]$. Let $r > T - T_0$ be a positive integer to be selected later. We define an increasing sequence of real numbers $t_i$ by $t_0 = T_0$ and $t_{i+1} - t_i = p_*$ where $p_* = (T - T_0)/r$. Let $\sigma_r$ be a switching sequence defined by $\sigma_r(t) = t_i$ when $t \in [t_i, t_{i+1})$ for all integers $i \geq 0$. Then

$$|\mathcal{E}(u) - \mathcal{E}(\sigma_r(u))| \leq K_\mathcal{E} |u - \sigma_r(u)| \leq K_\mathcal{E} p_* = K_\mathcal{E} \frac{T - T_0}{r} \tag{3.5}$$

for all $u \in [T_0, T]$.

Consider any constant $\delta > 0$. Let $\phi_r$ denote the fundamental solution of the time-varying linear system $\dot{\xi}(t) = [\mathcal{A} + \mathcal{E}(\sigma_r(t))]\,\xi(t)$. By letting $r \to +\infty$ in (3.5) and using Lemma A.5 from Appendix A (with the choices $N(t) = A + \mathcal{E}(t)$ and $\epsilon(t) = \mathcal{E}(\sigma_r(t)) - \mathcal{E}(t)$), we deduce that there is an integer $r_c > 0$ such that when $r \in \mathbb{N}$ is such that $r \geq r_c$, then

$$|\phi(a_1, a_2) - \phi_r(a_1, a_2)| \leq \delta \tag{3.6}$$

for all $a_1 \in [T_0, T]$ and $a_2 \in [T_0, a_1]$. Fix two constants $t \in (T_0, T]$ and $s \in [T_0, t)$. From (3.6), it follows that

$$\begin{aligned} \left|\phi(t,s) - e^{\mathcal{A}(t-s)}\right| &\leq |\phi(t,s) - \phi_r(t,s)| + \left|\phi_r(t,s) - e^{\mathcal{A}(t-s)}\right| \\ &\leq \delta + |\Lambda_r(t,s)| \end{aligned} \tag{3.7}$$

with $\Lambda_r(t,s) = \phi_r(t,s) - e^{\mathcal{A}(t-s)}$.

Since $t \in (T_0, T]$ and $s \in [T_0, t)$, there exists $l \in \mathbb{N}$ such that $t \in [t_l, t_{l+1})$ and there exists $p \in \mathbb{N}$ such that $s \in [t_p, t_{p+1})$ with $p \leq l$. If $p = l$, then

$$|\Lambda_r(t,s)| = |\phi_r(t,s) - e^{\mathcal{A}(t-s)}| = |e^{(t-s)(\mathcal{A}+\mathcal{E}(t_l))} - e^{\mathcal{A}(t-s)}|\,. \tag{3.8}$$

Using Lemma A.3 from Appendix A, it follows that

$$|\Lambda_r(t,s)| \leq \left(1 - e^{-(t-s)|\mathcal{E}|_\infty}\right) e^{(t-s)(|\mathcal{A}|+|\mathcal{E}|_\infty)}\,. \tag{3.9}$$

On the other hand, if $p < l$, then we obtain

$$\begin{aligned}
\Lambda_r(t,s) &= \phi_r(t,t_l)\phi_r(t_l,t_{l-1})....\phi_r(t_{p+1},s) - e^{\mathcal{A}(t-s)} \\
&= e^{(t-t_l)(\mathcal{A}+\mathcal{E}(t_l))}e^{p_*(\mathcal{A}+\mathcal{E}(t_{l-1}))}....e^{p_*(\mathcal{A}+\mathcal{E}(t_{p+1}))}e^{(t_{p+1}-s)(\mathcal{A}+\mathcal{E}(t_p))} \\
&\quad -e^{\mathcal{A}(t-s)} \qquad (3.10) \\
&= e^{\frac{t-t_l}{p_*}p_*(\mathcal{A}+\mathcal{E}(t_l))}e^{p_*(\mathcal{A}+\mathcal{E}(t_{l-1}))}....e^{p_*(\mathcal{A}+\mathcal{E}(t_{p+1}))}e^{\frac{t_{p+1}-s}{p_*}p_*(\mathcal{A}+\mathcal{E}(t_p))} \\
&\quad -e^{\mathcal{A}(t-s)} \ .
\end{aligned}$$

Since $(t-t_l)/p_*$, $(t_{p+1}-s)/p_*$, and $p_*$ are all contained in $[0,1]$, it follows from Lemma A.4 (Appendix A) with the choices $m = \ell - p + 1$, $G = p_*\mathcal{A}$ and $\nu_i = p_*\mathcal{E}(t_i)$ that

$$\begin{aligned}
|\Lambda_r(t,s)| &\le e^{(\ell-p+1)p_*(|\mathcal{A}|+|\mathcal{E}|_\infty)}\left(1-e^{-(\ell-p+1)p_*|\mathcal{E}|_\infty}\right) \\
&\le e^{2p_*(|\mathcal{A}|+|\mathcal{E}|_\infty)}e^{(t-s)(|\mathcal{A}|+|\mathcal{E}|_\infty)} \qquad (3.11) \\
&\quad \times\left(1-e^{-2p_*(|\mathcal{A}|+|\mathcal{E}|_\infty)}e^{-(t-s)|\mathcal{E}|_\infty}\right) \ ,
\end{aligned}$$

where we used the fact that $(\ell - p - 1)p_* \le t - s$. This inequality and (3.7) give

$$\begin{aligned}
\left|\phi(t,s) - e^{\mathcal{A}(t-s)}\right| &\le \delta + e^{2p_*(|\mathcal{A}|+|\mathcal{E}|_\infty)}e^{(t-s)(|\mathcal{A}|+|\mathcal{E}|_\infty)} \\
&\quad \times\left(1-e^{-2p_*(|\mathcal{A}|+|\mathcal{E}|_\infty)}e^{-(t-s)|\mathcal{E}|_\infty}\right) \ . 
\end{aligned} \qquad (3.12)$$

Since $\delta > 0$ is arbitrary, and since we can make $r$ as large as we wish (and therefore $p_*$ as small a positive value as we wish), we deduce that from (3.12) that

$$\left|\phi(t,s) - e^{\mathcal{A}(t-s)}\right| \le e^{(t-s)(|\mathcal{A}|+|\mathcal{E}|_\infty)}\left(1-e^{-(t-s)|\mathcal{E}|_\infty}\right) \qquad (3.13)$$

for all $t \in (T_0, T]$ and $s \in [T_0, t)$. Since the constants $T$ and $T_0$ are arbitrary, we can conclude because $1 - e^{-s} \le s$ for all $s \ge 0$. □

**Lemma 3.2** *Let*

$$\mathcal{M} = \begin{bmatrix} \alpha & \beta \\ \gamma & \omega \end{bmatrix}$$

*be a positive Schur stable matrix. Then for each constant $\mathcal{N} \ge 0$, we can find constants $c_i > 0$ for $i = 1$ and $2$ and $k \in \mathbb{N}$ such that for all piecewise continuous* *functions* $z_i : [0, +\infty) \to [0, +\infty)$ *for* $i = 1, 2$ *that satisfy*

$$\begin{aligned} z_1(t) &\leq \alpha \sup_{s\in[t-\mathcal{N},t]} z_1(s) + \beta \sup_{s\in[t-\mathcal{N},t]} z_2(s) \\ z_2(t) &\leq \gamma \sup_{s\in[t-\mathcal{N},t]} z_1(s) + \omega \sup_{s\in[t-\mathcal{N},t]} z_2(s) \end{aligned} \tag{3.14}$$

*for all* $t \geq \mathcal{N}$ *and that have finite left limits* $z_i(t^-)$ *at each point* $t \geq 0$ *for* $i = 1$ *and* 2*, we have* $|z(t)| \leq c_1 e^{-c_2 t} \sup_{s\in[0,k\mathcal{N}]} |z(s)|$ *for all* $t \geq k\mathcal{N}$*.*

**Proof.** Set $z = (z_1, z_2)^\top$. Let $V$ be a positive eigenvector associated with an eigenvalue $\lambda \in [0, 1)$ of $\mathcal{M}^\top$ (which exist by the Perron-Frobenius Theorem, since $\mathcal{M}^\top$ is a positive Schur stable matrix). Choose an integer $k \geq 1$ such that $\lambda^k < 1/2$, and set $R = \mathcal{M}^k$, $p = \lambda^k$, and $\Psi_k(t) = (\sup_{s\in[t-k\mathcal{N},t]} z_1(s), \sup_{s\in[t-k\mathcal{N},t]} z_2(s))^\top$ for all $t \geq k\mathcal{N}$. Then (3.14) can be rewritten as $z(t) \leq \mathcal{M}\Psi_1(t)$ (where inequalities of matrices are taken entry-wise), and we can prove (by induction on $k$) that $z(t) \leq R\Psi_k(t)$ for all $t \geq k\mathcal{N}$. We also have $\Psi_k(t) \leq 2 \sup_{s\in[t-k\mathcal{N},t]}(z_1(s), z_2(s))^\top$ for all $t \geq k\mathcal{N}$, which follows because $(z_1(s_1), z_2(s_2)) \leq (z_1(s_1), z_2(s_1)) + (z_1(s_2), z_2(s_2))$ for all $s_1$ and $s_2$ in $[t - k\mathcal{N}, t]$ and all $t \geq k\mathcal{N}$. Since $V^\top R = pV^\top$, we conclude that

$$V^\top z(t) \leq pV^\top \Psi_k(t) \leq 2p \sup_{s\in[t-k\mathcal{N},t]} V^\top z(s) \tag{3.15}$$

for all $t \geq k\mathcal{N}$. Since $2p \in [0, 1)$, we can apply [28, Lemma 1] (with the choice $w(\ell) = V^\top z(\ell + k\mathcal{N})$) to find a constant $c_2 > 0$ (that only depends on $k$, $\mathcal{N}$, and $p$) such that $V^\top z(t) \leq \sup\{V^\top z(\ell) : 0 \leq \ell \leq k\mathcal{N}\}e^{-c_2(t-k\mathcal{N})}$ for all $t \geq k\mathcal{N}$. The lemma now follows because all components of $V$ are positive. □

## 3.4 Finite Time Observer

### 3.4.1 Statement of Result

In this section, we complement the papers [30] and [80], where a finite time observer is provided, by allowing the more general class of systems (3.1) for general choices of the state dependent uncertainties $\epsilon$. We make the following assumptions.

**Assumption 3.1** *The function $f$ in (3.1) is locally Lipschitz with respect to $y$ and $u$ and piecewise continuous with respect to $t$. The pair $(A, C)$ is observable and (3.1) is forward complete. The function $\epsilon$ is bounded and locally Lipschitz. Finally, $u$ is piecewise continuous and locally bounded.* □

When Assumption 3.1 is satisfied, we can use [80, Lemma 1] to show that the observability of $(A, C)$ implies that we can select a matrix $L \in \mathbb{R}^{n\times q}$ with the following property: For each constant $\overline{\tau} > 0$, there is a constant $\tau \in (0, \overline{\tau})$ such that with the choice $F = A + LC$, the matrix

$$E(\tau) = e^{-A\tau} - e^{-F\tau} \tag{3.16}$$

is invertible. This follows from an analytic continuity argument by first using [80, Lemma 1] to find a constant $\tau_a > 0$ and a matrix $L$ such that (3.16) is invertible with the choices $F = A + LC$ and $\tau = \tau_a$, and then noting that if there were a $\overline{\tau} \in (0, \tau_a)$ such that $\det(E(r)) = 0$ for all $r \in (0, \overline{\tau})$, then real analyticity of $\det(E(r))$ as a function of $r$ would also give the contradiction $\det(E(\tau_a)) = 0$.

**Assumption 3.2** *There exist a positive constant $\tau$ and a constant matrix $L$ such that (i) the matrix $E(\tau)$ defined in (3.16) with the choice $F = A+LC$ is invertible and (ii) the bound*

$$|\epsilon|_\infty \tau \left[e^{(|A|+|\epsilon|_\infty)\tau} + e^{(|F|+|\epsilon|_\infty)\tau}\right] |E^{-1}(\tau)| < 1 \tag{3.17}$$

*is satisfied.* □

We are ready to state and prove the following result:

**Theorem 3.1** *Let the system (3.1) satisfy Assumptions 3.1-3.2. Let*

$$\hat{x}(t) = E^{-1}(\tau) \int_{t-\tau}^{t} \left[\left(e^{(t-\tau-\ell)A} - e^{(t-\tau-\ell)F}\right) f(\ell, y(\ell), u(\ell)) + e^{(t-\tau-\ell)F} L y(\ell)\right] d\ell \tag{3.18}$$

*for all $t \geq \tau$, where $E$, $F$ and $\tau$ are from Assumption 3.2. Then, in terms of the constants*

$$c(\tau) = |\epsilon|_\infty \left(e^{(|A|+|\epsilon|_\infty)\tau} + e^{(|F|+|\epsilon|_\infty)\tau}\right) \left[J(\tau) + \left(|\epsilon|_\infty J(\tau) + |E^{-1}(\tau)|\right)\tau\right] \tag{3.19}$$

*and*

$$J(\tau) = \frac{\tau \left[e^{(|A|+|\epsilon|_\infty)\tau} + e^{(|F|+|\epsilon|_\infty)\tau}\right] |E^{-1}(\tau)|^2}{1 - |\epsilon|_\infty \tau \left[e^{(|A|+|\epsilon|_\infty)\tau} + e^{(|F|+|\epsilon|_\infty)\tau}\right] |E^{-1}(\tau)|}\,, \tag{3.20}$$

*we have*

$$|\hat{x}(t) - x(t)| \leq c(\tau) \int_{t-\tau}^{t} \left(|f(\ell, y(\ell), u(\ell))| + |L||y(\ell)|\right) d\ell \tag{3.21}$$

*for all $t \geq \tau$ along all maximal solutions of the system (3.1).*

**Remark 3.1** *In general, $\tau \left[e^{|A|\tau} + e^{|F|\tau}\right] |E^{-1}(\tau)|$ does not converge to zero when $\tau$ converges to zero, so Assumption 3.2 is a constraint on $|\epsilon|_\infty$. Since the inequality (3.21) holds for all $t \geq \tau$, the function (3.18) is a finite time observer which gives an approximate value of the solution in finite time, and which agrees with the true state variable for all $t \geq \tau$ when $\epsilon = 0$. To simplify, and in contrast with [80], we do not assume that there are disturbances in $f$ and $y$. However, an extension to this case can be proved by combining the proof of Theorem 3.1 with the key ideas of [80].*

### 3.4.2 Proof of Theorem 3.1

Since the system (3.1) is forward complete, it follows that for any initial condition, both $x(t)$ and $\hat{x}(t)$ are defined over $[\tau, +\infty)$. Fix any maximal solution $x(t)$ of (3.1). The proof has two parts. In the first part, we consider the case where $\epsilon$ does not depend on $x$. In the second part, we use the result of the first part to show how the case where $\epsilon$ depends on $x$ can also be handled. To simplify the notation, throughout the proof, we write $f(t, y(t), u(t))$ and $g(t, y(t), u(t))$ as $f(t)$ and $g(t)$ respectively, where $g(t, y(t), u(t)) = f(t, y(t), u(t)) - Ly(t)$ and $L$ is from Assumption 3.2.

Let $\phi_1(t, s)$ and $\phi_2(t, s)$ denote the fundamental solutions of the systems

$$\dot{\xi}_1(t) = [A + \epsilon(t)]\xi_1(t) \quad \text{and} \quad \dot{\xi}_2(t) = [F + \epsilon(t)]\xi_2(t)$$

respectively. Let $\psi_1(t,s) = \phi_1(t,s)^{-1}$ and $\psi_2(t,s) = \phi_2(t,s)^{-1}$. It is well known that

$$\frac{\partial \psi_1^\top}{\partial t}(t,0) = -[A+\epsilon(t)]^\top \psi_1(t,0)^\top \tag{3.22}$$

and

$$\frac{\partial \psi_2^\top}{\partial t}(t,0) = -[F+\epsilon(t)]^\top \psi_2(t,0)^\top \tag{3.23}$$

hold for all $t \geq 0$; see [33, Appendix C.4]. Let $z_i(t) = \psi_i(t,0)x(t)$ for $i = 1, 2$. Then

$$\dot{z}_1(t) \;=\; \psi_1(t,0)[A+\epsilon(t)]x(t) + \psi_1(t,0)f(t) + \tfrac{\partial \psi_1}{\partial t}(t,0)x(t) \;=\; \psi_1(t,0)f(t) \tag{3.24}$$

for all $t \geq 0$. By integrating this equality, we obtain

$$z_1(t) \;=\; z_1(t-\tau) + \textstyle\int_{t-\tau}^{t} \psi_1(\ell,0)f(\ell)d\ell \,. \tag{3.25}$$

Here and in the sequel, all inequality and equalities are for all $t \geq \tau$ unless otherwise noted. Then the semigroup property of the flow map $\phi_1$ gives $\phi_1(t, t-\tau) = \phi_1(t,0)\phi_1^{-1}(t-\tau,0)$ and $\phi_1(t,0)\phi_1^{-1}(\ell,0) = \phi_1(t,\ell)$ for all $\ell \in [t-\tau, t]$, and therefore also

$$\begin{aligned} x(t) &= \psi_1(t,0)^{-1}\psi_1(t-\tau,0)x(t-\tau) + \textstyle\int_{t-\tau}^{t} \psi_1(t,0)^{-1}\psi_1(\ell,0)f(\ell)d\ell \\ &= \phi_1(t,t-\tau)x(t-\tau) + \textstyle\int_{t-\tau}^{t} \phi_1(t,\ell)f(\ell)d\ell \,. \end{aligned} \tag{3.26}$$

Observing that

$$\dot{x}(t) = [F+\epsilon(t)]x(t) + g(t,y(t),u(t)) \tag{3.27}$$

and using variation of parameters gives

$$x(t) = \phi_2(t,t-\tau)x(t-\tau) + \int_{t-\tau}^{t} \phi_2(t,\ell)g(\ell)d\ell \,. \tag{3.28}$$

Left multiplying the second line of (3.26) and (3.28) by $\psi_1(t,t-\tau)$ and $\psi_2(t,t-\tau)$ respectively and computing the difference of the results gives

$$\begin{aligned} \left[\psi_1(t,t-\tau) - \psi_2(t,t-\tau)\right] x(t) &= \textstyle\int_{t-\tau}^{t} \psi_1(t,t-\tau)\phi_1(t,\ell)f(\ell)d\ell \\ &\quad - \textstyle\int_{t-\tau}^{t} \psi_2(t,t-\tau)\phi_2(t,\ell)g(\ell)d\ell \\ &= \textstyle\int_{t-\tau}^{t} \psi_1(\ell,t-\tau)f(\ell)d\ell \\ &\quad - \textstyle\int_{t-\tau}^{t} \psi_2(\ell,t-\tau)g(\ell)d\ell \,, \end{aligned} \tag{3.29}$$

where the second equality used the semigroup property of the $\phi_i$'s. Therefore,

$$[E(\tau) + Z(t)]\, x(t) \;=\; \int_{t-\tau}^{t} \psi_1(\ell, t-\tau) f(\ell) d\ell - \int_{t-\tau}^{t} \psi_2(\ell, t-\tau) g(\ell) d\ell \tag{3.30}$$

holds with $Z(t) = \psi_1(t, t-\tau) - \psi_2(t, t-\tau) - E(\tau)$ and $E$ defined in (3.16). From (3.22)-(3.23), and by Lemma 3.1 with the choice $\mathcal{E} = \epsilon$, we deduce that

$$\begin{aligned} \left|\psi_1(\ell, t-\tau) - e^{-A(\ell-t+\tau)}\right| &\leq \bar{\epsilon}(\ell - t + \tau) e^{(|A|+\bar{\epsilon})(\ell-t+\tau)} \\ \left|\psi_2(\ell, t-\tau) - e^{-F(\ell-t+\tau)}\right| &\leq \bar{\epsilon}(\ell - t + \tau) e^{(|F|+\bar{\epsilon})(\ell-t+\tau)} \end{aligned} \tag{3.31}$$

for all $\ell \in [t-\tau, t]$, where we set $\bar{\epsilon} = |\epsilon|_\infty$ for brevity. It follows by setting $\ell = t$ in (3.31) that

$$|Z(t)| \leq \bar{\epsilon}\tau \left[e^{(|A|+\bar{\epsilon})\tau} + e^{(|F|+\bar{\epsilon})\tau}\right] \tag{3.32}$$

for all $t \geq \tau$. From this inequality and Lemma A.2 from Appendix A (applied with $M = E(\tau)$ and $N = Z(t)$) we deduce that (3.17) from Assumption 3.2 ensures that for all $t \geq \tau$, the matrix $E(\tau) + Z(t)$ is invertible. Also, Lemma A.2 from Appendix A implies that $|(E(\tau) + Z(t))^{-1} - E^{-1}(\tau)| \leq \bar{\epsilon} J(\tau)$ for all $t \geq \tau$, where $J$ is the constant defined in (3.20).

Then, omitting the argument $\tau$ of $E$ to keep our notation simple, (3.30) gives

$$\begin{aligned} x(t) \;=\; & [E + Z(t)]^{-1} \int_{t-\tau}^{t} \left[e^{(t-\tau-\ell)A} f(\ell) - e^{(t-\tau-\ell)F} g(\ell)\right] d\ell \\ & + [E + Z(t)]^{-1} \int_{t-\tau}^{t} \left(\psi_1(\ell, t-\tau) - e^{(t-\tau-\ell)A}\right) f(\ell) d\ell \\ & + [E + Z(t)]^{-1} \int_{t-\tau}^{t} \left(e^{(t-\tau-\ell)F} - \psi_2(\ell, t-\tau)\right) g(\ell) d\ell. \end{aligned} \tag{3.33}$$

We deduce that $\tilde{x}(t) = \hat{x}(t) - x(t)$ satisfies

$$\begin{aligned} |\tilde{x}(t)| \;\leq\; & |E^{-1} - [E + Z(t)]^{-1}| \int_{t-\tau}^{t} \left|e^{(t-\tau-\ell)A} f(\ell) - e^{(t-\tau-\ell)F} g(\ell)\right| d\ell \\ & + |[E + Z(t)]^{-1}| \int_{t-\tau}^{t} \left|\psi_1(\ell, t-\tau) - e^{(t-\tau-\ell)A}\right| |f(\ell)| d\ell \\ & + |[E + Z(t)]^{-1}| \int_{t-\tau}^{t} \left|e^{(t-\tau-\ell)F} - \psi_2(\ell, t-\tau)\right| |g(\ell)| d\ell \\ \leq\; & \bar{\epsilon} J \int_{t-\tau}^{t} \left|e^{(t-\tau-\ell)A} f(\ell) - e^{(t-\tau-\ell)F} g(\ell)\right| d\ell \\ & + (\bar{\epsilon} J + |E^{-1}|) \int_{t-\tau}^{t} \left|\psi_1(\ell, t-\tau) - e^{(t-\tau-\ell)A}\right| |f(\ell)| d\ell \\ & + (\bar{\epsilon} J + |E^{-1}|) \int_{t-\tau}^{t} \left|e^{(t-\tau-\ell)F} - \psi_2(\ell, t-\tau)\right| |g(\ell)| d\ell \end{aligned}$$

where we omitted the dependency of $J$ on $\tau$ for brevity. Using (3.31), and setting $\bar{\epsilon}^\sharp = \bar{\epsilon}(\bar{\epsilon}J + |E^{-1}|)$, we obtain

$$\begin{aligned} |\tilde{x}(t)| \;\leq\;& \bar{\epsilon}J \int_{t-\tau}^{t} \left|e^{(t-\tau-\ell)A} f(\ell) - e^{(t-\tau-\ell)F} g(\ell)\right| d\ell \\ &+ \bar{\epsilon}^\sharp \int_{t-\tau}^{t} (\ell - t + \tau) e^{(|A|+\bar{\epsilon})(\ell-t+\tau)} |f(\ell)| d\ell \\ &+ \bar{\epsilon}^\sharp \int_{t-\tau}^{t} (\ell - t + \tau) e^{(|F|+\bar{\epsilon})(\ell-t+\tau)} |g(\ell)| d\ell \\ \leq\;& \bar{\epsilon}J \int_{t-\tau}^{t} \left[e^{\tau|A|} |f(\ell)| + e^{\tau|F|} |f(\ell) - Ly(\ell)|\right] d\ell \\ &+ \bar{\epsilon}^\sharp \tau \int_{t-\tau}^{t} e^{(|A|+\bar{\epsilon})\tau} |f(\ell)| d\ell \\ &+ \bar{\epsilon}^\sharp \tau \int_{t-\tau}^{t} e^{(|F|+\bar{\epsilon})\tau} |f(\ell) - Ly(\ell)| d\ell \\ \leq\;& \bar{\epsilon}J \int_{t-\tau}^{t} \left[\left(e^{\tau|A|} + e^{\tau|F|}\right) |f(\ell)| + e^{\tau|F|} |L||y(\ell)|\right] d\ell \\ &+ \bar{\epsilon}^\sharp \tau \int_{t-\tau}^{t} \left(e^{(|A|+\bar{\epsilon})\tau} + e^{(|F|+\bar{\epsilon})\tau}\right) |f(\ell)| d\ell \\ &+ \bar{\epsilon}^\sharp \tau \int_{t-\tau}^{t} e^{(|F|+\bar{\epsilon})\tau} |L||y(\ell)| d\ell \,. \end{aligned} \tag{3.34}$$

This concludes the first part of the proof, by our choice (3.19) of the constant $c(\tau)$.

We now use the preceding result to cover the case where $\epsilon$ depends on both $t$ and $x$ to complete the proof of the theorem. Fix any specific solution of (3.1), which we denote by $x_\natural(t)$. Then we consider the system

$$\left\{ \begin{array}{rcl} \dot{X}(t) &=& [A + \epsilon(t, x_\natural(t))]X(t) + f(t, Y(t), u(t)) \\ Y(t) &=& CX(t) \,. \end{array} \right. \tag{3.35}$$

For the system (3.35), $\epsilon(t, x_\natural(t))$ depends only on $t$ and not on $X$. Moreover, it is bounded by $\bar{\epsilon}$. Thus, from our previous proof, we deduce that for all solutions $X$ of (3.35) and for all $t \geq \tau$, we have

$$|\hat{X}(t) - X(t)| \leq c(\tau) \int_{t-\tau}^{t} \left(|f(\ell, Y(\ell), u(\ell))| + |L||Y(\ell)|\right) d\ell \tag{3.36}$$

with $c(\tau)$ as defined in the first part of the proof and with

$$\hat{X}(t) = E^{-1}(\tau) \int_{t-\tau}^{t} \left[\left(e^{(t-\tau-\ell)A} - e^{(t-\tau-\ell)F}\right) f(\ell, Y(\ell), u(\ell)) + e^{(t-\tau-\ell)F} LY(\ell)\right] d\ell \,. \tag{3.37}$$

Since $x_\natural(t)$ is a solution of (3.1), it follows that (3.35) holds with $X = x_\natural$. From (3.36), it follows that

$$|\hat{x}_\natural(t) - x_\natural(t)| \le c(\tau) \int_{t-\tau}^{t} [|f(\ell, Cx_\natural(\ell), u(\ell))| + |L||Cx_\natural(\ell)|]\, d\ell \tag{3.38}$$

with

$$\begin{array}{rcl} \hat{x}_\natural(t) & = & E^{-1}(\tau)\int_{t-\tau}^{t} \left(e^{(t-\tau-\ell)A} - e^{(t-\tau-\ell)F}\right) f(\ell, Cx_\natural(\ell), u(\ell))d\ell \\ & & +E^{-1}(\tau)\int_{t-\tau}^{t} e^{(t-\tau-\ell)F} LCx_\natural(\ell)d\ell \, . \end{array} \tag{3.39}$$

Since $x_\natural$ was an arbitrary solution of (3.1), this concludes the proof of Theorem 3.1.

## 3.5 Stabilization of Systems with Temporary Loss of Measurements

### 3.5.1 Assumptions and Statement of Result

Theorem 3.1 relies on the assumption that the output is available for all $t \ge 0$ and the fact that the matrix $A$ in (3.1) does not depend on $t$. In this section, we relax these assumptions. We assume that the output is only available on some specific intervals of time, and we will consider systems whose linear approximation at the origin is time-varying even when no disturbance is acting and $y$ and $u$ are set to zero. Under these assumptions, complemented by a stabilizability assumption of ISS type and a mild restriction on $f$, we construct a globally exponentially stabilizing dynamic output feedback with an observer for the system

$$\dot{x}(t) = [\mathcal{M}(t) + \eta(t, x(t))]x(t) + \mathcal{B}(t)u(t) \tag{3.40}$$

with $x$ valued in $\mathbb{R}^n$ for any $n \in \mathbb{N}$ and $u$ valued in $\mathbb{R}^p$ for any $p \in \mathbb{N}$. The function $u$ represents a control. We assume that there are constants $P > 0$ and $\theta \in (0, P)$ such that the $\mathbb{R}^p$ valued output

$$y(t) = \mathcal{C}x(t) \tag{3.41}$$

is only available when $t$ is in the set

$$\mathcal{S}_{P,\theta} = \bigcup_{i \in \mathbb{N}} [iP, iP + \theta]. \tag{3.42}$$

**Assumption 3.3** *The functions $\eta$, $\mathcal{M}$, and $\mathcal{B}$ are locally Lipschitz and bounded, and $\mathcal{M}$ is not the zero function.* □

For any locally Lipschitz bounded function $\mathcal{Z} : [0, \infty) \to \mathbb{R}^{n \times p}$, we can therefore fix a constant $\bar{k} > 0$ and choose $s$ matrices $\mathcal{A}_j \in \mathbb{R}^{n \times n}$ for $j = 1, 2, \ldots, s$ such that the function

$$\mathcal{D}(t, j, x) = \mathcal{M}(t) - \mathcal{Z}(t)\mathcal{C} - \mathcal{A}_j + \eta(t, x) \ , \tag{3.43}$$

possesses the following property: For each $i \in \mathbb{N}$, there is a $j \in \{1, ..., s\}$ (depending on $i$) such that for all $t \in [iP, iP + \theta]$ and all $x \in \mathbb{R}^n$, the inequality

$$|\mathcal{D}(t, j, x)| \leq \overline{k} \tag{3.44}$$

is satisfied. In terms of the matrix $\mathcal{C}$ from (3.41), we also assume that the $\mathcal{A}_i$'s from Assumption 3.3 satisfy:

**Assumption 3.4** *For all $i \in \{1, ..., s\}$ the pair $(\mathcal{A}_i, \mathcal{C})$ is observable.* □

Assumption 3.4 ensures that there are matrices $\mathcal{L}_i$ and values $\tau_i \in (0, \theta)$ for $i = 1$ to $s$ such that each of the matrices

$$\mathcal{E}_i(\tau_i) = e^{-\mathcal{A}_i \tau_i} - e^{-\mathcal{F}_i \tau_i} \quad \text{where} \quad \mathcal{F}_i = \mathcal{A}_i + \mathcal{L}_i \mathcal{C} \tag{3.45}$$

is invertible for $i = 1, ..., s$. This follows from the analytic continuity argument from Section 3.4.1.

For later use, we introduce the matrices

$$\mathcal{H}_i = \mathcal{E}_i^{-1}(\tau_i) \tag{3.46}$$

and upper bounds for the matrices $\mathcal{A}_i$, $\mathcal{F}_i$, $\mathcal{L}_i$ and $\mathcal{H}_i$ of the form

$$|\mathcal{A}_i| \leq \overline{\mathcal{A}} \quad , \quad |\mathcal{F}_i| \leq \overline{\mathcal{F}} \quad , \quad |\mathcal{L}_i| \leq \overline{\mathcal{L}} \quad , \quad \text{and} \quad |\mathcal{H}_i| \leq \overline{\mathcal{H}} \tag{3.47}$$

for all $i \in \{1, ..., s\}$.

**Assumption 3.5** *There are a locally Lipschitz function $u_s$ and a constant $\mu > 0$ such that*

$$|u_s(t, x)| \leq \mu |x| \tag{3.48}$$

*for all $t \geq 0$ and $x \in \mathbb{R}^n$, and constants $\kappa \in (0, 1)$ and $\gamma > 0$, such that for any piecewise continuous function $\delta$, all solutions of the system*

$$\dot{x}(t) = [\mathcal{M}(t) + \eta(t, x(t))]x(t) + \mathcal{B}(t)u_s(t, x(t) + \delta(t)) \tag{3.49}$$

*satisfy*

$$|x(t)| \leq \kappa |x(s)| + \gamma \sup_{w \in [s,t]} |\delta(w)| \tag{3.50}$$

*for all $t \geq 0$ and $s \in [t - P - \theta, t - P]$.* □

Assumption 3.5 can be viewed as a generalized Hurwitzness type condition, because in the specific case where $u_s(t, x) = Kx$ for some matrix $K$ such that $\mathcal{M}(t)+\eta(t, x(t))+\mathcal{B}(t)K$ is constant and Hurwitz, condition (3.50) can be checked using variation of parameters. In terms of the constants

$$\bar{g} = \overline{k} \left( e^{(\overline{\mathcal{A}}+\overline{k})\bar{\tau}} + e^{(\overline{\mathcal{F}}+\overline{k})\bar{\tau}} \right) \left[ \mathbb{J} + (\overline{k}\mathbb{J} + \overline{\mathcal{H}})\bar{\tau} \right] \tag{3.51}$$

and

$$\mathbb{J} = \frac{\bar{\tau} \left[ e^{(\overline{\mathcal{A}}+\overline{k})\bar{\tau}} + e^{(\overline{\mathcal{F}}+\overline{k})\bar{\tau}} \right] \overline{\mathcal{H}}^2}{1 - \overline{k}\bar{\tau} \left[ e^{(\overline{\mathcal{A}}+\overline{k})\bar{\tau}} + e^{(\overline{\mathcal{F}}+\overline{k})\bar{\tau}} \right] \overline{\mathcal{H}}} \quad \text{and} \quad \bar{\tau} = \max\{\tau_i : 1 \leq i \leq s\}, \tag{3.52}$$

our final assumption is:

**Assumption 3.6** *The inequality*

$$\overline{k}\bar{\tau} \left[ e^{(\overline{\mathcal{A}}+\overline{k})\bar{\tau}} + e^{(\overline{\mathcal{F}}+\overline{k})\bar{\tau}} \right] \overline{\mathcal{H}} < 1 \tag{3.53}$$

*is satisfied. Moreover, with the choices*

$$\bar{p}_1 = \bar{g}|\mathcal{B}|_\infty \mu \quad \textit{and} \quad \bar{p}_2 = \bar{g}\big(|\mathcal{B}|_\infty \mu + (\overline{\mathcal{L}} + |\mathcal{Z}|_\infty)|\mathcal{C}|\big) , \tag{3.54}$$

*the matrix*

$$\bar{\mathcal{S}} = \begin{bmatrix} \bar{p}_1 e^{|\mathcal{M}|_\infty P}\bar{\tau} & \bar{p}_2 e^{|\mathcal{M}|_\infty P}\bar{\tau} + |\eta|_\infty e^{P\mathcal{M}_\infty}\frac{e^{|\mathcal{M}|_\infty P}-1}{|\mathcal{M}|_\infty} \\ \gamma & \kappa \end{bmatrix} \tag{3.55}$$

*is Schur stable.* □

Defining the switching signal $\sigma : [0, +\infty) \to \{1, ..., s\}$ by $\sigma(t) = j$ for all $t \in [iP, (i+1)P)$ and integers $i \geq 0$, where $j \in \{1, ..., s\}$ is any integer such that $|\mathcal{D}(t, j, x)| \leq \overline{k}$ for all $t \in [iP, iP + \theta]$ and all $x \in \mathbb{R}^n$, it follows from our choices of $\bar{k}$ and the $\mathcal{A}_i$'s that for all $t \in \mathcal{S}_{P,\theta}$ and $x \in \mathbb{R}^n$, we have $|\mathcal{D}(t, \sigma(t), x)| \leq \overline{k}$. Notice that the system (3.40) can be rewritten as

$$\dot{x}(t) = [\mathcal{A}_{\sigma(t)} + \mathcal{D}(t, \sigma(t), x(t))]x(t) + \mathcal{Z}(t)\mathcal{C}x(t) + \mathcal{B}(t)u(t) . \tag{3.56}$$

Thus, the following theorem can be interpreted as a result for switched systems:

**Theorem 3.2** *Let the system (3.40) satisfy Assumptions 3.3 to 3.6 and consider the continuous-discrete system*

$$\begin{aligned} \dot{\hat{x}}(t) &= \mathcal{M}(t)\hat{x}(t) + \mathcal{B}(t)u_s(t, \hat{x}(t)) , \quad t \in [t_i, t_{i+1}) \\ \hat{x}(t_i) &= \mathcal{H}_{\sigma(t_i)}(\tau_i)\int_{t_i-\tau_i}^{t_i}\mathcal{E}(t_i-\tau_i-\ell, \sigma(t_i))\big(\mathcal{Z}(\ell)y(\ell) + \mathcal{B}(\ell)u_s(\ell, \hat{x}(\ell))\big)d\ell \\ &\quad + \mathcal{H}_{\sigma(t_i)}(\tau_i)\int_{t_i-\tau_i}^{t_i} e^{(t_i-\tau_i-\ell)\mathcal{F}_{\sigma(t_i)}}\mathcal{L}_{\sigma(t_i)}y(\ell)d\ell \end{aligned} \tag{3.57}$$

*where* $\mathcal{E}(m, k) = e^{m\mathcal{A}_k} - e^{m\mathcal{F}_k}$*,* $\mathcal{H}_{\sigma(t_i)}$ *is the matrix defined in (3.46) and* $t_i = iP + \theta$ *for integers* $i \in \mathbb{N}$*. Then the system (3.40) in closed loop with the dynamic output feedback*

$$u_s(t, \hat{x}(t)) \tag{3.58}$$

*is such that the dynamics for* $(x, \tilde{x})$ *are globally exponentially stable to* $0$*, where* $\tilde{x}(t) = x(t) - \hat{x}(t)$*.*

**Remark 3.2** *Since we chose values* $\tau_i \in (0, \theta)$*, (3.57) only requires* $y(t)$ *values at times* $t \in \mathcal{S}_{P,\theta}$*. Theorem 3.2 implies that the dynamic output feedback (3.58)* *asymptotically stabilizes the closed loop system to* $0$*. Assumption 3.3 is inspired by the technique of [51] and [83], which makes it possible to stabilize time-varying systems with auxiliary switched systems whose subsystems are time-invariant systems that are affected by disturbances. Notice that if* $|\eta|_\infty = 0$*, and if* $\mathcal{M}$ *and* $\mathcal{Z}$ *are continuous and periodic, then one can have an arbitrarily small constant* $\overline{k}$ *by choosing a sufficiently large number of matrices* $\mathcal{A}_i$*, which can facilitate satisfying Assumption 3.6.*

**Remark 3.3** *Assumption 3.5 is a stabilizability assumption. Constructing a feedback* $u_s$ *such that (3.50) is satisfied can be challenging. Nevertheless, see [25] and [85] for techniques for constructing feedback controls for nonlinear time-varying systems. The main difference between (3.40) and the system studied in [84] is that* $\mathcal{M}$ *depends on* $t$*, but [84] does not provide exponentially stabilizing feedbacks and so does not apply to the problems we study here.*

**Remark 3.4** *If* $|\eta|_\infty = 0$ *and* $\overline{k}$ *can be chosen arbitrarily small, then by choosing both* $\bar{\tau}$ *and* $\overline{k}$ *sufficiently small, the matrix* $\bar{\mathcal{S}}$ *is Schur stable since* $\kappa \in (0, 1)$*. In this case, there exists a* $\bar{\eta} > 0$ *such that the corresponding matrix* $\bar{\mathcal{S}}$ *is Schur stable if* $|\eta|_\infty \leq \bar{\eta}$*.*

**Remark 3.5** *The size of the constant* $\bar{k}$ *depends on* $\theta$*, and our condition (3.53) is more easily satisfied when* $\bar{k}$ *is sufficiently small. Thus, since an arbitrarily small constant* $\theta$ *can be selected, it may be useful to choose a* $\theta$ *value resulting in a small constant* $\bar{k} > 0$*.*

**Remark 3.6** *Theorem 3.2 can be extended to the case where* $\mathcal{C}$ *in the output depends on* $t$*. For the sake of simplicity, we do not investigate this case. Notice that* $\mathcal{Z}(t)$ *can always be chosen identically equal to zero. However, to decrease the conservatism of the approach, it is worth introducing the function* $\mathcal{Z}$ *that can be freely chosen.*

### 3.5.2 Proof of Theorem 3.2

Let $i \geq 0$ be an integer. Bearing in mind (3.56), we deduce that the system (3.40) in closed loop with $u_s(t, \hat{x}(t))$ can be rewritten as

$$\dot{x}(t) = [\mathcal{A}_{\sigma(t_i)} + \mathcal{D}(t, \sigma(t_i), x(t))]x(t) + \mathcal{Z}(t)\mathcal{C}x(t) + \mathcal{B}(t)u_s(t, \hat{x}(t)) \tag{3.59}$$

for all $t \in [iP, (i+1)P)$ and $i \geq 0$, and our linear growth condition (3.48) on $u_s$ ensures that the closed loop system is forward complete. Assumptions 3.3 to 3.6 imply that for each choice of $i$, Assumptions 3.1-3.2 are satisfied by

$$\begin{cases} \dot{x}(t) &= [\mathcal{A}_{\sigma(t_i)} + \mathcal{D}^\flat(t, \sigma(t_i), x(t))]x(t) + \mathcal{Z}(t)y(t) + \mathcal{B}(t)u(t) \\ y(t) &= \mathcal{C}x(t) \ , \end{cases} \tag{3.60}$$

where $u(t) = u_s(t, \hat{x}(t))$, and where $\mathcal{D}^\flat$ is any locally Lipschitz function that equals $\mathcal{D}$ on $[iP, iP + \theta] \times \{\sigma(t_i)\} \times \mathbb{R}^n$ and that is bounded by $\bar{k}$. (We introduce the function $\mathcal{D}^\flat$ because $\bar{k}$ is not necessarily a global bound on $\mathcal{D}$ but provides the required bound for $t \in [iP, iP + \theta]$.) Therefore, since the $\hat{x}_i$ formula from (3.57) agrees with the $\hat{x}(t)$ formula from (3.18) for suitable choices of $f$ and other functions from (3.18), we deduce from Theorem 3.1 that

$$|\tilde{x}(t_i)| \leq \bar{g} \int_{t_i - \tau_i}^{t_i} \left[ |\mathcal{B}(\ell)u_s(\ell, \hat{x}(\ell))| + \left( |\mathcal{Z}|_\infty + |\mathcal{L}_{\sigma(t_i)}| \right) |y(\ell)| \right] d\ell \tag{3.61}$$

with $\bar{g}$ defined in (3.51) and $\tilde{x} = \hat{x} - x$. As an immediate consequence of the bounds (3.47), we obtain

$$\begin{aligned} |\tilde{x}(t_i)| &\leq \bar{g} \textstyle\int_{t_i - \tau_i}^{t_i} \left[ |\mathcal{B}|_\infty \mu |\hat{x}(\ell)| + (\overline{\mathcal{L}} + |\mathcal{Z}|_\infty)|\mathcal{C}||x(\ell)| \right] d\ell \\ &\leq \textstyle\int_{t_i - \tau_i}^{t_i} \left[ \bar{p}_1 |\tilde{x}(\ell))| + \bar{p}_2 |x(\ell)| \right] d\ell \end{aligned} \tag{3.62}$$

with $\bar{p}_1$ and $\bar{p}_2$ defined in (3.54).

On the other hand, (3.40) and (3.57) imply that, for all $t \in [t_i, t_{i+1})$ and integers $i \geq 0$, we have

$$\dot{\tilde{x}}(t) = \mathcal{M}(t)\tilde{x}(t) - \eta(t, x(t))x(t) \tag{3.63}$$

when we use the control (3.58). This gives $|\dot{\tilde{x}}(t)| \leq |\mathcal{M}|_\infty |\tilde{x}(t)| + |\eta|_\infty |x(t)|$, which we can integrate to obtain

$$|\tilde{x}(t)| \leq \left( |\tilde{x}(t_i)| + \textstyle\int_{t_i}^{t} |\eta|_\infty |x(\ell)| d\ell \right) + \textstyle\int_{t_i}^{t} |\mathcal{M}|_\infty |\tilde{x}(\ell)| d\ell. \tag{3.64}$$

Hence, Gronwall's inequality gives

$$\begin{aligned}
|\tilde{x}(t)| &\leq \left(|\tilde{x}(t_i)| + \int_{t_i}^{t} |\eta|_\infty |x(\ell)| d\ell\right) e^{|\mathcal{M}|_\infty (t-t_i)} \\
&\leq e^{|\mathcal{M}|_\infty (t-t_i)} |\tilde{x}(t_i)| + |\eta|_\infty \int_{t_i}^{t} |x(\ell)| e^{|\mathcal{M}|_\infty (t-\ell)} e^{|\mathcal{M}|_\infty (\ell-t_i)} d\ell \\
&\leq e^{|\mathcal{M}|_\infty (t-t_i)} |\tilde{x}(t_i)| + |\eta|_\infty e^{P|\mathcal{M}|_\infty} \int_{t_i}^{t} |x(\ell)| e^{|\mathcal{M}|_\infty (t-\ell)} d\ell.
\end{aligned} \tag{3.65}$$

We can combine the last inequality in (3.65) with (3.62) to obtain

$$\begin{aligned}
|\tilde{x}(t)| &\leq e^{|\mathcal{M}|_\infty (t-t_i)} \int_{t_i-\bar{\tau}}^{t_i} [\bar{p}_1 |\tilde{x}(\ell))| + \bar{p}_2 |x(\ell)|] \, d\ell \\
&\quad + |\eta|_\infty e^{P|\mathcal{M}|_\infty} \int_{t_i}^{t} e^{|\mathcal{M}|_\infty (t-\ell)} |x(\ell)| d\ell
\end{aligned}$$

for all $t \in [t_i, t_{i+1})$, and so also

$$\begin{aligned}
|\tilde{x}(t)| &\leq \bar{p}_1 e^{|\mathcal{M}|_\infty (t-t_i)} \bar{\tau} \sup_{w \in [t_i-\bar{\tau}, t_i]} |\tilde{x}(w)| \\
&\quad + \left[\bar{p}_2 e^{|\mathcal{M}|_\infty (t-t_i)} \bar{\tau} + |\eta|_\infty e^{P|\mathcal{M}|_\infty} \tfrac{e^{|\mathcal{M}|_\infty P} - 1}{|\mathcal{M}|_\infty}\right] \sup_{w \in [t_i-\bar{\tau}, t]} |x(w)| \, ,
\end{aligned}$$

where we used the fact that

$$\begin{aligned}
\int_{t_i}^{t} e^{|\mathcal{M}|_\infty (t-\ell)} |x(\ell)| \mathrm{d}\ell &\leq \tfrac{1}{|\mathcal{M}|_\infty} \left(e^{|\mathcal{M}|_\infty (t-t_i)} - 1\right) \sup_{w \in [t_i, t]} |x(w)| \\
&\leq \tfrac{1}{|\mathcal{M}|_\infty} \left(e^{|\mathcal{M}|_\infty P} - 1\right) \sup_{w \in [t_i-\bar{\tau}, t]} |x(w)|
\end{aligned}$$

for all $t \in [t_i, t_{i+1})$.

We deduce that for all $t \geq P + \bar{\tau}$,

$$\begin{aligned}
|\tilde{x}(t)| &\leq \bar{p}_1 e^{|\mathcal{M}|_\infty P} \bar{\tau} \sup_{w \in [t-P-\bar{\tau}, t]} |\tilde{x}(w)| \\
&\quad + \left[\bar{p}_2 e^{|\mathcal{M}|_\infty P} \bar{\tau} + |\eta|_\infty e^{P|\mathcal{M}|_\infty} \tfrac{e^{|\mathcal{M}|_\infty P} - 1}{|\mathcal{M}|_\infty}\right] \sup_{w \in [t-P-\bar{\tau}, t]} |x(w)| \, .
\end{aligned} \tag{3.66}$$

The system (3.40) in closed loop with (3.58) admits the representation

$$\dot{x}(t) = [\mathcal{M}(t) + \eta(t, x(t))] x(t) + \mathcal{B}(t) u_s(x(t) + \tilde{x}(t)) \, . \tag{3.67}$$

From Assumption 3.5 and the fact that $\bar{\tau} \leq \theta$, it follows that

$$|x(t)| \leq \kappa |x(t-P-\bar{\tau})| + \gamma \sup_{w \in [t-P-\bar{\tau}, t]} |\tilde{x}(w)| \tag{3.68}$$

for all $t \geq P + \bar{\tau}$. By grouping (3.66) and (3.68), we obtain

$$\begin{aligned}
|\tilde{x}(t)| &\leq \bar{p}_1 e^{|\mathcal{M}|_\infty P} \bar{\tau} \sup_{w \in [t-P-\bar{\tau}, t]} |\tilde{x}(w)| \\
&\quad + \left[\bar{p}_2 e^{|\mathcal{M}|_\infty P} \bar{\tau} + |\eta|_\infty e^{P|\mathcal{M}|_\infty} \tfrac{e^{|\mathcal{M}|_\infty P} - 1}{|\mathcal{M}|_\infty}\right] \sup_{w \in [t-P-\bar{\tau}, t]} |x(w)| \\
|x(t)| &\leq \kappa \sup_{w \in [t-P-\bar{\tau}, t]} |x(w)| + \gamma \sup_{w \in [t-P-\bar{\tau}, t]} |\tilde{x}(w)|
\end{aligned} \tag{3.69}$$

for all $t \geq P + \bar{\tau}$. The system (3.69) is of the form (3.14) from Lemma 3.2 (with $(z_1, z_2) = (|x(t)|, |\tilde{x}(t)|)$, $\mathcal{N} = P + \bar{\tau}$, and $\mathcal{M} = \bar{\mathcal{S}}$), save for the fact that the matrix $\bar{\mathcal{S}}$ from (3.55) is Schur stable but is not necessarily a positive Schur stable matrix (since some of the entries of $\bar{\mathcal{S}}$ can be zero). However, we can majorize all of the entries of $\bar{\mathcal{S}}$ by positive values, in such a way that the new positive matrix that we obtain is a positive Schur stable matrix. This follows from the continuity of the eigenvalues of a matrix as functions of the entries of the matrix, by increasing the entries of $\bar{\mathcal{S}}$ by adding small enough positive constants to the entries. Then it follows from Lemma 3.2 that $(x(t), \tilde{x}(t))$ converges exponentially to the origin, which proves the theorem.

## 3.6 Illustrations

The class of systems represented by (3.1) and (3.40) covers broad applications that are beyond the scope of existing finite time observer approaches. In this section, we illustrate above theorems using the controlled Mathieu equation

$$\ddot{q}(t) + (R_1 + R_2 \cos(t))q(t) + u(t) = 0 \tag{3.70}$$

for positive constants $R_1$ and $R_2$, which arises in the study of vibrations of an elliptic membrane [86]. The Mathieu equation has also been studied in [87] to illustrate parameter identification for a certain family of nonlinear and time varying systems using data over a limited time interval. See also [31] for the study of the uncontrolled Mathieu equation corresponding to cases where $u = 0$. The work [31] studied domains of stability and instability including Hopf bifurcations along the boundaries of the domains of stability.

### 3.6.1 Illustration of Theorem 3.1

The controlled Mathieu equation (3.70) can be written as

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & [A + \epsilon(t, x(t))]x(t) + f(t, y(t), u(t)) \\ y(t) & = & Cx(t) \end{array} \right. \tag{3.71}$$

with

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -R_1 & 0 \end{bmatrix}, \quad C = e_1^\top, \quad \epsilon(t) = \begin{bmatrix} 0 & 0 \\ -R_2\cos(t) & 0 \end{bmatrix}, \tag{3.72}$$

and $f(t, y, u) = -e_2 u$ where $e_i$ is the $i$th standard basis vector for $i = 1$ and 2. We consider the case where only an upper bound on $R_2$ is known.

Choosing

$$L = \begin{bmatrix} 0 \\ 2R_1 \end{bmatrix} \tag{3.73}$$

we can satisfy Assumptions 3.1-3.2 for many choices of the parameters $R_1$, $R_2$, and $\tau$. For instance, if we choose $R_1 = 1$, any $R_2 \in (0, 0.024]$, and $\tau = 1$, then the matrix $E(\tau) = e^{-A\tau} - e^{-F\tau}$ with the choice $F = A + LC$ has determinant $\det(E(\tau)) = 0.33254$ and so is invertible, and

$$|\epsilon|_\infty \tau \left[ e^{(|A|+|\epsilon|_\infty)\tau} + e^{(|F|+|\epsilon|_\infty)\tau} \right] |E^{-1}(\tau)| = 0.998418 < 1, \tag{3.74}$$

which ensures that Assumptions 3.1-3.2 are satisfied for any piecewise continuous locally bounded choice of $u$. Then the finite time observer is provided by (3.18) from Theorem 3.1.

To illustrate our result, Figure 3.1 shows a MATLAB simulation of our finite time observer (3.18), using an integration algorithm of the model (3.71) with a semi-implicit integration step of 0.001, and with the initial conditions $x_1(0) = x_2(0) = 2$ and $\hat{x}_1(0) = \hat{x}_2(0) = -2$, where $u(t) = \sin(2t)$, and using the preceding choices of $A$, $f$, and $L$. Since our simulation shows good tracking performance of the estimator $\hat{x}_2$ for the state component $x_2$ of (3.71), it helps to illustrate our general theory in the special case of the system (3.71).

### 3.6.2 Illustration of Theorem 3.2

We rewrite the controlled Mathieu equation (3.70) as

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & [\mathcal{M}(t) + \eta(t, x(t))]x(t) + \mathcal{B}(t)u(t) \\ y(t) & = & \mathcal{C}x(t) \end{array} \right. \tag{3.75}$$

with

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}, \quad \mathcal{M}(t) = \begin{bmatrix} 0 & 1 \\ -R_1 - R_2\cos(t) & 0 \end{bmatrix}, \quad \mathcal{C} = e_1^\top, \quad \eta(t,x) = 0, \qquad (3.76)$$

and $\mathcal{B}(t) = -e_2$. We choose $P = 2\pi$. We consider the case where $R_1 = 1$ and $R_2 = 0.024$, and we apply Theorem 3.2 with $s = 1$, and we use $\tau$ to denote the constant $\tau_1$ from Theorem 3.2. The preceding choices of $\eta$, $\mathcal{M}$, and $\mathcal{B}$ ensure that Assumption 3.3 is satisfied. Let

$$\mathcal{A}_1 = \begin{bmatrix} 0 & 1 \\ -R_1 - R_2 & 0 \end{bmatrix} \text{ and } \mathcal{Z}(t) = \begin{bmatrix} 0 \\ R_2(1-\cos(t)) \end{bmatrix}. \qquad (3.77)$$

Then Assumption 3.4 is satisfied, and our choice of $\mathcal{Z}$ gives $\mathcal{D}(t,j,x) = \mathcal{M}(t) - \mathcal{Z}(t)\mathcal{C} - \mathcal{A}_j + \eta(t,x) = 0$. Consequently, our requirements on the functions $\mathcal{D}$ are satisfied with $\bar{k} = 0$.

We choose

$$\mathcal{L}_1 = \begin{bmatrix} 0 \\ 2R_1 \end{bmatrix} \text{ and } \mathcal{F}_1 = \mathcal{A}_1 + \mathcal{L}_1\mathcal{C} = \begin{bmatrix} 0 & 1 \\ R_1 - R_2 & 0 \end{bmatrix}. \qquad (3.78)$$

Then the matrix $\mathcal{E}_1(\tau) = e^{-\mathcal{A}_1\tau} - e^{-\mathcal{F}_1\tau}$ is

$$\mathcal{E}_1(\tau) = \begin{bmatrix} \cos(\omega_1\tau) & -\frac{1}{\omega_1}\sin(\omega_1\tau) \\ \omega_1\sin(\omega_1\tau) & \cos(\omega_1\tau) \end{bmatrix} - \begin{bmatrix} \cosh(\omega_2\tau) & -\frac{1}{\omega_2}\sinh(\omega_2\tau) \\ -\omega_2\sinh(\omega_2\tau) & \cosh(\omega_2\tau) \end{bmatrix} \qquad (3.79)$$

where $\omega_1 = \sqrt{R_1+R_2}$ and $\omega_2 = \sqrt{R_1-R_2}$. This can be checked by noting that the terms in the difference in (3.79) are the fundamental matrix solutions of $\dot{M} = -\mathcal{A}_1 M$ and $\dot{M} = -\mathcal{F}_1 M$, respectively. Thus

$$\mathcal{E}_1(\tau) = \begin{bmatrix} \cos(\omega_1\tau) - \cosh(\omega_2\tau) & -\frac{1}{\omega_1}\sin(\omega_1\tau) + \frac{1}{\omega_2}\sinh(\omega_2\tau) \\ \omega_1\sin(\omega_1\tau) + \omega_2\sinh(\omega_2\tau) & \cos(\omega_1\tau) - \cosh(\omega_2\tau) \end{bmatrix}. \qquad (3.80)$$

Then

$$\det \mathcal{E}_1(\tau) = 2\left[1 - \cos(\omega_1\tau)\cosh(\omega_2\tau) - \frac{R_2}{\omega_1\omega_2}\sin(\omega_1\tau)\sinh(\omega_2\tau)\right].$$

One can easily check that $\det \mathcal{E}_1(0) = 0$ and

$$\begin{aligned} \tfrac{d}{d\tau}\det \mathcal{E}_1(\tau) &= 2R_1\Big(\tfrac{1}{\sqrt{R_1+R_2}}\sin(\sqrt{R_1+R_2}\tau)\cosh(\sqrt{R_1-R_2}\tau) \\ &\quad -\tfrac{1}{\sqrt{R_1-R_2}}\cos(\sqrt{R_1+R_2}\tau)\sinh(\sqrt{R_1-R_2}\tau)\Big) > 0 \end{aligned} \qquad (3.81)$$

for all

$$\tau \in \left(0, \tfrac{\pi}{2\sqrt{1.024}}\right) . \tag{3.82}$$

This follows by noting that (3.81) is equivalent to positivity of

$$S(\tau) = \tfrac{1}{\omega_1}\tan(\omega_1\tau) - \tfrac{1}{\omega_2}\tanh(\omega_2\tau)$$

when (3.82) holds, which follows because $S(0) = 0$ and $S' > 0$ on this interval. It follows that $\mathcal{E}_1(\tau)$ is invertible when $\tau \in (0, \pi/4]$. To check Assumption 3.5, we select the feedback $u_s(t, x) = [1 - R_1 - R_2\cos(t)]x_1 + 2x_2$. Then the system which corresponds to (3.49) is $\dot{x}(t) = Gx(t) - e_2\delta_a(t)$, with

$$G = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \text{ and } \delta_a(t) = (1 - R_1 - R_2\cos(t))\delta_1(t) + 2\delta_2(t) . \tag{3.83}$$

One can check that for all $\ell \geq 0$, we have

$$e^{G\ell} = e^{-\ell}\begin{bmatrix} 1+\ell & \ell \\ -\ell & 1-\ell \end{bmatrix} , \tag{3.84}$$

and therefore also

$$\left|e^{G\ell}\right| = e^{-\ell}\left|I + \begin{bmatrix} \ell & \ell \\ -\ell & -\ell \end{bmatrix}\right| \leq e^{-\ell}(1 + 2\ell) . \tag{3.85}$$

Our choice $P = 2\pi$ gives (3.50) with the choice

$$\kappa = \max_{\ell\in[P,P+\theta]} e^{-\ell}(1 + 2\ell) = e^{-P}(1 + 2P) = \frac{1 + 4\pi}{e^{2\pi}} . \tag{3.86}$$

In terms of the notation from Assumptions 3.3-3.6, one has and one can choose $\bar{k} = 0$, $\mu = 2.024$, $\overline{\mathcal{A}} = |\mathcal{M}|_\infty = 1.024$, $|\mathcal{B}|_\infty = |\mathcal{C}| = 1$, $\overline{\mathcal{L}} = 2$, and $\overline{\mathcal{F}} = 1$, which give

$$\bar{k}\tau\left[e^{(\overline{\mathcal{A}}+\bar{k})\tau} + e^{(\overline{\mathcal{F}}+\bar{k})\tau}\right]\overline{\mathcal{H}} = 0 < 1 \tag{3.87}$$

for any choices of $\tau \in (0, \theta)$. One can also compute $\bar{g} = \bar{p}_1 = \bar{p}_2 = 0$. We deduce that the matrix

$$\bar{\mathcal{S}} = \begin{bmatrix} 0 & 0 \\ \gamma & \frac{1+4\pi}{e^{2\pi}} \end{bmatrix} \tag{3.88}$$

is Schur stable, so Assumptions 3.3-3.6 are satisfied for any $\tau \in (0, \pi/4]$. Thus, we can apply Theorem 3.2.

To illustrate our result, Figure 3.2 shows a MATLAB simulation of the closed loop system with the controller (3.58) and the continuous-discrete observer (3.57) using an integration algorithm of the model (3.75) with a semi-implicit integration of step 0.001 with initial conditions $x_1(0) = 4$, $x_2(0) = -1$, $\hat{x}_1(0) = -3$, and $\hat{x}_2(0) = -2$, with $\theta = \pi/4$ and $\tau = 0.5$. We again show the estimate $\hat{x}_2$ tracking the state component $x_2$, and the closed loop system performance and the control values. Since our simulation shows good stabilization and tracking performance, it helps to illustrate our general theory in the special case of the system (3.75).

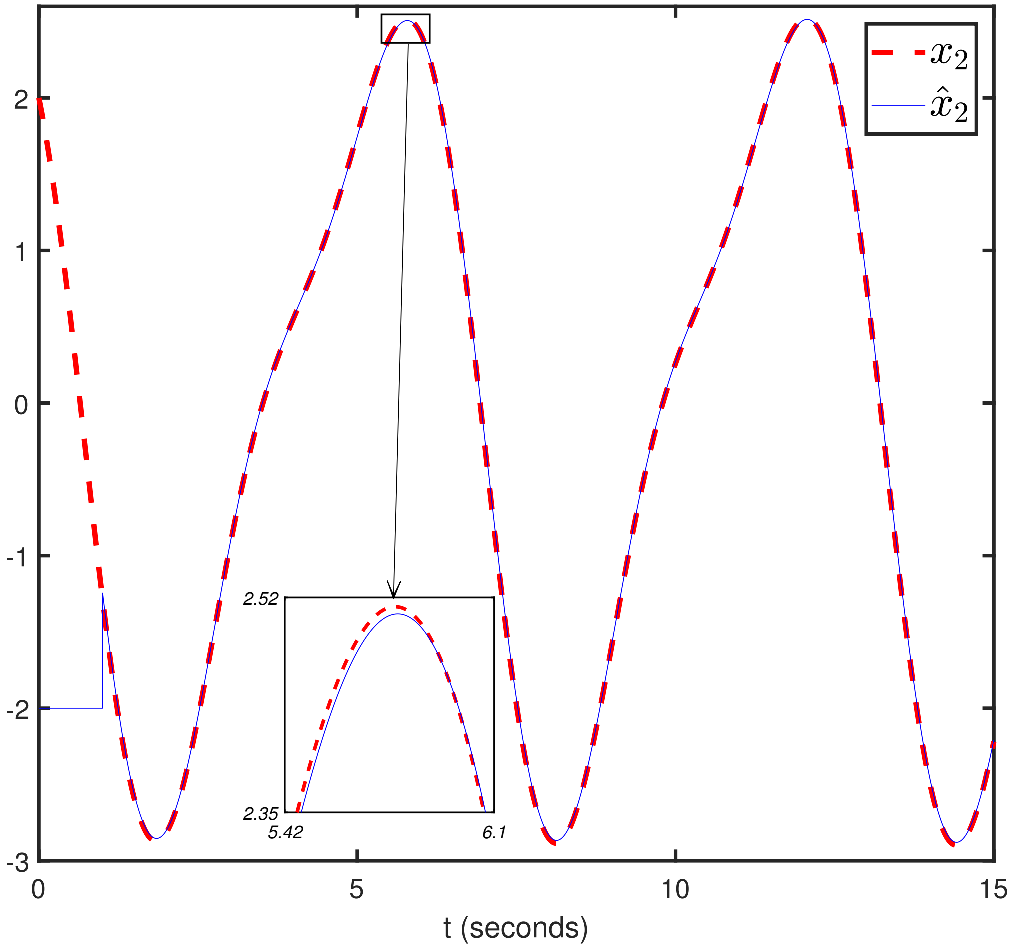


Figure 3.1: Simulations of finite-time observer (3.18) for (3.71): Component $x_2$ and its estimate $\hat{x}_2$

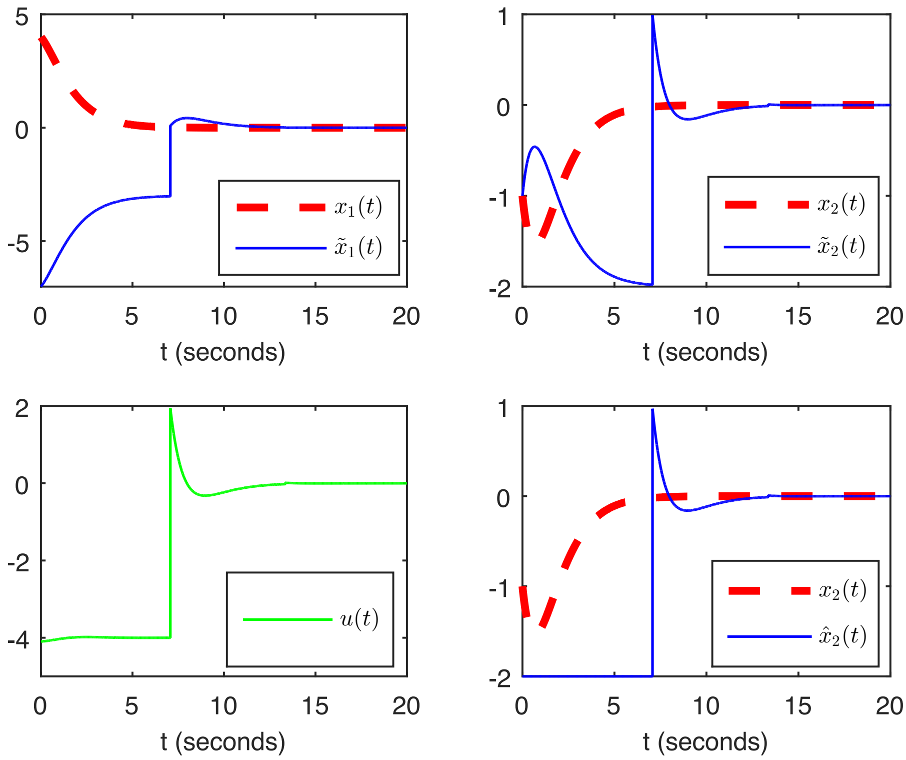


Figure 3.2: Simulation using controller (3.58) and continuous-discrete observer (3.57) for (3.75)

# Chapter 4

# Reduced Order Finite Time Observer Design and Stabilization of Nonlinear Systems

*The content of this chapter is based on the publication of the author [88].* In order to obtain an extact estimate of the solutions of a system in an arbitrary short amount of time, finite time observers have been proposed. Some of them use nonsmooth functions; see for instance [60] and [89]. Their designs are based on homogeneous properties which preclude the possibility of deriving smooth observers from this technique. Another type of finite time observers has been developed. They are smooth and use past values of the output or dynamic extensions. They have been proposed a few decades ago for linear systems; see in particular [78] and [90]. More recently, finite time observers were designed for classes of nonlinear systems; see [30], [73], and [80]. They apply to systems whose vector field is time-invariant when the output is set to zero and provide estimates of all the state variables. Since systems are frequently time-varying and since the measured components of the state do not need to be estimated, this chapter adapts the main results of [30] and [80] to construct finite-time reduced order observers for a family of nonlinear time-varying systems.

## 4.1 Contributions of this chapter

The observer we will build gives estimates only of the unmeasured variables, as does the asymptotic observer proposed for instance in [32, Chapt. 4, Sec. 4.4.3]. This feature presents the following technical advantages. The design is simpler and in some cases, due to the need to determine fundamental solutions of time-varying systems, considering systems with a smaller dimension than the studied one makes it possible to solve the problem, which would be impossible if we were attempting to construct a full-order observer, due to the difficulty of determining explicit formulas for fundamental solutions of systems of dimension larger than one. In addition, the reduced order observer we propose yield better performances than full order observers, in some cases. The family of the time-varying systems is of great insterest, because tracking problems can be recasted into stabilization problems of equilibria of time-varying systems.

In a second part, we give another new result where we show how the reduced order finite time observer we propose can be used to solve a dynamic output feedback stabilization problem for a MIMO nonlinear system.

The chapter is organized as follows. An introductory result is developed in Section 4.2. Section 4.3 is devoted to design reduced order finite-time observers for time-varying systems. In Section 4.4, the reduced order observer is used to stabilize a nonlinear system. In Section 4.5, we illustrate our results in an example derived from a tracking problem for nonholonomic systems in chained form.

## 4.2 An introductory result

In this section, we introduce some ideas to show how to construct reduced order finite-time observers for a simple family of systems. We consider the system

$$\dot{x}(t) = Ax(t) + \delta(t) \tag{4.1}$$

with $x$ valued in $\mathbb{R}^n$ and where $\delta : [0, +\infty) \to \mathbb{R}^n$ is a piecewise-continuous function.

We assume that the output is

$$y(t) = Cx(t) \in \mathbb{R}^p \tag{4.2}$$

with $p \leq n$ and $C$ is of full rank. Also, we assume that the pair $(A, C)$ is observable. Since $C$ is of full rank, there is a linear change of coordinates

$$x_\dagger = C_T x = \begin{bmatrix} y \\ x_r \end{bmatrix} \tag{4.3}$$

and matrices $A_1$ and $A_2$ and functions $\delta_i$, $i = 1, 2$ that are piecewise-continuous with respect to their first argument and linear with respect to $y$ such that

$$\left\{ \begin{array}{rcl} \dot{y}(t) & = & A_1 x_r(t) + \delta_1(t, y(t)) \\ \dot{x}_r(t) & = & A_2 x_r(t) + \delta_2(t, y(t)) \ . \end{array} \right. \tag{4.4}$$

Since the pair $(A, C)$ is observable, it follows that the pair $(A_2, A_1)$ is observable [32, Chapt. 4, Sec. 4.4.3]. This can be proved as follows. Since $(A, C)$ is observable,

$$\left(C_T A C_T^{-1}, C C_T^{-1}\right) = \left( \begin{bmatrix} \star & A_1 \\ \star & A_2 \end{bmatrix}, [I\ 0] \right) \tag{4.5}$$

is observable. Now, let us proceed by contradiction. Assume that $(A_2, A_1)$ is not observable. Then the Popov-Belevitch-Hautus Test for observability of a pair ensures that there is $v \neq 0$ and $\lambda \in \mathbb{C}$ such that $A_2 v = \lambda v$ and $A_1 v = 0$. Then $X = \begin{pmatrix} 0 \\ v \end{pmatrix}$ is such that $[I\ 0]X = 0$ and

$$\begin{bmatrix} \star & A_1 \\ \star & A_2 \end{bmatrix} X = \begin{pmatrix} A_1 v \\ A_2 v \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ v \end{pmatrix} = \lambda X \ .$$

Since $X \neq 0$, the Popov-Belevitch-Hautus Test implies that $(C_T A C_T^{-1}, C C_T^{-1})$ is not observable. This yields a contradiction.

Since $(A_2, A_1)$ is observable, one can prove (see [80, Lemma 1]) that there are a matrix $L \in \mathbb{R}^{n-p \times p}$ and a constant $\tau > 0$ such that the matrix

$$M_\tau = e^{-A_2 \tau} - e^{-H\tau} \tag{4.6}$$

with

$$H = A_2 + LA_1 \tag{4.7}$$

is invertible.

Let us introduce the variable:

$$x_s = x_r + Ly \; . \tag{4.8}$$

Then simple calculations give

$$\dot{x}_s(t) \;\; = \;\; (A_2 + LA_1)x_r(t) + \delta_2(t, y(t)) + L\delta_1(t, y(t)) \; . \tag{4.9}$$

From the definition of $H$, it follows that

$$\dot{x}_s(t) = Hx_s(t) + Ky(t) + \delta_3(t, y(t)) \tag{4.10}$$

with

$$K = -(A_2 + LA_1)L \tag{4.11}$$

and

$$\delta_3 = \delta_2 + L\delta_1 \; . \tag{4.12}$$

By integrating the second equation in (4.4) and (4.10), we obtain

$$x_r(t - \tau) = e^{-A_2 \tau} x_r(t) - \int_{t-\tau}^{t} e^{A_2(t-m-\tau)} \delta_2(m, y(m)) dm \tag{4.13}$$

and

$$x_s(t - \tau) = e^{-H\tau} x_s(t) - \int_{t-\tau}^{t} e^{H(t-m-\tau)} [Ky(m) + \delta_3(m, y(m))] dm \tag{4.14}$$

for all $t \geq \tau$. From the definition of $x_s$ in (4.8), we deduce that

$$\begin{aligned} x_r(t - \tau) \;\; = \;\; & e^{-H\tau} x_r(t) + e^{-H\tau} Ly(t) - Ly(t - \tau) \\ & - \textstyle\int_{t-\tau}^{t} e^{H(t-m-\tau)} [Ky(m) + \delta_3(m, y(m))] dm \; . \end{aligned} \tag{4.15}$$

Subtracting (4.15) from (4.13) and then using the definition of $M_\tau$ in (4.6), we deduce that

$$\begin{array}{rcl} M_\tau x_r(t) & = & -\int_{t-\tau}^{t} e^{H(t-m-\tau)}[Ky(m)+\delta_3(m,y(m))]dm \\ & & +\int_{t-\tau}^{t} e^{A_2(t-m-\tau)}\delta_2(m,y(m))dm \\ & & +e^{-H\tau}Ly(t)-Ly(t-\tau)\ . \end{array} \quad (4.16)$$

Since $M_\tau$ is invertible, for all $t \geq \tau$,

$$x_r(t) = \hat{x}_r(t) \quad (4.17)$$

with

$$\begin{array}{rcl} \hat{x}_r(t) & = & -M_\tau^{-1}\int_{t-\tau}^{t} e^{H(t-m-\tau)}[Ky(m) \\ & & +\delta_2(m,y(m))+L\delta_1(m,y(m))]dm \\ & & +M_\tau^{-1}\int_{t-\tau}^{t} e^{A_2(t-m-\tau)}\delta_2(m,y(m))dm \\ & & +M_\tau^{-1}\left[e^{-H\tau}Ly(t)-Ly(t-\tau)\right]\ . \end{array} \quad (4.18)$$

Thus, when $\delta_1$ and $\delta_2$ are known, the forumla (4.18) provides the exact value of $x_r(t)$ for all $t \geq \tau$.

## 4.3 Time-varying systems

In this section, we show how the finite-time observer design of the previous section adapts to time-varying nonlinear systems of the type

$$\left\{ \begin{array}{rcl} \dot{z}(t) & = & A_1(t)x_r(t)+\delta_1(t,z(t)) \\ \dot{x}_r(t) & = & A_2(t)x_r(t)+\delta_2(t,z(t)) \end{array} \right. \quad (4.19)$$

where $z$ is valued in $\mathbb{R}^p$, $x_r$ is valued in $\mathbb{R}^{n-p}$, the output is

$$y(t) = z(t)+\epsilon(t) \quad (4.20)$$

where $\epsilon(t)$ is a piecewise continuous function that is bounded by a constant $\overline{\epsilon} \geq 0$, the functions $A_i$ for $i=1,2$ are piecewise continuous and bounded, and $\delta_1$ and $\delta_2$ are functions that are piecewise continuous with respect to $t$ and locally Lipschitz with respect to $z$ and such that there is a nonnegative valued continuous function $\overline{\delta}$ such that

$$|\delta_1(t,z)|+|\delta_2(t,z)| \leq \overline{\delta}(|z|) \quad (4.21)$$

for all $t \geq 0$ and $z \in \mathbb{R}^p$.

**Remark 4.1** *The special structure of the system (4.19) does not limit the family of linear systems to which our approach applies because, as explained in the previous section, any system of the type*

$$\dot{X} = A(t)X + \mathcal{F}(t, Y)$$

*with an output* $Y = CX$ *with* $C$ *of full rank can be transformed through a linear time-invariant change of coordinates into a system of the form (4.19).*

**Remark 4.2** *The term* $\epsilon(t)$ *in (4.20) represents a measurement noise, which is of great practical interest.*

**Assumption 4.1** *There are a constant* $\tau > 0$ *and a bounded function* $L$ *of class* $C^1$ *with a bounded first derivative such that for all* $t \in \mathbb{R}$*, the matrix*

$$\Lambda(t) = \Phi_{A_2}(t, t-\tau) - \Phi_H(t, t-\tau) \tag{4.22}$$

*with* $H(t) = A_2(t) + L(t)A_1(t)$ *is invertible.*

Let us define

$$\delta_3(t, z) = L(t)\delta_1(t, z) + \delta_2(t, z) \quad \text{and} \quad \delta_4(t, z) = -[D(t)z + \delta_3(t, z)] \tag{4.23}$$

with

$$D(t) = \dot{L}(t) - H(t)L(t)\ . \tag{4.24}$$

Let

$$\begin{aligned} \hat{x}_r(t) \;&=\; \Lambda(t)^{-1}\textstyle\int_{t-\tau}^{t} [\Phi_{A_2}(m, t-\tau)\delta_2(m, y(m) - \epsilon(m)) \\ &\quad + \Phi_H(m, t-\tau)\delta_4(m, y(m) - \epsilon(m))]\, dm \\ &\quad + \Lambda(t)^{-1} [\Phi_H(t, t-\tau)L(t)(y(t) - \epsilon(t)) \\ &\quad - L(t-\tau)(y(t-\tau) - \epsilon(t-\tau))]\ . \end{aligned} \tag{4.25}$$

**Theorem 4.1** *Let the system (4.19) satisfy Assumption 4.1. Then*

$$x_r(t) = \hat{x}_r(t) \tag{4.26}$$

*for all* $t \geq \tau$*.*

**Remark 4.3** *In general, one can check easily that Assumption 4.1 is satisfied when* $n-p=1$ *because then* $\Phi_{A_2}$ *and* $\Phi_H$ *are:* $\Phi_{A_2}(t,t_0) = e^{-\int_{t_0}^{t} A_2(m)dm}$ *and* $\Phi_H(t,t_0) = e^{-\int_{t_0}^{t} H(m)dm}$. *When* $n-p>1$, *checking this assumption may be more difficult. When a full order finite time observer is constructed in this context, then fundamental solutions of matrices of dimension* $n \times n$ *have to be considered, whereas we only have to consider fundamental solutions of matrices* $(n-p)\times(n-p)$. *This can be a crucial advantage of the reduced-order approach over the full order one.*

**Remark 4.4** *If there is* $L$, $\tau > 0$ *and* $\varpi \in (0,1)$ *such that* $|\Phi_{A_2}(t,t-\tau)^{-1}\Phi_H(t,t-\tau)| \leq \varpi$ *for all* $t \geq 0$, *then Assumption 4.1 is satisfied and* $\Lambda^{-1}$ *is bounded. Indeed, in this case* $I - \Phi_{A_2}(t,t-\tau)^{-1}\Phi_H(t,t-\tau)$ *is invertible and*

$$\left[I - \Phi_{A_2}(t,t-\tau)^{-1}\Phi_H(t,t-\tau)\right]^{-1} = \sum_{k=0}^{\infty} \Phi_{A_2}(t,t-\tau)^{-1}\Phi_H(t,t-\tau)^k \quad (4.27)$$

*for all* $t \geq 0$ *which implies that*

$$\left|\left[I - \Phi_{A_2}(t,t-\tau)^{-1}\Phi_H(t,t-\tau)\right]^{-1}\right| \leq \frac{1}{1-\varpi} \quad (4.28)$$

*for all* $t \geq 0$. *We deduce that* $\Lambda(t)^{-1}$ *is well-defined for all* $t \geq 0$ *and*

$$\begin{array}{rcl} |\Lambda(t)^{-1}| & = & \left|\left(I - \Phi_{A_2}(t,t-\tau)^{-1}\Phi_H(t,t-\tau)\right)^{-1}\Phi_{A_2}(t,t-\tau)^{-1}\right| \\ & \leq & \frac{\left|\Phi_{A_2}(t,t-\tau)^{-1}\right|}{1-\varpi} \end{array} \quad (4.29)$$

*which is bounded by a constant because* $\Phi_{A_2}(t,t-\tau)^{-1}$ *is bounded.*

Notice also that the function $\Lambda^{-1}$ is bounded if the system is periodic because then $\Lambda^{-1}$ is continuous and periodic.

**Remark 4.5** *When* $\epsilon$ *is unknown, the exact estimate (4.26) cannot be used. Fortunately, we can deduce from equation (4.26) an approximate observer:*

$$\begin{array}{rcl} x_r^*(t) & = & \Lambda(t)^{-1}\int_{t-\tau}^{t}\left[\Phi_{A_2}(m,t-\tau)\delta_2(m,y(m)) + \Phi_H(m,t-\tau)\delta_4(m,y(m))\right]dm \\ & & +\Lambda(t)^{-1}\left[\Phi_H(t,t-\tau)L(t)y(t) - L(t-\tau)y(t-\tau)\right] . \end{array} \quad (4.30)$$

*Since the functions $A$ and $H$ are bounded,* $\sup_{m\in[t-\tau,t]} |\Phi_H(m,t-\tau)|$ *and* $\sup_{m\in[t-\tau,t]} |\Phi_{A_2}(m,t-\tau)|$ *are bounded. We deduce that when $\delta_1$ and $\delta_2$ are globally Lipschitz with respect to their second argument and $\Lambda(t)^{-1}$ is bounded, there is a constant $l_a > 0$ such that*

$$|x_r^*(t) - x_r(t)| \leq l_a|\epsilon|_\infty \tag{4.31}$$

*for all $t \geq \tau$.*

**Proof.** Let us introduce

$$x_v(t) = \Phi_{A_2}(t,0)x_r(t) \ . \tag{4.32}$$

An immediate calculation gives

$$\begin{aligned} \dot{x}_v(t) &= -\Phi_{A_2}(t,0)A_2x_r(t) + \Phi_{A_2}(t,0)[A_2(t)x_r(t) + \delta_2(t,z(t))] \\ &= \Phi_{A_2}(t,0)\delta_2(t,z(t)) \ . \end{aligned} \tag{4.33}$$

By integrating (4.33) between $t-\tau$ and $t$, we obtain

$$x_v(t) = x_v(t-\tau) + \int_{t-\tau}^{t} \Phi_{A_2}(m,0)\delta_2(m,z(m))dm \tag{4.34}$$

for all $t \geq \tau$. From the definition of $x_v$, it straightforwardly follows that

$$\begin{aligned} &\Phi_{A_2}(t-\tau,0)^{-1}\Phi_{A_2}(t,0)x_r(t) = \\ &x_r(t-\tau) + \textstyle\int_{t-\tau}^{t} \Phi_{A_2}(t-\tau,0)^{-1}\Phi_{A_2}(m,0)\delta_2(m,z(m))dm \ . \end{aligned} \tag{4.35}$$

For any continuous function $\Omega : \mathbb{R} \to \mathbb{R}^{n\times n}$, the function $\Psi$ defined as $\Psi_\Omega(t,t_0) = \Phi_\Omega(t,t_0)^\top$ satisfies

$$\frac{\partial \Psi_\Omega}{\partial t}(t,t_0) = -\Omega(t)^\top \Psi_\Omega(t,t_0) \ . \tag{4.36}$$

It follows from the semigroup property of flow maps that $\Psi_\Omega(t,0) = \Psi_\Omega(t,t-\tau)\Psi_\Omega(t-\tau,0)$. Consequently $\Psi_\Omega(t,0)^\top = \Psi_\Omega(t-\tau,0)^\top\Psi_\Omega(t,t-\tau)^\top$. Also, we have $\Psi_\Omega(m,0) = \Psi_\Omega(m,t-\tau)\Psi_\Omega(t-\tau,0)$, which implies that $\Psi_\Omega(m,0)^\top = \Psi_\Omega(t-\tau,0)^\top\Psi_\Omega(m,t-\tau)^\top$ for all $m \geq t-\tau$. It follows that

$$\Phi_\Omega(t,0) = \Phi_\Omega(t-\tau,0)\Phi_\Omega(t,t-\tau) \tag{4.37}$$

and

$$\Phi_\Omega(m,0) = \Phi_\Omega(t-\tau,0)\Phi_\Omega(m,t-\tau)\ . \tag{4.38}$$

From this identity and (4.35), we deduce that

$$\Phi_{A_2}(t,t-\tau)x_r(t) \quad = \quad x_r(t-\tau) + \int_{t-\tau}^{t} \Phi_{A_2}(m,t-\tau)\delta_2(m,z(m))dm\ . \tag{4.39}$$

Now, let

$$x_s(t) = x_r(t) + L(t)z(t)\ . \tag{4.40}$$

Simple calculations give

$$\begin{aligned} \dot{x}_s(t) &= A_2(t)x_r(t) + \delta_2(t,z(t)) + \dot{L}(t)z(t) \\ &\quad + L(t)[A_1(t)x_r(t) + \delta_1(t,z(t))] \\ &= H(t)x_r(t) + \dot{L}(t)z(t) + \delta_3(t,z(t)) \\ &= H(t)x_s(t) + [\dot{L}(t) - H(t)L(t)]z(t) + \delta_3(t,z(t))\ . \end{aligned} \tag{4.41}$$

Arguing as we did to prove (4.39), we obtain

$$\begin{aligned} &\Phi_H(t,t-\tau)x_s(t) \\ &= x_s(t-\tau) + \int_{t-\tau}^{t} \Phi_H(m,t-\tau)[D(m)z(m) + \delta_3(m,z(m))]dm \end{aligned} \tag{4.42}$$

where $D$ is the function defined in (4.24) for all $t \geq \tau$. From the definition of $x_s$, we deduce that

$$\begin{aligned} &\Phi_H(t,t-\tau)[x_r(t) + L(t)z(t)] \\ &= x_r(t-\tau) + L(t-\tau)z(t-\tau) \\ &+ \int_{t-\tau}^{t} \Phi_H(m,t-\tau)[D(m)z(m) + \delta_3(m,z(m))]dm\ . \end{aligned} \tag{4.43}$$

Consequently,

$$\begin{aligned} &\Phi_H(t,t-\tau)x_r(t) \\ &= x_r(t-\tau) - \Phi_H(t,t-\tau)L(t)z(t) + L(t-\tau)z(t-\tau) \\ &+ \int_{t-\tau}^{t} \Phi_H(m,t-\tau)[D(m)z(m) + \delta_3(m,z(m))]dm\ . \end{aligned} \tag{4.44}$$

By substracting (4.44) from (4.39), we obtain

$$\begin{aligned} \Lambda(t)x_r(t) &= \int_{t-\tau}^{t} \Phi_{A_2}(m,t-\tau)\delta_2(m,z(m))dm \\ &\quad - \int_{t-\tau}^{t} \Phi_H(m,t-\tau)[D(m)z(m) + \delta_3(m,z(m))]dm \\ &\quad + \Phi_H(t,t-\tau)L(t)z(t) - L(t-\tau)z(t-\tau) \end{aligned} \tag{4.45}$$

for all $t \geq \tau$. Assumption 4.1 ensures that for all $t \in \mathbb{R}$, $\Lambda(t)$ is invertible, which implies that (4.26) is satisfied. □

## 4.4 Output feedback stabilization

In this section, we show how the finite-time reduced order observer (4.25) can be used to asymptotically stabilize the origin of nonlinear systems. Consider the nonlinear system

$$\left\{\begin{array}{rcl} \dot{z}(t) & = & A_1(t)x_r(t) + B_1(t)u(t) + \varsigma_1(t, y(t)) + \varphi_1(t, \chi(t)) + f_1(t) \\ \dot{x}_r(t) & = & A_2(t)x_r(t) + B_2(t)u(t) + \varsigma_2(t, y(t)) + \varphi_2(t, \chi(t)) + f_2(t) \end{array}\right. \tag{4.46}$$

where the input $u$ is valued in $\mathbb{R}^q$ is the input, $z$ is valued in $\mathbb{R}^p$, $x_r$ is valued in $\mathbb{R}^{n-p}$, $\chi = (x_r, z) \in \mathbb{R}^n$, the output is

$$y(t) = z(t) + \epsilon(t) \tag{4.47}$$

where $\epsilon$, $f_1$ and $f_2$ are piecewise continuous functions, $\epsilon$ is bounded in norm by a constant $\overline{\epsilon}$, the functions $A_i$ and $B_i$ are piecewise continuous and bounded and the functions $\varphi_i$ and $\varsigma_i$ are piecewise continuous with respect to $t$ and locally Lipschitz with respect respectively to $\chi$ and $y$ and such that there is a function $\overline{\Delta}$ of class $\mathcal{K}$ such that

$$|\varsigma_1(t, y)| + |\varsigma_2(t, y)| + |\varphi_1(t, \chi)| + |\varphi_2(t, \chi)| \leq \overline{\Delta}(|\chi| + |y|) \tag{4.48}$$

for all $t \geq 0$, $\chi \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$.

We introduce several assumptions.

**Assumption 4.2** *The functions $A_1$ and $A_2$ of system (4.46) are such that Assumption 4.1 is satisfied. There is a constant $\Lambda_\sharp > 0$ is such that*

$$|\Lambda(t)^{-1}| \leq \Lambda_\sharp \tag{4.49}$$

*for all $t \in \mathbb{R}$. There are a function $u_s(t, \chi)$ that is piecewise continuous with respect to $t$ and Lipschitz continuous with respect to $\chi$, a constant $\alpha_c > 0$, a function $V$ of class $C^1$, functions $\gamma_i$ of class $\mathcal{K}$ and functions $\kappa_i$ of class $\mathcal{K}_\infty$ such that*

$$\kappa_1(|\chi|) \leq V(t, \chi) \leq \kappa_2(|\chi|) \quad , \quad |u_s(t, \chi)| \leq \kappa_3(|\chi|) \tag{4.50}$$

*hold for all* $t \geq 0$, $\chi \in \mathbb{R}^n$ *and such that the derivative of* $V$ *along all trajectories of*

$$\left\{ \begin{array}{rcl} \dot{z}(t) & = & A_1(t)x_r(t) + B_1(t)u_s(t, \chi(t) + \mu(t)) + \varsigma_1(t, z(t) \\ & & +\epsilon(t)) + \varphi_1(t, \chi(t)) + \mathfrak{h}_1(t) \\ \dot{x}_r(t) & = & A_2(t)x_r(t) + B_2(t)u_s(t, \chi(t) + \mu(t)) + \varsigma_2(t, z(t) \\ & & +\epsilon(t)) + \varphi_2(t, \chi(t)) + \mathfrak{h}_2(t) \end{array} \right. \tag{4.51}$$

*satisfies*

$$\dot{V}(t) \leq -\alpha_c V(t, \chi(t)) + \gamma_1(|\mu(t)|) + \gamma_2(|(\mathfrak{h}_1(t), \mathfrak{h}_2(t), \epsilon(t))|) \, . \tag{4.52}$$

*Moreover, the system (4.46) is forward complete.*

**Assumption 4.3** *There is a constant* $\mathfrak{c} \geq 0$ *such that*

$$4\mathfrak{c}\tau \leq \alpha_c \tag{4.53}$$

*and for all* $t \geq 2\tau$ *and* $\phi \in C_{in}$,

$$\gamma_1(2|\varrho(t, \phi)|) \leq \mathfrak{c} \int_{-\tau}^{0} V(m + t, \phi(m))dm \tag{4.54}$$

*where*

$$\begin{array}{rcl} \varrho(t, \phi) & = & \Lambda(t)^{-1} \int_{t-\tau}^{t} [\Phi_{A_2}(m, t-\tau) - \Phi_H(m, t-\tau)]\varphi_2(m, \phi(m))dm \\ & & -\Lambda(t)^{-1} \int_{t-\tau}^{t} \Phi_H(m, t-\tau)L(m)\varphi_1(m, \phi(m))dm \, . \end{array} \tag{4.55}$$

**Assumption 4.4** *There are two constants* $\overline{\varphi}_i$, *for* $i = 1, 2$ *such that*

$$|\varphi_i(t, y_1, x) - \varphi_i(t, y_2, x)| \ \leq \ \overline{\varphi}_i |y_1 - y_2| \tag{4.56}$$

*for all* $t \geq 0$, $x \in \mathbb{R}^{n-p}$, $y_1 \in \mathbb{R}^p$ *and* $y_2 \in \mathbb{R}^p$.

**Theorem 4.2** *Assume that the system (4.46) satisfies Assumptions 4.2 to 4.4. Then the system (4.46) in closed-loop with the control law* $u_s(t, \hat{x}_r(t), y(t))$ *from*

*Assumption 4.2 with*

$$
\begin{array}{rcl}
\hat{x}_r(t) & = & 0 \quad when \quad t \in [-\tau, \tau) \\
\hat{x}_r(t) & = & \Lambda(t)^{-1} \int_{t-\tau}^{t} [\Phi_{A_2}(m, t-\tau) - \Phi_H(m, t-\tau)] \\
& & \times [B_2(m) u_s(m, \hat{x}_r(m), y(m)) + \varsigma_2(m, y(m))] dm \\
& & -\Lambda(t)^{-1} \int_{t-\tau}^{t} \Phi_H(m, t-\tau) L(m) \\
& & \times [B_1(m) u_s(m, \hat{x}_r(m), y(m)) + \varsigma_1(m, y(m))] dm \\
& & +\Lambda(t)^{-1} [\Phi_H(t, t-\tau) L(t) y(t) - L(t-\tau) y(t-\tau)] \\
& & -\Lambda(t)^{-1} \int_{t-\tau}^{t} \Phi_H(m, t-\tau) D(m) y(m) dm \\
& & \quad when \quad t \geq \tau
\end{array}
\tag{4.57}
$$

*is ISS with respect to $\epsilon$ and $(f_1, f_2)$.*

**Remark 4.6** *Comparison functions for the ISS inequality can be deduced from the proof below.*

**Remark 4.7** *The function $\hat{x}_r(t)$ defined in (4.57) does not depend on the functions $\varphi_1$ and $\varphi_2$. Thus Theorem 4.2 applies in the absence of an exact knowledge of $\varphi_1$ and $\varphi_2$.*

**Remark 4.8** *The function $\hat{x}_r(t)$ is given as the solution of an integral equation. Since only past values of $\hat{x}_r$ are involved in the right hand side of (4.57), this is not an obstacle to the practical use of $\hat{x}_r$.*

**Proof.** The system (4.46) in closed-loop with $u_s(t, \hat{x}_r(t), y(t))$ is

$$
\left\{
\begin{array}{rcl}
\dot{z}(t) & = & A_1(t) x_r(t) + B_1(t) u_s(t, \hat{x}_r(t), y(t)) + \varsigma_1(t, y(t)) \\
& & +\varphi_1(t, \chi(t)) + f_1(t) \\
\dot{x}_r(t) & = & A_2(t) x_r(t) + B_2(t) u_s(t, \hat{x}_r(t), y(t)) + \varsigma_2(t, y(t)) \\
& & +\varphi_2(t, \chi(t)) + f_2(t) \; .
\end{array}
\right.
\tag{4.58}
$$

One can prove that the solutions of this system are well-defined by considering first the interval $[0, \tau]$, where the closed-loop system behaves as an ordinary differential equation and next the interval $[\tau, +\infty)$ where the closed-loop system presents a classical term with a distributed delay. According to Assumption 4.2, we can apply Theorem 4.1 to (4.58) with

$$\begin{array}{rcl} \delta_1(t,z) & = & B_1(t)u_s(t,\hat{x}_r(t),y(t)) + \varsigma_1(t,y(t)) + \varphi_1(t,z,x_r(t)) + f_1(t) \\ \delta_2(t,z) & = & B_2(t)u_s(t,\hat{x}_r(t),y(t)) + \varsigma_2(t,y(t)) + \varphi_2(t,z,x_r(t)) + f_2(t)\ . \end{array} \tag{4.59}$$

We obtain

$$\begin{array}{rcl} x_r(t) & = & \Lambda(t)^{-1}\int_{t-\tau}^{t}\Phi_{A_2}(m,t-\tau)\delta_2(m,y(m)-\epsilon(m))dm \\ & & +\Lambda(t)^{-1}\int_{t-\tau}^{t}\Phi_H(m,t-\tau)[-D(m)y(m)+D(m)\epsilon(m)]dm \\ & & -\Lambda(t)^{-1}\int_{t-\tau}^{t}\Phi_H(m,t-\tau)[L(m)\delta_1(m,y(m)-\epsilon(m)) \\ & & +\delta_2(m,y(m)-\epsilon(m))]dm \\ & & +\Lambda(t)^{-1}\left[\Phi_H(t,t-\tau)L(t)(y(t)-\epsilon(t))\right. \\ & & \left.-\ L(t-\tau)(y(t-\tau)-\epsilon(t-\tau))\right] \end{array} \tag{4.60}$$

for all $t \geq \tau$. By grouping terms, we obtain

$$\begin{array}{rcl} x_r(t) & = & \Lambda(t)^{-1}\int_{t-\tau}^{t}\left[\Phi_{A_2}(m,t-\tau)-\Phi_H(m,t-\tau)\right] \\ & & \times\delta_2(m,y(m)-\epsilon(m))dm \\ & & -\Lambda(t)^{-1}\int_{t-\tau}^{t}\Phi_H(m,t-\tau)L(m)\delta_1(m,y(m)-\epsilon(m))dm \\ & & +\Lambda(t)^{-1}\left[\Phi_H(t,t-\tau)L(t)y(t)-L(t-\tau)y(t-\tau)\right] \\ & & -\Lambda(t)^{-1}\int_{t-\tau}^{t}\Phi_H(m,t-\tau)D(m)y(m)dm \\ & & +\Upsilon_1(t,\epsilon_t) \end{array} \tag{4.61}$$

where

$$\begin{array}{rcl} \Upsilon_1(t,\epsilon_t) & = & \Lambda(t)^{-1}\int_{t-\tau}^{t}\Phi_H(m,t-\tau)D(m)\epsilon(m)dm \\ & & -\Lambda(t)^{-1}\Phi_H(t,t-\tau)L(t)\epsilon(t) \\ & & +\Lambda(t)^{-1}L(t-\tau)\epsilon(t-\tau)\ . \end{array} \tag{4.62}$$

Then

$$x_r(t) = \hat{x}_r(t) + \varrho(t,\chi_t) + \Upsilon_1(t,\epsilon_t) + \Upsilon_2(t) + \Upsilon_3(t,y_t,x_{r,t}) \tag{4.63}$$

with $\hat{x}_r$ defined in (4.57), $\varrho$ defined in (4.55) and

$$\begin{array}{rcl} \Upsilon_2(t) & = & \Lambda(t)^{-1}\int_{t-\tau}^{t}[\Phi_{A_2}(m,t-\tau)-\Phi_H(m,t-\tau)]f_2(m)dm \\ & & -\Lambda(t)^{-1}\int_{t-\tau}^{t}\Phi_H(m,t-\tau)L(m)f_1(m)dm \\ \Upsilon_3(t,y_t,x_{r,t}) & = & \Lambda(t)^{-1}\int_{t-\tau}^{t}\left[\Phi_{A_2}(m,t-\tau)-\Phi_H(m,t-\tau)\right] \\ & & \times[\varphi_2(m,y(m)-\epsilon(m),x_r(m))-\varphi_2(m,y(m),x_r(m))]dm \\ & & +\Lambda(t)^{-1}\int_{t-\tau}^{t}\Phi_H(m,t-\tau)L(m) \\ & & \times[\varphi_1(m,y(m),x_r(m))-\varphi_1(m,y(m)-\epsilon(m),x_r(m))]dm \end{array} \tag{4.64}$$

for all $t \geq \tau$.

By replacing $\hat{x}_r(t)$ in (4.58) by $x_r(t) - \varrho(t, \chi_t) - \Upsilon_4(t, \epsilon_t, y_t, x_{r,t})$ with

$$\Upsilon_4(t, \epsilon_t, y_t, x_{r,t}) = \Upsilon_1(t, \epsilon_t) + \Upsilon_2(t) + \Upsilon_3(t, y_t, x_{r,t}) \tag{4.65}$$

we obtain

$$\left\{ \begin{array}{rcl} \dot{z}(t) & = & A_1(t)x_r(t) + B_1(t)u_s(t, x_r(t) - \varrho(t, \chi_t) - \Upsilon_4(t, \epsilon_t, y_t, x_{r,t}), z(t) + \epsilon(t)) \\ & & +\varsigma_1(t, y(t)) + \varphi_1(t, \chi(t)) + f_1(t) \\ \dot{x}_r(t) & = & A_2(t)x_r(t) + B_2(t)u_s(t, x_r(t) - \varrho(t, \chi_t) - \Upsilon_4(t, \epsilon_t, y_t, x_{r,t}), z(t) + \epsilon(t)) \\ & & +\varsigma_2(t, y(t)) + \varphi_2(t, \chi(t)) + f_2(t) \; . \end{array} \right. \tag{4.66}$$

From Assumption 4.2, we deduce that the time derivative of the function $V$ along the trajectories of (4.66) satisfies

$$\begin{array}{rcl} \dot{V}(t) & \leq & -\alpha_c V(t, \chi(t)) + \gamma_1(\sqrt{|\varrho(t, \chi_t) + \Upsilon_4(t, \epsilon_t, y_t, x_{r,t})|^2 + |\epsilon(t)|^2}) \\ & & +\gamma_2(|(f_1(t), f_2(t), \epsilon(t))|) \\ & \leq & -\alpha_c V(t, \chi(t)) + \gamma_1(|\varrho(t, \chi_t)| + |\Upsilon_4(t, \epsilon_t, y_t, x_{r,t})| + |\epsilon(t)|) \\ & & +\gamma_2(|(f_1(t), f_2(t), \epsilon(t))|) \\ & \leq & -\alpha_c V(t, \chi(t)) + \gamma_1(2|\varrho(t, \chi_t)|) + \gamma_1(2|\Upsilon_4(t, \epsilon_t, y_t, x_{r,t})| \\ & & +2|\epsilon(t)|) + \gamma_2(|(f_1(t), f_2(t), \epsilon(t))|) \; . \end{array} \tag{4.67}$$

According to Assumption 4.3, it follows that

$$\begin{array}{rcl} \dot{V}(t) & \leq & -\alpha_c V(t, \chi(t)) + \mathfrak{c} \int_{t-\tau}^{t} V(m, \chi(m))dm + 2|\epsilon(t)|) \\ & & +\gamma_1(2|\Upsilon_4(t, \epsilon_t, y_t, x_{r,t})| + \gamma_2(|(f_1(t), f_2(t), \epsilon(t))|) \end{array} \tag{4.68}$$

for all $t \geq 2\tau$.

Now, observe that, using Assumption 4.4 and the boundedness of $A$, $J$ and $\Lambda^{-1}$, one can determine a constant $\mathfrak{k} \geq 0$ such that

$$\begin{array}{rcl} |\Upsilon_4(t, \epsilon_t, y_t, x_{r,t})| & \leq & |\Upsilon_1(t, \epsilon_t)| + |\Upsilon_2(t)| + |\Upsilon_3(t, y_t, x_{r,t})| \\ & \leq & \mathfrak{k} \sup\limits_{m[t-\tau,t]} (|\epsilon(m)| + |f_1(m)| + |f_2(m)|) \; . \end{array} \tag{4.69}$$

Thus, there is a function $\gamma_3$ of class $\mathcal{K}$ such that

$$\begin{array}{rcl} \dot{V}(t) & \leq & -\alpha_c V(t, \chi(t)) + \mathfrak{c} \int_{t-\tau}^{t} V(m, \chi(m))dm \\ & & +\gamma_3 \left( \sup\limits_{m[t-\tau,t]} (|\epsilon(m)| + |f_1(m)| + |f_2(m)|) \right) \end{array} \tag{4.70}$$

for all $t \geq 2\tau$. Set

$$U(t,\phi) \;=\; V(t,\phi(0)) + 2\mathfrak{c}\int_{-\tau}^{0}\int_{\ell}^{0} V(m+t,\phi(m))dmd\ell\,. \tag{4.71}$$

Then along the trajectories of (4.66),

$$U(t,\chi_t) \;=\; V(t,\chi(t)) + 2\mathfrak{c}\int_{t-\tau}^{t}\int_{\ell}^{t} V(m,\chi(m))dmd\ell \tag{4.72}$$

and

$$\begin{aligned}\dot{U}(t) \;\leq\; & -(\alpha_c - 2\mathfrak{c}\tau)V(t,\chi(t)) - \mathfrak{c}\int_{t-\tau}^{t} V(m,\chi(m))dm \\ & +\gamma_3\left(\sup_{m[t-\tau,t]}(|\epsilon(m)| + |f_1(m)| + |f_2(m)|)\right)\end{aligned} \tag{4.73}$$

for all $t \geq 2\tau$. From (4.53), it follows that

$$\begin{aligned}\dot{U}(t) \;\leq\; & -\tfrac{1}{2}\alpha_c V(t,\chi(t)) - \mathfrak{c}\int_{t-\tau}^{t} V(m,\chi(m))dm \\ & +\gamma_3\left(\sup_{m[t-\tau,t]}(|\epsilon(m)| + |f_1(m)| + |f_2(m)|)\right)\,.\end{aligned} \tag{4.74}$$

We deduce that there is $\beta_c > 0$ such that

$$\dot{U}(t) \;\leq\; -\beta_c U(t,\chi_t) + \gamma_3\left(\sup_{m[t-\tau,t]}(|\epsilon(m)| + |f_1(m)| + |f_2(m)|)\right)\,. \tag{4.75}$$

Noticing that (4.50) in Assumption 4.2 implies that

$$\kappa_1(|\phi(0)|) \leq U(t,\phi) \leq \kappa_2(|\phi(0)|) + 2\mathfrak{c}\tau^2 \sup_{m\in[-\tau,0]} \kappa_2\left(|\phi(m)|\right) \tag{4.76}$$

for all $\phi \in C_{in}$ and $t \in \mathbb{R}$, we conclude that $U$ is an ISS Lyapunov-Krasovskii functional. Then by considersing the intervals $[0, 2\tau]$ and $[2\tau, +\infty)$, we can complete the proof. □

## 4.5 Illustration

### 4.5.1 The studied problem

To illustrate Theorem 4.2, let us consider the system (see [25, p. 137]):

$$\left\{\begin{array}{rcl}\dot{\xi}_4 &=& \xi_3 v_1\\ \dot{\xi}_3 &=& \xi_2 v_1\\ \dot{\xi}_2 &=& v_2\\ \dot{\xi}_1 &=& v_1\end{array}\right. \tag{4.77}$$

with $(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4$ and the input $(v_1, v_2) \in \mathbb{R}^2$. The system (4.77) is a nonholonomic system in chained form.

We assume that the variables $\xi_4$, $\xi_3$ and $\xi_1$ are measured but not $\xi_3$ and that $\epsilon = 0$, so $y = z$. Let us design a dynamic output feedback making the system (4.77) track the trajectory

$$(\xi_{1r}(t), \xi_{2r}(t), \xi_{3r}(t), \xi_{4r}(t)) = \left(t + \frac{1}{2}\sin(t), 0, 0, 0\right) . \tag{4.78}$$

### 4.5.2 Control design

The time-varying change of variables

$$x_1 = \xi_1 - \xi_{1r}(t) \tag{4.79}$$

and the feedback

$$v_1(t, x_1) = -x_1 + 1 + \frac{1}{2}\cos(t) \tag{4.80}$$

result in

$$\left\{\begin{array}{rcl}\dot{\xi}_4 &=& \xi_3\left[-x_1 + 1 + \frac{1}{2}\cos(t)\right]\\ \dot{\xi}_3 &=& \xi_2\left[-x_1 + 1 + \frac{1}{2}\cos(t)\right]\\ \dot{\xi}_2 &=& v_2\\ \dot{x}_1 &=& -x_1 \ .\end{array}\right. \tag{4.81}$$

This prompts us to consider the problem of stabilizing three dimensional system

$$\begin{cases} \dot{\xi}_4 &= \xi_3 \left[1+\frac{1}{2}\cos(t)\right] \\ \dot{\xi}_3 &= \xi_2 \left[1+\frac{1}{2}\cos(t)\right] \\ \dot{\xi}_2 &= v_2 \;. \end{cases} \tag{4.82}$$

We adopt a backsteping strategy: in a first step, we consider the system

$$\begin{cases} \dot{z} &= x_r \left[1+\frac{1}{2}\cos(t)\right] \\ \dot{x}_r &= u \left[1+\frac{1}{2}\cos(t)\right] + f_2(t) \end{cases} \tag{4.83}$$

where $u$ is the input, and then in a second step, we complete the stabilization design of (4.82) by applying the backstepping approach.

Then, with the notation of the previous section, we have

$$\begin{cases} \dot{z}(t) &= A_1(t)x_r(t) \\ \dot{x}_r(t) &= A_2(t)x_r(t) + B_2(t)u(t) + f_2(t) \end{cases} \tag{4.84}$$

with

$$A_1(t) = B_2(t) = 1 + \frac{1}{2}\cos(t) \;,\; A_2(t) = 0 \tag{4.85}$$

and the output $z(t) = x_1(t)$.

Let us select the function $L(t) = -\frac{1}{3}$ for all $t \in \mathbb{R}$. Then $H(t) = A_2(t) + L(t)A_1(t) = -\frac{1}{3}\left(1+\frac{1}{2}\cos(t)\right)$ for all $t \in \mathbb{R}$. Let $\tau$ be a positive real number. Then

$$\Phi_{A_2}(t,t_0) = 1,\; \Phi_H(t,t_0) = e^{\frac{1}{3}(t-t_0)+\frac{1}{6}(\sin(t)-\sin(t_0))},\; \Lambda(t) = 1 - e^{\frac{\tau}{3}+\frac{1}{6}[\sin(t)-\sin(t-\tau)]},$$

where $\Lambda$ us defined in (4.22). Since $\frac{1}{3}\tau + \frac{1}{6}[\sin(t)-\sin(t-\tau)] \geq \frac{\tau}{6} > 0$ for all $t \in \mathbb{R}$, it follows that the matrix $\Lambda(t)$ is invertible for each $t \in \mathbb{R}$. Hence, Assumption 4.1 is satisfied by $A_1$ and $A_2$ defined in (4.85).

Let us check that Assumption 4.2 is satisfied. Since $\Lambda(t)^{-1} = \frac{1}{1-e^{\frac{\tau}{3}+\frac{1}{6}[\sin(t)-\sin(t-\tau)]}}$, this function is bounded by $\frac{1}{e^{\frac{\tau}{6}}-1}$. Let us choose:

$$u_s(t,\chi) = -2(x_r + z) \;. \tag{4.86}$$

With this choice, the system which corresponds to (4.51) is

$$\begin{cases} \dot{z}(t) &= \left[1+\frac{1}{2}\cos(t)\right] x_r + \mathfrak{h}_1(t) \\ \dot{x}_r(t) &= \left[1+\frac{1}{2}\cos(t)\right] \left[-2(x_r+z)+\mu(t)\right] + \mathfrak{h}_2(t)\ . \end{cases} \tag{4.87}$$

Consider the positive definite quadratic function

$$V(\chi) = z^2 + \frac{1}{2}x_r^2 + zx_r\ . \tag{4.88}$$

Then

$$\begin{aligned} \dot{V}(t) &= -2\left[1+\tfrac{1}{2}\cos(t)\right] V(\chi) \\ &\quad + \left\{(2z+x_r)\mathfrak{h}_1(t) + (x_r+z)\left[\mu(t)\left(1+\tfrac{1}{2}\cos(t)\right)+\mathfrak{h}_2(t)\right]\right\}\ . \end{aligned} \tag{4.89}$$

Consequently,

$$\begin{aligned} \dot{V}(t) &\le -V(\chi) + (2z+x_r)\mathfrak{h}_1(t) \\ &\quad +(x_r+z)\left[\mu(t)\left(1+\tfrac{1}{2}\cos(t)\right)+\mathfrak{h}_2(t)\right]\ . \end{aligned} \tag{4.90}$$

Using the quadratic inequality $zx_r \ge -\frac{1}{3}x_r^2 - \frac{3}{4}z^2$ to get the lower bound $V(\chi) \ge \frac{1}{6}x_r^2 + \frac{1}{4}z^2$, and then applying Young's inequality three times to upper bound the terms in curly braces in (4.89), we deduce easily that there is a constant $\aleph > 0$ such that

$$\dot{V}(t) \le -\tfrac{1}{2}V(\chi) + \aleph\mathfrak{h}_1(t)^2 + \aleph|\mu(t)|^2 + \aleph|\mathfrak{h}_2(t)|^2\ . \tag{4.91}$$

It follows that Assumption 4.2 is satisfied. Since the functions $\varphi_1$ and $\varphi_2$ are not present, Assumptions 4.3 and 4.4 are satisfied. We conclude that Theorem 4.2 applies to the system (4.84).

Then, with

$$u_s(t,\hat{x}_r,z) = -2(\hat{x}_r+z) \tag{4.92}$$

and $B_1 = 0$ and $D(t) = \dot{L}(t) - H(t)L(t) = -\frac{1}{q}\left(1+\frac{1}{2}\cos(t)\right)$ the estimator which corresponds to (4.57) is

$$\begin{aligned} \hat{x}_r(t) &= 0 \quad \text{when} \quad t\in[-\tau,\tau) \\ \hat{x}_r(t) &= \tfrac{-2}{\zeta(t)}\int_{t-\tau}^{t}\left[1-e^{\frac{1}{3}(m-t+\tau)+\frac{1}{6}(\sin(m)-\sin(t-\tau))}\right] \\ &\quad \times\left(1+\tfrac{1}{2}\cos(m)\right)(\hat{x}_r(m)+z(m))dm \\ &\quad +\tfrac{1}{3\zeta(t)}\left[-e^{\frac{1}{3}\tau+\frac{1}{6}(\sin(t)-\sin(t-\tau))}z(t)+z(t-\tau)\right] \\ &\quad +\tfrac{1}{9\zeta(t)}\int_{t-\tau}^{t} e^{\frac{1}{3}(m-t+\tau)+\frac{1}{6}(\sin(m)-\sin(t-\tau))} \\ &\quad \times\left(1+\tfrac{1}{2}\cos(m)\right)z(m)dm \\ &\qquad \text{when} \quad t\ge\tau \end{aligned} \tag{4.93}$$

with $\zeta(t) = 1 - e^{\frac{\tau}{3}+\frac{1}{6}[\sin(t)-\sin(t-\tau)]}$.

Returning to (4.82), we let

$$\omega = \xi_2 + 2(\hat{\xi}_3 + \xi_4) \tag{4.94}$$

with

$$\begin{array}{rcl} \hat{\xi}_3(t) & = & \frac{-2}{\zeta(t)} \int_{t-\tau}^{t} \left[1 - e^{\frac{1}{3}(m-t+\tau)+\frac{1}{6}(\sin(m)-\sin(t-\tau))}\right] \\ & & \times \left(1 + \frac{1}{2}\cos(m)\right)(\hat{\xi}_3(m) + \xi_4(m))dm \\ & & + \frac{1}{3\zeta(t)} \left[-e^{\frac{\tau}{3}+\frac{1}{6}(\sin(t)-\sin(t-\tau))}\xi_4(t) + \xi_4(t-\tau)\right] \\ & & + \frac{1}{9\zeta(t)} \int_{t-\tau}^{t} e^{\frac{1}{3}(m-t+\tau)+\frac{1}{6}(\sin(m)-\sin(t-\tau))} \\ & & \times \left(1 + \frac{1}{2}\cos(m)\right)\xi_4(m)dm \ . \end{array} \tag{4.95}$$

We obtain

$$\left\{ \begin{array}{rcl} \dot{\xi}_4(t) & = & \left(1 + \frac{1}{2}\cos(t)\right)\xi_3 \\ \dot{\xi}_3(t) & = & \left(1 + \frac{1}{2}\cos(t)\right)\left(-2\hat{\xi}_3 - 2\xi_4 + \omega\right) \\ \dot{\omega}(t) & = & v_2 + 2\left(\dot{\hat{\xi}}_3(t) + \hat{\xi}_3(t)\left(1 + \frac{\cos(t)}{2}\right)\right) \ . \end{array} \right. \tag{4.96}$$

By Theorem 4.2, the $(\xi_4, \xi_3)$-subsystem of (4.95) is ISS with respect to $\omega$. Thus, with

$$v_2 = -\omega(t) - 2\left(\dot{\hat{\xi}}_3(t) + \hat{\xi}_3(t)\left(1 + \frac{\cos(t)}{2}\right)\right) \tag{4.97}$$

we obtain

$$\left\{ \begin{array}{rcl} \dot{\xi}_4(t) & = & \left(1 + \frac{1}{2}\cos(t)\right)\xi_3 \\ \dot{\xi}_3(t) & = & \left(1 + \frac{1}{2}\cos(t)\right)\left(-2\hat{\xi}_3 - 2\xi_4 + \omega\right) \\ \dot{\omega}(t) & = & -\omega \\ \hat{\xi}_3(t) & = & \frac{-2}{\zeta(t)} \int_{t-\tau}^{t} \left[1 - e^{\frac{1}{3}(m-t+\tau)+\frac{1}{6}(\sin(m)-\sin(t-\tau))}\right] \\ & & \times \left(1 + \frac{1}{2}\cos(m)\right)(\hat{\xi}_3(m) + \xi_4(m))dm \\ & & + \frac{1}{3\zeta(t)} \left[-e^{\frac{\tau}{3}+\frac{1}{6}(\sin(t)-\sin(t-\tau))}\xi_4(t) + \xi_4(t-\tau)\right] \\ & & + \frac{1}{9\zeta(t)} \int_{t-\tau}^{t} e^{\frac{1}{3}(m-t+\tau)+\frac{1}{6}(\sin(m)-\sin(t-\tau))} \\ & & \times \left(1 + \frac{1}{2}\cos(m)\right)\xi_4(m)dm \end{array} \right. \tag{4.98}$$

which uniformly globally asymptotically stable to 0.

We performed simulations, which show the efficiency of our approach. Fig. 4.1 shows the time responses of the closed loop time varying system (4.98). This example shows good stabilization and tracking, and it helps illustrate our general theory, in the special case of the system (4.77). We choose $\tau = 2\pi$ which implies asymptotic convergence to zero for $t \geq 2\pi$. This is evident from the simulation as well.

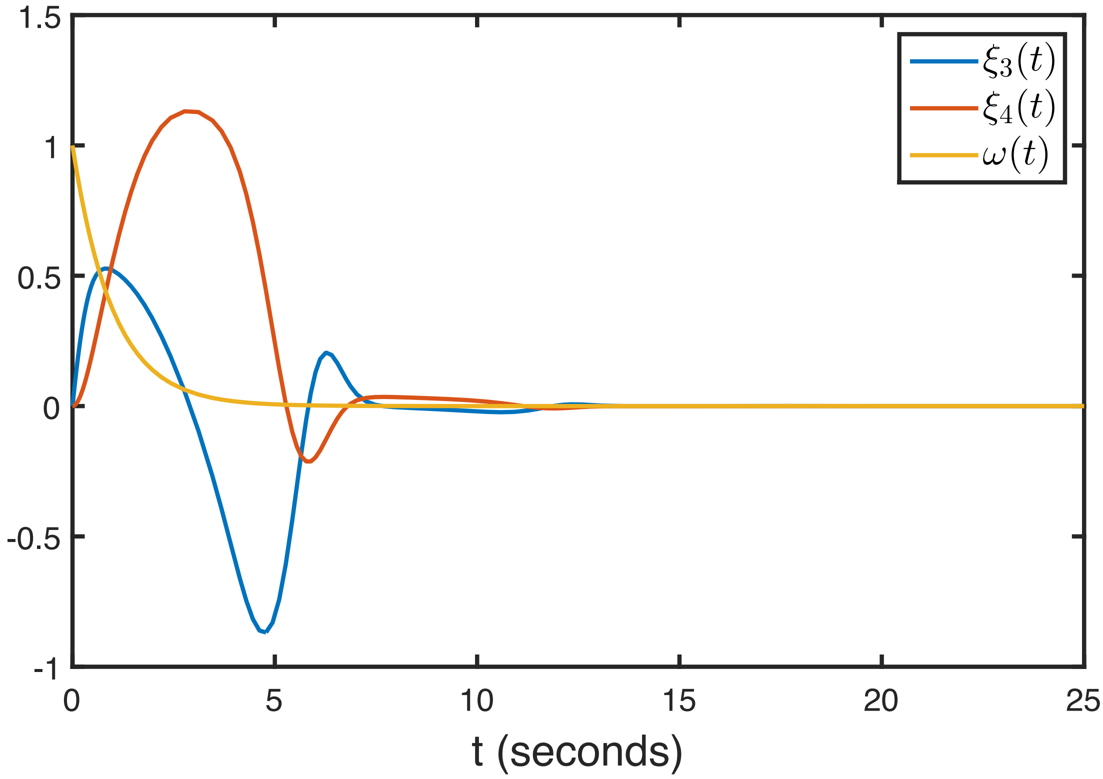


Figure 4.1: Simulation of the closed loop time varying system (4.98)

# Chapter 5

# Finite Time Observer Design for Continuous-Discrete Systems

*The content of this chapter is based on the publication of the author [91].* Most of the finite time observers discussed in the literature require continuous measurements. However, in many engineering applications, the measurements are collected at discrete time instants. These systems are called continuous-discrete systems where the system dynamics are continuous while the measurements are only available at discrete instants; see [92] and [93] for the notion of a continuous-discrete system. Various techniques in the literature design sampled data observers for continuous-discrete systems; see, e.g., [94]. Output feedback control using these sampled data observers is achieved via a discrete controller, and a generalized sampled-data hold function (GSHF) is used to reconstruct continuous-time control input based on sampled measurements which is then given to the actuator [95]. However, dynamic output feedback stabilization using the observers proposed in this chapter can be achieved with a continuous controller without requiring GSHF which is one of the significant contributions of this chapter.

## 5.1 Literature Review

There are several works on finite time observer design for cases where the measurements are continuous instead of being discrete; see, e.g., [78]; [79]; [80]; [90]; [96]; [56]. However, to the best of our knowledge, the finite time estimation we study in this work via sampled measurements has remained unsolved, even in the case of linear systems, due to the challenges of quantifying the effects of piecewise continuous disturbances on the observer performance. By contrast, for simpler cases where there are no such disturbances in the system, notable works on finite time observers include [97], which uses periodic sampling times in the outputs and an observability assumption that is similar to the one we use in this work.

## 5.2 Contributions of this chapter

We construct an observer to estimate the exact state of a linear continuous-time system from synchronously sampled outputs. We consider a sequence of real numbers $\{t_i\}$ and a constant $\nu > 0$ such that $t_0 = 0$ and $t_{i+1} - t_i = \nu$ for all integers $i \geq 0$. Then the $t_i$'s will serve as the measurement instants for the output and the sampling period $\nu$ will be a tuning parameter that will govern the estimation error. We will show that the smaller the tuning parameter $\nu$, the better the estimation. We also provide an approximate estimate of the system's state that overcomes the problem of determining explicit formulas for fundamental solutions. Our strategy has several steps. We use a classical prediction result, the finite time observer design technique of [80] and [56], and finally a novel construction of continuous-discrete observers to complete the observer design; see, e.g., [84] for the notion of continuous-discrete observer. We establish robust stability of the observer with respect to disturbances in the system dynamics. Since the disturbance is a general piecewise continuous function, this allows systems with a discontinuous right side which were beyond the scope of [97] and other works.

Our paper shares fundamental features with the significant work of [98]. The idea of repeatedly reconstructing the state values in a short amount of time is already present in [98], where a semi-globally stabilizing sampled output feedback for a nonlinear system is proposed. However, there are three key differences between the results of this chapter and [98]. First, in [98], the output is assumed to be known at any instant. Second, high gain observers are used in [98] to obtain approximate values of the state variable, while here we adopt a finite time reconstruction strategy. Third, although [98] covers nonlinear systems and the present paper is confined to linear systems, [98] imposes a limitation on the size of the sampling period of the feedback, while none of our results here rely on a restriction of this type. In particular, the piecewise continuous disturbances in our systems can capture the effects of sampled feedbacks with arbitrarily large sampling periods.

The rest of this chapter is organized as follows. In Section 5.3 we describe our objectives in detail and present a lemma that we will use to prove our main result in Section 5.4. Our illustration in Section 5.5 includes numerical simulations and demonstrates the utility of our theory.

## 5.3 Problem Statement and Preliminaries

Our objective in this section is to construct an observer for a linear continuous-time system with a sampled output such that the observer converges in predetermined finite time in the presence of a disturbance in the dynamics of the system. The observer is expressed in terms of the fundamental solution of suitable time-varying system. Then in the next section, we use ideas from this section to obtain more explicit formulas for finite time observers that do not contain the fundamental matrix and therefore may be better suited to implementations where the fundamental matrix is not available in explicit closed form.

Our systems have the form

$$\begin{cases} \dot{x}(t) &= Ax(t) + \delta(t) \\ y(t) &= Cx(t_i) \quad \text{for all } t \in [t_i, t_{i+1}) \text{ and } i \in N \end{cases} \tag{5.1}$$

with $x$ valued in $\mathbb{R}^n$ for any $n \in \mathbb{N}$, $y$ valued in $\mathbb{R}^q$ for any $q \in \mathbb{N}$, $t_i$'s will serve as the measurement times for the output in our systems, and $\delta : [0, +\infty) \to \mathbb{R}^n$ being a known bounded and piecewise continuous disturbance. We assume that $A$ and $C$ are known matrices of appropriate dimensions and the following assumption throughout this chapter:

**Assumption 5.1** *There is a constant $\nu > 0$ such that $t_{i+1} - t_i = \nu$ for all $i \geq 0$. Also, the pair $(A, C)$ is observable.* □

When Assumption 5.1 is satisfied, we can use [80, Lemma 1] to find a constant $T > 0$ and a constant matrix $L$ such that with the choice $F = A + LC$, the matrix

$$M_T = e^{-AT} - e^{-FT} \tag{5.2}$$

is invertible and such that $T/\nu$ is an integer.

The following lemma will be useful later in this chapter.

**Lemma 5.1** *Let $\mathcal{A} \in \mathbb{R}^{n \times n}$ be a constant matrix. Consider the system*

$$\dot{\zeta}(t) = [\mathcal{A} + \mathcal{E}(t)]\, \zeta(t) \tag{5.3}$$

*where $\zeta$ is valued in $\mathbb{R}^n$ and $\mathcal{E} : [0, +\infty) \to \mathbb{R}^{n \times n}$ is a bounded piecewise continuous function. Let $\phi$ denote the fundamental solution of the system (5.3). Then for all $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$ such that $t_2 \geq t_1$, the inequality*

$$\left|\phi(t_2, t_1) - e^{(t_2 - t_1)\mathcal{A}}\right| \leq |\mathcal{E}|_\infty e^{(t_2 - t_1)|\mathcal{A}|} \frac{e^{2|\mathcal{A}|(t_2 - t_1)} - 1}{2|\mathcal{A}|} exp\left(|\mathcal{E}|_\infty \frac{e^{2|\mathcal{A}|(t_2 - t_1)} - 1}{2|\mathcal{A}|}\right)$$

*is satisfied.* □

**Proof.** Let $\phi$ be the fundamental solution of the system

$$\frac{\partial \phi}{\partial t}(t,t_0) = [\mathcal{A} + \mu(t)]\phi(t,t_0)\ . \tag{5.4}$$

Here and in the sequel, $t_0 \geq 0$ and $t \geq t_0$ are arbitrary. Let $\psi(t,t_0) = e^{-\mathcal{A}(t-t_0)}\phi(t,t_0)$. Then

$$\frac{\partial \psi}{\partial t}(t,t_0) = \omega(t,t_0)\psi(t,t_0) \tag{5.5}$$

holds with

$$\omega(t,t_0) = e^{-\mathcal{A}(t-t_0)}\mathcal{E}(t)e^{\mathcal{A}(t-t_0)}. \tag{5.6}$$

For any vector $V \in \mathbb{R}^n$, we have

$$\frac{\partial}{\partial t}\left((\psi(t,t_0)V)^\top \psi(t,t_0)V\right) = V^\top \psi(t,t_0)^\top \omega(t,t_0)\psi(t,t_0)V\ . \tag{5.7}$$

Consequently,

$$\frac{\partial(|\psi(t,t_0)V|^2)}{\partial t} \leq |\omega(t,t_0)||\psi(t,t_0)V|^2\ . \tag{5.8}$$

Through a simple integration, we obtain

$$|\psi(t,t_0)V| \leq e^{\int_{t_0}^t |\omega(m,t_0)|\mathrm{d}m}|V|\ . \tag{5.9}$$

One can check readily that

$$|\omega(t,t_0)| \leq |\mathcal{E}|_\infty e^{2|\mathcal{A}|(t-t_0)}\ . \tag{5.10}$$

Consequently,

$$\textstyle\int_{t_0}^t |\omega(m,t_0)|\mathrm{d}m \leq |\mathcal{E}|_\infty \int_{t_0}^t e^{2|\mathcal{A}|(m-t_0)}\mathrm{d}m = |\mathcal{E}|_\infty \frac{e^{2|\mathcal{A}|(t-t_0)}-1}{2|\mathcal{A}|}. \tag{5.11}$$

Combining (5.9) and (5.11), we obtain

$$|\psi(t,t_0)V| \leq \exp\left(|\mathcal{E}|_\infty \frac{e^{2|\mathcal{A}|(t-t_0)}-1}{2|\mathcal{A}|}\right)|V|\ . \tag{5.12}$$

Since this inequality is valid for all $V \in \mathbb{R}^n$, we have

$$|\psi(t,t_0)| \leq \exp\left(|\mathcal{E}|_\infty \frac{e^{2|\mathcal{A}|(t-t_0)}-1}{2|\mathcal{A}|}\right)\ . \tag{5.13}$$

Again using (5.5), we deduce that

$$\int_{t_0}^t \frac{\partial \psi}{\partial t}(s,t_0)\mathrm{d}s = \int_{t_0}^t \omega(s,t_0)\psi(s,t_0)\mathrm{d}s\ . \tag{5.14}$$

It follows from the Fundamental Theorem of Calculus that

$$\psi(t,t_0) - I = \int_{t_0}^{t} \omega(s,t_0)\psi(s,t_0)\mathrm{d}s \; . \tag{5.15}$$

We deduce that

$$\begin{array}{rcl} |\psi(t,t_0) - I| & \leq & \int_{t_0}^{t} |\omega(s,t_0)||\psi(s,t_0)|\mathrm{d}s \\ & \leq & \int_{t_0}^{t} |\mathcal{E}|_\infty e^{2|\mathcal{A}|(s-t_0)} \exp\left( |\mathcal{E}|_\infty \frac{e^{2|\mathcal{A}|(s-t_0)}-1}{2|\mathcal{A}|} \right) \mathrm{d}s \end{array} \tag{5.16}$$

where the last inequality is a consequence of (5.10) and (5.13). We deduce that

$$\begin{array}{rcl} |\psi(t,t_0) - I| & \leq & |\mathcal{E}|_\infty \int_{t_0}^{t} e^{2|\mathcal{A}|(s-t_0)}\mathrm{d}s \, \exp\left( |\mathcal{E}|_\infty \frac{e^{2|\mathcal{A}|(t-t_0)}-1}{2|\mathcal{A}|} \right) \\ & = & |\mathcal{E}|_\infty \frac{e^{2|\mathcal{A}|(t-t_0)}-1}{2|\mathcal{A}|} \exp\left( |\mathcal{E}|_\infty \frac{e^{2|\mathcal{A}|(t-t_0)}-1}{2|\mathcal{A}|} \right) \; . \end{array} \tag{5.17}$$

We also have

$$\begin{array}{rcl} \left|\phi(t,t_0) - e^{(t-t_0)\mathcal{A}}\right| & = & \left| e^{(t-t_0)\mathcal{A}} \left( e^{-(t-t_0)\mathcal{A}}\phi(t,t_0) - I \right) \right| \\ & \leq & e^{(t-t_0)|\mathcal{A}|} \left| \psi(t,t_0) - I \right| \; . \end{array} \tag{5.18}$$

The inequality in conjunction with (5.17) gives

$$\left|\phi(t,t_0) - e^{(t-t_0)\mathcal{A}}\right| \leq |\mathcal{E}|_\infty e^{(t-t_0)|\mathcal{A}|} \frac{e^{2|\mathcal{A}|(t-t_0)}-1}{2|\mathcal{A}|} \exp\left( |\mathcal{E}|_\infty \frac{e^{2|\mathcal{A}|(t-t_0)}-1}{2|\mathcal{A}|} \right)$$

which is the desired conclusion. ■

## 5.4 Finite Time Observer Design

Throughout this section, we consider the system (5.1) and assume that Assumption 5.1 is satisfied.

### 5.4.1 Exact Estimate

We provide an exact estimate of the system's state that converges in a predetermined finite time, using the piecewise constant function $\varphi(t) = t_i$ when

$t \in [t_i, t_{i+1})$ and $i \geq 0$. Here and in what follows, all equalities and inequalities are for all $t \geq 0$, unless otherwise indicated. We have

$$\dot{x}(t) = Fx(t) + \delta(t) - Ly(t) + LC[x(\varphi(t)) - x(t)] \ .$$

We also have

$$x(\varphi(t)) = e^{A(\varphi(t)-t)}x(t) + \int_t^{\varphi(t)} e^{A(\varphi(t)-m)}\delta(m)\mathrm{d}m \ .$$

As an immediate consequence,

$$\dot{x}(t) = [F + \mu(t)]\, x(t) + \delta(t) - Ly(t) + LC \int_t^{\varphi(t)} e^{A(\varphi(t)-m)}\delta(m)\mathrm{d}m \qquad (5.19)$$

where $\mu(t) = LC\left(e^{A(\varphi(t)-t)} - I\right)$.

Let $\xi(t) = \Phi_{F+\mu}(t,0)x(t)$. Then

$$\dot{\xi}(t) = -\Phi_{F+\mu}(t,0)[F + \mu(t)]x(t) + \Phi_{F+\mu}(t,0)\dot{x}(t). \qquad (5.20)$$

Using (5.19) and (5.20), we obtain

$$\dot{\xi}(t) = \Phi_{F+\mu}(t,0)\left[\delta(t) - Ly(t) + LC \int_t^{\varphi(t)} e^{A(\varphi(t)-m)}\delta(m)\mathrm{d}m\right] \ . \qquad (5.21)$$

For any $T > 0$ and $t \geq T$, we can integrate (5.21) over $[t - T, t]$ to obtain

$$\begin{aligned} \xi(t) \;\; = \;\; & \xi(t-T) \\ & + \int_{t-T}^t \Phi_{F+\mu}(m,0)\left[\delta(m) - Ly(m) + LC \int_m^{\varphi(m)} e^{A(\varphi(m)-s)}\delta(s)\mathrm{d}s\right] \mathrm{d}m \ . \end{aligned}$$

From the definition of $\xi$, and from the semigroup property of flow maps applied to the flow map $\Psi_{F+\mu}^{-1}$ of the system $\dot{q} = (F + \mu(t))q$, we deduce that

$$\begin{aligned} & x(t) = \Phi_{F+\mu}(t,0)^{-1}\Phi_{F+\mu}(t-T,0)x(t-T) \\ & + \int_{t-T}^t \Phi_{F+\mu}(t,0)^{-1}\Phi_{F+\mu}(m,0)\left(\delta(m) - Ly(m) + LC \int_m^{\varphi(m)} e^{A(\varphi(m)-s)}\delta(s)\mathrm{d}s\right) dm \\ & = \Phi_{F+\mu}^{-1}(t, t-T)x(t-T) \\ & + \int_{t-T}^t \Phi_{F+\mu}^{-1}(t,m)\left(\delta(m) - Ly(m) + LC \int_m^{\varphi(m)} e^{A(\varphi(m)-s)}\delta(s)\mathrm{d}s\right) dm \ . \end{aligned} \qquad (5.22)$$

Notice that (5.22) gives the exact value of the solution of the system (5.1) in a predetermined finite time $T$. In other words, the right hand side of (5.22) provides a finite time observer. However, finding an explicit expression for $\Phi_{F+\mu}$ may be difficult, which motivates our work in the next section.

### 5.4.2 Approximate Estimate

It is often difficult to determine explicit expressions for fundamental solutions in order to estimate the system's state using (5.22). Our next objective is to provide an approximate estimate of the system's state that overcomes the problem of determining explicit formulas for the fundamental solutions, under Assumption 5.1. In terms of the functions

$$\begin{aligned}
\Sigma(T,\nu) &= \mathcal{G}\big(T, |LC|(e^{\nu|A|}-1)\big),\\
\mathcal{G}(T,s) &= se^{T|F|}\frac{e^{2|F|T}-1}{2|F|}\exp\left(s\frac{e^{2|F|T}-1}{2|F|}\right),\\
\bar{G}(T,\nu) &= \frac{|e^{-FT}|^2\Sigma(T,\nu)}{1-|e^{-FT}|\Sigma(T,\nu)},\\
\bar{\alpha}(T,\nu) &= \frac{|M_T^{-1}|^2\bar{G}(T,\nu)}{1-|M_T^{-1}|\bar{G}(T,\nu)},
\end{aligned} \tag{5.23}$$

$$\bar{\beta}(T,\nu) = \left|e^{-FT}\right|\bar{\alpha}(T,\nu) + \left[|M_T^{-1}| + \bar{\alpha}(T,\nu)\right]\bar{G}(T,\nu), \tag{5.24}$$

and

$$\bar{\gamma}(T,\nu) = \left[|M_T^{-1}| + \bar{\alpha}(T,\nu)\right]\left[\left|e^{-FT}\right| + \bar{G}(T,\nu)\right], \tag{5.25}$$

we prove the following result:

**Theorem 5.1** *Let the system (5.1) satisfy Assumption 5.1, where A, B, and C are known constant matrices. Let F and T be such that $M_T$ as defined in (5.2) is invertible and such that $T/\nu$ is an integer, where the constant $\nu > 0$ is such that*

$$\max\left\{|e^{-FT}|\Sigma(T,\nu), |M_T^{-1}|\bar{G}(T,\nu)\right\} < 1. \tag{5.26}$$

*Let*

$$\hat{x}(t_i) = M_T^{-1}\int_{t_i-T}^{t_i} e^{A(t_i-m-T)}\delta(m)\mathrm{d}m - M_T^{-1}e^{-FT}\tilde{\mathcal{T}}(t_i,\delta,y) \tag{5.27}$$

*where*

$$\tilde{\mathcal{T}}(t_i,\delta,y) = \int_{t_i-T}^{t_i} e^{F(t_i-m)}\left(\delta(m) - Ly(m) + LC\int_m^{\varphi(m)} e^{A(\varphi(m)-s)}\delta(s)\mathrm{d}s\right)\mathrm{d}m\,.$$

*Then*

$$\begin{aligned}
|x(t_i)-\hat{x}(t_i)| \;\le\; & \bar{\alpha}(T,\nu)\int_{t_i-T}^{t_i} e^{A(t-m-T)}\delta(m)\mathrm{d}m + \bar{\beta}(T,\nu)|\tilde{\mathcal{T}}(t_i,\delta,y)|\\
& + \bar{\gamma}(T,\nu)\Sigma(T,\nu)\mathcal{T}_{\triangle}(t_i,\delta,y)
\end{aligned}$$

*holds with the choice*

$$\mathcal{T}_{\triangle}(t_i,\delta,y)=\int_{t_i-T}^{t_i}\left|\delta(m)-Ly(m)+LC\int_m^{\varphi(m)}e^{A(\varphi(m)-s)}\delta(s)\mathrm{d}s\right|\mathrm{d}m \tag{5.28}$$

*for all integers $i\in\mathbb{N}$ such that $t_i>T$.* □

**Proof.** Set $k=T/\nu$, which is a positive integer, by our assumptions. By integrating (5.1), we obtain

$$e^{-AT}x(t_i)=x(t_{i-k})+\int_{t_i-T}^{t_i}e^{A(t_i-m-T)}\delta(m)\mathrm{d}m\ . \tag{5.29}$$

Using (5.22) and Lemma 5.1 (applied with $\mathcal{A}=F$ and $\mathcal{E}=\mu$), we obtain

$$x(t_i)=\left(e^{FT}+\kappa(t)\right)x(t_{i-k})+\mathcal{T}(t_i,\delta,y) \tag{5.30}$$

with

$$\mathcal{T}(t_i,\delta,y)=\int_{t_i-T}^{t_i}\Phi_{F+\mu}^{-1}(t_i,m)\left(\delta(m)-Ly(m)+LC\int_m^{\varphi(m)}e^{A(\varphi(m)-s)}\delta(s)\mathrm{d}s\right)\mathrm{d}m$$

and

$$|\kappa(t)|\le\mathcal{G}(T,|\mu|_\infty). \tag{5.31}$$

Here and in the sequel, all equalities and inequalities should be understood to hold for all $t\ge t_i$ and all $i$ such that $t_i>T$.

The definition $\mu(t)=LC\left(e^{A(\varphi(t)-t)}-I\right)$ gives

$$\begin{aligned}|\mu|_\infty &= \left|LC\left(e^{A(\varphi(t)-t)}-I\right)\right|_\infty\le|LC|\left|\sum_{k=1}^{\infty}\frac{A^k(\varphi(t)-t)^k}{k!}\right|_\infty\\ &\le |LC|\sum_{k=1}^{\infty}\frac{|A|^k|\varphi(t)-t|^k}{k!}\ .\end{aligned}$$

Since $|t-\varphi(t)|_\infty\le\nu$, we deduce that

$$|\mu|_\infty\le|LC|\,(e^{\nu|A|}-1)\ . \tag{5.32}$$

Using (5.31) and (5.32), we have

$$|\kappa(t)|\le\mathcal{G}\left(T,|LC|\left(e^{\nu|A|}-1\right)\right)=\Sigma(T,\nu) \tag{5.33}$$

for all $t \geq 0$. Since the condition (5.26) on $\nu$ gives $|e^{-|F|T}|\Sigma(T,\nu) < 1$, we can use the inequality (5.33) and Lemma A.2 from Appendix A (applied with $M = e^{FT}$ and $N = \kappa(t)$) to deduce that $e^{FT} + \kappa(t)$ is invertible for all $t$. Then (5.30) gives

$$\left(e^{FT} + \kappa(t)\right)^{-1} x(t_i) = x(t_{i-k}) + \left(e^{FT} + \kappa(t)\right)^{-1} \mathcal{T}(t_i, \delta, y) \ . \tag{5.34}$$

Combining (5.29) and (5.34), we obtain

$$\begin{aligned} &\left[e^{-AT} - \left(e^{FT} + \kappa(t)\right)^{-1}\right] x(t_i) \\ &= \textstyle\int_{t_i-T}^{t_i} e^{A(t_i-m-T)}\delta(m)\mathrm{d}m - \left(e^{FT} + \kappa(t)\right)^{-1} \mathcal{T}(t_i, \delta, y) \ . \end{aligned}$$

Using the definition of $M_T$, we have

$$\begin{aligned} &\left[M_T + G(t,T)\right] x(t_i) \\ &= \textstyle\int_{t_i-T}^{t_i} e^{A(t_i-m-T)}\delta(m)\mathrm{d}m - \left(e^{FT} + \kappa(t)\right)^{-1} \mathcal{T}(t_i, \delta, y) \end{aligned} \tag{5.35}$$

where $G(t,T) = e^{-FT} - \left(e^{FT} + \kappa(t)\right)^{-1}$. Lemma A.2 from Appendix A (applied with $M = e^{FT}$ and $N = \kappa(t)$) also ensures that

$$|G(t,T)| \leq \bar{G}(T,\nu) \tag{5.36}$$

where $\bar{G}$ is from (5.23). Since $M_T$ is invertible, it follows from our condition (5.26) and the inequality (5.36) and Lemma A.2 from Appendix A (applied with $M = M_T$, $N = G(t,T)$, and $\bar{n} = \bar{G}(T,\nu)$) that $M_T + G(t,T)$ is invertible and from (5.35), we have

$$\begin{aligned} x(t_i) \ &= \ [M_T + G(t,T)]^{-1} \textstyle\int_{t_i-T}^{t_i} e^{A(t-m-T)}\delta(m)\mathrm{d}m \\ &\quad - [M_T + G(t,T)]^{-1} \left(e^{FT} + \kappa(t)\right)^{-1} \mathcal{T}(t_i, \delta, y) \ . \end{aligned} \tag{5.37}$$

From (5.27) and (5.37), we deduce that

$$\begin{aligned} |x(t_i) - \hat{x}(t_i)| \ \leq \ &\beta(t,T)|\tilde{\mathcal{T}}(t_i, \delta, y)| \\ &+\alpha(t,T)\left|\textstyle\int_{t_i-T}^{t_i} e^{A(t-m-T)}\delta(m)\mathrm{d}m\right| \\ &+\gamma(t,T)|\mathcal{T}(t_i, \delta, y) - \tilde{\mathcal{T}}(t_i, \delta, y)| \end{aligned} \tag{5.38}$$

where $\alpha(t,T) = |[M_T + G(t,T)]^{-1} - M_T^{-1}|$,

$$\beta(t,T) = \left|M_T^{-1}e^{-FT} - [M_T + G(t,T)]^{-1}\left(e^{FT} + \kappa(t)\right)^{-1}\right| \ ,$$

and

$$\gamma(t,T) = \left|[M_T + G(t,T)]^{-1}\left(e^{FT} + \kappa(t)\right)^{-1}\right| .$$

Lemma A.2 from Appendix A (applied with $M = M_T$ and $N = G(t,T)$) ensures that

$$\alpha(t,T) \le \bar{\alpha}(T,\nu) \tag{5.39}$$

where $\bar{\alpha}$ was defined in (5.23). We have

$$\begin{aligned}
\beta(t,T) &= \Big|\left(M_T^{-1} - [M_T + G(t,T)]^{-1}\right)e^{-FT} \\
&\quad + [M_T + G(t,T)]^{-1}\left(e^{-FT} - \left(e^{FT} + \kappa(t)\right)^{-1}\right)\Big| \\
&\le \left|M_T^{-1} - [M_T + G(t,T)]^{-1}\right|\left|e^{-FT}\right| \\
&\quad + \left|[M_T + G(t,T)]^{-1}\right|\left|e^{-FT} - \left(e^{FT} + \kappa(t)\right)^{-1}\right| \\
&\le \bar{\beta}(T,\nu)
\end{aligned}$$

with $\bar{\beta}$ also as defined in (5.23). We also have

$$\begin{aligned}
\gamma(t,T) &= \left|[M_T + G(t,T)]^{-1}\left(e^{FT} + \kappa(t)\right)^{-1}\right| \\
&\le \left|[M_T + G(t,T)]^{-1}\right|\left|\left(e^{FT} + \kappa(t)\right)^{-1}\right| \\
&\le \bar{\gamma}(T,\nu)
\end{aligned}$$

where $\bar{\gamma}$ is also from (5.23). Observe that Lemma 5.1 gives

$$\begin{aligned}
&|\mathcal{T}(t_i,\delta,y) - \tilde{\mathcal{T}}(t_i,\delta,y)| \\
&\le \textstyle\int_{t_i-T}^{t_i}\left|\Phi_{F+\mu}^{-1}(t_i,m) - e^{F(t_i-m)}\right|\left|\delta(m) - Ly(m) + LC\int_m^{\varphi(m)} e^{A(\varphi(m)-s)}\delta(s)\mathrm{d}s\right|\mathrm{d}m \\
&\le \Sigma(T,\nu)|\mathcal{T}_\triangle(t_i,\delta,y)|
\end{aligned}$$

with $\mathcal{T}_\triangle$ as defined in (5.28). It follows from (5.38)-(5.39) that

$$\begin{aligned}
|x(t_i) - \hat{x}(t_i)| &\le \bar{\beta}(T,\nu)|\tilde{\mathcal{T}}(t_i,\delta,y)| + \bar{\alpha}(T,\nu)\int_{t_i-T}^{t_i} e^{A(t-m-T)}\delta(m)\mathrm{d}m \\
&\quad + \bar{\gamma}(T,\nu)\Sigma(T,\nu)\mathcal{T}_\triangle(t_i,\delta,y),
\end{aligned} \tag{5.40}$$

which is our desired estimate. This concludes the proof. □

## 5.5 Illustrative Example

We illustrate Theorem 5.1 with the system

$$\dot{x}(t) = \begin{bmatrix} 0 & 0.15 \\ -0.15 & 0 \end{bmatrix} x(t) + \begin{bmatrix} d(t) \\ 0 \end{bmatrix} \tag{5.41}$$

where $x = (x_1, x_2)$ is valued in $\mathbb{R}^2$, $d$ is scalar valued and represents a perturbation, and the measurement is

$$y(t) = \begin{bmatrix} 0.3 & 0 \end{bmatrix} x(t_i) \tag{5.42}$$

where $t_i = i\nu$ for all $i \in \mathbb{N}$. One can easily check that Assumption 5.1 is satisfied with $C = [0.3 \;\; 0]$, $T = 6$, and that with the choice (5.41), we have

$$e^{At} = \begin{bmatrix} \cos(0.15t) & \sin(0.15t) \\ -\sin(0.15t) & \cos(0.15t) \end{bmatrix} \tag{5.43}$$

where

$$A = \begin{bmatrix} 0 & 0.15 \\ -0.15 & 0 \end{bmatrix}. \tag{5.44}$$

Hence, choosing

$$L = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \quad \text{and} \quad F = A + LC = \begin{bmatrix} 0 & 0.15 \\ -0.12 & 0 \end{bmatrix},$$

we obtain

$$e^{-FT} = \begin{bmatrix} \cos\left(\frac{\sqrt{35}T}{50}\right) & -\frac{\sqrt{5}}{2}\sin\left(\frac{\sqrt{35}T}{50}\right) \\ \sin\left(\frac{\sqrt{35}T}{50}\right) & \cos\left(\frac{\sqrt{35}T}{50}\right) \end{bmatrix}, \tag{5.45}$$

e.g., by checking that (5.45) has derivative $-Fe^{-FT}$ with respect to $T$. Choosing $T = 6$, we have

$$M_T = e^{-AT} - e^{-FT} = \begin{bmatrix} -0.0715 & 0.0226 \\ 0.1386 & -0.0715 \end{bmatrix}$$

which has a nonzero determinant equal to 0.0020. Then $M_T$ is invertible and

$$M_T^{-1} = \begin{bmatrix} -36.0244 & -11.3718 \\ -69.8228 & -36.0244 \end{bmatrix}. \tag{5.46}$$

Now choosing the sampling rate to be $\nu = 0.05$, one can corroborate that (5.26) is satisfied with $|e^{-FT}| = 1.0838$, $\Sigma(T, \nu) = 0.0094$, $|M_T{}^{-1}| = 86.9858$, and $\bar{G}(T, \nu) = 0.0111$. Therefore, we can use (5.46) in the formula (5.27) for the continuous-discrete observer from Theorem 5.1 for the system (5.41) with $t_i = 0.05i$ for all $i \in \mathbb{N}$.

To illustrate our result, Fig. 5.1 shows MATLAB simulation of our observer (5.27) for the system (5.41) under a piecewise continuous perturbation $d(t) = 0.5u(t)$ with initial conditions $x_1(0) = -2$, $\hat{x}_1(0) = \hat{x}_2(0) = 0$, and $x_2(0) = 2$. We have also include a zoomed plot in Fig. 5.1 to depict that we have used a zero-order hold with $\nu = 0.05$ to construct the piecewise continuous estimate $\hat{x}_2$ from its discrete samples. The fundamental sampling rate of our simulation is 0.1 kHz. The simulation results corroborate convergence of our estimate after $T = 6$ seconds. Since our simulations show good tracking performance, they help illustrate our general theory in the special case of the system (5.41) with the measurement (5.42).

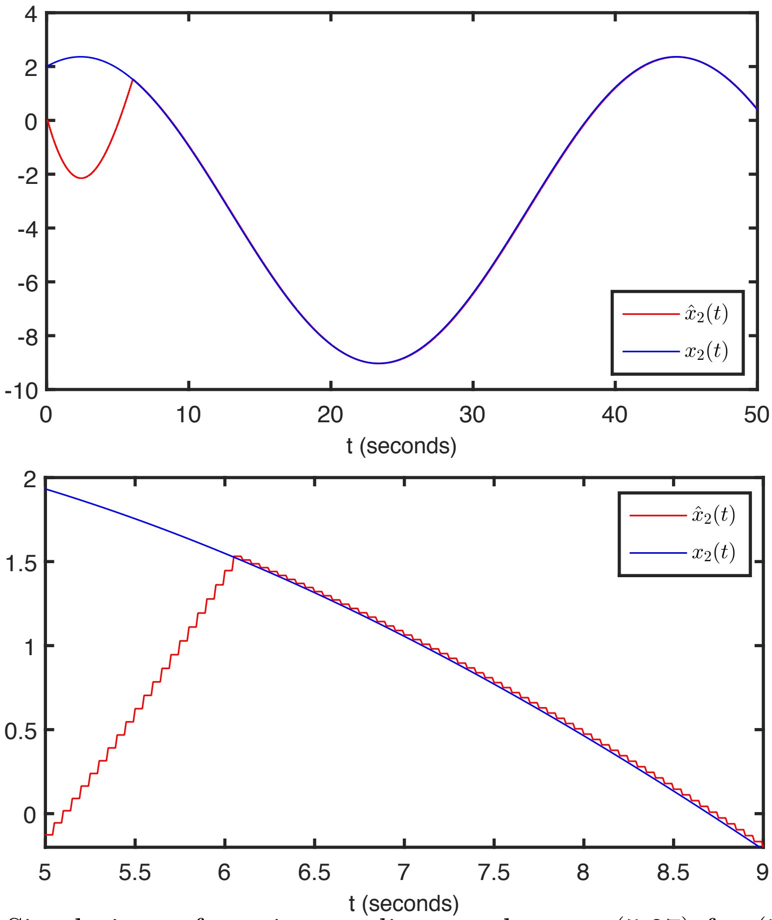


Figure 5.1: Simulations of continuous-discrete observer (5.27) for (5.41): Component $x_2$ and its estimate $\hat{x}_2$

# Chapter 6

# Conclusion and Future Perspectives

In Chapter 2, we presented dynamic output feedback stabilization results for systems with switches in the difficult case where a time-varying pointwise delay in the output is present. The technique of proof we proposed is based on the recent trajectory based approach. To solve the conservatism problem we encountered in [34], we developed an extension of the main result of [28], which is of interest for its own sake. Many extensions of the results of the present paper are possible, pertaining for instance to design of $K_i$ and $L_i$ for maximization of the delay bound, robustness issues with respect to disturbances, the presence of a delay in the input, the design of reduced order observers and extensions to families of nonlinear systems.

In Chapter 3, we provided new constructions of observers and output feedback controls for time-varying nonlinear systems with intermittent output observations and disturbances. Our feedback control result proved exponentially stable convergence of the closed loop system to the desired equilibrium. This is valuable, because it is common in engineering to encounter systems for which there are periods during which no output measurements are available for use in the control. The presence of the disturbances and nonlinearities makes the observer design much more challenging than standard observer design problems. The main strategy to overcome these challenges combined finite time observers and a switched systems approach. Applications of the main results to the Mathieu equation exhibited the good performance of our methods in simulations. Many extensions can be expected for systems with delay, and for local stability of broader classes of systems. An accompanying MATLAB based software titled Finite-time Estimation Meysam Mazenc Et Saeed (FEMMES) to simulate the observer we constructed in Chapter 3 is available at: https://gitlab.inria.fr/fmazenc/our-software-femmes

In Chapter 4, we proposed a new type of reduced order finite time observers and used them to solve a stabilization problem. The results apply to time-varying systems. We conjecture that the proposed observer design can be used to solve a problem of constructing interval observers that is similar to those in [80]. We also plan to combine the observer design discussed in this chapter with the result of [56] to solve a stabilization problem for the case where there are a delay and a disturbance in the input, and where the outputs are only available on some finite time intervals.

In Chapter 5, we proposed an observer of a new type for linear continuous-time systems with a piecewise constant output, estimating the system state in a predetermined finite time in the presence of a disturbance in the dynamics of the system. It provides an exact estimate which in general is not given by an explicit formula. This led us to propose an approximate formula, which is given by an explicit formula and whose accuracy is proportional to the size of the sampling interval. We also provided an approximate estimate to overcome the problem of computing the explicit expressions of the fundamental solutions. Many extensions of this observer design are possible, pertaining for instance to the design of reduced order observers and extensions to families of globally Lipschitz nonlinear time-varying systems and asynchronous sampling.

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# Appendix A

# Technical Lemmas

**Lemma A.1** *Let* $R \in \mathbb{R}^{m \times m}$ *be a nonnegative matrix. Let us consider functions* $w_j : [0, +\infty) \to [0, +\infty)$, $j = 1, \ldots, m$, *and a constant* $h > 0$ *such that for all* $t \geq h$, $w = (w_1 \ldots w_m)^\top$ *satisfies*

$$w(t) \leq R\zeta(t) \tag{A.1}$$

*with* $\zeta(t) = \left( \sup_{\ell \in [t-h,t]} w_1(\ell) \; \ldots \; \sup_{\ell \in [t-h,t]} w_m(\ell) \right)^\top$. *Then, for all integer* $k$ *larger than* 1, *and all* $t \geq kh$, *we have*

$$w(t) \leq R^k \Psi_k(t)$$

*with*

$$\Psi_k(t) = \left( \sup_{\ell \in [t-kh,t]} w_1(\ell) \; \ldots \; \sup_{\ell \in [t-kh,t]} w_m(\ell) \right)^\top .$$

**Proof.** We prove the lemma by induction:
*Induction Assumption:* There is $l \in \mathbb{N}$, $l > 0$ such that the result of Lemma A.1 holds for all $k \in \{1, ..., l\}$.
*Step 1:* The assumption is satisfied at the step 1.
*Step l:* Let us assume that it is satisfied at the step $l \geq 1$. Then the inequalities

$$w(t) \leq R^l \Psi_l(t) \tag{A.2}$$

hold for all $t \geq lh$. From (A.1), we deduce that for all $t \geq (l+1)h$ and $\ell \in [t-lh, t]$, the inequalities

$$\begin{pmatrix} w_1(\ell) \\ \vdots \\ w_m(\ell) \end{pmatrix} \leq R \begin{pmatrix} \sup\limits_{s\in[\ell-h,\ell]} w_1(s) \\ \vdots \\ \sup\limits_{s\in[\ell-h,\ell]} w_m(s) \end{pmatrix} \leq R\Psi_{l+1}(t)$$

hold. It follows that

$$\Psi_l(t) \leq R\Psi_{l+1}(t) \ . \tag{A.3}$$

By combining (A.2) and(A.3), we deduce that

$$w(t) \leq R^{l+1}\Psi_{l+1}(t)$$

for all $t \geq (l+1)h$. Thus the induction assumption is satisfied at the step $l+1$. This concludes the proof. □

**Lemma A.2** *Let $M \in \mathbb{R}^{n\times n}$ be an invertible matrix. Let $N \in \mathbb{R}^{n\times n}$ be a matrix. Let $\bar{n}$ and $\bar{m}$ be two constants such that $|M^{-1}| \leq \bar{m}$ and $|N| \leq \bar{n}$. Assume that*

$$\bar{m}\bar{n} < 1 \ . \tag{A.4}$$

*Then the matrix $M + N$ is invertible and*

$$\left|(M+N)^{-1} - M^{-1}\right| \leq \frac{\bar{m}^2\bar{n}}{1-\bar{m}\bar{n}} \tag{A.5}$$

*is satisfied.*

**Proof.** To prove that the matrix $M+N$ is invertible, let us proceed by contradiction. We suppose that it is not invertible. Then there is a nonzero vector $V \in \mathbb{R}^n$ such that $V^\top(M+N) = 0$, so invertibility of $M$ gives $V^\top = -V^\top NM^{-1}$, and so also $|V| \leq |V|\bar{m}\bar{n}$. Since $V \neq 0$, we conclude that $1 \leq \bar{m}\bar{n}$, which contradicts (A.4). We deduce that $M+N$ is invertible. To prove the inequality (A.5), we first set $R = (M+N)^{-1} - M^{-1}$. By multiplying $R$ by $M+N$ and $M$, we obtain $(M+N)RM = M - (M+N) = -N$, and so also $MRM = -N - NRM$. We deduce that $R = -M^{-1}NM^{-1} - M^{-1}NR$. As an immediate consequence, we obtain $|R| \leq \bar{m}^2\bar{n} + \bar{m}\bar{n}|R|$, which allows us to conclude. □

**Lemma A.3** *The inequality*

$$|e^{M+N} - e^{M}| \leq \left(1 - e^{-|N|}\right) e^{|M|+|N|} \tag{A.6}$$

*is satisfied for any matrices* $M \in \mathbb{R}^{n\times n}$ *and* $N \in \mathbb{R}^{n\times n}$.

**Proof.** Let $U_1 = M$ and $U_2 = N$. Then for each pair of integers $(i, j)$ with $i \geq 0$ and $j \geq 1$, there is a function $\lambda_{j,i} : \{1, ..., j\} \to \{1, 2\}$ such that $\lambda_{j,1}(k) = 1$ for all $k \in \{1, ..., j\}$ and such that

$$(M + N)^j = \sum_{i=1}^{2^j} U_{\lambda_{j,i}(1)}...U_{\lambda_{j,i}(j)} \tag{A.7}$$

which gives

$$(M + N)^j - M^j = \sum_{i=2}^{2^j} U_{\lambda_{j,i}(1)}...U_{\lambda_{j,i}(j)} \ . \tag{A.8}$$

It follows that

$$|(M + N)^j - M^j| \leq \sum_{i=2}^{2^j} |U_{\lambda_{j,i}(1)}|...|U_{\lambda_{j,i}(j)}| = (|M| + |N|)^j - |M|^j \ . \tag{A.9}$$

Observing that

$$\left|e^{M+N} - e^{M}\right| = \left|\sum_{j=1}^{+\infty} \frac{(M+N)^j - M^j}{j!}\right| \leq \sum_{j=1}^{+\infty} \frac{\left|(M+N)^j - M^j\right|}{j!} \tag{A.10}$$

we deduce that

$$\left|e^{M+N} - e^{M}\right| \leq \sum_{j=1}^{+\infty} \frac{(|M|+|N|)^j - |M|^j}{j!} = e^{|M|+|N|} - e^{|M|} \ . \tag{A.11}$$

This allows us to conclude.

**Lemma A.4** *Let* $m \geq 1$ *be an integer. Let* $G \in \mathbb{R}^{n\times n}$ *and* $\nu_i \in \mathbb{R}^{n\times n}$ *for all* $i \in \{1, ..., m\}$ *be matrices such that* $|G| \leq \overline{g}$ *and* $\max_i |\nu_i| \leq \overline{\nu}$, *where* $\overline{g} > 0$ *and* $\overline{\nu} > 0$ *are two constants. Let* $\alpha_i \in [0, 1]$ *be a constant for each* $i \in \{1, ..., m\}$. *Then*

$$\left|e^{\alpha_m(G+\nu_m)}.e^{\alpha_{m-1}(G+\nu_{m-1})}....e^{\alpha_1(G+\nu_1)} - e^{\alpha_\triangle G}\right| \leq e^{m(\overline{g}+\overline{\nu})}\left(1 - e^{-m\overline{\nu}}\right) \tag{A.12}$$

*is satisfied with the choice* $\alpha_\triangle = \alpha_1 + \ldots + \alpha_m$.

**Proof.** We start the proof with some definitions. Set $S(i) = (\alpha_1 + \ldots + \alpha_i)G$ for all $i \geq 1$, and set

$$\Omega_0 = I \;\text{ and }\; \Omega_i = e^{\alpha_i(G+\nu_i)}.e^{\alpha_{i-1}(G+\nu_{i-1})}....e^{\alpha_1(G+\nu_1)} \;\text{ for }\; i \geq 1, \tag{A.13}$$

and set $\xi_0 = 0$ and $\xi_i = \Omega_i - e^{S(i)}$ for all $i \in \{1, ..., m\}$. Then the left side of (A.12) is $|\xi_m|$. Consider any $i \in \{1, ..., m\}$. Elementary calculations give

$$\xi_i = e^{\alpha_i(G+\nu_i)}\Omega_{i-1} - e^{S(i)} = \left[e^{\alpha_i(G+\nu_i)} - e^{\alpha_i G}\right]\Omega_{i-1} + e^{\alpha_i G}\xi_{i-1} \tag{A.14}$$

and therefore also

$$|\xi_i| \;\leq\; \left|e^{\alpha_i(G+\nu_i)} - e^{\alpha_i G}\right| |\Omega_{i-1}| + e^{\alpha_i \overline{g}}|\xi_{i-1}| \;\leq\; \left(1 - e^{-\overline{\nu}}\right) e^{\overline{g}+\overline{\nu}}|\Omega_{i-1}| + e^{\overline{g}}|\xi_{i-1}| \tag{A.15}$$

where the last inequality is a consequence of Lemma A.3 and the fact that $\alpha_i \in [0, 1]$ for all $i$. Since for all $i \in \{0, ..., m\}$, we have $|\Omega_i| \leq e^{i(\overline{g}+\overline{\nu})}$, we obtain

$$\begin{aligned} |\xi_m| &\leq \left(1 - e^{-\overline{\nu}}\right) e^{m\beta} + e^{\overline{g}}|\xi_{m-1}| \\ |\xi_{m-1}| &\leq \left(1 - e^{-\overline{\nu}}\right) e^{(m-1)\beta} + e^{\overline{g}}|\xi_{m-2}| \\ &\vdots \\ |\xi_1| &\leq \left(1 - e^{-\overline{\nu}}\right) e^{\beta} \end{aligned} \tag{A.16}$$

with $\beta = \overline{g} + \overline{\nu}$. A simple induction argument then gives

$$|\xi_m| \;\leq\; \left(1 - e^{-\overline{\nu}}\right) \sum_{k=0}^{m-1} e^{k\overline{g}} e^{(m-k)\beta} = \left(1 - e^{-\overline{\nu}}\right) e^{m\beta} \sum_{k=0}^{m-1} e^{-k\overline{\nu}}\,. \tag{A.17}$$

In fact, if we have

$$|\xi_j| \;\leq\; \left(1 - e^{-\overline{\nu}}\right) \sum_{k=0}^{j-1} e^{k\overline{g}} e^{(j-k)\beta} \tag{A.18}$$

for some $j \in \{1, 2, \ldots, m - 1\}$, then (A.16) gives

$$\begin{aligned} |\xi_{j+1}| &\leq e^{\bar{g}}(1 - e^{-\bar{\nu}}) \sum_{k=0}^{j-1} e^{k\overline{g}} e^{(j-k)\beta} + (1 - e^{-\bar{\nu}})e^{(j+1)\beta} \\ &= \left(1 - e^{-\overline{\nu}}\right) \sum_{k=1}^{j} e^{k\overline{g}} e^{(j-k+1)\beta} + (1 - e^{-\bar{\nu}})e^{(j+1)\beta} \\ &= \left(1 - e^{-\overline{\nu}}\right) \sum_{k=0}^{j} e^{k\overline{g}} e^{(j+1-k)\beta}, \end{aligned} \tag{A.19}$$

which proves the inductive step. Then the geometric sum formula implies that the inequality (A.12) holds.

**Lemma A.5** *Consider the system*

$$\dot{X}(t) = N(t)X(t) \tag{A.20}$$

*with $X$ valued in $\mathbb{R}^n$ and where $N : [0, +\infty) \to \mathbb{R}^{n\times n}$ is a continuous function. Let $\epsilon : \mathbb{R} \to \mathbb{R}^{n\times n}$ be a piecewise continuous function that is bounded everywhere by some constant $\bar{\epsilon} \geq 0$. Consider*

$$\dot{Y}_\epsilon(t) = [N(t) + \epsilon(t)]Y_\epsilon(t) \; . \tag{A.21}$$

*Let $\Phi$ and $\Phi_\epsilon$ denote the fundamental solutions of the systems (A.20) and (A.21), respectively. Let $T_0$ and $T \geq T_0$ be two real numbers. Let $\delta > 0$ be a real number. There exists a constant $\bar{\bar{\epsilon}} > 0$ such that if $\bar{\epsilon} \leq \bar{\bar{\epsilon}}$, then for all $t \in [T_0, T]$ and $s \in [T_0, t]$, the inequality*

$$|\Phi(t, s) - \Phi_\epsilon(t, s)| \leq \delta \tag{A.22}$$

*holds.*

**Proof.** Observe for later use that the continuity of $\Phi$ and $\Phi^{-1}$ implies that there is a constant $\overline{\Phi} \geq 0$ such that for all $a \in [T_0, T]$ and $b \in [T_0, a]$, we have

$$\max\{|\Phi^{-1}(a, b)|, |\Phi(a, b)|\} \leq \overline{\Phi} \; . \tag{A.23}$$

Set $\lambda_\epsilon(a, b) = \Phi^{-1}(a, b)\Phi_\epsilon(a, b) - I$. Through simple calculations, we obtain

$$\begin{aligned}
\frac{\partial \lambda_\epsilon}{\partial a}(a, b) &= \frac{\partial \Phi^{-1}}{\partial a}(a, b)\Phi_\epsilon(a, b) \\
&\quad + \Phi^{-1}(a, b)[N(a) + \epsilon(a)]\Phi_\epsilon(a, b) \\
&= -\Phi^{-1}(a, b)N(a)\Phi_\epsilon(a, b) \\
&\quad + \Phi^{-1}(a, b)[N(a) + \epsilon(a)]\Phi_\epsilon(a, b) \\
&= \Phi^{-1}(a, b)\epsilon(a)\Phi_\epsilon(a, b) \\
&= \Phi^{-1}(a, b)\epsilon(a)\Phi(a, b)[\lambda_\epsilon(a, b) + I] \; ,
\end{aligned} \tag{A.24}$$

where we also used [33, Lemma C.4.1]. Let $v(a, b) = \frac{1}{2}|\lambda_\epsilon(a, b)V|^2$, where $V \in \mathbb{R}^n$ is any vector satisfying $|V| = 1$. Then the chain rule and (A.24) give

$$\frac{\partial v}{\partial a}(a, b) \; = \; V^\top \lambda_\epsilon(a, b)^\top \Phi^{-1}(a, b)\epsilon(a)\Phi(a, b)[\lambda_\epsilon(a, b) + I]V \; . \tag{A.25}$$

Using (A.23) and the upper bound of $\epsilon$, we deduce that for all $t \in [T_0, T]$ and $s \in [T_0, t]$, we have

$$\begin{aligned} \frac{\partial v}{\partial t}(t,s) &\leq \overline{\Phi}^2 \bar{\epsilon} |\lambda_\epsilon(t,s) V|^2 + \overline{\Phi}^2 \bar{\epsilon} |\lambda_\epsilon(t,s) V| \\ &= 2\overline{\Phi}^2 \bar{\epsilon} v(t,s) + \sqrt{2}\, \overline{\Phi}^2 \bar{\epsilon} \sqrt{v(t,s)}\,. \end{aligned} \tag{A.26}$$

Using Young's inequality $ab \leq a^2 + \frac{b^2}{4}$ with $b = \sqrt{2}$ and $a = \sqrt{v(t,s)}$, we obtain

$$\frac{\partial v}{\partial t}(t,s) \leq 3\overline{\Phi}^2 \bar{\epsilon} v(t,s) + \frac{1}{2} \overline{\Phi}^2 \bar{\epsilon}\,. \tag{A.27}$$

By integrating the last inequality between $s$ and $t$ and bearing in mind that $v(s,s) = 0$, we deduce that

$$v(t,s) \leq \frac{1}{6} \left[ e^{3\overline{\Phi}^2 \bar{\epsilon}(t-s)} - 1 \right]\,. \tag{A.28}$$

Since (A.28) holds for any vector $V$ such that $|V| = 1$, we deduce that

$$|\lambda_\epsilon(t,s)|^2 \leq \frac{1}{3} \left[ e^{3\overline{\Phi}^2 \bar{\epsilon}(t-s)} - 1 \right]\,. \tag{A.29}$$

Also, $|\Phi(t,s) - \Phi_\epsilon(t,s)| \leq |\Phi(t,s)|\, |I - \Phi(t,s)^{-1} \Phi_\epsilon(t,s)|$. From the definition of $\lambda_\epsilon$ and (A.23), we have

$$|\Phi(t,s) - \Phi_\epsilon(t,s)| \leq |\Phi(t,s)||\lambda_\epsilon(t,s)| \leq \frac{\overline{\Phi}}{\sqrt{3}} \sqrt{e^{3\overline{\Phi}^2 \bar{\epsilon} T} - 1}\,. \tag{A.30}$$

We conclude that if

$$\bar{\epsilon} \leq \bar{\bar{\epsilon}} = \frac{1}{3\overline{\Phi}^2 T} \ln\left(1 + \frac{3\delta^2}{\overline{\Phi}^2}\right) \tag{A.31}$$

then (A.22) is satisfied.