Journal of Graph Algorithms and Applications
http://jgaa.info/ vol. 16, no. 3, pp. 759–782 (2012)
DOI: 10.7155/jgaa.00266

Adjacent Crossings Do Matter

Radoslav Fulek[1] *Michael J. Pelsmajer*[2] *Marcus Schaefer*[3]
Daniel Štefankovič[4]

[1]Ecole Polytechnique Fédérale de Lausanne, Lausanne, Switzerland
[2]Department of Applied Mathematics Illinois Institute of Technology, Chicago, Illinois 60616, USA
[3]Department of Computer Science DePaul University, Chicago, Illinois 60604, USA
[4]Computer Science Department University of Rochester, Rochester, NY 14627-0226

Abstract

In a drawing of a graph, two edges form an *odd pair* if they cross each other an odd number of times. A pair of edges is *independent* if the two edges share no endpoint. For a graph G, let $\text{ocr}(G)$ be the smallest number odd pairs in a drawing of G and let $\text{iocr}(G)$ be the smallest number of independent odd pairs in a drawing of G. We construct a graph G with $\text{iocr}(G) < \text{ocr}(G)$, answering an open question of Székely. The same graph G also separates two notions of algebraic crossing numbers that Tutte expected to be the same.

The graph G was found via considering monotone drawings of ordered graphs. A drawing of a graph is *x-monotone* if every edge intersects every vertical line at most once and every vertical line contains at most one vertex. In an *ordered graph*, the vertices have a left-to-right ordering that must be preserved in x-monotone drawings. For every integer $n > 0$ we construct an ordered graph G such that for x-monotone drawings, the monotone variant of ocr and iocr satisfy $2 = \text{mon-iocr}(G) \leq \text{mon-ocr}(G) - n$. We can also separate mon-ocr from its variant in which crossings of adjacent edges are prohibited.

We also offer a general translation result from monotone separations to non-monotone separations. This could prove useful in settling several open separation problems, such as pair crossing number versus crossing number.

Submitted:	Reviewed:	Revised:	Accepted:	Final:
November 2011	February 2012	February 2012	April 2012	April 2012
		Published: September 2012		

Article type: Regular paper — Communicated by: M. van Kreveld and B. Speckmann

An earlier version of this paper was presented at Graph Drawing 2011 [3]. The first author gratefully acknowledges support from the Swiss National Science Foundation Grant No. 200021-125287/1. The second author gratefully acknowledges the support from NSA Grant H98230-08-1-0043 and the Swiss National Science Foundation Grant No. 200021-125287/1.
E-mail addresses: radoslav.fulek@epfl.ch (Radoslav Fulek) pelsmajer@iit.edu (Michael J. Pelsmajer) mschaefer@cs.depaul.edu (Marcus Schaefer) stefanko@cs.rochester.edu (Daniel Štefankovič)

1 Introduction

The crossing number of a graph, $\mathrm{cr}(G)$, is the smallest number of crossings in any drawing of G. When defining the crossing number, one often restricts drawings by requiring that edges do not cross each other more than once and adjacent edges do not cross at all. The reason is simple: in any optimal drawing, neither of these phenomena will occur, so there is no harm in excluding them from the beginning. It is understandable then, that researchers in crossing numbers have often treated adjacent crossings as pathologies that can be defined away[1]. However, doing so hides some of the more mysterious aspects of the crossing number.

Say, for example, we are tempted to think that adjacent crossings are inconsequential and we define them away by counting only the crossings between pairs of independent edges (edges that do not share an endpoint). Minimizing the number of crossings between pairs of independent edges defines what is known as the *independent crossing number*, cr_-. As far as we know, it is open whether $\mathrm{cr} = \mathrm{cr}_-$, even for complete graphs. Until very recently, it was not even known whether the gap between the two could be arbitrarily large, although we now know that $\mathrm{cr}_-(G) \geq c \cdot \mathrm{cr}(G)^{1/2}$, where $c = 2^{-1/2}$ [15].

For the standard crossing number, we are left with an unsolved mystery, but in this paper we show that for some other well-known crossing numbers, not counting adjacent crossings or prohibiting adjacent crossings does make a provable difference.

The *odd crossing number* of a graph, $\mathrm{ocr}(G)$, is the smallest number of pairs of edges that cross an odd number of times in any drawing of G. If we only count pairs of independent edges, we obtain $\mathrm{ocr}_-(G)$ (also known as $\mathrm{iocr}(G)$), the *independent odd crossing number*. If we restrict ourselves to drawings in which crossings between adjacent edges are entirely prohibited, we get another variant, ocr_+.[2]

We can show that there are graphs G for which $\mathrm{ocr}_-(G) < \mathrm{ocr}(G)$: for the odd crossing number it does matter whether adjacent edges are counted or not.

Theorem 1 *For every n, there is a graph G with* $\mathrm{ocr}_-(G) + n < \mathrm{ocr}(G)$.

In short, adjacent crossings matter.[3] This particular separation was mentioned as a specific open problem on crossing numbers by Székely [18].

Our proof of Theorem 1 is based on a separation in a more restrictive model: monotone drawings of ordered graphs. A drawing of a graph is *x-monotone* if every edge intersects every vertical line at most once and every vertical line

[1]This somewhat psychological claim is born out by the names used for drawings that do not have adjacent crossings or edges that cross more than once: "nice", "good".

[2]A $-$ subscript means that only crossings of independent edges are counted. A subscript $+$ denotes variants in which crossings of adjacent edges are prohibited. We did not introduce cr_+ since $\mathrm{cr}_+ = \mathrm{cr}$ (adjacent edges do not cross in drawings minimizing the total number of crossings).

[3]Among other things, Theorem 1 justifies the rather baroque **NP**-completeness proof for ocr_- in [16]; if it had turned out that $\mathrm{ocr}_- = \mathrm{ocr}$, then ocr_- would have been **NP**-complete by virtue of the far simpler proof of **NP**-completeness of ocr [9].

contains at most one vertex. An *ordered graph* is a graph with a total ordering of its vertices. For vertices u, v in an ordered graph G with $u < v$, we require that the x-coordinates satisfy $x(u) < x(v)$ in every drawing of G. We write u *is to the left of* v or v *is to the right of* u instead of $u < v$ even if we are not considering a particular drawing of G. Similarly, notions like *left-degree* and *right-degree* are well-defined for vertices of ordered graphs.

If we only consider x-monotone drawings of ordered graphs G, we obtain a monotone version of each crossing number variant, which we denote mon-cr(G), mon-ocr(G), mon-ocr$_-$(G), mon-ocr$_+$(G), etc. One may also define monotone crossing numbers of graphs without ordering (by allowing any ordering of the vertices); such monotone crossing numbers of (unordered) graphs were introduced by Valtr [23] and were studied more recently by Pach and Tóth [10]. In this paper we will only consider monotone crossing numbers of ordered graphs.

In Section 4.2.1 we show that there can be an arbitrary gap between mon-ocr and mon-ocr$_-$:

> For every $n \geq 3$, there is an ordered graph G with $\text{mon-ocr}_-(G) = 2 < n = \text{mon-ocr}(G)$.

In Section 4.2.2 we show that prohibiting crossings between adjacent edges can increase mon-ocr:

> For every $n \geq 2$, there is an ordered graph G with $\text{mon-ocr}(G) \leq n^2 + n < n^3 \leq \text{mon-ocr}_+(G)$.

We then prove a translation result, Lemma 7, that implies that a separation of the monotone crossing numbers (in most cases) leads to a corresponding separation of the crossing numbers in the unrestricted (i.e. non-monotone) case. Indeed, we can prove this translation result for a much wider variety of crossing numbers including the pair (pcr) and algebraic (acr) crossing numbers (which we will define in the next section).

Theorem 2 *If* mon-ψ *and* mon-ϕ *separate, then so do* ψ *and* ϕ*; more precisely, for every ordered* G *there is a* G' *(without ordering) so that* $\psi(G') - \phi(G') = \text{mon-}\psi(G) - \text{mon-}\phi(G)$ *where* ψ *and* ϕ *are among the crossing numbers* $\{\text{ocr}, \text{acr}, \text{pcr}, \text{cr}\}$*, the independent crossing numbers* $\{\text{ocr}_-, \text{acr}_-, \text{pcr}_-, \text{cr}_-\}$ *and* pcr_+*.*

As explained in Remark 12, the translation result fails for ocr$_+$ and acr$_+$; hence, the separation of mon-ocr from mon-ocr$_+$ has no implications for the unrestricted case.

However, Theorem 2 gives us one more separation: a graph G for which $\text{acr}_-(G) < \text{acr}(G)$. Thus, the independent algebraic and the algebraic crossing numbers differ. It was with respect to the algebraic crossing number that Tutte [22] wrote that "we are taken the view that crossings of adjacent edges are trivial, and easily got rid of". If taken literally, that view is now demonstrably false.[4]

[4]One of the anonymous referees asked whether Tutte may have meant acr = acr$_+$, which

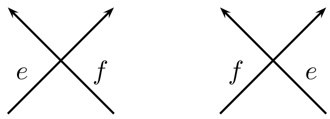


Figure 1: The crossing contributes +1 or −1 to $\lambda(e, f)$, depending on the relative orientation of the crossing.

Theorem 3 *For every n, there is a graph G with* $\mathrm{acr}_-(G) + n < \mathrm{acr}(G)$.

The following section will review crossing number variants and what is known about their relationships.

2 Crossing Number Variants

Pach and Tóth describe in "Which Crossing Number is It Anyway?" how researchers in the past have used (consciously or not) different notions of crossing numbers, including the following (see [8, 17]):

pair crossing number: $\mathrm{pcr}(G)$, the smallest number of pairs of edges crossing in a drawing of G,

odd crossing number: $\mathrm{ocr}(G)$, the smallest number of pairs of edges crossing oddly (*odd pairs*) in a drawing of G.

We write $\mathrm{ocr}(e, f)$ for the number of times that edges e and f cross modulo 2, so e and f form an odd pair if and only if $\mathrm{ocr}(e, f) = 1$.

Tutte introduced another type of crossing number: orient every edge arbitrarily, assign a +1 or −1 to each crossing depending on the relative orientation of the edges at the crossing (see Figure 1), and let $\lambda(e, f)$ be the sum of these values over all crossings between e and f. Changing the orientation of e or f will only change the sign of $\lambda(e, f)$, so $\mathrm{acr}(e, f) := |\lambda(e, f)|$ is well-defined and one can define:

algebraic crossing number: $\mathrm{acr}(G)$, the minimum of $\sum |\lambda(e, f)|$ in a drawing of G, where the sum is taken over pairs of edges e, f.

By definition $\mathrm{ocr}(G) \leq \mathrm{pcr}(G) \leq \mathrm{cr}(G)$ and $\mathrm{ocr}(G) \leq \mathrm{acr}(G) \leq \mathrm{cr}(G)$.

For each of these notions, one can ask whether adjacent crossings matter. Pach and Tóth [7] suggested a systematic study of this issue (see also [1, Section 9.4]) by introducing two rules: "Rule +" restricts the drawings to drawings in which adjacent edges are not allowed to cross. "Rule −" allows crossings of adjacent edges, but does not count them towards the crossing number. Each of

the quote seems to suggest. Tutte's actions, however, suggest that he meant $\mathrm{acr}_- = \mathrm{acr}$: he does not prohibit incident crossings, but rather he defines his algebraic crossing chains so that incident crossings do not count. Tutte may have thought that $\mathrm{acr}_- = \mathrm{acr} = \mathrm{acr}_+$, not realizing that there might be three distinct variants.

the crossing numbers ocr, pcr, acr, and cr can be modified by either rule (indicated by a + or − in the subscript), but since $\mathrm{cr}_+ = \mathrm{cr}$ (as discussed earlier) this yields up to eleven possible distinct variants.

Remark 4 *When defining Rule + we entirely prohibit adjacent edges to cross; however, one can argue that it would make sense to define the + notion in such a way that there are no adjacent crossings in the sense of the given crossing number, ψ. In other words, we require that $\psi(e,f) = 0$ for all pairs of adjacent edges e and f, where $\psi(e,f)$ is the contribution of the crossings between e and f to ψ. For* cr *and* pcr *this is the same as Rule +, but for* ocr *it means that every pair of adjacent edges must cross evenly. We call this new convention Rule ±. By definition, we have $\psi \le \psi_\pm \le \psi_+$ for $\psi \in \{\mathrm{cr}, \mathrm{ocr}, \mathrm{acr}, \mathrm{pcr}\}$. We will occasionally discuss Rule ±, and* $\mathrm{ocr}_\pm$ *in particular.*

The tables below are based on a figure from [1]. The notion of ocr_- is now called *independent odd crossing number* [17]. The variants acr and acr_- are implicit in Tutte's paper [22].[5]

Rule +	ocr_+	pcr_+	cr
	ocr	pcr	
Rule −	$\mathrm{ocr}_- = \mathrm{iocr}$	pcr_-	cr_-

ocr_+	acr_+	cr
ocr	acr	
$\mathrm{ocr}_- = \mathrm{iocr}$	$\mathrm{acr}_- = \mathrm{iacr}$	cr_-

It immediately follows from the definitions that the values in each table increase monotonically as one moves from the left to the right and from the bottom to the top. Not much more is known about the relationships between these crossing number variants. Pach and Tóth [7] write, "We cannot prove anything else about $\mathrm{ocr}_-(G)$, $\mathrm{pcr}_-(G)$, and $\mathrm{cr}_-(G)$. We conjecture that these values are very close to $\mathrm{cr}(G)$, if not the same. That is, we believe that by letting pairs of incident edges cross an arbitrary number of times, we cannot effectively reduce the total number of crossings between pairs of independent edges."[6]

There are situations when the entire hierarchy of crossing number variants collapses. The classic Hanani-Tutte theorem states that if a graph can be drawn in the plane so that no pair of independent edges crosses an odd number of times, then it is planar [2, 22]. In other words, $\mathrm{ocr}_-(G) = 0$ implies that $\mathrm{cr}(G) = 0$, so all of the eleven variants are equal (to zero). We also know that all eleven variants are equal if $\mathrm{ocr}_-(G) \le 2$ [15]. Székely gave an explicit criterion for when all variants are equal [19]. It is also known that all eleven variants are within a square of each other, since $\mathrm{cr}(G) \le \binom{2\,\mathrm{ocr}_-(G)}{2}$ [15]. For drawings of G on the projective plane N_1, we know that $\mathrm{ocr}_-(G, N_1) = 0$ implies that $\mathrm{cr}(G, N_1) = 0$, so again all variants are equal (to zero) [11].

Temporarily setting aside the Rule − variants, there are stronger results for the remaining seven crossing numbers (ocr, ocr_+, pcr, pcr_+, acr, acr_+,

[5] In other papers we use iocr and iacr, but in the current paper we prefer ocr_- and acr_- to emphasize their nature as Rule − variants of ocr and acr.

[6] Some authors write *incident edges* to mean two edges that share an endpoint; we use *adjacent edges*. Non-adjacent edges are also called *independent edges*.

and cr). If $\text{ocr}(G) \leq 3$ then all seven variants are equal [12]. Valtr [23] showed that $\text{cr}(G) = O(\text{pcr}^2(G)/\log \text{pcr}(G))$, which Tóth later improved twice, first [20] to $\text{cr}(G) = O(\text{pcr}^2(G)/\log^2 \text{pcr}(G))$ and only recently [21] to $\text{cr}(G) = O(\text{pcr}^{7/4}(G)\log^{3/2} \text{pcr}(G))$. For drawings on any surface S, if $\text{ocr}(G, S) = 0$ then all seven variants are equal (to zero) [14].

On the other hand, we know that ocr and pcr differ: there is an infinite family of graphs with $\text{ocr}(G) < 0.867 \cdot \text{pcr}(G)$ [13]. Tóth improved this by giving a family of graphs with $\text{acr}(G) < 0.855 \cdot \text{pcr}(G)$ [20] (so $\text{ocr}(G) < 0.855 \cdot \text{pcr}(G)$ as well). For such G it immediately follows that $\text{ocr}(G) < \text{cr}(G)$ and $\text{acr}_-(G) < \text{cr}(G)$, answering questions of Pach and Tóth [8] and Tutte [22]; additional consequences can be deduced from the tables above. However, none of these results address the intuitions expressed in the quotes earlier from Tutte and by Pach and Tóth, which are concerned with how Rule $-$ and Rule $+$ may or may not affect cr, pcr, ocr, or acr. Our separations of ocr_- vs ocr, acr_- vs acr and mon-ocr vs mon-ocr_+ seem to be the first such results.

For monotone crossing numbers of ordered graphs, Pach and Tóth proved that $\text{mon-ocr}(G) = 0$ implies $\text{mon-cr}(G) = 0$ [9]. We recently strengthened this by showing that $\text{mon-ocr}_-(G) = 0$ implies $\text{mon-cr}(G) = 0$ [4], which had been left as an open problem in [9]. On the other hand, in the same paper we showed that for every n there is an ordered graph G so that $\text{mon-cr}(G) \geq n$ and $\text{mon-ocr}(G) = 1$. This result with Theorem 2 gives us a new example that separates ocr from cr, in addition to those from [13] and [20].

3 Redrawing Tools

In this section we introduce the redrawing tools used in this paper. For any crossing number (variant) ψ and any graph G, a *ψ-optimal drawing of G* is a drawing realizing $\psi(G)$.

3.1 Self-Crossings

It is often pointed out that self-crossings can be eliminated easily using the method suggested by Figure 2 (originally from [13]).

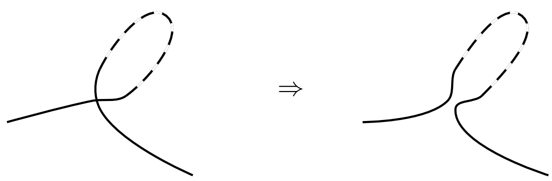


Figure 2: Removing a self-intersection.

This method can be used to show that there are no self-crossings in a ψ-optimal drawing, where $\psi \in \{\text{ocr}, \text{pcr}, \text{cr}\}$ or one of its Rule $+$ or Rule $-$ variants.

For pcr and cr, one could simply shorten the curve by deleting the middle part, but there is a little subtlety here: shortening the curve like that could change the value of ocr (and ocr_- and ocr_+) for the drawing. Neither method works for acr-variants, so we must prove that for acr and acr_-, we can assume that self-crossings do not occur in an optimal drawing.

Lemma 1 *An edge e with self-crossings in a drawing of a graph can be redrawn so that it has no self-crossings, without affecting* $\text{acr}(f, g)$ *for any pair of edges* f, g.

Proof: Suppose an edge $e = uv$ has a self-crossing, as in Figure 3(a). Travel along uv starting at u until you encounter the first self-crossing. Cut the edge at the crossing and reconnect the four ends of the edge respecting orientation without reintroducing the self-crossing (there will always be exactly one way of doing this); see Figure 3(b). The edge now consists of an arc and a closed

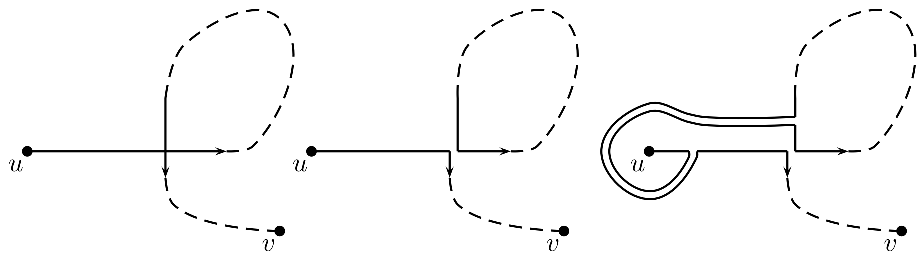


Figure 3: (a) A self-crossing of an oriented edge uv; (b) removing crossing and reconnecting ends; (c), reconnecting the oriented components of uv.

component. Extend a narrow tunnel from the closed component—near the location of the removed crossing—along one side of e toward u. Continue the tunnel around u until it touches e from the other side. At this point the tunnel and e pass each other in opposing direction, so we can cut them and reconnect them so that e is represented by a single curve again (see Figure 3(c)). This does not introduce any self-crossings of e and any crossings it creates with other edges balance each other out so that $\text{acr}(e, g)$ is unchanged for all g. Repeating this operation removes the remaining self-crossings. □

The lemma allows us to conclude that the values of acr and acr_- are not affected by allowing self-crossings or not. In other words, we can assume that acr-optimal and acr_--optimal drawings do not contain self-crossings. We leave open the question whether self-crossings can be removed from acr_+-optimal drawings. [7]

[7] Of course, this question only makes sense if we do not interpret the prohibition of crossings between adjacent edges as a prohibition on self-crossings.

3.2 Bigons

We also need a simple and well-known lemma about *bigons*, a region homeomorphic to a disk that has a boundary formed by two arcs belonging to two different edges. We allow the case that one of the intersection points of the arcs is a common endpoint of the two edges. An *empty* bigon is a bigon that does not contain any vertices in its interior. A segment α of some edge *crosses* the bigon if α lies in the interior of the bigon with the exception of its two endpoints which lie on the boundary; if α has an endpoint on each of the two arcs forming the bigon we call this crossing *transversal*.

Lemma 2 *Let* $\psi \in \{\mathrm{pcr}_-, \mathrm{pcr}, \mathrm{pcr}_+, \mathrm{cr}, \mathrm{cr}_-\}$. *We can assume that a* ψ*-optimal drawing does not contain any empty bigons.*

Proof: Among the ψ-optimal drawings fix one that minimizes the number of (standard) crossings. We claim that this drawing does not contain any empty bigons. If it did, we could pick a minimal (with respect to containment) empty bigon. Since the bigon is empty and minimal, all edge segments crossing the bigon must do so transversally, that is, cross both boundary arcs. But then we can switch the boundary arcs without changing ψ while reducing the total number of standard crossings by two or one (depending on whether the arcs share a common endpoint), contradicting the choice of drawing. □

3.3 Removing Even Crossings

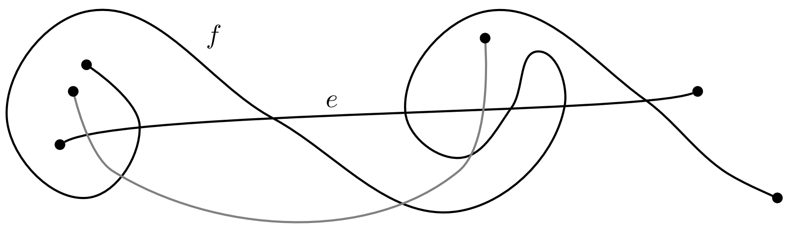
Call an edge *even* if it crosses every other edge an even number of times (perhaps zero times). It is known that crossings with even edges can be removed without introducing odd pairs.

Lemma 3 (Pelsmajer, Schaefer, Štefankovič [12]) *If* D *is a drawing of* G *in the plane and* E_0 *is the set of even edges in* D*, then* G *has a redrawing with the same rotation system, in which all edges in* E_0 *are crossing-free and there are no new odd pairs.*

We include a new (and shorter) proof of this result.

Proof: Fix some $e \in E_0$. We can assume that e has no self-crossings (as explained in Section 3.1). Pick an edge f that crosses e, see Figure 4(a). Since f has to cross e an even number of times, we can match up the crossings of f with e in consecutive pairs along e. Cut f at those crossings, which creates four ends for each matched pair. Reconnect the severed ends by drawing curves along each side of e, according to the matching. Since e has no self-crossings, this creates neither self-crossings of f nor crossings of f with e. See Figure 4(b).

This process does not change the parity of crossings between any pair of edges, however, f may now consist of multiple components; one of those components (the *arc-part* of f) still connects the endpoints of f, while any further components are closed curves without vertices. Repeat this process for all edges that cross e. As a result e will be entirely free of crossings, but edges that



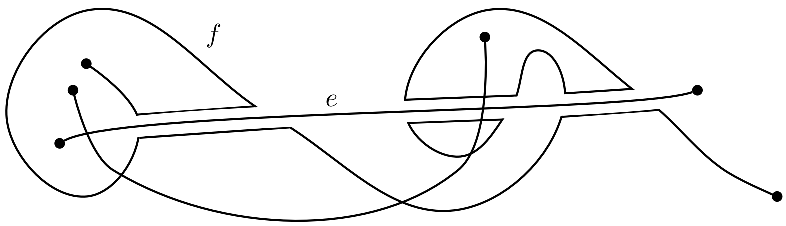


Figure 4: (a) An edge e crossed evenly by an edge f; (b) crossings of f with e are removed, resulting ends of f paired up and reconnected, resulting in an *arc-component* of f and (possibly) closed components of f.

crossed it may now consist of multiple components. Let f be one such edge. If any component of f can be reconnected to the arc-part of f without crossing any crossing-free edges (which now includes e), we do so using two parallel curves that run close to each other; consequently, the parity of crossing between f and any other edge does not change. Any remaining component of f must be separated from the arc-part of f by a cycle of crossing-free edges. Perform this reconnection step for all such edges f.

At this point we would like to drop all remaining closed components (of all edges), but we first have to argue that this does not introduce a new odd pair. Suppose that f and g cross oddly after dropping all their remaining closed components. In other words, the arc-part of f crosses the arc-part of g oddly. Since any closed component of f is separated from the arc-part of f by a crossing-free cycle, it follows that such a closed component cannot cross the arc-part of g (since that has to lie on the same side for the crossing-free cycle as the arc-part of f to cross it). So any remaining closed component of f crosses g evenly (since it can only cross closed components of g). Hence dropping all remaining closed components of f and g does not change their parity of crossing, so f and g were an odd pair to begin with. Thus dropping all closed components does not introduce any new odd pairs. This gives us a drawing of G in which e has been freed of crossings and all previously crossing-free edges have remained crossing-free. Proceeding in this way, we can eliminate all crossings with edges in E_0. □

4 Monotone Crossing Numbers

For each crossing number notion ψ there is, as we saw, a corresponding monotone crossing number mon-ψ which is obtained by restricting the drawings to be monotone drawings of an ordered graph. This restricted model gives us a new handle on crossing numbers.

Example 5 *In [4] we showed that for every n there is an ordered graph G such that* $\text{mon-cr}(G) \geq n$ *and* $\text{mon-ocr}(G) = 1$. *Theorem 2 now immediately implies that there is a graph G with* $\text{ocr}(G) < \text{cr}(G)$*; separating* ocr *from* cr *was a long-standing open problem that was solved earlier in [13] (based on a graph with rotation). A stronger separation was given in [20].*

Below we will see how to separate the three monotone ocr variants mon-ocr_-, mon-ocr, and mon-ocr_+. Our hope is that other separations can be obtained by use of this model. Even if one believes that two crossing number variants do not separate, e.g. pcr = cr (which has been conjectured), the monotone model can serve as a testing ground: by Theorem 2, pcr = cr implies that mon-pcr = mon-cr. So if one believes that pcr = cr, one could try establishing the presumably easier mon-pcr = mon-cr first.[8]

In the monotone model there is no difference between counting odd and algebraic crossings:

Lemma 4 *For every ordered graph G, we have* $\text{mon-acr}(G) = \text{mon-ocr}(G)$, $\text{mon-acr}_+(G) = \text{mon-ocr}_+(G)$, *and* $\text{mon-acr}_-(G) = \text{mon-ocr}_-(G)$.

Proof: Consider two edges e and f in a monotone drawing; if e and f cross, their direction of crossing must alternate between left to right and right to left. So $\text{ocr}(e, f) = \text{acr}(e, f)$ for any two edges of the graph. From this the lemma follows. □

Remark 6 *If desired, we can assume that all the vertices lie on the x-axis: gradually move a vertex v along the vertical line $x = x(v)$ to the x-axis, deforming edges so that they are pushed ahead of v rather than crossed by v; in this way, no crossings are lost or gained. We never actually use this observation in the current paper.*

Lemma 4 simplifies the proof of the translation result for acr-variants considerably. On the downside, it implies that we cannot use monotone crossing numbers to separate odd from algebraic crossing number variants; this is somewhat mitigated by the fact that we already know that some of these variants differ, e.g. ocr and acr, and ocr_- and acr_- [13].

Section 4.2 contains the examples separating the monotone ocr-variants (and thus, by Lemma 4, the monotone acr-variants as well). The examples will be given as weighted graphs, so Section 4.1 contains a short discussion on weighted graphs and how to remove the weights.

[8] The "presumably easier" question whether pcr = cr for two-vertex graphs with rotation is still open as far as we know [13].

4.1 Weighted Crossings

We first generalize the crossing number definitions for graphs with weighted edges. Suppose that G is a graph (with or without ordering) and each edge e has weight $w(e)$. A crossing between edges e and f is assigned *crossing weight* equal to the product $w(e)w(f)$. For a drawing D of G and crossing number ψ, let

$$\psi(D) := \sum w(e)w(f) \cdot \psi(e, f),$$

where the sum is taken over all unordered pairs of edges e, f. With this, we define $\psi(G) := \min_D \psi(D)$ and mon-$\psi(G') := \min_{D'} \psi(D')$, where D ranges over all drawings of G and D' ranges over all monotone drawings of the ordered graph G'. If we let all weights equal 1 these definitions revert back to their original, unweighted versions.

It is an often-used fact that weighted edges can be replaced by multiple edges or parallel paths without changing the crossing number; here we want to show that this remains true for the monotone odd and algebraic crossing numbers and their Rule $+$ and Rule $-$ variants. For the Rule $+$ variants, we cannot prove the result in general, but we can prove it for a special class of graphs, which will be sufficient for the intended application. To state the result for the Rule $+$ variants, we introduce a new notion. We call an ordered graph a *semi-matching* if for every edge uv with $u < v$ either u has right-degree 1 or v has left-degree 1. In other words, every edge is alone within the left of right rotation at one of its ends.[9]

Lemma 5 *Suppose we are given an ordered graph G with edges of positive integer weights.*

- (i) *We can build an ordered graph G' in which each edge of weight w is replaced by w (unweighted) paths (of arbitrary length) so that* mon-$\psi(G) =$ mon-$\psi(G')$ *for* $\psi \in \{\text{ocr}, \text{acr}, \text{ocr}_-, \text{acr}_-\}$.
- (ii) *If G is a semi-matching, then we can build an ordered graph G' in which each edge of weight w is replaced by w (unweighted) paths of length* 2 *so that* mon-$\psi(G) =$ mon-$\psi(G')$ *for* $\psi \in \{\text{ocr}_+, \text{acr}_+\}$.

For the proof of (i) we will use the following lemma.

Lemma 6 *Suppose G is an ordered graph (with multiple edges allowed). Then any edge of G can be subdivided so that the resulting ordered graph G' satisfies* mon-$\psi(G) =$ mon-$\psi(G')$ *for* $\psi \in \{\text{ocr}, \text{acr}, \text{ocr}_-, \text{acr}_-\}$.

Remark 7 *Lemma 6 is trivial for* mon-cr *and* mon-pcr *(subdivide near an endpoint), but it remains open for* mon-cr_-, mon-pcr_-, mon-pcr_+ *(and* cr_-, pcr_-, pcr_+*); these cases do not seem trivial.*

[9] There are various notions of semi-matchings in the literature. While ours is the first for ordered graphs, it does resemble semi-matchings as defined in [5].

Proof of Lemma 6: By Lemma 4, we may assume $\psi \in \{\text{ocr}, \text{ocr}_-\}$. Fix a monotone drawing of G, and choose any edge uv. Subdivide uv with a vertex z, which is added to the drawing of uv near the endpoint u. Then for each edge $e \neq uv$, e crosses zv oddly if and only if e crossed uv oddly, and e does not cross uz at all. Hence mon-ocr is unchanged; mon-ocr$_-$ is also unchanged, unless e shares an endpoint with uv but not with zv, which means that e is incident to u but not v. In this case, we can deform a small section of e (while maintaining its monotonicity) and push it over z; e now crosses zv evenly (and uz oddly, which is fine). Do this for all such e. This yields a drawing with mon-ocr$_-$ no larger than in the initial drawing.

Now consider any monotone drawing of G'. We can erase z from that drawing to obtain a drawing of G. Erasing z does not increase mon-ocr or mon-ocr$_-$. □

Erasing z in the last proof may create a new adjacent crossing, so the proof of Lemma 6 does not work for Rule $+$ or Rule $\pm$ variants of $\psi \in \{\text{ocr}, \text{acr}\}$. We do not know whether a result like Lemma 6 can be proved for ocr_+ and acr_+ in general, but we can handle these two crossing numbers when the graph is a semi-matching.

Proof of Lemma 5: We first prove (i). Let G' be created from G by replacing an edge of weight w in G with w parallel edges. Consider an mon-ψ-optimal drawing D' of G' and suppose that $e_1, \ldots, e_w$ are w parallel edges. Without loss of generality assume that e_1 has the smallest contribution to mon-$\psi(D')$. That is, it minimizes mon-$\psi(D')$ − mon-$\psi(D' - e_i)$, so it maximizes mon-$\psi(D' - e_i)$. Then removing $e_2, \ldots, e_w$ and giving e_1 weight w leads to a drawing D'' of G for which mon-$\psi(D'') \leq$ mon-$\psi(D')$. Hence, mon-$\psi(G) \leq$ mon-$\psi(D'') \leq$ mon-$\psi(D')$ = mon-$\psi(G')$. Since any drawing of G yields a drawing of G' with the same value of mon-ψ, we get mon-$\psi(G') \leq$ mon-$\psi(G)$. Thus, mon-$\psi(G')$ = mon-$\psi(G)$. Repeat this for all weighted edges and apply Lemma 6 to turn every edge into a path of any given length. This completes the proof of (i).

The argument for (ii) is similar, but the construction of G' has to be done slightly more carefully. Let $e = uv$ be an edge of weight w in G. Without loss of generality, $u < v$ and uv is the only edge with u as its left endpoint. Replace e by w parallel paths $P_1, \ldots, P_w$ of length 2 so that all the middle vertices of these paths occur very close to u. Call the resulting graph G'. Consider a mon-ψ-optimal drawing D' of G'. We can assume that mon-$\psi(D' - P_i)$ is maximized by $P_1 = uu'v$ with $u < u' < v$. Removing $P_2, \ldots, P_w$ and assigning a weight of w to both edges of P_1 leads to a drawing D'' for which mon-$\psi(D'') \leq$ mon-$\psi(D')$. Now suppressing u' gives us a drawing of G without affecting mon-ψ, unless we introduce an adjacent crossing. This, however, cannot happen since u has right-degree 1, so $u'v$ cannot be crossed by any edge incident with u. This shows that mon-$\psi(G) \leq$ mon-$\psi(G')$. As earlier, we can use a drawing of G to get a drawing of G' showing that mon-$\psi(G') \leq$ mon-$\psi(G)$, so again mon-$\psi(G')$ = mon-$\psi(G)$. Note that G' as constructed is also a semi-matching, so we can keep replacing weighted edges by parallel paths to obtain a simple, ordered graph G' with mon-$\psi(G')$ = mon-$\psi(G)$. □

4.2 Monotone Separations

For convenience, in this section we say that two monotone drawings of G are *essentially the same* if for all vertex-edge pairs x, uv with $u < x < v$, their above/below relationship is the same in both drawings. Note that mon-ocr$_-$ is the same for two drawings that are essentially the same. If, in addition, the two drawings have the same rotation system, then mon-ocr is the same as well.

4.2.1 Separating mon-ocr$_-$ from mon-ocr

Consider the ordered graph $G = (V, E)$, where $V = \{1, \ldots, 16\}$, and $E = B \cup T$, with bold edges $B = \{13, 24, 2{\cdot}15, 34, 45, 46, 58, 69, 7{\cdot}10, 8{\cdot}11, 10{\cdot}13, 12{\cdot}14, 13{\cdot}14, 13{\cdot}15, 14{\cdot}16\}$ of weight x and thin edges $T = \{35, 47, 9{\cdot}12, 11{\cdot}15\}$ of weight 1; see Figure 5 for drawings of G.[10]

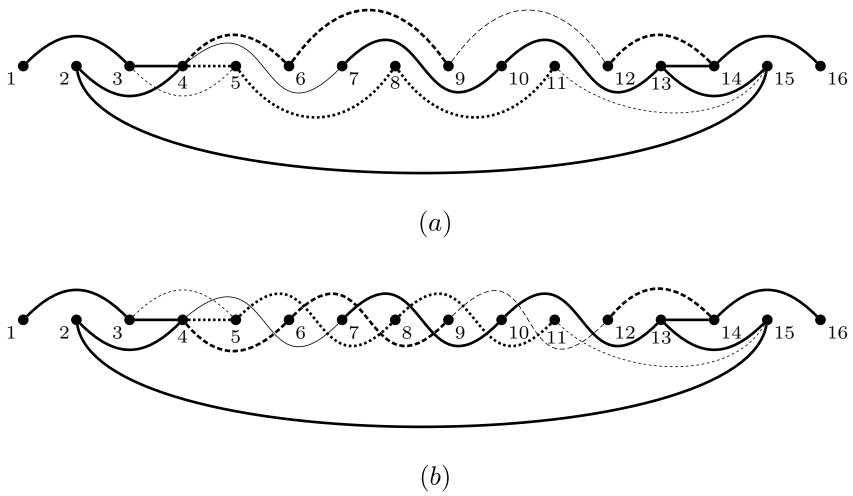


Figure 5: (a) Drawing of the ordered graph G with monotone odd crossing number x; (b) Drawing of the ordered graph G with monotone independent odd crossing number 2. Thick edges have weight x and thin edges weight 1. The different edges styles (solid, dashed, dotted) will be helpful in the proof, but have no other meaning.

Theorem 8 *For the weighted ordered graph G in Figure 5 with $x \geq 3$, we have*

$$\text{mon-ocr}_-(G) = 2 < x = \text{mon-ocr}(G).$$

Combining Theorem 8 with Lemma 5(i) and Theorem 2 immediately yields Theorem 1. Together with Lemma 4 we obtain a proof of Theorem 3.

[10]We use $u{\cdot}v$ instead of uv whenever uv could be ambiguous.

Proof: The drawings in Figure 5 show that mon-ocr$(G) \leq x$ and mon-ocr$_{-}(G) \leq 2$.

Suppose that D is a drawing of G with mon-ocr$_{-}(G) < x$. Then each thick edge must cross all non-adjacent edges evenly. Assuming without loss of generality that 13 passes above 2, we will show that $G - 45$ is drawn essentially the same as shown in Figure 6(c).

First consider the restriction of D to G_1, the subgraph of G shown in Figure 6(a). 13 passes above 2, so 24 passes below 3. Then 2·15 has to pass below 4 and (because of 47, 7·10, 10·13 and 13·14) below 7, 10, 13, 14. But this forces 14·16 above 15 which in turn means that 13·15 passes below 14. Thus the drawing of G_1 is essentially the same as in Figure 6(a).

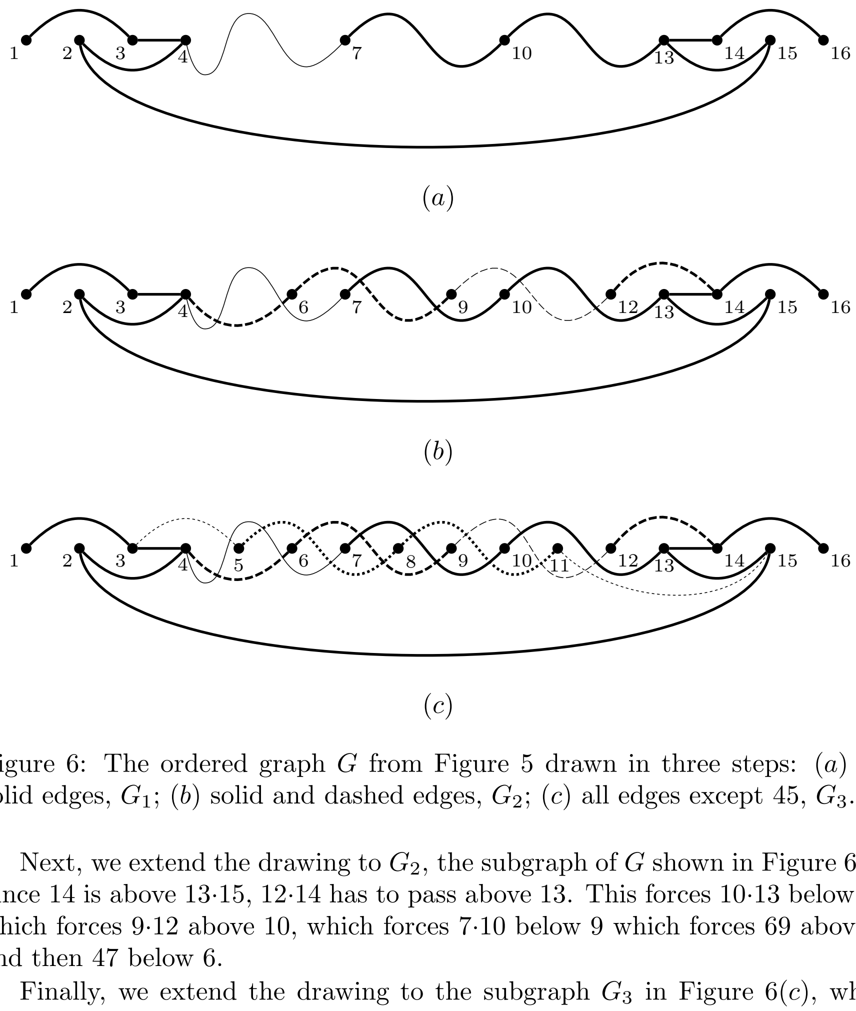


Figure 6: The ordered graph G from Figure 5 drawn in three steps: (a) the solid edges, G_1; (b) solid and dashed edges, G_2; (c) all edges except 45, G_3.

Next, we extend the drawing to G_2, the subgraph of G shown in Figure 6(b). Since 14 is above 13·15, 12·14 has to pass above 13. This forces 10·13 below 12, which forces 9·12 above 10, which forces 7·10 below 9 which forces 69 above 7, and then 47 below 6.

Finally, we extend the drawing to the subgraph G_3 in Figure 6(c), which equals $G - 45$. Since 3 is above 24, the edge 35 must pass above 4. This forces

46 below 5, then 58 above 6, then 69 below 8, then 8·11 above 9, and 11 above 9·12. Since 14·16 is above 15 and 15, 13, 14, 12 is a path, the edge 11·15 must pass below all its intermediate vertices. This forces 11 to be below 10·13, then 8·11 passes below 10, then 8 is below 7·10, then 58 passes below 7, then 5 is below 47.

We have shown that without loss of generality, $G - 45$ is drawn essentially the same as shown in Figure 6(c). Since 5 is below 47 and 4 is below 35, the edges 35 and 47 must cross oddly. Similarly, the edges 9·12 and 11·15 cross oddly, which shows that $\text{mon-ocr}_-(G) \geq 2$.

Suppose that $\text{mon-ocr}(G) < x$; then $\text{mon-ocr}_-(G) < x$ as well, so we still have essentially the same drawing of $G - 45$. Also, since 47 passes below 6, 47 must be below 46 in the right rotation at 4. Adding 45 to the drawing forces an odd crossing between 45 and 47 or between 45 and 46; thus, there is no drawing of G with $\text{mon-ocr}(G) < x$. □

Remark 9 *In the conference version of this paper [3], we presented a weighted ordered graph G on 7 vertices that achieves the weaker separation* $\text{mon-ocr}_-(G) = 3x < x^2 + x = \text{mon-ocr}(G)$ *for every $x \geq 3$. Experimental evidence suggests that there is no smaller weighted graph that gives a separation. We do not have any interesting bounds on the smallest unweighted graph for which* mon-ocr_- *and* mon-ocr *separate.*

4.2.2 Separating mon-ocr from mon-ocr_+

Consider the ordered graph $G = (V, E)$, where $V = \{1, \ldots, 14\}$, and $E = B \cup M \cup T$, with bold edges $B = \{1{\cdot}11, 34, 39, 3{\cdot}14, 45, 67, 6{\cdot}12, 7{\cdot}11, 8{\cdot}10, 9{\cdot}10\}$ of weight x^3, medium edges $M = \{36, 5{\cdot}13\}$ of weight x and thin edges $T = \{25, 10{\cdot}11\}$ of weight 1, see Figure 7.

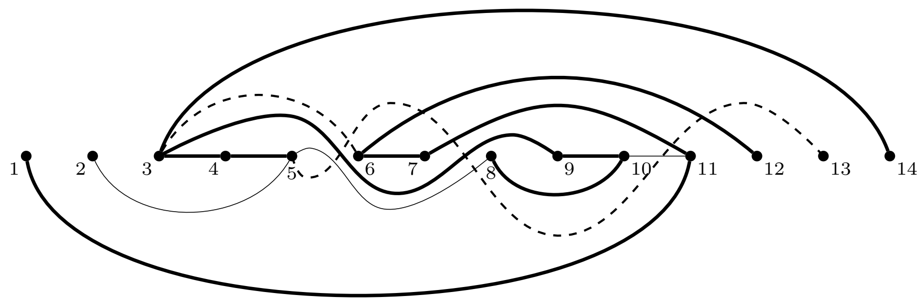


Figure 7: A weighted ordered semi-matching G with $\text{mon-ocr}(G) < \text{mon-ocr}_+(G)$; thick solid (bold) edges have weight x^3, thick dashed (medium) edges have weight x, and the thin solid (thin) edges have weight 1.

Theorem 10 *For the weighted ordered graph G in Figure 7 with $x \geq 2$, we have*

$$\text{mon-ocr}(G) \leq x^2 + x < x^3 = \text{mon-ocr}_+(G).$$

Since the graph G of Theorem 10 is a semi-matching, we can apply Lemma 5(ii) to obtain the following corollary; the mon-acr part follows from Lemma 4.

Corollary 11 *For every n, there is a (simple) graph G with* $\text{mon-ocr}(G)+n < \text{mon-ocr}_+(G)$ *and* $\text{mon-acr}(G)+n < \text{mon-acr}_+(G)$.

Proof of Theorem 10: The drawing in Figure 7 shows that $\text{mon-ocr}(G) \leq x^2+x$. Consider an mon-ocr$_+$-optimal drawing of G. Suppose $\text{mon-ocr}_+(G) < x^3$. Then no edge of weight x^3 can be involved in an odd crossing. Assume, without loss of generality, that 7·11 passes above 10. Then it passes above 9 (because of 9·10) and 8 (because of 8·10) as well. This implies that 58 has to pass below 7 (because of 7·11) and thus below 6 (because of 67). Now 6·12 cannot pass below 7. If it did, it would also pass below 11 (7·11), 10 (10·11), 8 (8·10), and thus would cross 58 oddly, which it cannot. Hence 6·12 passes above all vertices between its endpoints. But then 1·11 has to pass below 6 (6·12) and thus below all its intermediate vertices (as 6 is connected to all the intermediate points by paths only using edges in the range 2 through 10). A similar argument shows that 3·14 has to lie above all its intermediate vertices (starting with 11). At this point, the remaining edges fall into place: 25 runs below 3 and 4, so 39 has to run above 5, 4, below 6 and 7 (because of 6·12 and 7·11 that end to the right of 39), and above 8 (58), so 8·10 passes below 9, and 36 runs above 4 and 5. Consider the final edge 5·13. It has to pass above 11 (1·11), and thus above 7 (7·11), 6 (67), and 12 (6·12). However, because 39 passes above 5, 5·13 has to pass below 9 and thus below 10 (9·10), and 8 (8·10). So 5·13 must leave 5 below 58, but pass above 6, while 58 passes below 6. Hence 58 and 5·13 have to cross which is not allowed in a Rule + drawing. □

5 Translation Result

Given an ordered graph $G=(V,E)$ with $V=\{v_1 < v_2 \cdots < v_n\}$ let G' be obtained from G by adding the following framework: start with a cycle C_{4n} formed by two paths $s, u_1, x_1, u_2, \ldots, x_{n-1}, u_n, t$ and $s, w_1, y_1, w_2, \ldots, y_{n-1}, w_n, t$; call this the *outer framework*. Into the outer framework we insert n paths $Q_i = u_i v_i w_i$, $1 \leq i \leq n$; call this the *inner framework*. Assign a weight of $w_I = n^4+1$ to the edges in the inner framework and a weight of $w_O = n^4 + n^3 w_I + 1$ to the edges in the outer framework. Edges originally in G have weight 1. From the weighted graph G' we obtain the unweighted graph G'' by replacing each edge of weight $w > 1$ in G' by w copies of P_3 (3-vertex paths) with the same endpoints. See Figure 8 for an illustration of a graph with added outer and inner framework.

Lemma 7 *With G'' as defined above we have* $\psi(G'') = \text{mon-}\psi(G) + c$ *for any connected graph G, where* $\psi \in \{\text{ocr}, \text{ocr}_-, \text{acr}, \text{acr}_-, \text{pcr}, \text{pcr}_-, \text{pcr}_+, \text{cr}, \text{cr}_-\}$ *and* $c = w_I \sum_{v_i v_j \in E(G), i<j} (j-i-1)$.

Theorem 2 is an immediate consequence of Lemma 7.

Remark 12 *Variants* ocr_+ *and* acr_+ *are noticeably absent from Lemma 7, indeed, the translation result itself (not just the proof) fails with the current framework. We have experimented with other frameworks (based on triangles which cannot self-intersect in drawings conforming to Rule +), but it is very difficult to control* ocr_+*. One other problem we face anew for* ocr_+ *and* acr_+ *is weighted edges. It seems non-trivial to replace a weighted edge with some graph construct that does not affect* ocr_+*, certainly parallel paths will not be sufficient. All of this is unfortunate, because we do have a monotone separation of* ocr *and* ocr_+ *(Theorem 10). As an alternative approach one may try separating the monotone versions of* ocr *and* $\mathrm{ocr}_\pm$*. For* $\mathrm{ocr}_\pm$ *the translation result can be made to work with the current framework. However, in the example from Theorem 10,* $\text{mon-ocr} = \text{mon-ocr}_\pm$*, so this still does not give us a separation. Indeed, it is tempting to conjecture that* $\text{mon-ocr} = \text{mon-ocr}_\pm$*.*

In the proof of Lemma 7 we stress the similarities between the 9 different cases by arguing them in parallel, although the arguments differ significantly in the details, since redrawing methods for ocr, say, will not be appropriate for pcr (or vice versa). As many of the redrawing tools employed here have been used in earlier papers (for example [12, 14, 15]), the main challenge lies in identifying the right strategy to deal with the drawings.

Proof of Lemma 7: First note that $\psi(G'') \leq \text{mon-}\psi(G)+c$ is immediate: take a monotone drawing realizing $\text{mon-}\psi(G)$ and overlay it with a planar drawing of the framework, call the resulting drawing D'. Then $\psi(D') = \text{mon-}\psi(G)+c$ since all newly added crossings are on pairs of independent edges that cross exactly once. From D' we can obtain a drawing D'' of G'' by replacing the weighted edges in the drawing by parallel P_3s; still we only have single crossings between independent edges, so $\psi(D'') = \psi(D')$. Hence, $\psi(G'') \leq \psi(D'') = \text{mon-}\psi(G)+c$.

It remains to prove $\psi(G'') \geq \text{mon-}\psi(G) + c$. It is easy to see that $\psi(G'') \geq \psi(G')$: fix a ψ-optimal drawing D'' of G''. Consider the w parallel P_3s that were used to replace an edge of weight w in G'. Pick one of these paths P that contributes minimally to $\psi(D'')$ (in the sense that $\psi(D'') - \psi(D'' - P)$ is minimized). Now redraw the remaining $w - 1$ paths to run very close to P and without crossing each other. This redrawing cannot increase $\psi(D'')$. But now we can bundle the parallel paths into a single weighted edge to obtain a drawing D' of G' with $\psi(D') \leq \psi(G'')$. So $\psi(G') \leq \psi(G'')$.[11]

Hence, to establish the lemma it is sufficient to show that $\psi(G') \geq \text{mon-}\psi(G)+c$. We postpone $\psi \in \{\mathrm{acr}, \mathrm{acr}_-\}$ to the end of the proof. We proceed in three steps; first, we show that there is a ψ-optimal drawing of G' in which the edges of the outer framework are crossing-free. In the second step we show that we can also assume that the edges of the inner framework do not cross each other. In the third step we show that from such a drawing of G', we can construct a monotone drawing of G with at most $\psi(G') - c$ crossings. It follows that $\text{mon-}\psi(G) \leq \psi(G') - c$.

[11] For the standard crossing number, arguments like this go back at least as far as Kainen [6].

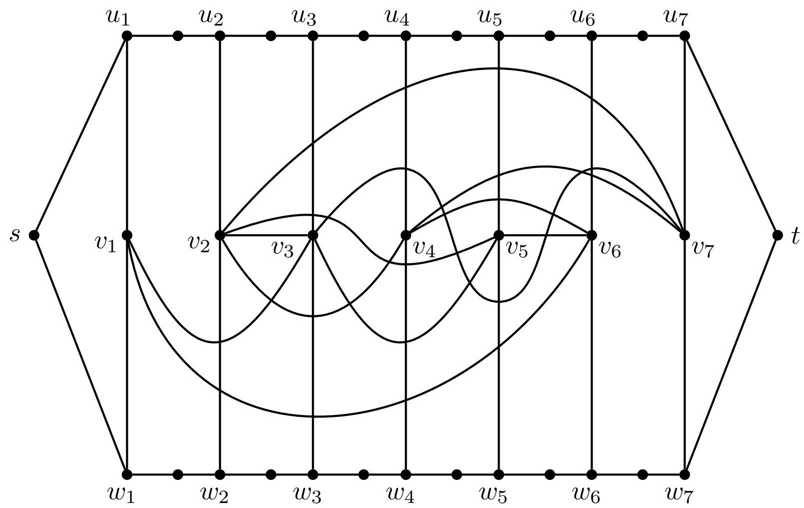


Figure 8: A graph with outer and inner framework added.

First Step. Fix a ψ-optimal drawing of G'. For $\psi \in \{\text{pcr}, \text{pcr}_+, \text{cr}\}$ the claim is immediate: any edge crossing an edge of the outer framework contributes at least w_O to $\psi(G')$. However, we already proved that $\psi(G') \leq \text{mon-}\psi(G) + c \leq n^4 + n^3 w_I < w_O$, so all edges of the outer framework must be crossing-free. If $\psi = \text{ocr}$ then edges of the outer framework cannot be part of an odd pair, since any such pair would contribute w_O to ocr and, as above, $\psi(G') < w_O$. So all the edges in the outer framework are even. We can then apply Lemma 3 to make all edges in the outer framework crossing-free without introducing any new pair of edges crossing oddly—in particular, ψ does not increase. For $\psi = \text{ocr}_-$, edges of the outer framework cannot be part of an independent odd pair, so all odd pairs including these edges have to be pairs of adjacent edges. However, all vertices in the outer framework have degree 2 or 3, so we can redraw the edges near these vertices so that all the edges in the outer framework are even.[12] We then proceed as in the case of ocr.

This leaves the case $\psi \in \{\text{pcr}_-, \text{cr}_-\}$. Let e be an edge of the outer framework. As above we can argue that any crossings with e must be with edges adjacent to e. Choose a closed curve γ that surrounds e closely (we can choose γ as the boundary of an ε-neighborhood of e for sufficiently small ε). Then γ can only cross the three edges incident to e and it has to cross all three of them (since each has an endpoint outside of the region bounded by γ). For each of these three, select the endpoint on the exterior of γ and the arc from it to the first intersection with γ, and erase the rest of the edge. Erase e and its endpoints as well. Now that the interior of γ is empty, we can redraw e and

[12]If two edges are consecutive in the rotation at a common endpoint, their crossing parity can be flipped swapping their ends at the vertex. At a vertex of degree 2 or 3, every pair of edges is consecutive in its rotation, so we can make them cross evenly pairwise. This is not true for vertices of higher degree.

the ends of the three edges adjacent to e within the interior of γ, without any crossings. Repeating this procedure, we can ensure that all edges of the outer framework are free of crossings.

This completes the first step: we can now assume (by modifying our ψ-optimal drawing, if necessary) that the outer framework is entirely free of crossings. Since G is connected, all vertices of G must lie in the same face of C_{4n}; without loss of generality, it is the inner face. Since every edge not in the outer framework is incident to a vertex of G, this also implies that all edges lie in the inner face and the outer face is empty.

Second Step. We show that we can assume that edges of the inner framework do not cross each other. Recall that $Q_i = u_iv_iw_i$ are the paths of the inner framework, with v_i in G and endpoints u_i and w_i on C_{4n}, for $1 \leq i \leq n$.

For $\psi \in \{\text{pcr}, \text{pcr}_+, \text{cr}\}$ the claim is immediate since any such crossing would contribute $w_I^2 = w_I(n^4+1) = n^4w_I + w_I > n^3w_I + n^4 + 1 = w_O$ to $\psi(G')$, but we already know that $\psi(G') \leq w_O$.

For $\psi \in \{\text{pcr}_-, \text{cr}_-\}$ two independent inner framework edges cannot cross each other, since they would contribute w_I^2 to $\psi(G')$. However, it is possible that two adjacent inner framework edges cross each other; say u_iv_i crosses v_iw_i. Then they form a bigon (which may include v_i). Note that this bigon cannot contain any vertex in its interior since any such vertex would have a path of framework edges to s or t that avoids $\{u_i, v_i, w_i\}$, causing a contribution to pcr_- or cr_- of at least w_I^2 (which is too much). By Lemma 2 we can assume that there is no such bigon. Hence, no two adjacent edges of the inner framework cross when $\psi \in \{\text{pcr}_-, \text{cr}_-\}$.

For $\psi \in \{\text{ocr}, \text{ocr}_-\}$ we first argue that any two edges of the inner framework cross an even number of times. For $\psi = \text{ocr}$ this is true since an odd pair would contribute w_I^2. For $\psi = \text{ocr}_-$, edges u_iv_i and v_iw_i could cross oddly for some i. In that case, we redraw u_iv_i near v_i so that it wraps once around v_i (very close to v_i); this does not affect ocr_- and ensures that u_iv_i and v_iw_i cross evenly. Thus, we may assume that for $\psi \in \{\text{ocr}, \text{ocr}_-\}$, any two edges of the inner framework cross an even number of times. We next show how to remove even crossings between edges of the inner framework.

Let us consider $Q_1 = u_1v_1w_1$. Let e be an edge of the inner framework that crosses u_1v_1 (we allow the case $e = v_1w_1$). Push any crossings of e along u_1v_1 toward v_1 and then over v_1 (see Figure 9). Performing this for all such edges e of the inner framework leaves u_1v_1 free of crossings with edges of the inner framework. Since each inner framework edge e crossed u_1v_1 an even number of times, e is pushed over v_1 an even number of times, so the value of ψ of the drawing does not change. During the process, we may have introduced self-crossings of v_1w_1 which we remove (as discussed in Section 3.1) without affecting ψ. At this point, u_1v_1 crosses no edge of the inner framework and v_1w_1 crosses every other edge of the inner framework evenly. Q_1 does not cross itself, so it divides the interior of C_{4n} into two regions; let R_1 be the region bounded by $su_1v_1w_1s$. The vertex t lies outside of R_1, and every vertex is connected to t by a path with edges of weight at least w_I; such a path cannot end in R_1 since it would cross it boundary oddly, contributing at least w_I^2 to ψ. Therefore R_1 contains no vertices.

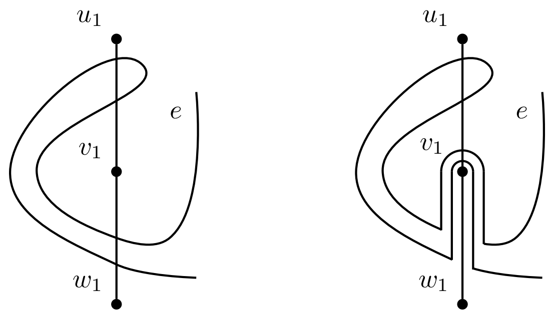


Figure 9: Pushing crossings of e with u_1v_1 along u_1v_1 toward v_1 and then over v_1.

We now remove crossings of inner framework edges with v_1w_1 using the procedure from Lemma 3. Cut each edge e of the inner framework where it crosses v_1w_1. We can partition the crossings of e and v_1w_1 into consecutive pairs since e crosses v_1w_1 an even number of times. For each pair we add curves that run along each side of v_1w_1 to reconnect the severed ends. Thus, e is replaced by a curve which may have more than one component, one of which is an arc connecting the endpoints of e while any remaining components are closed curves. None of the components intersect v_1w_1. The components lying in R_1 are all closed curves. Since there is no vertex within R_1, all those closed curves are even, so they can be deleted without affecting ψ. The closed components on the other side of Q_1 can be reconnected to the arc part of e by using two parallel curves that avoid $Q_1 \cup C_{4n}$. After performing this process for all such edges e, $Q_1 = u_1v_1w_1$ does not have any crossing with edges of the inner framework. We repeat this argument with Q_2, Q_3, and so on, making Q_i free of self-crossing and letting R_i be the interior face of $C_{4n} \cup Q_i$ that is incident to s, eventually showing that none of the Q_is have crossings with any edges of the inner framework. This completes the second step.

Hence, for the third step, we can assume that every crossing is between two edges of G or between an edge of G and an edge of the inner framework.

Third Step. At this point, let us deform the whole drawing so that $C_{4n} \cup \{Q_1, Q_n\} - \{s, t\}$ is a rectangle and all the Q_i are parallel straight-line segments orthogonal to the outer framework.

For $\psi \in \{\text{pcr}_-, \text{pcr}, \text{pcr}_+, \text{cr}, \text{cr}_-\}$ we argue as follows: a G-edge e connecting v_i to v_j must cross all Q_k with $i < k < j$, so the edges of G contribute at least c to $\psi(G')$. This leaves $\psi(G') - c \leq \text{mon-}\psi(G) \leq n^4 < w_I$ crossings counting towards $\psi(G')$. Since a crossing with an edge of the inner framework contributes at least w_I to $\psi(D')$ the only remaining crossings with inner framework edges are when an edge v_iv_j crosses u_iv_i, v_iw_i, u_jv_j, or v_jw_j and $\psi \in \{\text{pcr}_-, \text{cr}_-\}$ (because adjacent crossings do not count in these cases). Lemma 2 allows us to assume that v_iv_j forms no empty bigon with any edge. Follow v_iv_j as a curve from v_i to v_j and let L be the list of indices of Q_ks that are crossed, in order. We will show that $L = i+1, i+2, \ldots, j-1$, which implies that $|L| = j-i-1$.

Due to the drawing of the outer framework, consecutive indices in L differ by at most 1. So to establish the claim it suffices to show that (i) no index occurs twice and (ii) i and j are not in L. If there is a k that appears twice in L, pick such a k that reoccurs in the smallest number of steps. The two crossings of v_iv_j with Q_k cannot be consecutive: v_iv_j cannot have two consecutive crossings with u_kv_k or v_kw_k since that would form an empty bigon. And it cannot cross both u_kv_k and v_kw_k since that would contribute an extra w_I to $\psi(D')$, which is not possible. So the crossing with Q_k must be followed by a crossing with Q_{k+1} or Q_{k-1}. But if v_iv_j crosses Q_k again, it must first cross Q_{k+1} or Q_{k-1} again, contradicting the choice of Q_k as the quickest reoccurring crossing. It remains to show (ii). The first index of L cannot be i since in that case v_iv_j would form a bigon with Q_i. But then Q_i cannot occur in L at all, since that would require that earlier index occurs twice, which we already excluded as a possibility. Similarly, Q_j cannot be in L. Hence, $L = i+1, i+2, \ldots, j-1$, so v_iv_j crosses $Q_{i+1}, \ldots, Q_{j-1}$ once each in that order. The actual behavior of v_iv_j between two neighboring Q_ks is irrelevant: within each such region we can replace v_iv_j by a straight-line segment connecting its crossings between neighboring Q_ks, and it does not increase the value of ψ. Removing the framework (which contributes at least c to ψ) results in a monotone drawing of G, proving that mon-$\psi(G) \leq \psi(G') - c$, which is what we had to prove.

Next, suppose that $\psi \in \{\text{ocr}, \text{ocr}_-\}$. In these cases, a G-edge e connecting v_i to v_j must cross all Q_k with $i < k < j$ oddly, contributing at least c to ψ. This leaves at most $\psi(G') - c \leq$ mon-$\psi(G) < w_I$ in $\psi(G')$ unaccounted for. So there cannot be any independent odd pairs that include an edge of the inner framework except those absolutely necessary to connect the endpoints of every edge in G. Odd pairs that include inner framework edges may occur in the ocr_- case (where such odd pairs do not contribute to the crossing number) when an edge v_iv_j $(i < j)$ crosses an adjacent inner framework edge u_iv_i, v_iw_i, u_jv_j, or v_jw_j oddly. In this case we redraw v_iv_j near each of its endpoints (if necessary) so that the ends of v_iv_j at v_i and v_j lie between Q_i and Q_j; this does not affect ocr_- and results in v_iv_j crossing both Q_i and Q_j an even number of times. It is possible at this point that v_iv_j crosses both u_kv_k and v_kw_k oddly (where k is i or j). In that case we wrap v_iv_j once around v_k. This does not affect ocr_- and ensures that v_iv_j crosses both u_kv_k and v_kw_k evenly.

Thus for $\psi \in \{\text{ocr}, \text{ocr}_-\}$ we can now assume that if an edge $e = v_iv_j$ crosses u_kv_k or v_kw_k with $k \leq i$ or $k \geq j$ it must do so evenly. As we did above for the inner framework edges (as seen in Figure 9), we push all crossings of e with u_kv_k along u_kv_k and over v_k to v_kw_k so that u_kv_k does not cross e at all; pushing e off u_kv_k does not affect the value of ψ, since e crossed u_kv_k evenly. For all $k \leq i$ and $k \geq j$, cut e at each crossing with v_kw_k, partition the crossings into pairs, and reconnect severed ends of e on both sides of v_kw_k according to the pairs.

Closed components of e between Q_i and Q_j can be reconnected to the arc-component of e without affecting ψ. Every other closed component of e is entirely contained in a region which does not contain a vertex, so such components are even and can be dropped without affecting ψ. In the end, all of e lies in the region in C_{4n} between Q_i and Q_j.

For any $i < k < j$, since e crosses Q_k oddly, we either have $\text{ocr}(e, u_k v_k) = 0$ and $\text{ocr}(e, v_k w_k) = w_I$ or $\text{ocr}(e, u_k v_k) = w_I$ and $\text{ocr}(e, v_k w_k) = 0$. For every k push all crossings of e with Q_k from its edge with $\text{ocr} = 0$ to the other edge, which does not affect the value of ψ; then e avoids one of the edges of Q_k for every $i < k < j$. Let e' be any curve in the region in C_{4n} between Q_i and Q_j that shares ends with e (here, an end is an endpoint together with a small, crossing-free part of the edge incident to the endpoint); and avoids the same edge in each Q_k that e avoids. Then $\text{ocr}(e, g) = \text{ocr}(e', g)$ for every edge g (other than e), since e can be continuously deformed to e' without passing over any vertex. In particular, we can replace e with a monotone polygonal arc without changing the value of ψ. Repeating this for all edges of G gives us a monotone drawing of G with mon-ψ crossings. This completes the argument for $\psi \in \{\text{ocr}, \text{ocr}_-\}$.

We still need to derive the result for acr and acr_-, but this is now easy. Consider $\psi = \text{acr}$, for example. Then

$$\begin{aligned}
\text{acr}(G') &\geq \text{ocr}(G') && \text{since ocr} \leq \text{acr} \\
&= \text{mon-ocr}(G) + c && \text{this is the ocr-case} \\
&= \text{mon-acr}(G) + c && \text{by Lemma 4,}
\end{aligned}$$

and likewise for acr_-. This completes the proof. □

Acknowledgments We would like to thank the anonymous referees for useful feedback that improved the presentation of the paper.

References

[1] P. Brass, W. Moser, and J. Pach. *Research Problems in Discrete Geometry.* Springer, New York, 2005.

[2] C. Chojnacki (Haim Hanani). Über wesentlich unplättbare Kurven im dreidimensionalen Raume. *Fundamenta Mathematicae*, 23:135–142, 1934.

[3] R. Fulek, M. Pelsmajer, M. Schaefer, and D. Štefankovič. Adjacent crossings do matter. In M. J. van Kreveld and B. Speckmann, editors, *Graph Drawing*, volume 7034 of *Lecture Notes in Computer Science*, pages 343–354. Springer, 2011.

[4] R. Fulek, M. Pelsmajer, M. Schaefer, and D. Štefankovič. Hanani-Tutte, monotone drawings, and level-planarity. In J. Pach, editor, *Thirty Essays on Geometric Graph Theory*, pages xx+430. Springer, 2012.

[5] N. J. A. Harvey, R. E. Ladner, L. Lovász, and T. Tamir. Semi-matchings for bipartite graphs and load balancing. In *Algorithms and data structures*, volume 2748 of *Lecture Notes in Comput. Sci.*, pages 294–306. Springer, Berlin, 2003.

[6] P. C. Kainen. A lower bound for crossing numbers of graphs with applications to K_n, $K_{p,q}$, and $Q(d)$. *J. Combinatorial Theory Ser. B*, 12:287–298, 1972.

[7] J. Pach and G. Tóth. Thirteen problems on crossing numbers. *Geombinatorics*, 9(4):194–207, 2000.

[8] J. Pach and G. Tóth. Which crossing number is it anyway? *J. Combin. Theory Ser. B*, 80(2):225–246, 2000.

[9] J. Pach and G. Tóth. Monotone drawings of planar graphs. *J. Graph Theory*, 46(1):39–47, 2004.

[10] J. Pach and G. Tóth. Monotone crossing number. In *Graph drawing 2011*, to appear.

[11] M. J. Pelsmajer, M. Schaefer, and D. Stasi. Strong Hanani–Tutte on the projective plane. *SIAM Journal on Discrete Mathematics*, 23(3):1317–1323, 2009.

[12] M. J. Pelsmajer, M. Schaefer, and D. Štefankovič. Removing even crossings. *J. Combin. Theory Ser. B*, 97(4):489–500, 2007.

[13] M. J. Pelsmajer, M. Schaefer, and D. Štefankovič. Odd crossing number and crossing number are not the same. *Discrete Comput. Geom.*, 39(1):442–454, 2008.

[14] M. J. Pelsmajer, M. Schaefer, and D. Štefankovič. Removing even crossings on surfaces. *European J. Combin.*, 30(7):1704–1717, 2009.

[15] M. J. Pelsmajer, M. Schaefer, and D. Štefankovič. Removing independently even crossings. *SIAM Journal on Discrete Mathematics*, 24(2):379–393, 2010.

[16] M. J. Pelsmajer, M. Schaefer, and D. Štefankovič. Crossing numbers of graphs with rotation systems. *Algorithmica*, 60:679–702, 2011. 10.1007/s00453-009-9343-y.

[17] L. A. Székely. A successful concept for measuring non-planarity of graphs: the crossing number. *Discrete Math.*, 276(1-3):331–352, 2004.

[18] L. A. Székely. Progress on crossing number problems. In P. Vojtás, M. Bieliková, B. Charron-Bost, and O. Sýkora, editors, *SOFSEM 2005: Theory and Practice of Computer Science, 31st Conference on Current Trends in Theory and Practice of Computer Science, Liptovský Ján, Slovakia, January 22-28, 2005, Proceedings*, volume 3381 of *Lecture Notes in Computer Science*, pages 53–61. Springer, 2005.

[19] L. A. Székely. An optimality criterion for the crossing number. *Ars Math. Contemp.*, 1(1):32–37, 2008.

[20] G. Tóth. Note on the pair-crossing number and the odd-crossing number. *Discrete Comput. Geom.*, 39(4):791–799, 2008.

[21] G. Tóth. A better bound for pair-crossing number. In J. Pach, editor, *Thirty Essays on Geometric Graph Theory*, pages xx+430. Springer, 2012.

[22] W. T. Tutte. Toward a theory of crossing numbers. *J. Combinatorial Theory*, 8:45–53, 1970.

[23] P. Valtr. On the pair-crossing number. In *Combinatorial and Computational Geometry*, volume 52 of *Math. Sci. Res. Inst. Publ.*, pages 569–575. Cambridge University Press, Cambridge, 2005.