Tetrahedral Picture Languages and their Applications in Imaging

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Abstract

In formal languages, a Picture language is a set of pictures, where a picture is a two-dimensional arrangement of symbols over an alphabet. Many families of picture languages namely rectangular, hexagonal and iso-picture languages were introduced in the literatre. Initially the study of picture languages was motivated by pattern recognition and image processing. We have introduced tetrahedral picture languages and studied its recognizability using tetrahedral tiles. Various interesting patterns are generated using 3D-array token Petri nets generating tetrahedral picture languages. In this paper, we give a survey of tetrahedral picture languages and we propose domino recognizability of tetrahedral picture languages.

Introduction

In formal language theory, two dimensional picture languages plays a vital role in Pattern recognition and image processing. Many recognizing devices such as grammars, automata, pasting systems and Petri nets have been used in the literature (Siromoney, Siromoney, and Krithivasan 1972, 1973). Recognizability of two-dimensional picture languages have been introduced by Gimmarresi and Restivo (Giammarresi and Restivo 1992). M. Latteux and Simplot have introduced hv-local picture languages in which horizontal and vertical dominoes were used (Latteux and Simplot 1997). K.S. Dersanambika et al. (Dersanambika et al. 2005) introduced the recognizability of hexagonal picture languages. T. Kalyani et al. (Kalyani, Dare, and Thomas 2004) have introduced the recognizability of iso picture languages.

Extending these ideas to three dimensions D.G. Thomas et al. (Thomas et al. 2008) have introduced the recognizability of three dimensional rectangular picture languages. Motivated by these studies in the literature F. Sweety et al. (Sweety et al. 2019) have introduced the recognizabilty of tetrahedral picture languages. T. Kalyani et al. (Kalyani et al. 2020) have generated tetrahedral picture languages using Petri nets. Some interesting 3D-patterns were generated using this system. Tetrahedral picture languages can be used to generate some interesting floor and wall designs.

In this paper we give a study on tetrahedral picture languages and its recognizability is studied using domino system.

Recognizable Tetrahedral Picture Languages

In this section, we recall the notion of a tetrahedral tile, tetrahedral picture languages, local and recognizable tetrahedral picture languages (Dharani, Maragatham, and Siromoney 2017; Sweety et al. 2019).

Definition 1. *(Sweety et al. 2019) A Parallel Space Filling Grammar (PSFG) is a 5-tuple (S, S_0, Σ, P, C) where S is the set of three dimensional polyhedral. $S_0 \in S$ is the initial polyhedron. Σ is the set of alphabets representing vertices v, edge positions e_p, face positions f_p or any combination of these in a three dimensional polyhedron. The production rules of P are of the following types*

$$\begin{array}{lll} (i) v \to v & (ii) e_p \to e_p & (iii) f_p \to f_p \\ (iv) v \to e_p & (v) f_p \to e_p & (vi) v \to f_p \\ (vii) e_p \to v & (viii) e_p \to f_p & (ix) f_p \to v. \end{array}$$

C is the control language over P. The family of all possible 3D pictures generated by PSFG is denoted by $\mathcal{L}^*$-PSFG. The subset of $\mathcal{L}^*$-PSFG is called picture language $\mathcal{L}$-PSFG.

Definition 2. *(Sweety et al. 2019) A tetrahedral tile is a polyhedral which has four vertices, four faces and six edges. Each face is an equilateral triangle. f_4 is the base of the tetrahedron. The directions along the vertices v_1, v_2 and v_3 are denoted by D_1, D_2 and D_3 respectively.*

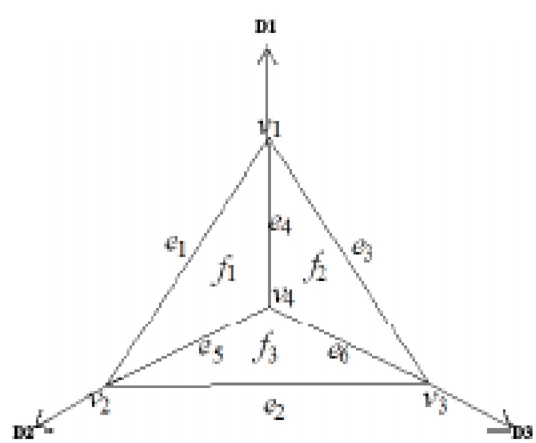
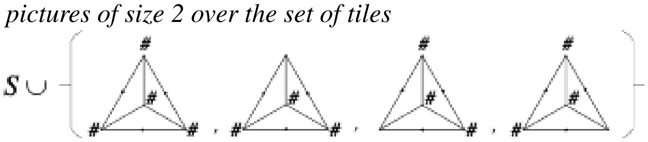


Figure 1: A tetrahedron

Definition 3. *A tetrahedral picture is a picture generated by PSFG where S is the set of regular tetrahedron. S^{**T} is the set of all tetrahedral pictures over the set S. A tetrahedral picture language over S is a subset of S^{**T}.*

Definition 4. *Let S be a finite set of regular tetrahedral tiles. A tetrahedral picture language over $(v \to e_p)$ $\mathcal{L} \subseteq S^{**T}$ is called local if there exists a finite set Δ of tetrahedral pictures of size 2 over the set of tiles*



such that $\mathcal{L} = \{p \in S^{**T} / B_2(\hat{p}) \subseteq \Delta\}$. $B_2(\hat{p})$ denotes the set of all sub pictures of $\hat{p}$ of size 2. The family of local tetrahedral picture languages will be denoted by TrLoc$(v \to e_p)$.

Definition 5. *A tetrahedral tiling system T is a 6-tuple $(S, S', \Sigma, \Gamma, \pi, \theta)$ where S and S' are finite sets of tetrahedral tiles and Σ and Γ are two finite sets of symbols representing vertices v, edge positions e_p, and face positions f_p are any combinations of these. $\pi : \Gamma \to \Sigma$ is a projection and θ is a set of tetrahedral pictures of size 2 over the alphabet $\Gamma \cup \{\#\}$.*

Definition 6. *The tetrahedral picture language $\mathcal{L} \subseteq S^{**T}$ is tiling recognizable if there exists a tetrahedral tiling system $T = (S, S', \Sigma, \Gamma, \pi, \theta)$ such that $\mathcal{L} = \pi(\mathcal{L}'(\theta))$. TrREC is exactly the family of tetrahedral picture languages recognizable by tetrahedral tiling systems (TrTS).*

Theorem 1. *The family TrREC is closed under projection.*

Theorem 2. *The family TrREC is closed under right and left catenations with the production rules $v \to e_p$ and $e_p \to v$.*

Petri Nets Generating Tetrahedral Picture Languages

In this section, we recall the notion of a Petri net structure generating tetrahedral picture languages (Kalyani et al. 2020).

Definition 7. *A 3D Tetrahedral Tile Array Token Petri Net (3D-TetATPN) is a six tuple $N = (\Sigma, C, \mu, S, \sigma, F)$ where Σ is an alphabet of tetrahedral tiles or extended tetrahedral tiles (3D-picture made up of tetrahedral tiles), C is a Petri Net structure, μ is an initial marking of 3D-pictures made up of tetrahedral tiles or extended tetrahedral tiles kept in some places of the net, S is a set of catenation rules, σ is a partial mapping which attaches rules to the various transitions of the Petri Net of the form $\sigma(t_i) = P$ (cat) Q, F is a subset of the set of places of the Petri Net where the final 3D-tetrahedral picture is stored after all the firing of the various possible transitions of the Petri Net.*

Definition 8. *The language generated by 3D-TetATPN is the set of all 3D-tetrahedral pictures stored in the final places of the Petri Net structure and is denoted by $\mathcal{L}(N)$.*

Example 1. *Consider the 3D-TetATPN $N_2 = (\Sigma, C, \mu, S, \sigma, F)$ where,*
$\Sigma = \{H, A, B, C, D\}$, $C = (P, T, I, O)$, $P = \{P_1, P_2, P_3, \ldots P_8\}$,
$T = \{t_1, t_2, t_3, \ldots t_7\}$.
The initial marking μ is the hexagonal polyhedral H in the place of P_1.

S = { H (ru) B, H (rd) A , B (rd) A, D (ru) C, C (ru) H, C (rd) H, H (rd) H }

σ the mapping from the set of transitions to the set of rules is shown in Fig. 2 and $F = \{P_8\}$

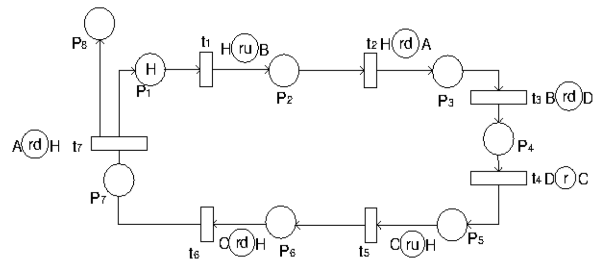


Figure 2: 3D-Array token Petri Net generating increasing sequence of hexagonal polyhedrals

Starting with H on firing the sequence $t_1t_2t_3t_4$ the tetrahedral tiles B, A, D and C are catenated to H according to the catenation rules respectively and the resultant 3D-array is sent out to place P_5. On firing the sequence t_5t_6 Hexagonal polyhedrals are catenated to C-tetrahedral tile in parallel in the right up and right down directions and then firing t_7 hexagonal polyhedrals are catenated to hexagonal polyhedrals in the right down direction in parallel and finally the resultant sequence of hexagonal polyhedral language is sent to the final place P_8.

The first member of the language generated is shown in Fig. 3. In the first member two hexagonal polyhedrals are catenated twice. In the n^{th} member $n + 1$ hexagonal polyhedrals are catenated twice.

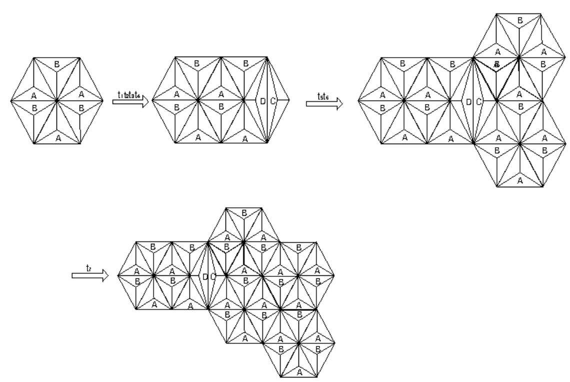
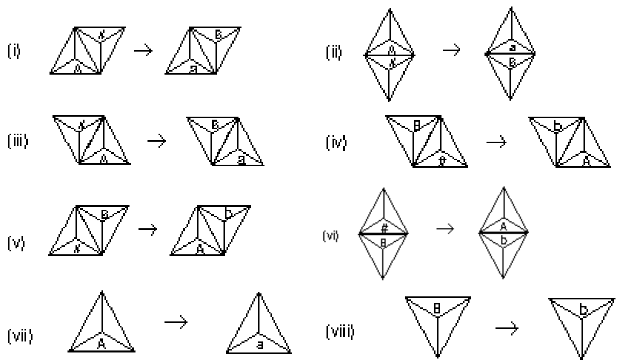
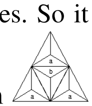


Figure 3: First member of the language generated by N_2

Theorem 3. $\mathcal{L}(3D - TetATPN) - RTAL \neq \emptyset$.

Proof. We consider a tetrahedral language whose boundary is an equilateral triangle. This language cannot be generated by any Regular Tetrahedral Array Grammar (RTAG) (Raman, Kalyani, and Thomas 2020). Since the rules in RTAG are of the following forms:



Similar rules can be given for the other two tetrahedral tiles C and D, where A and B are non terminal symbols and a and b are terminal symbols. Starting with a tetrahedral tile A, RTAG can generate at most three connected tiles. So it cannot generate an equilateral triangle of the form  but this language can be generated by the following 3D-TetATPN.

Consider a 3D-TetATPN $N_4 = (\Sigma, C, \mu, S, \sigma, F)$, where $\Sigma = \{A, B\}$, $C = (P, T, I, O)$ where $P = \{P_1, P_2, P_3, P_4\}$, $T = \{t_1, t_2, t_3\}$. The initial marking μ is the tetrahedral tile A in place P_1.

S = {A(ru)B, B(u)A, B(rd)A}

σ the mapping from the set of transitions to the set of rules is shown in Fig. 4, $F = \{P_4\}$ and the language generated by N_4 is shown in Fig. 5.

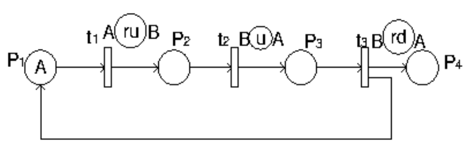


Figure 4: 3D-TetATPN of the language of equilateral triangle tetrahedral

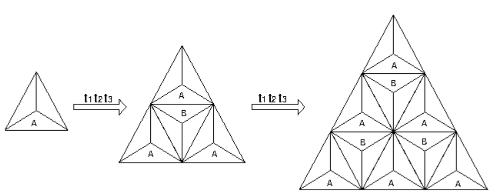


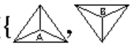
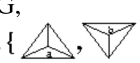
Figure 5: language of equilateral triangle tetrahedral

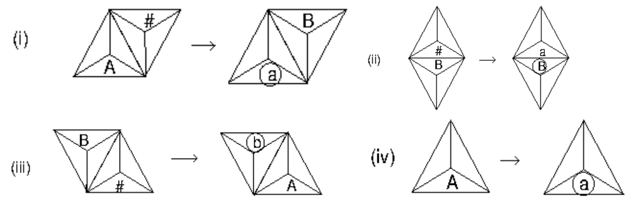
□

Theorem 4. *$\mathcal{L}(3DTetATPN)$ and $BPTAL$ are incomparable but not disjoint.*

Proof. Consider a tetrahedral language whose boundary is an equilateral triangle of size 2. This language is generated by both systems Basic Puzzle Tetrahedral Array Grammar (BPTAG) (**?**) as well as by 3D-TetATPN.

Consider a BPTAG,

G = ({  }, {  }, **P**, **S**) where P consists of the following rules:



The language generated by G is an equilateral triangle tetrahedral of size 2 which is shown in Fig. 6.

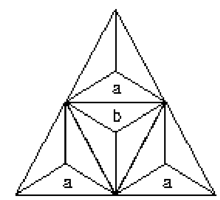


Figure 6: Equilateral triangle tetrahedral picture of size 2

This language can be generated by the following 3D-TetATPN:

Consider a 3DTetATPN $N_5 = (\Sigma, C, \mu, S, \sigma, F)$, where $\Sigma = \{A, B\}$, $C = (P, T, I, O)$ where $P = \{P_1, P_2, P_3, P_4\}$, $T = \{t_1, t_2, t_3\}$. The initial marking μ is the tetrahedral tile A in place $P1$.

S = { A(ru)B, B(u) A ,B(rd)A }

, σ the mapping from the set of transitions to the set of rules is shown in Fig. 7 and $F = \{P_4\}$

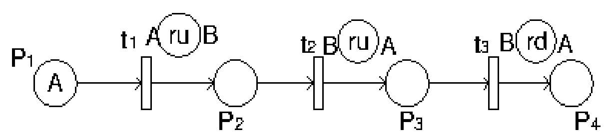


Figure 7: 3D-TetATPN generating equilateral triangle tetrahedral of size 2

Equilateral triangle tetrahedral of size more than 2 cannot be generated by BPTAG, whereas it can be generated by 3D-TetATPN (as in Theorem 3). On the other hand the sequence of over lapping equilateral triangle tetrahedral can be generated by the above BPTAG, whereas it cannot be generated by any 3D-TetATPN as the catenation rules are applied in parallel wherever possible. □

Domino Recognizable Tetrahedral Picture Languages

In this section, we define vertical, right and left dominoes of the types shown in Fig. 8.

Figure 8: Vertical, right and left dominoes of size (2, 1)

We have defined local tetrahedral picture languages as languages given by a finite set of authorized tetrahedral pictures of size 2. Here, vertical, right and left controls are done at the same time. In domino system the three controls are done separately.

Given a tetrahedral picture p of size n, we denote by $B_{h,k}(p)$, the set of all sub pictures of p of size (h, k), i.e., h tetrahedral pictures of size k, $1 \leq h, k \leq n$ are generated by the production rule $v \to e_p$.

Definition 9. *A three dimensional tetrahedral picture language $L \subseteq S^{**T}$ is vrl-local if there exists a finite set Δ of dominoes over the alphabet $\Sigma \cup \{\#\}$ such that the language $L = \{p \in S^{**T} / B_{2,1}(\hat{p}) \subseteq \Delta\}$. The family of vrl-local tetrahedral picture languages is denoted by vrl-Loc$(v \to e_p)$.*

The family of vrl-local tetrahedral picture languages over $(v \to e_p)$ is strictly included in local tetrahedral picture languages over $(v \to e_p)$.

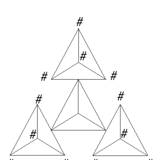
Definition 10. *A Tetrahedral Domino System (TDS) is a 4-tuple $TD = (\Sigma, \Gamma, \Delta, \pi)$, where Σ and Γ are two finite alphabets of tetrahedral tiles, Δ is a finite set of vertical, right and left dominoes over the alphabet $\Gamma \cup \{\#\}$ and $\pi : \Gamma \to \Sigma$ is a projection.*

The tetrahedral domino system recognizes a tetrahedral picture language L over the alphabet Σ and is defined as $L = \pi(L')$, where $L' = L(\Delta)$ is the vrl-local tetrahedral picture language over $(v \to e_p)$ of Γ. The family of tetrahedral picture languages recognizable by tetrahedral domino system is denoted by $\mathcal{L}(TDS)$.

Proposition 1. *If $L \subseteq S^{**T}$ is vrl-local tetrahedral picture language over $(v \to e_p)$ then L is local tetrahedral picture language over $(v \to e_p)$. That is $\mathcal{L}(TDS) \subseteq TrLoc(v \to e_p)$.*

Proof. Let $L \subseteq S^{**T}$ be a vrl-local tetrahedral picture language over $(v \to e_p)$. Then $L = L(\Delta)$ where Δ is a finite set of vertical, right and left dominoes of size (2, 1). We will construct a finite set of tetrahedral pictures θ of size 2 and show that $L = L(\theta)$. The set of tiles θ will be defined in a way that all sub-pictures of size (2, 1) of each tile in θ should belong to the set of dominoes Δ. We define θ as follows.

$$\theta = \left\{ \theta \in (\Sigma \cup \{\#\})^2 \Big/ \theta \neq \right.$$



$$\left. , B_{(2,1)}(\theta) \subseteq \Delta \right\}$$

Let $L' = L(\theta)$. We now show that $L' = L$. Let $p \in L'$. Then by definition $B_2(\hat{p}) \in \theta$. This implies that $B_{2,1}(\hat{p}) \subseteq B_{2,1}(B_2(\hat{p})) \subseteq B_{2,1}(\theta) \subseteq \Delta$. Hence $p \in L$.

Conversely, let $p \in L$ and $q \in B_2(\hat{p})$. Then $B_{2,1}(q) \subseteq B_2(\hat{p}) \subseteq \Delta$. Therefore $q \in \theta$ and $p \in L'$. Hence $L = L'$. □

Lemma 1. *Let L be a local tetrahedral picture language over an alphabet Σ. Then there exists a vrl-Loc tetrahedral picture language L' over an alphabet Γ and a mapping $\pi : \Gamma \to \Sigma$ such that $L = \pi(L')$.*

Theorem 5. $\mathcal{L}(TrTS) = \mathcal{L}(TDS)$.

Proof. The inclusion $\mathcal{L}(TDS) \subseteq \mathcal{L}(TrTS)$ is an immediate consequence of Proposition 1. The inverse inclusion follows from Lemma 1. □

Conclusion

In this paper we have defined domino recognizability of tetrahedral picture languages over the production rule $(v \to e_p)$. It is compared with local and recognizable tetrahedral picture languages. We have proposed $CTTv \to e_p$ & $e_p \to vTPS$ and $TetTv \to e_p$ & $e_p \to vPPS$ and these two systems are compared with local tetrahedral picture languages. This work can also be applied to other production rules of PSFG. Wang recognizability of tetrahedral picture languages can be studied. This is our future work.

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