

# Distributed Kernel Ridge Regression with Communications

**Shao-Bo Lin**

SBLIN1983@GMAIL.COM

*Center of Intelligent Decision-Making and Machine Learning  
School of Management  
Xi'an Jiaotong University  
Xi'an, China*

**Di Wang\***

WANGDI@WZU.EDU.CN

*Center of Intelligent Decision-Making and Machine Learning  
School of Management  
Xi'an Jiaotong University  
Xi'an, China*

**Ding-Xuan Zhou**

MAZHOU@CITYU.EDU.HK

*School of Data Science and Department of Mathematics  
City University of Hong Kong  
Kowloon, Hong Kong, China*

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## Abstract

This paper focuses on generalization performance analysis for distributed algorithms in the framework of learning theory. Taking distributed kernel ridge regression (DKRR) for example, we succeed in deriving its optimal learning rates in expectation and providing theoretically optimal ranges of the number of local processors. Due to the gap between theory and experiments, we also deduce optimal learning rates for DKRR in probability to essentially reflect the generalization performance and limitations of DKRR. Furthermore, we propose a communication strategy to improve the learning performance of DKRR and demonstrate the power of communications in DKRR via both theoretical assessments and numerical experiments.

**Keywords:** learning theory, distributed learning, kernel ridge regression, communication

## 1. Introduction

Commonly in this era, data of huge size are stored in numerous machines and cannot be shared for protecting data privacy. Typical examples include clinical data in medicine where medical data are collected in different hospitals and financial data in business where commercial data are generated in different companies. These distributively stored data bring a new challenge for machine learning in the sense that every one would like to use the other's data but is unwilling to share his own data. Nonparametric distributed learning (NDL) (Zhang et al., 2015; Lin et al., 2017) presents a preferable approach to conquer this challenge by means of combining the prediction results from many local processors without sharing individual data each other.

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\*. Corresponding author

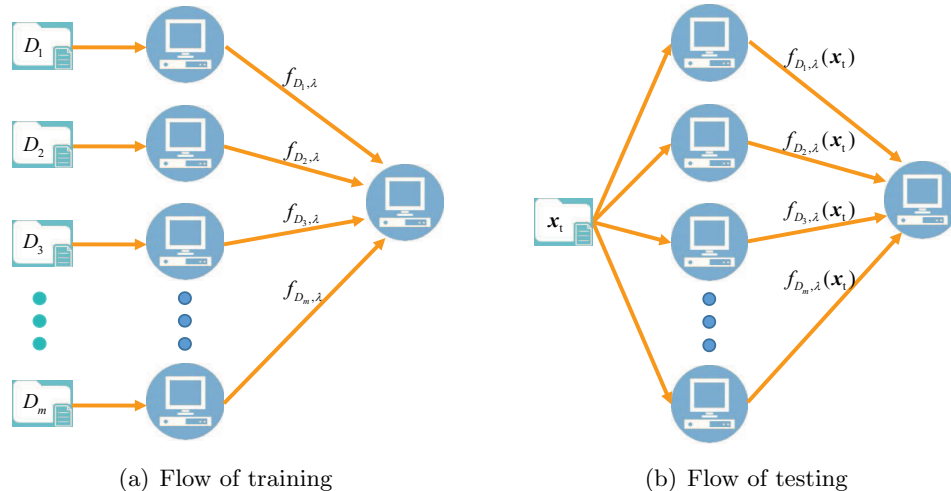


Figure 1: Training and testing flows of the divide-and-conquer learning.

There are three ingredients of NDL: local processing, communication, and synthesization. The local processing issue refers to applying a particular learning algorithm such as the kernel ridge regression (Zhang et al., 2015), local average regression (Chang et al., 2017), multi-penalty regularization (Guo et al., 2019), coefficient-based regularization (Pang and Sun, 2018; Shi, 2019), and spectral algorithms (Guo et al., 2017; Mücke and Blanchard, 2018; Lin et al., 2020) to tackle the data subset in a local machine and produce a local estimator. The communication issue focuses on exchanging exclusive information such as the data (Bellet et al., 2015), gradients (Zeng and Yin, 2018) and local estimator (Huang and Huo, 2019) between different local machines. The synthesization issue devotes to producing a global estimator by combining local estimators and communicated information on the global machine, typical strategies of which are the majority voting (Mann et al., 2009), weighted average (Chang et al., 2017) and gradient-based algorithms (Bellet et al., 2015).

One of the most popular NDL approaches is the divide-and-conquer learning, in which the communication is not required and the weighted average is used in the synthesization issue. Figure 1 presents the training and testing flows of the divide-and-conquer learning. As shown in Figure 1, to give a prediction of a query point  $x_t$ , only a real number  $f_{D_j, \lambda}(x_t)$  is communicated to the global machine, which succeeds in protecting the data privacy of each local machine. Generalization performances of the divide-and-conquer learning have been proved to be similar to running the corresponding algorithm processing the whole data on a single but large enough machine (Zhang et al., 2015; Chang et al., 2017; Lin et al., 2017; Guo et al., 2017; Mücke and Blanchard, 2018; Pang and Sun, 2018; Shi, 2019; Lin et al., 2020). The theoretical problem is, however, that there is a strict restriction on the number of local machines to guarantee the optimal generalization performance, which is difficult to be satisfied in real applications.

In this paper, taking the DKRR to be the specific algorithm, we aim at enlarging the number of local machines by considering communications among different local machines. There are three purposes in our study. At first, we improve the existing results for DKRR

in expectation in the sense of removing the eigen-function assumption in (Zhang et al., 2015) and relaxing the regularity assumption in (Lin et al., 2017). Our main tool to achieve this goal is a tight operator product estimate based on a recently developed concentration inequality for self-adjoint operators (Minsker, 2017). These estimates improve the results in (Lin et al., 2017; Guo et al., 2017), where the second order decomposition of operator differences and a classical concentration inequality in (Pinelis, 1994) are used.

Since generalization error estimates in probability quantify the generalization performance of DKRR in a single trial while estimates in expectation describe the average error, it is highly desired to deduce optimal learning rates for DKRR in probability. However, almost all existing results for DKRR are established in expectation (Zhang et al., 2015; Lin et al., 2017; Chang et al., 2017). The main reason is that the power of averaging in DKRR can be directly reflected by expectation, provided the samples are assumed to be drawn independently and identically according to some distribution. Our second purpose is to derive optimal learning rates for DKRR in probability, by means of a novel error decomposition technique motivated by Lin and Zhou (2018). Since the advantage of averaging cannot be directly used, the restriction on the number of local machines is a bit strict. Our estimates in probability support numerical observations that cannot be seen from the estimates in expectation.

Our last purpose is to develop a communication strategy to improve the performance of DKRR. Combining the recently developed integral approach (Lin et al., 2017; Guo et al., 2017) with a Newton-Raphson iteration, we design a communication strategy to enlarge the number of local machines to guarantee optimal learning rates for DKRR. Our basic idea is to communicate gradients of each local machine and use the Newton-Raphson iteration in the global machine to synthesize the global machine. Both theoretical analysis and numerical results are conducted to verify the power of communications. Theoretically, we prove that, in the sense of probability, DKRR with communications can reach the optimal learning rates, while the restriction to the number of local machines is the same as that in expectation. Numerically, we exhibit that communications enlarge the number of local machines of DKRR and thus essentially improve its learning performance.

The rest of the paper is organized as follows. In the next section, we present the communication strategy as well as its motivation. In Section 3, theoretical results including optimal learning rates for DKRR in expectation, optimal learning rates for DKRR in probability and optimal learning rates for DKRR with communications in probability are given. Section 4 makes some comparisons between our results and related work. In Section 5, we provide the main tool in our analysis, where a novel concentration inequality is used to bound the difference between integral operators and their empirical counterparts and some novel error decomposition strategies are adopted to quantify the generalization error. In Section 6, we prove our theoretical results presented in Section 3. In the last section, we conduct a series of numerical studies to verify the outperformance of DKRR with communications.

## 2. DKRR with Communications

In this section, we propose a novel communication strategy for DKRR to improve the learning performance.

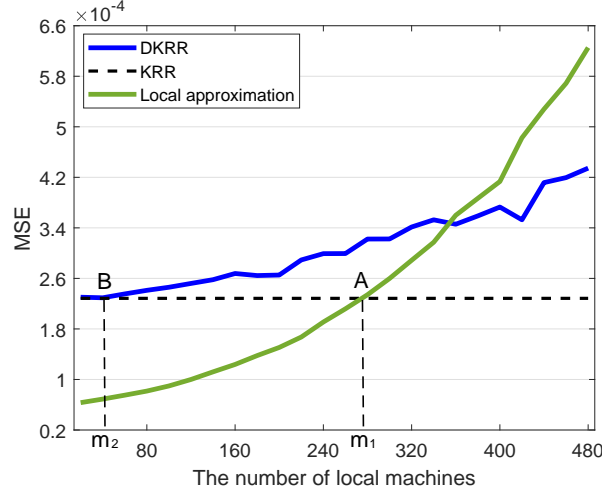


Figure 2: The kernel is  $K(t, t') = 1 + \min\{t, t'\}$  for  $t, t' \in [0, 1]$ . Data  $\{(t_i, y_i)\}_{i=1}^{10000}$  are generated with  $\{t_i\}_{i=1}^{10000}$  being drawn i.i.d. according to the uniform distribution on  $[0, 1]$  and  $y = f(t) + \varepsilon$ , where  $f(t) = \min\{t, 1 - t\}$  and  $\varepsilon \sim \mathcal{N}(0, 0.2)$ . The green line (local approximation) denotes the performance of running KRR on a local machine with a noiseless data subset (of size  $10000/m$ ),  $\{(t_i, f(t_i))\}$ .

### 2.1. Limitations of DKRR without Communications

Let  $m$  be the number of local machines,  $D_j = \{(x_{i,j}, y_{i,j})\}_{i=1}^{|D_j|}$  be the data subset stored in the  $j$ -th local machine with  $1 \leq j \leq m$  and  $D = \bigcup_{j=1}^m D_j$  be the disjoint union of  $\{D_j\}_{j=1}^m$ , where  $|D_j|$  denotes the cardinality of  $D_j$ . Write  $D_j(x) := \{x : (x, y) \in D_j\}$ . Let  $(\mathcal{H}_K, \|\cdot\|_K)$  be the reproduced kernel Hilbert space (RKHS) induced by a Mercer kernel  $K$  on a compact metric (input) space  $\mathcal{X}$ . DKRR is defined (Zhang et al., 2015) with a regularization parameter  $\lambda > 0$  by

$$\bar{f}_{D,\lambda}^0 = \sum_{j=1}^m \frac{|D_j|}{|D|} f_{D_j,\lambda}, \quad (1)$$

where

$$f_{D,\lambda} = \arg \min_{f \in \mathcal{H}_K} \left\{ \frac{1}{|D|} \sum_{(x,y) \in D} (f(x) - y)^2 + \lambda \|f\|_K^2 \right\}. \quad (2)$$

The limitations of DKRR were studied in (Shang and Cheng, 2017) and (Liu et al., 2018) by presenting a sharp upper bound of  $m$  to guarantee the comparable performances for DKRR and kernel ridge regression (KRR). Their core idea is that the weighted average in (1) cannot improve the approximation ability of KRR in each local machine. The representer theorem shows

$$f_{D_j,\lambda} \in \mathcal{H}_{K,D_j} := \left\{ \sum_{x \in D_j(x)} a_x K_x : a_x \in \mathbb{R} \right\},$$

where  $K_x := K(x, \cdot)$ . Since  $\mathcal{H}_{K, D_j}$  is a  $|D_j|$ -dimensional linear space, its approximation ability becomes worse when  $m$  increases, just as the trend of green line in Figure 2 shows. Therefore, it is impossible to derive comparable generalization errors of DKRR (blue line in Figure 2) and KRR with whole data (black line in Figure 2) when  $m$  is larger than  $m_1$ . An ideal range of  $m$  to guarantee the optimal generalization performance of DKRR is  $[1, m_1]$ . However, as shown in Figure 2, the practical range  $[1, m_2]$  is much narrower than  $[1, m_1]$ . This phenomenon says that the main bottleneck for DKRR is not due to the approximation ability but the fact that the weighted averaging is not good enough to compensate the loss of samples. Thus, efficient communication strategies and synthesization methods are required to enlarge the range of  $m$  to guarantee the best generalization performance of distributed learning.

## 2.2. Motivations from Operator Representations

Before presenting our communication strategy, we give the motivation first. Let  $S_D : \mathcal{H}_K \rightarrow \mathbb{R}^{|D|}$  be the sampling operator (Smale and Zhou, 2007) defined by

$$S_D f := (f(x))_{(x,y) \in D}.$$

Its scaled adjoint  $S_D^T : \mathbb{R}^{|D|} \rightarrow \mathcal{H}_K$  is given by

$$S_D^T \mathbf{c} := \frac{1}{|D|} \sum_{i=1}^{|D|} c_i K_{x_i}, \quad \mathbf{c} := (c_1, c_2, \dots, c_{|D|})^T \in \mathbb{R}^{|D|}.$$

Define

$$L_{K,D} f := S_D^T S_D f = \frac{1}{|D|} \sum_{(x,y) \in D} f(x) K_x.$$

Then, it can be found in (Smale and Zhou, 2007) and (Lin et al., 2017) respectively that

$$f_{D,\lambda} = (L_{K,D} + \lambda I)^{-1} S_D^T y_D \tag{3}$$

and

$$\bar{f}_{D,\lambda}^0 = \sum_{j=1}^m \frac{|D_j|}{|D|} (L_{K,D_j} + \lambda I)^{-1} S_{D_j}^T y_{D_j},$$

where  $y_D := (y_1, \dots, y_{|D|})^T$ . For an arbitrary  $f \in \mathcal{H}_K$ , we have

$$f_{D,\lambda} = f - (L_{K,D} + \lambda I)^{-1} [(L_{K,D} + \lambda I) f - S_D^T y_D], \tag{4}$$

and

$$\bar{f}_{D,\lambda}^0 = f - \sum_{j=1}^m \frac{|D_j|}{|D|} (L_{K,D_j} + \lambda I)^{-1} [(L_{K,D_j} + \lambda I) f - S_{D_j}^T y_{D_j}]. \tag{5}$$

Since the (half) gradient of the empirical risk in (2) over  $\mathcal{H}_K$  on  $f$  is

$$G_{D,\lambda,f} = \frac{1}{|D|} \sum_{(x,y) \in D} (f(x) - y) K_x + \lambda f = (L_{K,D} + \lambda I) f - S_D^T y_D$$

and the Hessian is

$$H_{D,\lambda} = \frac{1}{|D|} \sum_{(x,y) \in D} \langle \cdot, K_x \rangle_K K_x + \lambda I = L_{K,D} + \lambda I,$$

(4) and (5) can be regarded as the well known Newton-Raphson iteration. Comparing (4) with (5) and noting that the global gradients can be achieved via communications, i.e.,  $G_{D,\lambda,f} = \sum_{j=1}^m \frac{|D_j|}{|D|} G_{D_j,\lambda,f}$ , we aim at designing a communication strategy via the Newton-Raphson iteration formed as

$$\bar{f}_{D,\lambda}^\ell = \bar{f}_{D,\lambda}^{\ell-1} - \sum_{j=1}^m \frac{|D_j|}{|D|} (L_{K,D_j} + \lambda I)^{-1} [(L_{K,D} + \lambda I) \bar{f}_{D,\lambda}^{\ell-1} - S_D^T y_D], \quad \ell \in \mathbb{N}. \quad (6)$$

### 2.3. DKRR with Communications

In order to derive an estimator with the operator representation (6), we propose a communication strategy for DKRR( $\ell$ ) by iterating the following procedure for  $\ell = 1, \dots, L$ .

Step 1. Communicate the global estimator  $\bar{f}_{D,\lambda}^{\ell-1}$  to local machines and get the local gradient function  $G_{D_j,\lambda,\ell} := G_{D_j,\lambda,\bar{f}_{D,\lambda}^{\ell-1}}$ .

Step 2. Communicate back  $\{G_{D_j,\lambda,\ell} : j = 1, \dots, m\}$  to the global machine and synthesize the global gradient by  $G_{D,\lambda,\ell} := \sum_{j=1}^m \frac{|D_j|}{|D|} G_{D_j,\lambda,\ell}$ .

Step 3. Communicate the gradient function  $G_{D,\lambda,\ell}$  to each local machine and generate the gradient data  $G_{j,\ell} = \{(x, G_{D,\lambda,\ell}(x)) : x \in D_j(x)\}$ . Then run KRR on the data  $G_{j,\ell}$  to obtain a function

$$g_{D_j,\lambda,\ell} := \arg \min_{f \in \mathcal{H}_K} \left\{ \frac{1}{|D_j|} \sum_{(x,y) \in G_{j,\ell}} (f(x) - y)^2 + \lambda \|f\|_K^2 \right\}, \quad j = 1, \dots, m. \quad (7)$$

Step 4. Communicate back  $g_{D_j,\lambda,\ell}$  to the global machine and get

$$\bar{f}_{D,\lambda}^\ell = \bar{f}_{D,\lambda}^{\ell-1} - \frac{1}{\lambda} \left[ G_{D,\lambda,\ell} - \sum_{j=1}^m \frac{|D_j|}{|D|} g_{D_j,\lambda,\ell} \right]. \quad (8)$$

Due to (3) and (7), we have

$$g_{D_j,\lambda,\ell} = (L_{K,D_j} + \lambda I)^{-1} L_{K,D_j} G_{D,\lambda,\ell}.$$

This together with the identity

$$\frac{1}{\lambda} [I - (L_{K,D_j} + \lambda I)^{-1} L_{K,D_j}] G_{D,\lambda,\ell} = (L_{K,D_j} + \lambda I)^{-1} G_{D,\lambda,\ell}$$

and (8) yields (6). The training and testing flows of DKRR( $\ell$ ) are exhibited in Figure 3. Comparing Figure 1 with Figure 3, communications are required in both training and testing stages of DKRR( $\ell$ ). Noticing that communicating functions are infeasible in practice, Appendix B presents a simple realization for DKRR( $\ell$ ) by communicating input data. We believe that there are other efficient implementations of DKRR( $\ell$ ) and leave it as future studies.

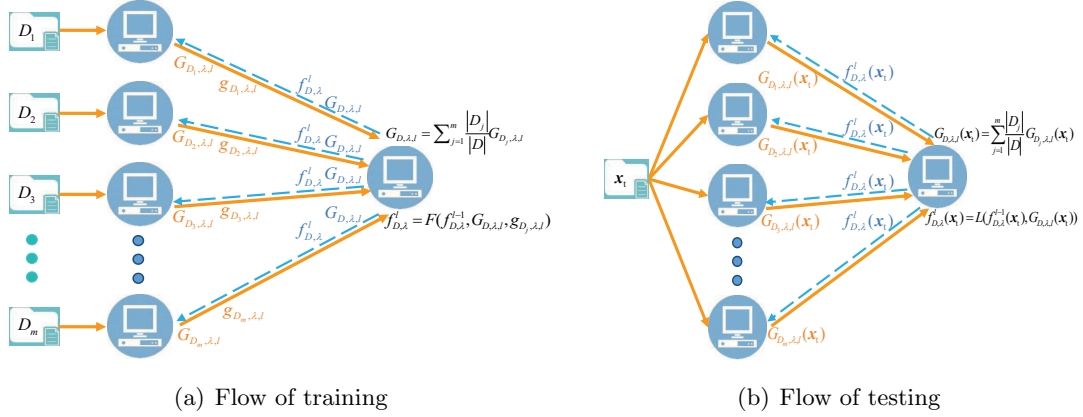


Figure 3: Training and testing flows of distributed learning with communications.

### 3. Main Results

In this section, we analyze the generalization performances of the proposed algorithm as well as DKRR in a standard regression setting (Cucker and Zhou, 2007). Let a sample  $D = \{(x_i, y_i)\}_{i=1}^N$  be independently drawn according to  $\rho$ , a Borel probability measure on  $\mathcal{Z} := \mathcal{X} \times \mathcal{Y}$  with  $\mathcal{Y} = \mathbb{R}$ . The primary objective is the regression function defined by

$$f_\rho(x) = \int_{\mathcal{Y}} y d\rho(y|x), \quad x \in \mathcal{X},$$

where  $\rho(y|x)$  denotes the conditional distribution at  $x$  induced by  $\rho$ . Throughout the paper, we assume that  $K$  is a Mercer kernel and  $\mathcal{X}$  is compact, which implies  $\kappa := \sqrt{\sup_{x \in \mathcal{X}} K(x, x)} < \infty$ .

#### 3.1. Optimal Learning Rates for DKRR in Expectation

To derive optimal learning rates for DKRR, we need some assumptions on the decay of the outputs, regularity of the regression function and capacity of  $\mathcal{H}_K$ .

**Assumption 1** We assume  $\int_{\mathcal{Y}} y^2 d\rho < \infty$  and

$$\int_{\mathcal{Y}} \left( e^{\frac{|y-f_\rho(x)|}{M}} - \frac{|y-f_\rho(x)|}{M} - 1 \right) d\rho(y|x) \leq \frac{\gamma^2}{2M^2}, \quad \forall x \in \mathcal{X}, \quad (9)$$

where  $M$  and  $\gamma$  are positive constants.

Condition (9) is satisfied if the noise is uniformly bounded, Gaussian or sub-Gaussian (Caponnetto and De Vito, 2007). Let  $\rho_X$  be the marginal distribution of  $\rho$  and  $L_{\rho_X}^2$  be the Hilbert space of  $\rho_X$  square integrable functions on  $\mathcal{X}$ , with norm denoted by  $\|\cdot\|_\rho$ . The Mercer kernel  $K : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{R}$  defines an integral operator  $L_K$  on  $\mathcal{H}_K$  (or  $L_{\rho_X}^2$ ) by

$$L_K f = \int_{\mathcal{X}} K_x f(x) d\rho_X, \quad f \in \mathcal{H}_K \quad (\text{or } f \in L_{\rho_X}^2).$$

Our second assumption is the capacity assumption measured by the effective dimension (Guo et al., 2017; Lin et al., 2017),

$$\mathcal{N}(\lambda) = \text{Tr}((\lambda I + L_K)^{-1} L_K), \quad \lambda > 0.$$

**Assumption 2** *There exists some  $s \in (0, 1]$  such that*

$$\mathcal{N}(\lambda) \leq C_0 \lambda^{-s}, \quad (10)$$

where  $C_0 \geq 1$  is a constant independent of  $\lambda$ .

Condition (10) with  $s = 1$  is always satisfied by taking the constant  $C_0 = \text{Tr}(L_K) \leq \kappa^2$ . For  $0 < s < 1$ , it was shown in (Guo et al., 2017, Page 7) that (10) is slightly more general than the eigenvalue decaying assumption in (Caponnetto and De Vito, 2007) and has been employed in (Blanchard and Krämer, 2016; Guo et al., 2017; Lin et al., 2017; Chang et al., 2017; Mücke and Blanchard, 2018) to derive optimal learning rates for kernel-based algorithms.

**Assumption 3** *For  $r > 0$ , assume*

$$f_\rho = L_K^r h_\rho, \quad \text{for some } h_\rho \in L_{\rho_X}^2, \quad (11)$$

where  $L_K^r$  denotes the  $r$ -th power of  $L_K : L_{\rho_X}^2 \rightarrow L_{\rho_X}^2$  as a compact and positive operator.

The regularity condition (11) describes the regularity of  $f_\rho$  and has been adopted in a large literature to quantify learning rates for some algorithms (Smale and Zhou, 2007; Caponnetto and De Vito, 2007; Blanchard and Krämer, 2016; Guo et al., 2017; Lin et al., 2017). Based on the above three assumptions, we can derive the following optimal learning rates for DKRR in expectation.

**Theorem 1** *Under Assumptions 1-3 with  $\frac{1}{2} \leq r \leq 1$  and  $0 < s \leq 1$ , if  $\lambda = |D|^{-\frac{1}{2r+s}}$ ,  $|D_1| = \dots = |D_m|$  and*

$$m \leq C_1 |D|^{\frac{2r+s-1}{2r+s}} (\log |D|)^{-3}, \quad (12)$$

then

$$E[\|\bar{f}_{D,\lambda}^0 - f_\rho\|_\rho^2] \leq C_2 |D|^{-\frac{2r}{2r+s}},$$

where  $C_1, C_2$  are constants independent of  $|D|$  or  $m$ , whose values will be given explicitly in the proof.

Theorem 1 exhibits optimal learning rates for DKRR in expectation under the restriction (12). In previous studies (Lin et al., 2017; Guo et al., 2017; Mücke and Blanchard, 2018), optimal learning rates for DKRR are built upon the restriction

$$m \leq |D|^{\frac{2r-1}{2r+s}}. \quad (13)$$

A direct consequence is that if  $r = 1/2$ , then DKRR with  $m = |D|^\theta$  for an arbitrarily small  $\theta > 0$  may not achieve the optimal learning, according to the existing work. In particular, it was shown in (Lin et al., 2017) that a parameter value for  $\lambda$  larger than  $|D|^{-1/(2r+s)}$  is



required under this circumstance, which leads to a sub-optimal learning rate. Comparing (13) with (12), we relax the restriction on  $m$  so that optimal learning rates for DKRR also hold for  $r = 1/2$ . The restriction (12) with  $r = 1/2$  is similar to that in (Zhang et al., 2015). However, we remove the eigenfunction assumption in (Zhang et al., 2015) and derive optimal learning rates for DKRR under Assumption 3 with  $\frac{1}{2} \leq r \leq 1$ . It should be mentioned that removing the eigenfunction assumption in (Zhang et al., 2015) was already made in a series of previous papers (Lin et al., 2017; Guo et al., 2017; Mücke and Blanchard, 2018; Lin et al., 2020). However, an additional level of regularity,  $r > 1/2$  is imposed due to (13), excluding  $r = 1/2$  for Zhang et al. (2015). Our study in Theorem 1 successfully fills this gap.

### 3.2. Optimal Learning Rates for DKRR in Probability

Theorem 1 presented optimal learning rates for DKRR in expectation. However, the expectation describes the average information for multiple trails and fails to capture the learning performance of DKRR for a single trail. This explains the inconsistency between the theoretical result in Theorem 1 and numerical observation in Figure 2, where  $m_2$  is much smaller than  $m_1$ . In the following theorem, we deduce learning rates for DKRR in probability.

**Theorem 2** *Let  $0 < \delta < 1$ . Under Assumptions 1-3 with  $\frac{1}{2} \leq r \leq 1$  and  $0 < s \leq 1$ , if  $\lambda = |D|^{-\frac{1}{2r+s}}$ ,  $|D_1| = \dots = |D_m|$ ,*

$$m \leq \frac{|D|^{\frac{2r+s-1}{4r+2s}}}{C_3 \log^3 |D|}, \quad (14)$$

and

$$16|D|^{\frac{2r+s-1}{4r+2s}} \exp \left\{ -C_4 |D|^{\frac{2r+s-1}{8r+4s}} \log |D| \right\} \leq \delta, \quad (15)$$

then with confidence at least  $1 - \delta$ , there holds

$$\|\bar{f}_{D,\lambda}^0 - f_\rho\|_\rho \leq C_5 |D|^{\frac{-r}{2r+s}} \log^2 \frac{8}{\delta}, \quad (16)$$

where  $C_3, C_4, C_5$  are constants independent of  $|D|, m$  or  $\delta$ , whose values will be given explicitly in the proof.

It has been a difficult task to derive optimal learning rates for DKRR in probability as shown in (16). Compared with the classical error decomposition in expectation (Chang et al., 2017), where the generalization error is decomposed into approximation error, sample error and distributed error, the error decomposition in probability is totally different. In particular, it is not easy to separate a distributed error in probability to control the number of local machines. As a consequence, the upper bound of (14) is tighter than that of (12), showing a stricter restriction on  $m$  to guarantee the optimal learning rate in probability.

Neglecting the logarithmic fact, we have  $|D|^{\frac{2r+s-1}{4r+2s}} = \sqrt{|D|^{\frac{2r+s-1}{2r+s}}}$ . Noting that  $m_2 \sim \sqrt{m_1}$  in Figure 2, the error estimate in probability coincides with the numerical results, showing the power of estimate in probability. Based on the confidence-based error estimate in Theorem 2, we can derive almost sure convergence of DKRR.

**Corollary 3** *Under Assumption 1, Assumption 3 with  $\frac{1}{2} \leq r \leq 1$  and Assumption 2 with  $0 < s \leq 1$ , if  $\lambda = |D|^{-\frac{1}{2r+s}}$ ,  $|D_1| = \dots = |D_m|$  and (14) holds, then for any  $\varepsilon > 0$ , there holds*

$$\lim_{|D| \rightarrow \infty} |D|^{\frac{r}{2r+s}(1-\varepsilon)} \|\bar{f}_{D,\lambda}^0 - f_\rho\|_\rho = 0.$$

### 3.3. Optimal Learning Rates for DKRR with Communications

In the previous subsection, we presented theoretical limitations of DKRR in terms of a small range of  $m$  to guarantee the optimal learning rate in probability. In the following theorem, we show that our proposed communication strategy can improve the performance of DKRR.

**Theorem 4** *Let  $0 < \delta < 1$ . Under Assumptions 1-3 with  $\frac{1}{2} \leq r \leq 1$  and  $0 < s \leq 1$ , if  $\lambda = |D|^{-\frac{1}{2r+s}}$ ,  $|D_1| = \dots = |D_m|$ ,*

$$m \leq \frac{|D|^{\frac{(2r+s-1)(\ell+1)}{(2r+s)(\ell+2)}}}{C_6 \log^3 |D|}, \quad (17)$$

and

$$\frac{|D|^{\frac{(2r+s-1)(\ell+1)}{(2r+s)(\ell+2)}}}{C_7 \log^3 |D|} \exp\{-|D|^{\frac{2r+s-1}{2(2r+s)(\ell+2)}} \log |D|\} \leq \delta < 1,$$

then with confidence at least  $1 - \delta$ , there holds

$$\|\bar{f}_{D,\lambda}^\ell - f_\rho\|_\rho \leq C_8 |D|^{\frac{-r}{2r+s}} \log^{\ell+2} \frac{4}{\delta},$$

where  $C_6, C_7, C_8$  are constants independent of  $m, \delta$  or  $|D|$ , whose value will be given explicitly in the proof.

Comparing (17) with (14), the proposed communication strategy relaxes the restriction on  $m$  from order  $|D|^{\frac{2r+s-1}{4r+2s}}$  to  $|D|^{\frac{(2r+s-1)(\ell+1)}{(2r+s)(\ell+2)}}$ . Furthermore, the upper bound of  $m$  is monotonically increasing with the number of communications, showing the power of communications in DKRR. As  $\ell \rightarrow \infty$ , up to a logarithmic factor, the restriction tends to the best one in (12). At the first glance, the restriction (17) is always worse than (12), contradicting our assertions on the outperformance of communications. However, it should be highlighted that (12) only guarantees the error bound in expectation. This means that if  $m$  satisfies (12), we cannot conduct feasibility analysis of DKRR via a single (or finite many) trial. Figure 2 numerically shows the drawback of analysis in expectation. Theorem 4 conducts the error analysis in probability, a totally different theoretical framework from Theorem 1. In this framework, Theorem 4 shows that communications improve the performance of DKRR since (17) is better than (14). It will be shown in Proposition 12 below that under (17), the error of  $\text{DKRR}(\ell)$  converges exponentially fast with respect to the number of communications, meaning that only a small  $\ell$  in  $\text{DKRR}(\ell)$  is required to get a satisfactory error bound in probability. Based on Theorem 4, we present almost sure convergence of DKRR with communications.

**Corollary 5** *Under Assumptions 1-3 with  $\frac{1}{2} \leq r \leq 1$  and  $0 < s \leq 1$ , if  $\lambda = |D|^{-\frac{1}{2r+s}}$ ,  $|D_1| = \dots = |D_m|$  and (17) holds, then for any  $\varepsilon > 0$ , there holds*

$$\lim_{|D| \rightarrow \infty} |D|^{\frac{r}{2r+s}(1-\varepsilon)} \|\bar{f}_{D,\lambda}^\ell - f_\rho\|_\rho = 0.$$

#### 4. Related Work and Discussions

KRR is a classical learning algorithm for regression and has been extensively studied in statistics and learning theory. Optimal learning rates for KRR were established in (Caponetto and De Vito, 2007; Steinwart et al., 2009; Lin et al., 2017, 2019). For DKRR, optimal learning rates were deduced in (Zhang et al., 2015) under Assumption 1, Assumption 3 with  $r = 1/2$ , some eigenvalue decaying assumption that is similar to Assumption 2, and an additional boundedness assumption for the eigenfunctions. Lin et al. (2017) removed the eigenfunction assumption by introducing a novel integral operator approach as well as a second-order decomposition for operator difference. However, optimal learning rates for DKRR (Lin et al., 2017) were derived under Assumption 3 with  $\frac{1}{2} < r \leq 1$ , excluding the most popular case  $r = 1/2$ , i.e.,  $f_\rho \in \mathcal{H}_K$ . Although several recent work (Guo et al., 2017; Chang et al., 2017; Mücke and Blanchard, 2018; Shi, 2019; Lin et al., 2020) focused on conquering this theoretical drawback, there is no essential improvement in presenting a good bound for the number of local machines according to the theory in (Shang and Cheng, 2017; Liu et al., 2018). In this paper, we succeed in deriving a tight bound for the number of local machines as (12) by applying the concentration inequality established in (Minsker, 2017) to describe the similarity of different operators (see the next section for detailed descriptions). Different from Lin et al. (2017), Theorem 1 in this paper removes the eigenfunction assumption of Zhang et al. (2015) without presenting additional regularity assumption.

Previous optimal learning rates for DKRR (Zhang et al., 2015; Lin et al., 2017; Guo et al., 2017; Chang et al., 2017; Mücke and Blanchard, 2018; Shi, 2019; Lin et al., 2020) were built in expectation. Technically, the generalization error in expectation can be divided into the approximation error, sample error and distributed error (Chang et al., 2017) by using the unbiasedness property  $E[y|x] = f_\rho(x)$ . The approximation error, independent of the sample, describes the approximation capability of the hypothesis space. The sample error connects the synthesized estimator (1) with the estimator (2) by showing an additional  $\frac{|D_j|}{|D|}$  in the sample error for local estimators. The distributed error measures the limitation of the distributed learning algorithm (1) and presents the upper bound of  $m$  to guarantee optimal learning rates for DKRR. Our error estimate in expectation also follows from this classical error decomposition (see Lemma 8 below). However, this widely used error decomposition is not applicable to DKRR in probability since there lacks an expectation operator to realize the unbiasedness  $E[y|x] = f_\rho(x)$ . Thus, it requires novel approaches to deduce optimal learning rates for DKRR in probability. According to an explicit operator representation for the kernel-based gradient descent algorithm, optimal learning rates as well as a novel error decomposition based on its operator representation for distributed gradient descent algorithms were established in probability (Lin and Zhou, 2018). Using the similar error decomposition as Lin and Zhou (2018), a minimum error entropy algorithm with distributed gradient descents was proposed in (Hu et al., 2019) and a tight learning rate was derived. However, DKRR requires the computation of the inverse matrix (or operator), the error

decomposition of gradient descent algorithm in (Lin and Zhou, 2018; Hu et al., 2019) is not suitable for DKRR. In this paper, as shown in Proposition 10 below, we succeed in deriving a novel error decomposition for DKRR by introducing some measurements to quantify the difference between integral operators and their empirical counterparts. Then, applying the recently developed concentration inequalities for positive operators (Minsker, 2017), we derive optimal learning rates for DKRR in probability under much looser restriction on  $m$  than (Lin and Zhou, 2018; Hu et al., 2019).

Numerous communication strategies (Li et al., 2014; Shamir et al., 2014; Bellet et al., 2015; Huang and Huo, 2019) were proposed to improve the learning performance of distributed learning algorithms in the framework of parametric regression (linear regression). To the best of our knowledge, our proposed communication strategy is the first work focusing on improving the performance of learning algorithms in nonparametric regression. As shown in Figure 1, nonparametric regression transmits function values and protects the privacy of local machines, while parametric regression (Zhang et al., 2013) transmits coefficients that may disclose the detailed information for local estimators. The most related work is Huang and Huo (2019), where a communication strategy based on Newton-Raphson iterations is proposed to equip ridge regression in linear regression. Our work differs from Huang and Huo (2019) in the following three aspects. Firstly, our analysis is carried out in nonparametric regression rather than linear regression. Secondly, the communication strategy is based on the operator representation, which is exclusive for kernel approaches. Thirdly, our theory focuses on enlarging the range of number of local machines rather than improving the learning rate of distributed learning algorithms, since DKRR without communications is already optimal for not so large  $m$  based on previous studies (Lin et al., 2017). It would be interesting to extend our communication strategy to other distributed learning schemes such as distributed learning with convolutional neural networks in deep learning (Zhou, 2018a,b, 2020).

## 5. Operator Similarities and Error Decomposition

We analyze the learning performance of  $\text{DKRR}(\ell)$  by using the integral operator approach (Smale and Zhou, 2007; Lin et al., 2017; Guo et al., 2017; Guo and Shi, 2019). Our novelty in analysis is tight bounds on quantifying the similarity between different operator. These bounds together with the exclusive error decomposition yield optimal learning rates for DKRR and show the advantage of communications in distributed learning.

### 5.1. Similarities of Operators

The similarity between  $\bar{f}_{D,\lambda}^\ell$  and  $f_{D,\lambda}$  depends heavily on that between the operator  $L_K$  and  $L_{K,D}$ . The classical method for analyzing similarity between  $L_K$  and  $L_{K,D}$  is to bound the norm of operator difference  $L_K - L_{K,D}$ . By using a concentration inequality in Hilbert spaces from Pinelis (1994), it can be found (Caponnetto and De Vito, 2007; Blanchard and Krämer, 2016) that for any  $\delta \in (0, 1)$ , with confidence at least  $1 - \delta$ , there holds

$$\mathcal{S}_{D,\lambda} := \|(L_K + \lambda I)^{-1/2}(L_K - L_{K,D})\| \leq 2\kappa(\kappa + 1)\mathcal{A}_{D,\lambda} \log \frac{2}{\delta}, \quad (18)$$

where

$$\mathcal{A}_{D,\lambda} := \frac{1}{\sqrt{|D|}} \left( \frac{1}{\sqrt{|D|\lambda}} + \sqrt{\mathcal{N}(\lambda)} \right). \quad (19)$$

The bound in (18) is tight. However, in estimating the difference between  $f_{D,\lambda}$  and  $\bar{f}_{D,\lambda}^\ell$ , one also needs to estimate

$$\mathcal{R}_{D,\lambda} := \|(L_K + \lambda I)^{-1/2}(L_K - L_{K,D})(L_K + \lambda I)^{-1/2}\|.$$

A classical approach (Lin et al., 2017; Guo et al., 2017) is to use  $\mathcal{R}_{D,\lambda} \leq \frac{1}{\sqrt{\lambda}} \mathcal{S}_{D,\lambda}$  and get that

$$\mathcal{R}_{D,\lambda} \leq 2\kappa(\kappa + 1)\lambda^{-1/2} \mathcal{A}_{D,\lambda} \log \frac{2}{\delta} \quad (20)$$

holds with confidence  $1 - \delta$ . The leading term in (20) is  $\frac{\sqrt{\mathcal{N}(\lambda)}}{\sqrt{|D|\lambda}}$  for  $\lambda \geq |D|^{-1}$ . In the following lemma, which will be proved in Appendix A, we reduce the leading term for bounding  $\mathcal{R}_{D,\lambda}$  from  $\frac{\sqrt{\mathcal{N}(\lambda)}}{\sqrt{|D|\lambda}}$  to  $\frac{\sqrt{\log \mathcal{N}(\lambda)}}{\sqrt{|D|\lambda}}$  by using a new concentration inequality for self-adjoint operators (Minsker, 2017).

**Lemma 6** *Let  $0 < \delta \leq 1$ . If  $0 < \lambda \leq 1$  and  $\mathcal{N}(\lambda) \geq 1$ , then with confidence  $1 - \delta$ , there holds*

$$\mathcal{R}_{D,\lambda} \leq C_1^* \mathcal{B}_{D,\lambda} \log \frac{4}{\delta}, \quad (21)$$

where  $C_1^* := \max\{(\kappa^2 + 1)/3, 2\sqrt{\kappa^2 + 1}\}$  and

$$\mathcal{B}_{D,\lambda} := \frac{1 + \log \mathcal{N}(\lambda)}{\lambda|D|} + \sqrt{\frac{1 + \log \mathcal{N}(\lambda)}{\lambda|D|}}. \quad (22)$$

Besides the differences between  $\mathcal{R}_{D,\lambda}$  and  $\mathcal{S}_{D,\lambda}$ , another quantity to measure the similarity between  $L_{K,D}$  and  $L_K$  is the operator product  $\|(L_K + \lambda I)^{1/2}(L_{K,D} + \lambda I)^{-1/2}\|$ . A recently developed second order decomposition for positive operators (Lin et al., 2017; Guo et al., 2017) asserts that if  $A$  and  $B$  are invertible operators on a Banach space, then

$$A^{-1} - B^{-1} = B^{-1}(B - A)B^{-1}(B - A)A^{-1} + B^{-1}(B - A)B^{-1}.$$

This implies the following decomposition of the operator product

$$BA^{-1} = (B - A)B^{-1}(B - A)A^{-1} + (B - A)B^{-1} + I. \quad (23)$$

Inserting  $A = L_{K,D} + \lambda I$  and  $B = L_K + \lambda I$  to (23) and noting (18), it is easy to derive the following upper bound for  $\|(L_K + \lambda I)(L_{K,D} + \lambda I)^{-1}\|$  (e.g., Guo et al., 2017): for any  $0 < \delta < 1$ , with confidence at least  $1 - \delta$ , there holds

$$\|(L_K + \lambda I)(L_{K,D} + \lambda I)^{-1}\| \leq 2 \left[ \left( \frac{2\kappa}{\sqrt{|D|\lambda}} \left( \frac{\kappa}{\sqrt{|D|\lambda}} + \sqrt{\mathcal{N}(\lambda)} \right) \log \frac{2}{\delta} \right)^2 + 1 \right].$$

Hence, according to the Cordes inequality (Bhatia, 1997)

$$\|A^\tau B^\tau\| \leq \|AB\|^\tau, \quad 0 < \tau \leq 1,$$

we have

$$\begin{aligned} \mathcal{Q}_{D,\lambda} &:= \|(L_K + \lambda I)^{1/2}(L_{K,D} + \lambda I)^{-1/2}\| \\ &\leq \sqrt{2 \left[ \left( \frac{2\kappa}{\sqrt{|D|\lambda}} \left( \frac{\kappa}{\sqrt{|D|\lambda}} + \sqrt{\mathcal{N}(\lambda)} \right) \log \frac{2}{\delta} \right)^2 + 1 \right]}. \end{aligned} \quad (24)$$

The leading term of the right-hand side of (24) is  $\frac{\sqrt{\mathcal{N}(\lambda)}}{\sqrt{|D|\lambda}}$ . In the following lemma, whose proof is postponed to Appendix A, we improve (24) by using Lemma 6.

**Lemma 7** *Assume  $0 < \lambda \leq 1$  and  $\mathcal{N}(\lambda) \geq 1$ . For  $\delta \geq 4 \exp\{-1/(2C_1^* \mathcal{B}_{D,\lambda})\}$ , with confidence  $1 - \delta$ , there holds*

$$\mathcal{Q}_{D,\lambda} \leq \sqrt{2}.$$

In (24), to guarantee the boundedness of  $\mathcal{Q}_{D,\lambda}$ , it requires  $\frac{\sqrt{\mathcal{N}(\lambda)}}{\sqrt{|D|\lambda}} \log \frac{2}{\delta} \leq C_2^*$  for some  $C_2^* > 0$  depending only on  $\kappa$ . However, in Lemma 7, recalling (22), it is sufficient that  $\frac{\log(\mathcal{N}(\lambda))}{\sqrt{|D|\lambda}} \log \frac{2}{\delta} \leq C_3^*$  for some  $C_3^* > 0$  depending only on  $\kappa$ .

## 5.2. Error Decomposition for DKRR in Expectation

We use the error decomposition for DKRR in (Chang et al., 2017), where the data-free limit and noise-free version of  $f_{D_j,\lambda}$ ,

$$f_\lambda = \arg \min_{f \in \mathcal{H}_K} \left\{ \int_{\mathcal{X}} (f(x) - f_\rho(x))^2 d\rho_X + \lambda \|f\|_K^2 \right\} = (L_K + \lambda I)^{-1} L_K f_\rho, \quad (25)$$

and

$$f_{D_j,\lambda}^\diamond := E[f_{D_j,\lambda} | D_j(x)] = (L_{K,D_j} + \lambda I)^{-1} L_{K,D_j} f_\rho \quad (26)$$

are used. The following lemma can be found in (Chang et al., 2017).

**Lemma 8** *Let  $\bar{f}_{D,\lambda}^0$  be defined by (1). We have*

$$\frac{1}{2} E \left[ \|\bar{f}_{D,\lambda}^0 - f_\rho\|_\rho^2 \right] \leq \|f_\lambda - f_\rho\|_\rho^2 + \sum_{j=1}^m \frac{|D_j|^2}{|D|^2} E \left[ \|f_{D_j,\lambda} - f_\lambda\|_\rho^2 \right] + \sum_{j=1}^m \frac{|D_j|}{|D|} E \left[ \left\| f_{D_j,\lambda}^\diamond - f_\lambda \right\|_\rho^2 \right]. \quad (27)$$

The three terms on the right-hand side of (27) are respectively the approximation error, sample error and distributed error. Based on Lemma 8, we can derive the following error decomposition for DKRR in expectation.

**Proposition 9** Let  $\bar{f}_{D,\lambda}^0$  be defined by (1). If  $f_\rho \in \mathcal{H}_K$ , then

$$\begin{aligned} \frac{1}{2}E \left[ \|\bar{f}_{D,\lambda}^0 - f_\rho\|_\rho^2 \right] &\leq \|f_\lambda - f_\rho\|_\rho^2 + \sum_{j=1}^m \frac{|D_j|^2}{|D|^2} E \left[ \mathcal{Q}_{D_j,\lambda}^4 (\mathcal{P}_{D_j,\lambda} + \mathcal{S}_{D_j,\lambda} \|f_\lambda\|_K)^2 \right] \\ &+ \sum_{j=1}^m \frac{|D_j|}{|D|} E \left[ \mathcal{Q}_{D_j,\lambda}^4 \mathcal{R}_{D_j,\lambda}^2 \|(L_K + \lambda I)^{1/2} (f_\lambda - f_\rho)\|_K^2 \right], \end{aligned} \quad (28)$$

where

$$\mathcal{P}_{D,\lambda} := \left\| (L_K + \lambda I)^{-1/2} (L_K f_\rho - S_D^T y_D) \right\|_K.$$

**Proof.** We first bound the sample error. It follows from (3) and (25) that

$$\begin{aligned} f_{D,\lambda} - f_\lambda &= (L_{K,D} + \lambda I)^{-1} S_D^T y_D - (L_K + \lambda I)^{-1} L_K f_\rho \\ &= (L_{K,D} + \lambda I)^{-1} (S_D^T y_D - L_K f_\rho) + [(L_{K,D} + \lambda I)^{-1} - (L_K + \lambda I)^{-1}] L_K f_\rho \\ &= (L_{K,D} + \lambda I)^{-1/2} (L_{K,D} + \lambda I)^{-1/2} (L_K + \lambda I)^{1/2} (L_K + \lambda I)^{-1/2} (S_D^T y_D - L_K f_\rho) \\ &+ (L_{K,D} + \lambda I)^{-1/2} (L_{K,D} + \lambda I)^{-1/2} (L_K + \lambda I)^{1/2} (L_K + \lambda I)^{-1/2} (L_K - L_{K,D}) f_\lambda. \end{aligned}$$

So

$$\|(L_K + \lambda I)^{1/2} (f_{D_j,\lambda} - f_\lambda)\|_K \leq \mathcal{Q}_{D_j,\lambda}^2 (\mathcal{P}_{D_j,\lambda} + \mathcal{S}_{D_j,\lambda} \|f_\lambda\|_K). \quad (29)$$

Then, we bound the distributed error. Due to (3) and (26), we get

$$\begin{aligned} f_{D,\lambda}^\circ - f_\lambda &= (L_{K,D} + \lambda I)^{-1} L_{K,D} f_\rho - (L_K + \lambda I)^{-1} L_K f_\rho \\ &= (L_{K,D} + \lambda I)^{-1} (L_{K,D} - L_K) f_\rho + [(L_{K,D} + \lambda I)^{-1} - (L_K + \lambda I)^{-1}] L_K f_\rho \\ &= (L_{K,D} + \lambda I)^{-1} (L_{K,D} - L_K) f_\rho + (L_{K,D} + \lambda I)^{-1} (L_K - L_{K,D}) f_\lambda \\ &= (L_{K,D} + \lambda I)^{-1} (L_{K,D} - L_K) (f_\rho - f_\lambda) \\ &= (L_{K,D} + \lambda I)^{-1} (L_K + \lambda I)^{1/2} (L_K + \lambda I)^{-1/2} (L_{K,D} - L_K) (L_K + \lambda I)^{-1/2} \\ &\quad (L_K + \lambda I)^{1/2} (f_\rho - f_\lambda). \end{aligned}$$

Combining this with  $f_\rho \in \mathcal{H}_K$  yields

$$\|(L_K + \lambda I)^{1/2} (f_{D_j,\lambda}^\circ - f_\lambda)\|_K \leq \mathcal{Q}_{D_j,\lambda}^2 \mathcal{R}_{D_j,\lambda} \|(L_K + \lambda I)^{1/2} (f_\lambda - f_\rho)\|_K. \quad (30)$$

Plugging (29) and (30) into (27), we have

$$\begin{aligned} \frac{1}{2}E \left[ \|\bar{f}_{D,\lambda}^0 - f_\rho\|_\rho^2 \right] &\leq \|f_\lambda - f_\rho\|_\rho^2 + \sum_{j=1}^m \frac{|D_j|^2}{|D|^2} E \left[ \mathcal{Q}_{D_j,\lambda}^4 (\mathcal{P}_{D_j,\lambda} + \mathcal{S}_{D_j,\lambda} \|f_\lambda\|_K)^2 \right] \\ &+ \sum_{j=1}^m \frac{|D_j|}{|D|} E \left[ \mathcal{Q}_{D_j,\lambda}^4 \mathcal{R}_{D_j,\lambda}^2 \|(L_K + \lambda I)^{1/2} (f_\lambda - f_\rho)\|_K^2 \right]. \end{aligned}$$

This completes the proof of Proposition 9. ■

### 5.3. Error Decomposition for DKRR in Probability

To deduce learning rates for DKRR in probability, we need the following error decomposition. It holds for  $\|\bar{f}_{D,\lambda}^0 - f_{D,\lambda}\|_\rho$ , which is totally different from Proposition 9 focusing on the expectation.

**Proposition 10** *Let  $\bar{f}_{D,\lambda}^0$  be defined by (1). Then*

$$\begin{aligned} \|\bar{f}_{D,\lambda}^0 - f_{D,\lambda}\|_\rho &\leq \|(L_K + \lambda I)^{1/2}(\bar{f}_{D,\lambda}^0 - f_{D,\lambda})\|_K \\ &\leq \mathcal{Q}_{D,\lambda}^2 \sum_{j=1}^m \frac{|D_j|}{|D|} (\mathcal{R}_{D_j,\lambda} + \mathcal{R}_{D,\lambda}) \mathcal{Q}_{D_j,\lambda}^2 (\mathcal{P}_{D_j,\lambda} + \mathcal{S}_{D_j,\lambda} \|f_\lambda\|_K). \end{aligned} \quad (31)$$

**Proof.** From the definition of  $\bar{f}_{D,\lambda}^0$ , we see

$$\begin{aligned} \bar{f}_{D,\lambda}^0 - f_{D,\lambda} &= \sum_{j=1}^m \frac{|D_j|}{|D|} (L_{K,D_j} + \lambda I)^{-1} S_{D_j}^T y_{D_j} - (L_{K,D} + \lambda I)^{-1} S_D^T y_D \\ &= \sum_{j=1}^m \frac{|D_j|}{|D|} [(L_{K,D_j} + \lambda I)^{-1} - (L_{K,D} + \lambda I)^{-1}] S_{D_j}^T y_{D_j} \\ &= \sum_{j=1}^m \frac{|D_j|}{|D|} (L_{K,D} + \lambda I)^{-1} (L_{K,D} - L_{K,D_j}) f_{D_j,\lambda} \\ &= \sum_{j=1}^m \frac{|D_j|}{|D|} (L_{K,D} + \lambda I)^{-1} (L_{K,D} - L_K) (f_{D_j,\lambda} - f_\lambda) \\ &\quad + \sum_{j=1}^m \frac{|D_j|}{|D|} (L_{K,D} + \lambda I)^{-1} (L_K - L_{K,D_j}) (f_{D_j,\lambda} - f_\lambda) \\ &= (L_{K,D} + \lambda I)^{-1} (L_K + \lambda I)^{1/2} (L_K + \lambda I)^{-1/2} (L_{K,D} - L_K) (L_K + \lambda I)^{-1/2} \\ &\quad \sum_{j=1}^m \frac{|D_j|}{|D|} (L_K + \lambda I)^{1/2} (f_{D_j,\lambda} - f_\lambda) \\ &\quad + (L_{K,D} + \lambda I)^{-1} (L_K + \lambda I)^{1/2} \\ &\quad \sum_{j=1}^m \frac{|D_j|}{|D|} (L_K + \lambda I)^{-1/2} (L_{K,D_j} - L_K) (L_K + \lambda I)^{-1/2} (L_K + \lambda I)^{1/2} (f_{D_j,\lambda} - f_\lambda). \end{aligned}$$

Then,

$$\begin{aligned} \|(L_K + \lambda I)^{1/2}(\bar{f}_{D,\lambda}^0 - f_{D,\lambda})\|_K &\leq \mathcal{Q}_{D,\lambda}^2 \mathcal{R}_{D,\lambda} \sum_{j=1}^m \frac{|D_j|}{|D|} \|(L_K + \lambda I)^{1/2}(f_{D_j,\lambda} - f_\lambda)\|_K \\ &\quad + \mathcal{Q}_{D,\lambda}^2 \sum_{j=1}^m \frac{|D_j|}{|D|} \mathcal{R}_{D_j,\lambda} \|(L_K + \lambda I)^{1/2}(f_{D_j,\lambda} - f_\lambda)\|_K. \end{aligned}$$



This together with (29) gives

$$\begin{aligned} \|\bar{f}_{D,\lambda}^0 - f_{D,\lambda}\|_\rho &\leq \mathcal{Q}_{D,\lambda}^2 \mathcal{R}_{D,\lambda} \sum_{j=1}^m \frac{|D_j|}{|D|} \mathcal{Q}_{D_j,\lambda}^2 (\mathcal{P}_{D_j,\lambda} + \mathcal{S}_{D_j,\lambda} \|f_\lambda\|_K) \\ &\quad + \mathcal{Q}_{D,\lambda}^2 \sum_{j=1}^m \frac{|D_j|}{|D|} \mathcal{R}_{D_j,\lambda} \mathcal{Q}_{D_j,\lambda}^2 (\mathcal{P}_{D_j,\lambda} + \mathcal{S}_{D_j,\lambda} \|f_\lambda\|_K). \end{aligned}$$

This completes the proof of Proposition 10. ■

#### 5.4. Error Decomposition for DKRR( $\ell$ )

In this subsection, we derive an error decomposition for DKRR( $\ell$ ). At first, we show in the following proposition the power of communications.

**Proposition 11** *Let  $\ell \geq 0$ . We have*

$$\begin{aligned} &\|(L_K + \lambda I)^{1/2}(\bar{f}_{D,\lambda}^\ell - f_{D,\lambda})\|_K \\ &\leq \left( \sum_{j=1}^m \frac{|D_j|}{|D|} \mathcal{Q}_{D_j,\lambda}^2 (\mathcal{R}_{D_j,\lambda} + \mathcal{R}_{D,\lambda}) \right)^\ell \|(L_K + \lambda I)^{1/2}(\bar{f}_{D,\lambda}^0 - f_{D,\lambda})\|_K. \end{aligned}$$

**Proof.** Since

$$f_{D,\lambda} = \bar{f}_{D,\lambda}^{\ell-1} - (L_{K,D} + \lambda I)^{-1}[(L_{K,D} + \lambda I)\bar{f}_{D,\lambda}^{\ell-1} - S_{D,D}^T y_D],$$

we have

$$\begin{aligned} f_{D,\lambda} - \bar{f}_{D,\lambda}^\ell &= \bar{f}_{D,\lambda}^{\ell-1} - \sum_{j=1}^m \frac{|D_j|}{|D|} (L_{K,D_j} + \lambda I)^{-1} [(L_{K,D} + \lambda I)\bar{f}_{D,\lambda}^{\ell-1} - S_{D,D}^T y_D] \\ &\quad - \bar{f}_{D,\lambda}^{\ell-1} + (L_{K,D} + \lambda I)^{-1} [(L_{K,D} + \lambda I)\bar{f}_{D,\lambda}^{\ell-1} - S_{D,D}^T y_D] \\ &= \sum_{j=1}^m \frac{|D_j|}{|D|} [(L_{K,D_j} + \lambda I)^{-1} - (L_{K,D} + \lambda I)^{-1}] [(L_{K,D} + \lambda I)\bar{f}_{D,\lambda}^{\ell-1} - S_{D,D}^T y_D] \\ &= \sum_{j=1}^m \frac{|D_j|}{|D|} (L_{K,D_j} + \lambda I)^{-1} (L_{K,D} - L_{K,D_j}) (L_{K,D} + \lambda I)^{-1} [(L_{K,D} + \lambda I)\bar{f}_{D,\lambda}^{\ell-1} - S_{D,D}^T y_D] \\ &= \sum_{j=1}^m \frac{|D_j|}{|D|} (L_{K,D_j} + \lambda I)^{-1} (L_{K,D} - L_{K,D_j}) (\bar{f}_{D,\lambda}^{\ell-1} - f_{D,\lambda}) \\ &= \sum_{j=1}^m \frac{|D_j|}{|D|} (L_{K,D_j} + \lambda I)^{-1} (L_{K,D} - L_K) (\bar{f}_{D,\lambda}^{\ell-1} - f_{D,\lambda}) \\ &\quad + \sum_{j=1}^m \frac{|D_j|}{|D|} (L_{K,D_j} + \lambda I)^{-1} (L_K - L_{K,D_j}) (\bar{f}_{D,\lambda}^{\ell-1} - f_{D,\lambda}) \\ &=: \mathcal{U}_{D,\lambda,1} + \mathcal{U}_{D,\lambda,2}. \end{aligned}$$

Since

$$\begin{aligned}
 & (L_K + \lambda I)^{1/2} \mathcal{U}_{D,\lambda,1} \\
 = & \sum_{j=1}^m \frac{|D_j|}{|D|} (L_K + \lambda I)^{1/2} (L_{K,D_j} + \lambda I)^{-1} (L_K + \lambda I)^{1/2} (L_K + \lambda I)^{-1/2} (L_{K,D} - L_K) \\
 \times & (L_K + \lambda I)^{-1/2} (L_K + \lambda I)^{1/2} (\bar{f}_{D,\lambda}^{\ell-1} - f_{D,\lambda}),
 \end{aligned}$$

we have

$$\|(L_K + \lambda I)^{1/2} \mathcal{U}_{D,\lambda,1}\|_K \leq \sum_{j=1}^m \frac{|D_j|}{|D|} \mathcal{Q}_{D_j,\lambda}^2 \mathcal{R}_{D,\lambda} \|(L_K + \lambda I)^{1/2} (\bar{f}_{D,\lambda}^{\ell-1} - f_{D,\lambda})\|_K.$$

Similarly, we find

$$\|(L_K + \lambda I)^{1/2} \mathcal{U}_{D,\lambda,2}\|_K \leq \sum_{j=1}^m \frac{|D_j|}{|D|} \mathcal{Q}_{D_j,\lambda}^2 \mathcal{R}_{D_j,\lambda} \|(L_K + \lambda I)^{1/2} (\bar{f}_{D,\lambda}^{\ell-1} - f_{D,\lambda})\|_K.$$

Then,

$$\begin{aligned}
 & \|(L_K + \lambda I)^{1/2} (\bar{f}_{D,\lambda}^\ell - f_{D,\lambda})\|_K \\
 \leq & \left( \sum_{j=1}^m \frac{|D_j|}{|D|} \mathcal{Q}_{D_j,\lambda}^2 \mathcal{R}_{D_j,\lambda} + \sum_{j=1}^m \frac{|D_j|}{|D|} \mathcal{Q}_{D_j,\lambda}^2 \mathcal{R}_{D,\lambda} \right) \|(L_K + \lambda I)^{1/2} (\bar{f}_{D,\lambda}^{\ell-1} - f_{D,\lambda})\|_K \\
 \leq & \left( \sum_{j=1}^m \frac{|D_j|}{|D|} \mathcal{Q}_{D_j,\lambda}^2 \mathcal{R}_{D_j,\lambda} + \sum_{j=1}^m \frac{|D_j|}{|D|} \mathcal{Q}_{D_j,\lambda}^2 \mathcal{R}_{D,\lambda} \right)^\ell \|(L_K + \lambda I)^{1/2} (\bar{f}_{D,\lambda}^0 - f_{D,\lambda})\|_K.
 \end{aligned}$$

This completes the proof of Proposition 11.  $\blacksquare$

Combining Proposition 11 with Proposition 10, we can derive the following error decomposition for DKRR( $\ell$ ).

**Proposition 12** *Let  $\ell \geq 0$ . We have*

$$\begin{aligned}
 & \|(L_K + \lambda I)^{1/2} (\bar{f}_{D,\lambda}^\ell - f_{D,\lambda})\|_K \leq \left( \sum_{j=1}^m \frac{|D_j|}{|D|} \mathcal{Q}_{D_j,\lambda}^2 (\mathcal{R}_{D_j,\lambda} + \mathcal{R}_{D,\lambda}) \right)^\ell \\
 \times & \mathcal{Q}_{D,\lambda}^2 \sum_{j=1}^m \frac{|D_j|}{|D|} (\mathcal{R}_{D,\lambda} + \mathcal{R}_{D_j,\lambda}) \mathcal{Q}_{D_j,\lambda}^2 (\mathcal{P}_{D_j,\lambda} + \mathcal{S}_{D_j,\lambda} \|f_\lambda\|_K).
 \end{aligned}$$

## 6. Proofs

In this section, we present proofs of our main results.

### 6.1. Optimal Learning Rates for DKRR in Expectation

In this subsection, we prove optimal learning rates for DKRR in expectation. We need the following general theorem based on Assumption 1 and Assumption 3.

**Theorem 13** *Under Assumption 3 with  $\frac{1}{2} \leq r \leq 1$  and Assumption 1, if  $0 < \lambda \leq 1$  and  $\mathcal{N}(\lambda) \geq 1$ , then*

$$\begin{aligned} E \left[ \|\bar{f}_{D,\lambda}^0 - f_\rho\|_\rho^2 \right] &\leq 2\lambda^{2r} + 24 \sum_{j=1}^m \frac{|D_j|^2}{|D|^2} \left( \exp\{-1/(2C_1^* \mathcal{B}_{D_j,\lambda})\} + \tilde{C}_1^2 \mathcal{A}_{D_j,\lambda}^2 \right) \\ &\quad + \sum_{j=1}^m \frac{|D_j|}{|D|} \left( 16 \exp\{-1/(2C_1^* \mathcal{B}_{D_j,\lambda})\} + 32\tilde{C}_2^2 \mathcal{B}_{D_j,\lambda}^2 \lambda^{2r} \right), \end{aligned} \quad (32)$$

where  $\tilde{C}_1 := 4((\kappa M + \gamma)(\kappa + 1) + \kappa^{2r}(2\kappa + 1)\|h_\rho\|_\rho)$  and  $\tilde{C}_2 := 4\kappa(\kappa + 1)\|h_\rho\|_\rho$ .

**Proof.** Due to Assumption 3 with  $r \geq 1/2$ , we obtain

$$\|f_\lambda\|_K = \|(L_K + \lambda I)^{-1} L_K L_K^r h_\rho\| \leq \kappa^{2r-1} \|L_K^{1/2} h_\rho\|_K = \kappa^{2r-1} \|h_\rho\|_\rho. \quad (33)$$

Moreover, (9) implies (Blanchard and Krämer, 2016; Lin and Zhou, 2018) that with confidence at least  $1 - \delta$ , there holds

$$\mathcal{P}_{D_j,\lambda} \leq 2(\kappa M + \gamma)(\kappa + 1) \mathcal{A}_{D_j,\lambda} \log \frac{2}{\delta}. \quad (34)$$

Thus, for  $\delta \geq 12 \exp\{-1/(2C_1^* \mathcal{B}_{D_j,\lambda})\}$ , it follows from Lemma 7, (18), (33) and (34) that with confidence  $1 - \delta$ , there holds

$$\mathcal{Q}_{D_j,\lambda}^2 (\mathcal{P}_{D_j,\lambda} + \mathcal{S}_{D_j,\lambda} \|f_\lambda\|_K) \leq \tilde{C}_1 \mathcal{A}_{D_j,\lambda} \log \frac{6}{\delta}, \quad \forall j = 1, \dots, m. \quad (35)$$

Using the probability to expectation formula

$$E[\xi] = \int_0^\infty P[\xi > t] dt \quad (36)$$

to the positive random variable  $\xi_{1,j} = \mathcal{Q}_{D_j,\lambda}^4 (\mathcal{P}_{D_j,\lambda} + \mathcal{S}_{D_j,\lambda} \|f_\lambda\|_K)^2$  for any  $j = 1, \dots, m$ , we have

$$\begin{aligned} E[\xi_{1,j}] &= \int_0^\infty P[\xi_{1,j} > t] dt \\ &= \int_0^{12 \exp\{-1/(2C_1^* \mathcal{B}_{D_j,\lambda})\}} P[\xi_{1,j} > t] dt + \int_{12 \exp\{-1/(2C_1^* \mathcal{B}_{D_j,\lambda})\}}^\infty P[\xi_{1,j} > t] dt \\ &\leq 12 \exp\{-1/(2C_1^* \mathcal{B}_{D_j,\lambda})\} + \int_{12 \exp\{-1/(2C_1^* \mathcal{B}_{D_j,\lambda})\}}^\infty P[\xi_{1,j} > t] dt. \end{aligned}$$

When  $t \geq 12 \exp\{-1/(2C_1^* \mathcal{B}_{D_j,\lambda})\}$ , it follows from (35) that

$$P[\xi_{1,j} > t] \leq 6 \exp\{-\tilde{C}_1^{-1} \mathcal{A}_{D_j,\lambda}^{-1} t^{1/2}\}, \quad \forall j = 1, \dots, m,$$

and

$$\begin{aligned} & \int_{12 \exp\{-1/(2C_1^* \mathcal{B}_{D_j, \lambda})\}}^{\infty} P[\xi_{1,j} > t] dt \leq 6 \int_0^{\infty} \exp\{-\tilde{C}_1^{-1} \mathcal{A}_{D_j, \lambda}^{-1} t^{1/2}\} dt \\ & \leq 12 \tilde{C}_1^2 \mathcal{A}_{D_j, \lambda}^2 \int_0^{\infty} u e^{-u} du = 12 \tilde{C}_1^2 \mathcal{A}_{D_j, \lambda}^2. \end{aligned}$$

Thus,

$$E \left[ \mathcal{Q}_{D_j, \lambda}^4 (\mathcal{P}_{D_j, \lambda} + \mathcal{S}_{D_j, \lambda} \|f_\lambda\|_K)^2 \right] \leq 12 \exp\{-1/(2C_1^* \mathcal{B}_{D_j, \lambda})\} + 12 \tilde{C}_1^2 \mathcal{A}_{D_j, \lambda}^2. \quad (37)$$

Due to Assumption 3 with  $r \geq 1/2$ , we have

$$\|(L_K + \lambda I)^{1/2} (f_\lambda - f_\rho)\|_K \leq \lambda \|(L_K + \lambda I)^{-1/2} L_K^{r+1/2}\| \|h_\rho\|_\rho \leq \lambda^r \|h_\rho\|_\rho. \quad (38)$$

Then, Lemma 6 and Lemma 7 with  $\delta \geq 8 \exp\{-1/(2C_1^* \mathcal{B}_{D_j, \lambda})\}$  yield that with confidence  $1 - \delta$ , there holds

$$\mathcal{Q}_{D_j, \lambda}^2 \mathcal{R}_{D_j, \lambda} \|(L_K + \lambda I)^{1/2} (f_\lambda - f_\rho)\|_K \leq \tilde{C}_2 \mathcal{B}_{D_j, \lambda} \lambda^r \log \frac{8}{\delta}, \quad \forall j = 1, \dots, m. \quad (39)$$

Then, applying (36) to  $\xi_{2,j} = \mathcal{Q}_{D_j, \lambda}^4 \mathcal{R}_{D_j, \lambda}^2 \|(L_K + \lambda I)^{1/2} (f_\lambda - f_\rho)\|_K^2$  and using the same approach as above, we can derive for any  $j = 1, \dots, m$ ,

$$E[\mathcal{Q}_{D_j, \lambda}^4 \mathcal{R}_{D_j, \lambda}^2 \|(L_K + \lambda I)^{1/2} (f_\lambda - f_\rho)\|_K^2] \leq 8 \exp\{-1/(2C_1^* \mathcal{B}_{D_j, \lambda})\} + 16 \tilde{C}_2^2 \mathcal{B}_{D_j, \lambda}^2 \lambda^{2r}. \quad (40)$$

Plugging (37), (38) and (40) into (28), we get (32) directly. This completes the proof of Theorem 13.  $\blacksquare$

Based on Theorem 13, we can prove Theorem 1 as follows.

**Proof of Theorem 1.** For  $C_1 := \min \left\{ \frac{2r+s}{2s}, \frac{(2r+s)^3}{2rs \max\{(\kappa^2+1)/3, 2\sqrt{\kappa^2+1}\}} \right\}$ , due to (22),  $|D_1| = \dots = |D_m|$ ,  $\lambda = |D|^{-\frac{1}{2r+s}}$  and (10), we have for any  $j = 1, \dots, m$ ,

$$\mathcal{B}_{D_j, \lambda} \leq \frac{2s}{2r+s} m |D|^{-\frac{2r+s-1}{2r+s}} \log |D| + \sqrt{\frac{2s}{2r+s} m |D|^{-\frac{2r+s-1}{2r+s}} \log |D|}. \quad (41)$$

Noting  $C_1^* = \max\{(\kappa^2+1)/3, 2\sqrt{\kappa^2+1}\}$ , we see that (12) implies

$$\mathcal{B}_{D_j, \lambda} \leq \frac{2r+s}{4rC_1^* \log |D|} \quad (42)$$

and

$$\exp\{-1/(2C_1^* \mathcal{B}_{D_j, \lambda})\} \leq |D|^{-\frac{2r}{2r+s}}, \quad \forall j = 1, \dots, m. \quad (43)$$

According to (19),  $|D_1| = \dots = |D_m|$ ,  $\lambda = |D|^{-\frac{1}{2r+s}}$  and (10), we also get

$$\mathcal{A}_{D_j, \lambda} \leq m |D|^{-\frac{4r+2s-1}{4r+2s}} + \sqrt{m} |D|^{-\frac{r}{2r+s}}, \quad \forall j = 1, \dots, m. \quad (44)$$

Inserting (42), (43) and (44) into (32) and noting  $\lambda = |D|^{-\frac{1}{2r+s}}$  and (12), we have

$$E[\|f_{D, \lambda}^0 - f_\rho\|_\rho^2] \leq C_1 |D|^{-\frac{2r}{2r+s}},$$

where  $C_2 := 42 + 48 \tilde{C}_1^2 + 32 \tilde{C}_2^2 \left( \frac{2r+s}{4rC_1^*} \right)^2$ . This completes the proof of Theorem 1.  $\blacksquare$

## 6.2. Optimal Learning Rates for DKRR in Probability

In this subsection, we prove Theorem 2. To this end, we need the following theorem for DKRR in probability.

**Theorem 14** *Under Assumption 1 and Assumption 3 with  $\frac{1}{2} \leq r \leq 1$ , if  $0 < \lambda \leq 1$ ,  $\mathcal{N}(\lambda) \geq 1$ , and*

$$16m \exp\{-1/(2C_1^* \mathcal{B}_{D_j, \lambda})\} < 1, \quad \forall j = 1, \dots, m, \quad (45)$$

then for

$$16m \exp\{-1/(2C_1^* \mathcal{B}_{D_j, \lambda})\} \leq \delta < 1, \quad \forall j = 1, \dots, m, \quad (46)$$

with confidence  $1 - \delta$  there holds

$$\|\bar{f}_{D, \lambda}^0 - f_{D, \lambda}\|_\rho \leq \tilde{C}_3 \log^2(4m) \sum_{j=1}^m \frac{|D_j|}{|D|} \mathcal{A}_{D_j, \lambda} \mathcal{B}_{D_j, \lambda} \log^2 \frac{4}{\delta}, \quad (47)$$

where  $\tilde{C}_3 = 16C_1^* \tilde{C}_1$ .

**Proof.** Let  $j \in \{1, \dots, m\}$  be fixed. It follows from Lemma 6 and  $\mathcal{B}_{D, \lambda} \leq \mathcal{B}_{D_j, \lambda}$  that with confidence  $1 - \frac{\delta}{4}$ , there holds

$$\mathcal{R}_{D, \lambda} + \mathcal{R}_{D_j, \lambda} \leq 2C_1^* \mathcal{B}_{D_j, \lambda} \log \frac{16}{\delta}. \quad (48)$$

Since  $\delta \geq 16 \exp\{-1/(2C_1^* \mathcal{B}_{D_j, \lambda})\}$ , Lemma 7 together with (35) shows that with confidence  $1 - 3\delta/4$ , there holds

$$\mathcal{Q}_{D, \lambda}^2 \mathcal{Q}_{D_j, \lambda}^2 (\mathcal{P}_{D_j, \lambda} + \mathcal{S}_{D_j, \lambda} \|f_\lambda\|_K) \leq 2\tilde{C}_1 \mathcal{A}_{D_j, \lambda} \log \frac{16}{\delta}. \quad (49)$$

Plugging (48) and (49) into (31), for fixed  $j \in \{1, \dots, m\}$ , we see with confidence  $1 - \delta$ , there holds

$$\mathcal{Q}_{D, \lambda}^2 (\mathcal{R}_{D_j, \lambda} + \mathcal{R}_{D, \lambda}) \mathcal{Q}_{D_j, \lambda}^2 (\mathcal{P}_{D_j, \lambda} + \mathcal{S}_{D_j, \lambda} \|f_\lambda\|_K) \leq \tilde{C}_3 \log^2 \frac{16}{\delta} \mathcal{A}_{D_j, \lambda} \mathcal{B}_{D_j, \lambda}. \quad (50)$$

Thus, for  $\delta \geq 16 \exp\{-1/(2C_1^* \mathcal{B}_{D_j, \lambda})\}$ , the above estimate implies that with confidence at least  $1 - m\delta$ , there holds

$$\max_{1 \leq j \leq m} \mathcal{Q}_{D, \lambda}^2 (\mathcal{R}_{D_j, \lambda} + \mathcal{R}_{D, \lambda}) \mathcal{Q}_{D_j, \lambda}^2 (\mathcal{P}_{D_j, \lambda} + \mathcal{S}_{D_j, \lambda} \|f_\lambda\|_K) \leq \tilde{C}_3 \log^2 \frac{16}{\delta} \mathcal{A}_{D_j, \lambda} \mathcal{B}_{D_j, \lambda}.$$

Scaling  $m\delta$  to  $\delta$ , for  $\delta \geq 16m \exp\{-1/(2C_1^* \mathcal{B}_{D_j, \lambda})\}$ , with confidence  $1 - \delta$ , there holds

$$\|\bar{f}_{D, \lambda}^0 - f_{D, \lambda}\|_\rho \leq 4C_1^* \tilde{C}_1 \log^2 \frac{16m}{\delta} \sum_{j=1}^m \frac{|D_j|}{|D|} \mathcal{A}_{D_j, \lambda} \mathcal{B}_{D_j, \lambda}.$$

Note that

$$\log \frac{16m}{\delta} = \log \frac{4}{\delta} + \log(4m) \leq (\log(4m) + 1) \log \frac{4}{\delta} \leq 2 \log(4m) \log \frac{4}{\delta},$$

which follows from (47). Then the proof of Theorem 14 is complete.  $\blacksquare$

**Proof of Theorem 2.** Due to the triangle inequality, we have

$$\|\bar{f}_{D,\lambda}^0 - f_\rho\|_\rho \leq \|f_{D,\lambda} - \bar{f}_{D,\lambda}^0\|_\rho + \|f_{D,\lambda} - f_\rho\|_\rho. \quad (51)$$

To bound  $\|f_{D,\lambda} - f_\rho\|_\rho$ , under (11) with  $\frac{1}{2} \leq r \leq 1$ , we obtain from (29) and (38) that

$$\|f_{D,\lambda} - f_\rho\|_\rho \leq \lambda^r + \mathcal{Q}_{D,\lambda}^2(\mathcal{P}_{D,\lambda} + \mathcal{S}_{D,\lambda}\|f_\lambda\|_K).$$

Then, (14) with  $C_3 := \max\{4C_1^*(4\log 2 + 1), 16\log^2 2\}$  and (15) with  $C_4 := 4C_1^*$  imply  $\delta \geq \exp\{2C_1^*\mathcal{B}_{D,\lambda}\}$ , and then (39) with  $D_j$  replaced by  $D$  shows that with confidence  $1 - \delta/2$ , there holds

$$\|f_{D,\lambda} - f_\rho\|_\rho \leq \lambda^r + \tilde{C}_1 \mathcal{A}_{D,\lambda} \log \frac{16}{\delta}. \quad (52)$$

Noting (14) and (15) imply (45) and (46), and plugging (47) and (52) into (51), with confidence  $1 - \delta$ , there holds

$$\|\bar{f}_{D,\lambda}^0 - f_\rho\|_\rho \leq \lambda^r + \tilde{C}_1 \mathcal{A}_{D,\lambda} \log \frac{16}{\delta} + \tilde{C}_3 \log^2(4m) \log^2 \frac{8}{\delta} \sum_{j=1}^m \frac{|D_j|}{|D|} \mathcal{A}_{D_j,\lambda} \mathcal{B}_{D_j,\lambda}. \quad (53)$$

Since  $|D_1| = \dots = |D_m|$ ,  $\lambda = |D|^{-\frac{1}{2r+s}}$  and  $2s/(2r+s) \leq 1$ , we have from (41) and (44) that

$$\begin{aligned} \mathcal{A}_{D_j,\lambda} \mathcal{B}_{D_j,\lambda} &\leq \left( m|D|^{-\frac{4r+2s-1}{4r+2s}} + \sqrt{m}|D|^{-\frac{r}{2r+s}} \right) \\ &\times \left( m|D|^{-\frac{2r+s-1}{2r+s}} \log |D| + \sqrt{m|D|^{-\frac{2r+s-1}{2r+s}} \log |D|} \right). \end{aligned} \quad (54)$$

This together with (14) shows

$$\log^2(4m) \mathcal{A}_{D_j,\lambda} \mathcal{B}_{D_j,\lambda} \leq 4|D|^{-\frac{r}{2r+s}}.$$

Inserting the above inequality into (53) and noting

$$\mathcal{A}_{D,\lambda} \leq |D|^{-\frac{4r+2s-1}{4r+2s}} + |D|^{-\frac{r}{2r+s}} \leq 2|D|^{-\frac{r}{2r+s}}, \quad (55)$$

we see with confidence  $1 - \delta$ , there holds

$$\|\bar{f}_{D,\lambda}^0 - f_\rho\|_\rho \leq C_5 |D|^{-\frac{r}{2r+s}} \log^2 \frac{8}{\delta},$$

where we use  $\log \frac{16}{\delta} \leq \log^2 \frac{8}{\delta}$  and

$$C_5 = 1 + 2\tilde{C}_1 + 4\tilde{C}_3.$$

This completes the proof of Theorem 2.  $\blacksquare$

To prove Corollary 3, we need the following Borel-Cantelli Lemma (Dudley, 2002, page 262). The Borel-Cantelli Lemma asserts for a sequence  $\{\eta_n\}_n$  of events that if the sum of the probabilities is finite, i.e.,  $\sum_{n=1}^{\infty} P[\eta_n] < \infty$ , then the probability that infinitely many of them occur is 0.

**Lemma 15** *Let  $\{\eta_n\}$  be a sequence of events in some probability space and  $\{\varepsilon_n\}$  be a sequence of positive numbers satisfying  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ . If*

$$\sum_{n=1}^{\infty} \text{Prob}[|\eta_n - \eta| > \varepsilon_n] < \infty,$$

*then  $\eta_n$  converges to  $\eta$  almost surely.*

**Proof of Corollary 3.** Let  $N := |D|$ , it is easy to check that  $\delta = \delta_N = \tilde{C}_4 N^{-2}$  satisfies (15) for some  $\tilde{C}_4 > 0$  independent of  $N$ . Set  $\Psi_N = N^{-\frac{r}{2r+s}}$ . By Theorem 2, if  $\lambda = |D|^{-\frac{1}{2r+s}}$ ,  $|D_1| = \dots = |D_m|$  and (14) holds, then for any  $N$  and  $\varepsilon > 0$ ,

$$P \left[ \Psi_N^{-1+\varepsilon} \|\bar{f}_{D,\lambda}^0 - f_\rho\|_\rho > C_5 \Psi_N^\varepsilon \left( \log \frac{8}{\delta_N} \right)^2 \right] \leq \delta_N.$$

Denote  $\mu_N = C_5 \Psi_N^\varepsilon \left( \log \frac{8}{\delta_N} \right)^2$ . Obviously,

$$\sum_{N=2}^{\infty} P \left[ \Psi_N^{-1+\varepsilon} \|\bar{f}_{t,D} - f_\rho\|_\rho > \mu_N \right] \leq \sum_{N=2}^{\infty} \delta_N < \infty$$

and  $\mu_N \rightarrow 0$  when  $N \rightarrow 0$ . Then our conclusion follows from Lemma 15. This completes the proof of Corollary 3.  $\blacksquare$

### 6.3. Optimal Learning Rates for DKRR( $\ell$ )

In this subsection, we present the proof of Theorem 4. To this end, we prove the following theorem.

**Theorem 16** *Under Assumption 1 and Assumption 3 with  $\frac{1}{2} \leq r \leq 1$ , if  $0 < \lambda \leq 1$ ,  $\mathcal{N}(\lambda) \geq 1$ , and (45) holds, then for  $\delta$  satisfying (46), with confidence  $1 - \delta$  there holds*

$$\|\bar{f}_{D,\lambda}^\ell - f_{D,\lambda}\|_\rho \leq 2\tilde{C}_3 (4C_1^*)^\ell \left( \log(4m) \log \frac{4}{\delta} \right)^{\ell+2} \left( \sum_{j=1}^m \frac{|D_j|}{|D|} \mathcal{B}_{D_j,\lambda} \right)^\ell \sum_{j=1}^m \frac{|D_j|}{|D|} \mathcal{A}_{D_j,\lambda} \mathcal{B}_{D_j,\lambda}.$$

**Proof.** Under (45) and (46), we obtain from Lemma 7, (48) and (50) that with confidence  $1 - m\delta$ , there holds

$$\max_{1 \leq j \leq m} \mathcal{Q}_{D_j,\lambda}^2 (\mathcal{R}_{D_j,\lambda} + \mathcal{R}_{D,\lambda}) \leq 4C_1^* \mathcal{B}_{D_j,\lambda} \log \frac{16}{\delta},$$

and

$$\max_{1 \leq j \leq m} \mathcal{Q}_{D,\lambda}^2 (\mathcal{R}_{D,\lambda} + \mathcal{R}_{D_j,\lambda}) \mathcal{Q}_{D_j,\lambda}^2 (\mathcal{P}_{D_j,\lambda} + \mathcal{S}_{D_j,\lambda} \|f_\lambda\|_K) \leq \tilde{C}_3 \log^2 \frac{16}{\delta} \mathcal{A}_{D_j,\lambda} \mathcal{B}_{D_j,\lambda}.$$

Scaling  $m\delta$  to  $\delta$ , we obtain from (32) that

$$\begin{aligned} \|\bar{f}_{D,\lambda}^\ell - f_{D,\lambda}\|_\rho &\leq \|(L_K + \lambda I)^{1/2}(\bar{f}_{D,\lambda}^\ell - f_{D,\lambda})\|_K \\ &\leq \left(4C_1^* \log \frac{16m}{\delta}\right)^\ell \left(\sum_{j=1}^m \frac{|D_j|}{|D|} \mathcal{B}_{D_j,\lambda}\right)^\ell \tilde{C}_3 \log^2 \frac{16m}{\delta} \sum_{j=1}^m \frac{|D_j|}{|D|} \mathcal{A}_{D_j,\lambda} \mathcal{B}_{D_j,\lambda}. \end{aligned}$$

Noting further  $\log \frac{16}{\delta} \leq 2 \log 4m$ , we have with confidence  $1 - \delta$ , there holds

$$\|\bar{f}_{D,\lambda}^\ell - f_{D,\lambda}\|_\rho \leq 2\tilde{C}_3 (4C_1^*)^\ell \log^{\ell+2}(4m) \log^{\ell+2} \frac{4}{\delta} \left(\sum_{j=1}^m \frac{|D_j|}{|D|} \mathcal{B}_{D_j,\lambda}\right)^\ell \sum_{j=1}^m \frac{|D_j|}{|D|} \mathcal{A}_{D_j,\lambda} \mathcal{B}_{D_j,\lambda}.$$

This completes the proof of Theorem 16.  $\blacksquare$

**Proof of Theorem 4.** Since  $|D_1| = \dots = |D_m|$  and  $\lambda = |D|^{-\frac{1}{2r+s}}$ , it follows from (41) and  $2s/(2r+s) \leq 1$  that

$$\left(\sum_{j=1}^m \frac{|D_j|}{|D|} \mathcal{B}_{D_j,\lambda}\right)^\ell \leq \left(m|D|^{-\frac{2r+s-1}{2r+s}} \log |D| + \sqrt{m \log |D|} |D|^{-\frac{2r+s-1}{4r+2s}}\right)^\ell.$$

This together with (54) shows that with confidence  $1 - \delta$ , there holds

$$\begin{aligned} &\left(\sum_{j=1}^m \frac{|D_j|}{|D|} \mathcal{B}_{D_j,\lambda}\right)^\ell \sum_{j=1}^m \frac{|D_j|}{|D|} \mathcal{B}_{D_j,\lambda} \mathcal{A}_{D_j,\lambda} \\ &\leq \left(m|D|^{-\frac{2r+s-1}{2r+s}} \log |D| + \sqrt{m \log |D|} |D|^{-\frac{2r+s-1}{4r+2s}}\right)^{\ell+1} \left(m|D|^{-\frac{4r+2s-1}{4r+2s}} + \sqrt{m} |D|^{-\frac{r}{2r+s}}\right). \end{aligned}$$

Since (17) with  $C_6 := (4C_1^*(2+4\log 2))^2$  and (15) with  $C_7 = C_6/4$ , we obtain (45) and (46). Then, it follows from Theorem 16 that with confidence  $1 - \delta$ , there holds

$$\|\bar{f}_{D,\lambda}^\ell - f_{D,\lambda}\|_\rho \leq 2\tilde{C}_3 |D|^{\frac{-r}{2r+s}} \log^{\ell+2} \frac{4}{\delta}.$$

Hence, we obtain from (52) and (55) that with confidence  $1 - \delta$ , there holds

$$\|\bar{f}_{D,\lambda}^\ell - f_\rho\|_\rho \leq \|\bar{f}_{D,\lambda}^\ell - f_{D,\lambda}\|_\rho + \|f_{D,\lambda} - f_\rho\|_\rho \leq C_8 |D|^{\frac{-r}{2r+s}} \log^{\ell+2} \frac{4}{\delta},$$

where  $C_8 = 1 + \tilde{C}_1 + 2\tilde{C}_3$ , which completes the proof of Theorem 4.  $\blacksquare$

**Proof of Corollary 5.** The proof of Corollary 5 is the same as that of Corollary 3 with (14) replaced by (17).  $\blacksquare$

## 7. Experiments

In this section, we report numerical results to verify our theoretical statements. We employ three criteria for comparisons. The first criterion is the global mean squared error (GMSE)



which is the mean square error (MSE) of KRR with training all samples in a batch mode. GMSE provides a baseline to assess the performances of DKRR and DKRR( $\ell$ ). The second criterion is the average error (AE) which is the MSE of DKRR. The third criterion is the average error with communications (AEC), which is the MSE of DKRR ( $\ell$ ). Regularization parameters in all experiments are selected by grid search.

We carry out three simulations to verify our theoretical statements. The first simulation is devoted to illustrating the power of communications in DKRR. The second simulation is performed to demonstrate the relation between the generalization ability of DKRR( $\ell$ ) and the number of training samples for fixed numbers of local machines and communications. The third simulation focuses on comparisons on the training complexities of DKRR and DKRR( $\ell$ ).

Before carrying out simulations, we describe how the synthetic data is generated. The inputs  $\{x_i\}_{i=1}^N$  of training samples are independently drawn according to the uniform distribution on the (hyper-)cube  $[0, 1]^d$  with  $d = 1$  or  $d = 3$ . The corresponding outputs  $\{y_i\}_{i=1}^N$  are generated from the regression models  $y_i = g_j(x_i) + \varepsilon_i$  for  $i = 1, 2, \dots, N$  and  $j = 1, 2$ , where  $\varepsilon_i$  is the independent Gaussian noise  $\mathcal{N}(0, 0.2)$ ,

$$g_1(x) = \begin{cases} x & \text{if } 0 < x \leq 0.5, \\ 1 - x & \text{if } 0.5 < x \leq 1 \end{cases}$$

for 1-dimensional data, and

$$g_2(x) = \begin{cases} (1 - \|x\|_2)^6(35\|x\|_2^2 + 18\|x\|_2 + 3) & \text{if } 0 < \|x\|_2 \leq 1, \\ 0 & \text{if } \|x\|_2 > 1 \end{cases}$$

for 3-dimensional data. The inputs  $\{x'_i\}_{i=1}^{N'}$  of testing samples are also drawn independently according to the uniform distribution on the (hyper-)cube  $[0, 1]^d$  but the corresponding outputs  $\{y'_i\}_{i=1}^{N'}$  are generated by  $y'_i = g_j(x'_i)$ . It can be found in (Wu, 1995; Schaback and Wendland, 2006) that  $g_1 \in W_1^1$  and  $g_2 \in W_3^4$ , where  $W_d^\alpha$  represents the  $\alpha$ -order Sobolev space on  $[0, 1]^d$ . If we define  $K_1(x, x') = 1 + \min(x, x')$  and  $K_2(x, x') = h(\|x - x'\|_2)$  with

$$h(r) = \begin{cases} (1 - r)^4(4r + 1) & \text{if } 0 < r \leq 1, \\ 0 & \text{if } r > 1, \end{cases}$$

then we know (Wu, 1995; Schaback and Wendland, 2006) that  $K_1$  and  $K_2$  are reproducing kernels for  $W_1^1$  and  $W_3^2$ , respectively. Obviously,  $g_1 \in \mathcal{H}_{K_1}$  and  $g_2 \in \mathcal{H}_{K_2}$ . In the training process of DKRR and DKRR( $\ell$ ), we uniformly distribute  $N$  training samples to  $m$  local machines.

**Simulation 1:** We generate 10000 samples for training and 1000 samples for testing. The number  $m$  of local machines varies from  $\{20, 40, 60, \dots, 480\}$  for 1-dimensional data, and varies from  $\{2, 4, 6, \dots, 60\}$  for 3-dimensional data. The testing results are shown in Figure 4 and Figure 5. Figure 4 shows the relation between MSE and the number of local machines by different numbers of communications. From Figure 4, we can conclude the following four assertions. 1) When  $m$  is not too large, AEs are always comparable to GMSEs. There exists an upper bound of  $m$ , denoted by  $m_B$  (e.g.,  $m_B \approx 40$  for  $d = 1$  and  $m_B \approx 6$  for  $d = 3$ ), larger than which AE curves increase dramatically and far from the GMSE curves. This verifies the theoretical statement in Theorem 2. 2) AECs also

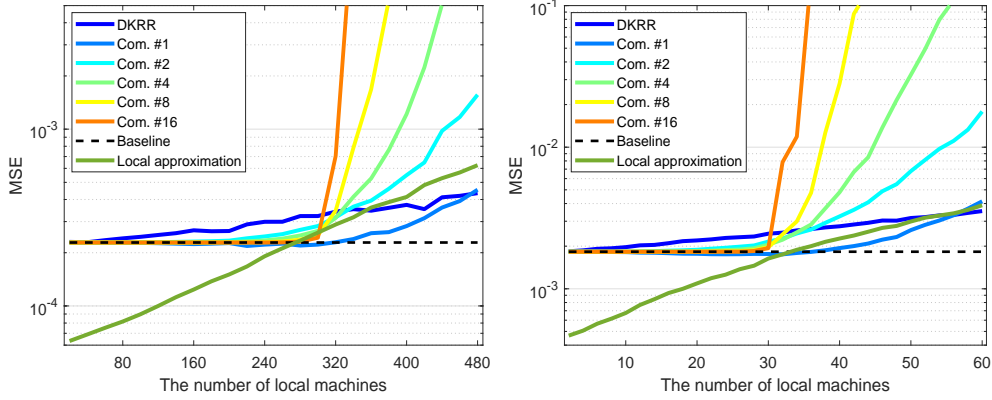


Figure 4: The relation between MSE and the number of local machines for fixed numbers of communications. The left figure and right figure are respectively the results on the 1-dimensional data and the 3-dimensional data. ‘Com. #’ represents the number of communications.

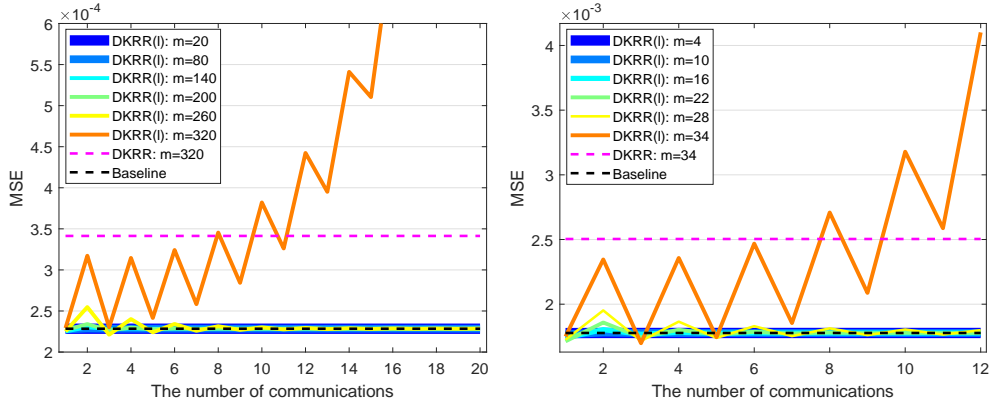


Figure 5: The relation between MSE and the number of communications for fixed numbers of local machines. The left figure and right figure are respectively the results on the 1-dimensional data and the 3-dimensional data.

have the upper bound, such as  $m_B \approx 200$  for  $d = 1$  and  $\ell = 2$ , which is much larger than that of AEs. This result confirms Theorem 4 by showing the power of communications in the sense that communications in DKRR can help to relax the restriction on  $m$ . 3) The upper bound  $m_B$  of AECs increases with the number of communication increasing, which implies the necessity of communications and verifies Theorem 4 by showing that the upper bound in (17) is monotonically increasing with respect to the number of communications. 4) Different from DKRR which cannot sufficiently embody the best approximation ability of local estimator,  $\text{DKRR}(\ell)$  succeeds in presenting an upper bound of  $m$  which is close to  $m_1$  in Figure 2.

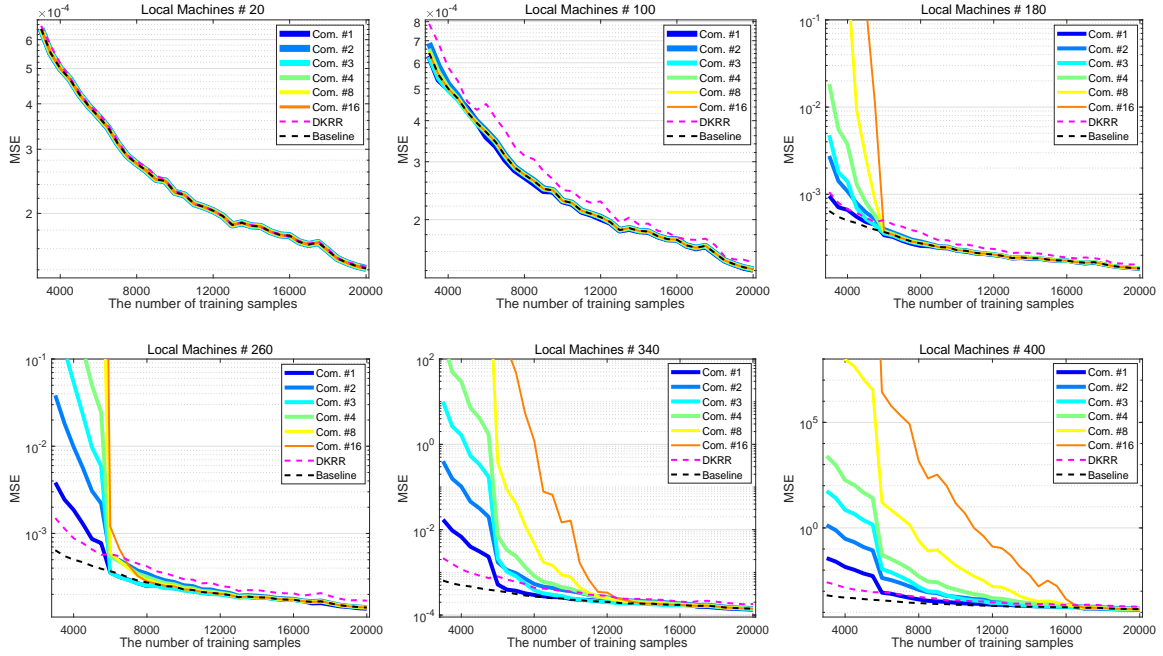


Figure 6: The relation between the number of training samples and MSE on the 1-dimensional data.

The relation between MSE and the number of communications by different numbers of local machines is illustrated in Figure 5. We can see that there exists an upper bound of  $m$  (e.g., about 280 for 1-dimensional data and 30 for 3-dimensional data), less than which AECs are guaranteed to converge to GMSEs with a fast rate, and larger than which AECs diverge dramatically. This verifies Proposition 12 by comparing  $\sum_{j=1}^m \frac{|D_j|}{|D|} \mathcal{Q}_{D_j, \lambda}^2 (\mathcal{R}_{D_j, \lambda} + \mathcal{R}_{D, \lambda})$  with 1.

**Simulation 2:** This simulation investigates the generalization performance of DKRR and  $\text{DKRR}(\ell)$  with varying the number of training samples  $N \in \{3000, 3500, 4000, \dots, 20000\}$  and fixing the number of testing samples  $N' = 1000$ . The regression results are shown in Figure 6 and Figure 7, from which it can be drawn the following conclusions. 1) For fixed numbers of local machines and communications, AE curves are getting closer and closer to GMSE curves, and AEC curves are converging to GMSE curves with the number of training samples increasing. This verifies our theoretical assertions of Theorem 2 and Theorem 4. 2) For each fixed number of local machines, there exists a lower bound for the number of training samples, denoted by  $N_B$  (e.g.,  $N_B \approx 6000$  for  $d = 1$  and  $m = 180$ , and  $N_B \approx 13500$  for  $d = 3$  and  $m = 40$ ), larger than which, the generalization performance of  $\text{DKRR}(\ell)$  is significantly superior than that of DKRR, and AECs converge to GMSEs with the number of communications increasing. Obviously, the bound  $N_B$  increases as the number of local machines increases. This also confirms the conclusion in Theorem 4.

**Simulation 3:** In this simulation, we compare the training complexities of DKRR and  $\text{DKRR}(\ell)$ . The computational complexity analysis for the training flow in Appendix B

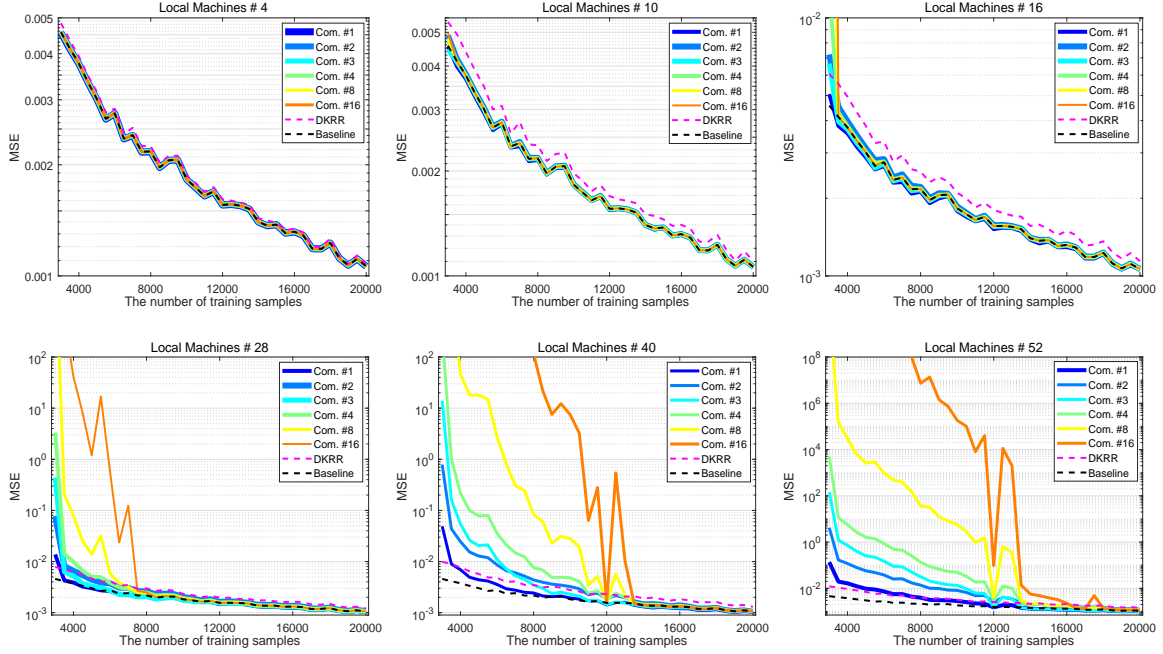


Figure 7: The relation between the number of training samples and MSE on the 3-dimensional data.

is given in Table 1, in which the computational complexities of Step 1, Step 4 and Step 6 refer to those on each local machine, provided that parallelization is implemented. It should be noted that the time of data transmission in  $\text{DKRR}(\ell)$  is not considered, and the computational complexity of Step 1 is also that of DKRR. Thus, the training complexities of DKRR and  $\text{DKRR}(\ell)$  are  $O(N^2\tau/m + N^3/m^3)$  and  $O(N^2\tau/m + N^3/m^3 + N^2\ell/m + mN\ell)$ , respectively. We define

$$\Omega_{\text{DKRR}} = N^2\tau/m + N^3/m^3, \quad \Omega_{\text{DKRR}(\ell)} = N^2\tau/m + N^3/m^3 + N^2\ell/m + mN\ell.$$

It can be seen that  $\Omega_{\text{DKRR}}$  monotonously decreases with  $m$  increasing;  $\Omega_{\text{DKRR}(\ell)}$  first decreases and then increases with  $m$  increasing, and the number  $m$  with minimum  $\Omega_{\text{DKRR}(\ell)}$  is given by

$$m^* = \sqrt{\frac{\sqrt{(\tau + \ell)^2 + 12\ell} + (\tau + \ell)}{2\ell}} \sqrt{N}.$$

Once data and the kernel function are given, the complexity  $\tau$  of calculating a kernel value can be considered as a constant. Provided  $\ell$  is large enough, we have  $m^* \approx \sqrt{N}$ , from which a rough conclusion can be drawn as follows: The training complexity of  $\text{DKRR}(\ell)$  mainly focuses on the local process (i.e., Step 1 of the training flow in Appendix B) when  $m < \sqrt{N}$ , while it mainly focuses on the communications (i.e., from Step 4 to Step 7 of the training flow in Appendix B) when  $m > \sqrt{N}$ . In the following, some numerical results are reported to verify the above analysis of training complexity.

Item	Step 1				Step2	Step4	Step5	Step6		Step7
	$\mathbb{K}_{D_k, D_j}$	$M_{D_j, \lambda}$	$\vec{\alpha}_{D_j}$	$\mathbb{K}_{D_k, D_j} \vec{\alpha}_{D_j}$	$\vec{f}_{D, \lambda, D_k}^0$	$\vec{G}_{D_j, \lambda, k, \ell}$	$\vec{G}_{D, \lambda, k, \ell}$	$\vec{\beta}_{j, \ell}$	$\vec{H}_{D_j, \lambda, k, \ell}$	$\vec{f}_{D, \lambda, D_k}^\ell$
INDV	$\frac{N^2 \tau}{m^2}$	$\frac{N^3}{m^3}$	$\frac{N^2}{m^2}$	$\frac{N^2}{m^2}$	$N$	$\frac{N^2}{m^2}$	$N$	$\frac{N^2}{m^2}$	$\frac{N^2}{m^2}$	$N$
TOT	$O\left(\frac{N^2 \tau}{m} + \frac{N^3}{m^3}\right)$				$O(mN)$	$O\left(mN\ell + \frac{N^2 \ell}{m}\right)$				

Table 1: Computational complexity of the training flow in Appendix B. ‘INDV’ represents the complexity of calculating each individual item, and ‘TOT’ represents the total complexities of Step 1, Step 2 and the loop from Step 4 to Step 7.  $\tau$  represents the complexity of calculating a value of a kernel function  $K(\cdot, \cdot)$ .

We start by introducing some notations that are used below.  $\text{AE}_{N,m}$  represents the MSE of DKRR with  $m$  local machines for  $N$  training samples.  $\text{AEC}_{N,m,\ell}$  represents the MSE of DKRR( $\ell$ ) with  $m$  local machines and  $\ell$  communications for  $N$  training samples.  $\text{GMSE}_N$  represents the MSE of KRR with training  $N$  samples in a batch mode. The relative errors for DKRR and DKRR( $\ell$ ) are defined by  $\text{RE}_{N,m} = \frac{|\text{AE}_{N,m} - \text{GMSE}_N|}{\text{GMSE}_N}$  and  $\text{REC}_{N,m,\ell} = \frac{|\text{AEC}_{N,m,\ell} - \text{GMSE}_N|}{\text{GMSE}_N}$ , respectively. For DKRR with  $N$  training samples, we denote the maximum number  $m$  of local machines that satisfies  $\text{RE}_{N,m} < \varepsilon$  as  $\bar{m}_B^N$ , i.e.,  $\bar{m}_B^N = \max\{m | \text{RE}_{N,m} < \varepsilon\}$ . For DKRR( $\ell$ ) with  $N$  training samples and  $\ell$  communications, we denote the maximum number  $m$  of local machines that satisfies  $\text{REC}_{N,m,\ell} < \varepsilon$  as  $\hat{m}_B^{N,\ell}$ , i.e.,  $\hat{m}_B^{N,\ell} = \max\{m | \text{REC}_{N,m,\ell} < \varepsilon\}$ . In the experiments,  $\varepsilon$  is set as 0.05, and the computing capability of each local machine is assumed to be the same for simplicity.

We generate 20000 training samples and 1000 testing samples. The number  $m$  varies from  $\{20, 40, 60, \dots, 600\}$  for the 1-dimensional data, and varies from  $\{2, 4, 6, \dots, 60\}$  for the 3-dimensional data. The training time of DKRR with the number  $m$  changing from the minimum number (i.e., 20 for 1-dimensional data and 2 for 3-dimensional data) to  $\bar{m}_B^N$  and the training time of DKRR( $\ell$ ) with the number  $m$  changing from the minimum number to  $\hat{m}_B^{N,\ell}$  by different numbers  $\ell \in \{2, 4, 8, 16, 32\}$  are shown in Figure 8. The following observations can be made from these results. 1) Because there are no calculations for communication steps, DKRR naturally consumes the least time when  $m < \bar{m}_B^N$ . 2) For the two simulation datasets, we have  $\bar{m}_B^N \ll \hat{m}_B^{N,\ell}$ , resulting that DKRR( $\ell$ ) can further reduce the least training time of DKRR with the maximum number  $\bar{m}_B^N$  when the number  $m$  approaches to  $\hat{m}_B^{N,\ell}$ . However, the improvement is very limited for the 1-dimensional data, because the number  $\bar{m}_B^N$  of DKRR is sufficiently large ( $\bar{m}_B^N = 120$ ), and thus the marginal utility for the reduction of training time is very small when DKRR( $\ell$ ) further enlarges the maximum number of local machines. 3) With the number  $m$  increasing, the training time curves of DKRR( $\ell$ ) ( $\ell = 4, 8, 16, 32$ ) first decrease quickly and then increase slowly for the 1-dimensional data, and monotonously decrease for the 3-dimensional data. This phenomenon coincides with the aforementioned theoretical analysis of training complexity, because  $\hat{m}_B^{N,\ell} \approx 450 > \sqrt{N}$  for the 1-dimensional data, and  $\hat{m}_B^{N,\ell} \approx 50 < \sqrt{N}$  for the 3-dimensional data. In practical applications, we estimate the optimal number of local

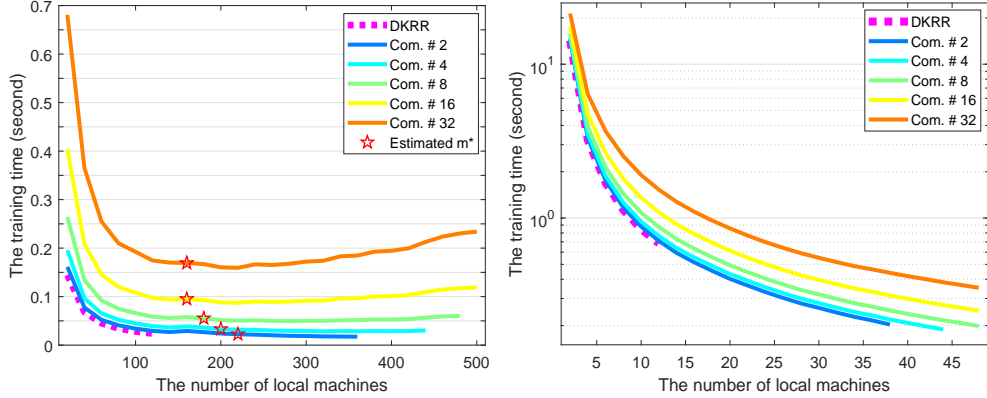


Figure 8: The relation between the number of local machines and the training time for fixed numbers of communications. The left figure and right figure are respectively the results on the 1-dimensional data and the 3-dimensional data with  $N = 20000$  training samples.

machines by the formula

$$\hat{m}_{N,\ell}^* = \begin{cases} m^*, & \text{if } \hat{m}_{\mathbb{B}}^{N,\ell} > m^*, \\ \hat{m}_{\mathbb{B}}^{N,\ell}, & \text{otherwise} \end{cases} \quad (56)$$

to expect it to have the lowest time consumption. The estimated  $m^*$  and the corresponding training time for  $\text{DKRR}(\ell)$  with  $\ell \in \{2, 4, 8, 16, 32\}$  on the 1-dimensional data are marked by red pentagrams in the left figure of Figure 8. It can be seen that the training time with the estimated number  $m^*$  is very close to the minimum of training time, which validates the correctness and reliability of equation (56).

We also record the results with varying the number of training samples  $N \in \{3000, 3500, 4000, \dots, 20000\}$  and fixing the number of testing samples  $N' = 1000$ . Figure 9 shows the maximum numbers  $\bar{m}_{\mathbb{B}}^N$  of DKRR and the maximum numbers  $\hat{m}_{\mathbb{B}}^{N,\ell}$  of  $\text{DKRR}(\ell)$  with varying the numbers of training samples and communications, and Figure 10 shows the training time of DKRR with  $\bar{m}_{\mathbb{B}}^N$  local machines and the training time of  $\text{DKRR}(\ell)$  with  $\hat{m}_{N,\ell}^*$  local machines. From these results, we have the following observations. 1) The maximum number of local machines increases as the number of training samples increases for both DKRR and  $\text{DKRR}(\ell)$ , but the growth of  $\text{DKRR}(\ell)$  is faster than that of DKRR. 2) The maximum number  $\hat{m}_{\mathbb{B}}^{N,\ell}$  of  $\text{DKRR}(\ell)$  is robust to the number  $\ell$  of communications when  $\ell > 4$ . This verifies the fast convergence as shown in Figure 5, and also implies that  $\text{DKRR}(\ell)$  can achieve a satisfactory generalization performance in a small number of communications. 3) For the 1-dimensional data,  $\text{DKRR}(\ell)$  does not have much advantage on the training time, and is even worse than DKRR when the number of training samples  $N \geq 10000$  and the number of communications  $\ell \geq 4$ . Furthermore, the training time increases significantly as the number of communications increases. The main reason has two folds. On one hand, the marginal utility of enlarging the number of local machines diminishes when  $\bar{m}_{\mathbb{B}}^N$  of DKRR is large enough to efficiently deal with large-scale data. On

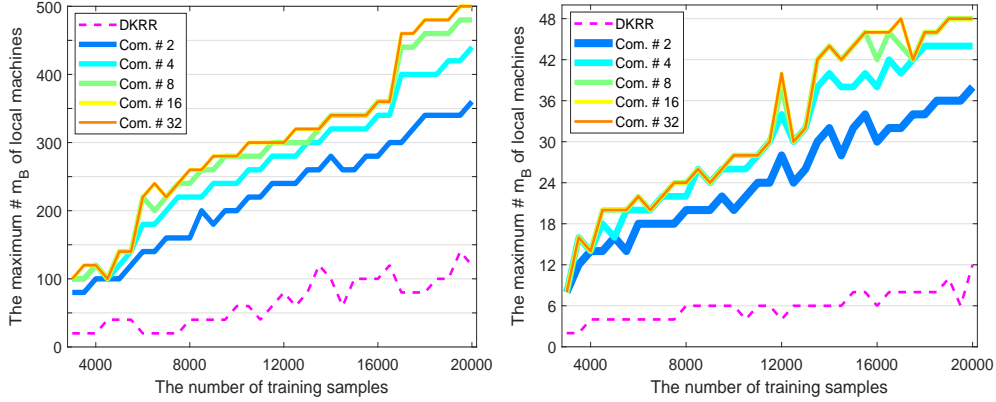


Figure 9: The relation between the number of training samples and the maximum number of local machines. The left figure and right figure are respectively the results on the 1-dimensional data and the 3-dimensional data.

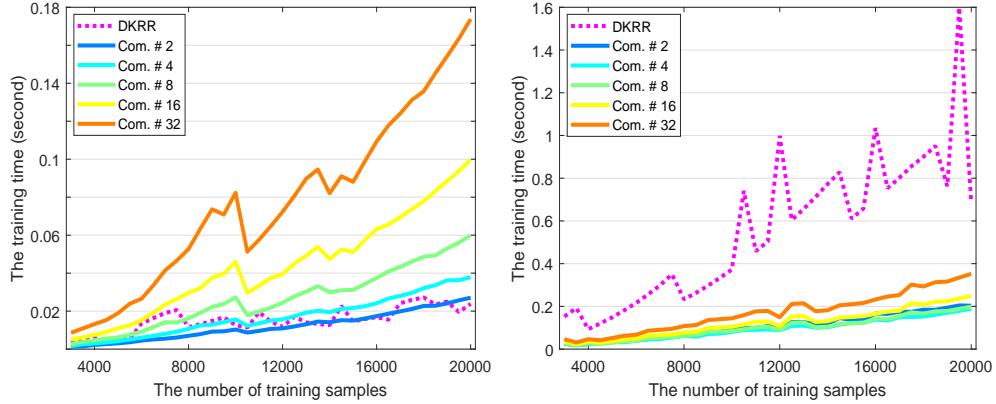


Figure 10: The relation between the number of training samples and the training time with the optimal number of local machines. The left figure and right figure are respectively the results on the 1-dimensional data and the 3-dimensional data.

the other hand, the training complexity of  $\text{DKRR}(\ell)$  mainly focuses on the communications when the number  $m$  is sufficiently large. 4) For the 3-dimensional data, the number  $\bar{m}_B^N$  of  $\text{DKRR}$  is relatively small (e.g.,  $\bar{m}_B^N = 12$  for  $N = 20000$  training samples).  $\text{DKRR}(\ell)$  shows the significantly superior performance on the training time when compared to  $\text{DKRR}$ , even for the large number  $\ell = 32$ . This is the main focus of this paper, on an enlargement of the maximum number of local machines guaranteeing optimal learning rates when  $\text{DKRR}$  has a small number  $\bar{m}_B^N$  of local machines.

All these simulations verify our theoretical statements and show the power of communications in distributed learning.

## Acknowledgments

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## Appendix A. Proofs of Lemmas in Section 5

We use a new concentration inequality for positive operators (Minsker, 2017), which was a refined estimate for the well known Bernstein inequality for matrix (Tropp, 2015). This lemma has been adopted in (Dicker et al., 2017) and (Guo and Shi, 2019) to derive optimal learning rates for kernel-based spectral algorithms and coefficient-regularization algorithms.

**Lemma 17** *Let  $R > 0$  be a positive real constant and consider a finite sequence of self-adjoint Hilbert-Schmidt operators  $\{\xi_i\}_{i=1}^n$  satisfying  $E[\xi_i] = 0$  and  $\|\xi_i\| \leq R$  almost surely. Suppose there are constants  $V, W > 0$  such that  $\|E[(\sum_{i=1}^n \xi_i)^2]\| \leq V$  and  $\text{Tr}(E[(\sum_{i=1}^n \xi_i)^2]) \leq VW$ . For all  $t \geq V^{1/2} + R/3$ , there holds*

$$P\left(\left\|\sum_{i=1}^n \xi_i\right\| \geq t\right) \leq 4W \exp\left(-\frac{t^2}{2(V + Rt/3)}\right).$$

**Proof of Lemma 6.** Define the random variable

$$\eta(x) = (L_K + \lambda I)^{-1/2} K_x \otimes K_x (L_K + \lambda I)^{-1/2}, \quad x \in \mathcal{X}.$$

It is easy to see that  $\eta(x)$  is a self-adjoint operator for  $x \in \mathcal{X}$ . Furthermore,

$$E[\eta] = (L_K + \lambda I)^{-1/2} L_K (L_K + \lambda I)^{-1/2},$$

and

$$\frac{1}{|D|} \sum_{(x,y) \in D} \eta(x) = (L_K + \lambda I)^{-1/2} L_{K,D} (L_K + \lambda I)^{-1/2}.$$

Set

$$\xi(x) = \eta(x) - E[\eta], \quad x \in \mathcal{X}.$$

Then, we have  $E[\xi] = 0$ . If  $\{\sigma_\ell, \psi_\ell\}_{\ell \geq 1}$  are normalized eigenpairs of the integral operator  $L_K$  on  $L^2_{\rho_X}$ , then  $\|\sqrt{\sigma_\ell} \psi_\ell\|_K = 1$  when  $\sigma_\ell > 0$ . Thus,  $\{\sqrt{\sigma_\ell} \psi_\ell : \sigma_\ell > 0\}$  forms an orthonormal basis of  $\mathcal{H}_K$ . This implies that

$$\begin{aligned} \|\eta(x)\|^2 &= \sup_{\|f\|_K=1} \|\eta(x)f\|_K^2 \leq \sum_{\ell} \|(L_K + \lambda I)^{-1/2} K_x \otimes K_x (L_K + \lambda I)^{-1/2} \sqrt{\sigma_\ell} \psi_\ell\|_K^2 \\ &= \sum_{\ell} \frac{\sigma_\ell \psi_\ell^2(x)}{\lambda + \sigma_\ell} \|(L_K + \lambda I)^{-1/2} K_x\|_K^2 = \left[ \sum_{\ell} \frac{\sigma_\ell \psi_\ell^2(x)}{\lambda + \sigma_\ell} \right]^2. \end{aligned}$$



Then,

$$\|\xi(x)\| = \|\eta(x) - E[\eta]\| \leq \max_{x \in \mathcal{X}} \|\eta(x)\| + 1 \leq \max_{x \in \mathcal{X}} \sum_{\ell} \frac{\sigma_{\ell} \psi_{\ell}^2(x)}{\lambda + \sigma_{\ell}} + 1 \leq (\kappa^2 + 1)\lambda^{-1},$$

where we use  $\|E[\eta]\| = \|(L_K + \lambda I)^{-1} L_K\| \leq 1$  and  $0 < \lambda \leq 1$ . For an arbitrary  $f \in \mathcal{H}_K$ , it follows from  $[\eta(x)]^2 f = \eta(x)[\eta(x)f]$  that

$$[\eta(x)]^2 = \langle K_x, (L_K + \lambda I)^{-1} K_x \rangle_K \eta(x) = \sum_{\ell} \frac{\sigma_{\ell} \psi_{\ell}^2(x)}{\lambda + \sigma_{\ell}} \eta(x).$$

Then, we get from  $\xi(x) = \eta(x) - E[\eta]$  that

$$\begin{aligned} \left\| E \left[ \left( \sum_{(x,y) \in D} \xi(x) \right)^2 \right] \right\| &\leq |D| (\|E[(\eta)^2]\| + \|(E[\eta])^2\|) \\ &\leq |D| (\kappa^2 \lambda^{-1} \|E[\eta]\| + \|E[\eta]\|^2) \leq |D| \lambda^{-1} (\kappa^2 + 1). \end{aligned}$$

Moreover,

$$\begin{aligned} \text{Tr} \left( E \left[ \left( \sum_{(x,y) \in D} \xi(x) \right)^2 \right] \right) &\leq |D| \text{Tr} (E[(\eta)^2] - (E[\eta])^2) \\ &\leq |D| (\kappa^2 \lambda^{-1} \text{Tr}(E[\eta]) + \text{Tr}(E[\eta])^2) \leq |D| \lambda^{-1} \mathcal{N}(\lambda) (\kappa^2 + 1). \end{aligned}$$

Plugging  $R = \lambda^{-1}(\kappa^2 + 1)$ ,  $V = |D| \lambda^{-1}(\kappa^2 + 1)$  and  $W = \mathcal{N}(\lambda)$ , we have that for all  $t \geq \frac{\sqrt{(\kappa^2 + 1)}}{\sqrt{\lambda|D|}} + \frac{\kappa^2 + 1}{3\lambda|D|}$ , there holds

$$P[\mathcal{R}_{D,\lambda} \geq t] = P \left[ \left\| \sum_{(x,y) \in D} \xi(x) \right\| \geq |D|t \right] \leq 4\mathcal{N}(\lambda) \exp \left( -\frac{\lambda|D|t^2}{2(\kappa^2 + 1) + (\kappa^2 + 1)t/3} \right).$$

Set  $4\mathcal{N}(\lambda) \exp \left\{ -\frac{\lambda|D|\varepsilon^2}{(\kappa^2 + 1)(2 + \varepsilon/3)} \right\} = \delta$ . With confidence  $1 - \delta$ ,

$$\varepsilon \leq \frac{(\kappa^2 + 1) \log \frac{4\mathcal{N}(\lambda)}{\delta}}{3\lambda|D|} + \sqrt{\frac{2(\kappa^2 + 1) \log \frac{4\mathcal{N}(\lambda)}{\delta}}{\lambda|D|}}.$$

Setting

$$t = \frac{(\kappa^2 + 1) \log \frac{4\mathcal{N}(\lambda)}{\delta}}{3\lambda|D|} + \sqrt{\frac{2(\kappa^2 + 1) \log \frac{4\mathcal{N}(\lambda)}{\delta}}{\lambda|D|}} \geq \frac{\sqrt{(\kappa^2 + 1)}}{\sqrt{\lambda|D|}} + \frac{\kappa^2 + 1}{3\lambda|D|},$$

we then get that (21) holds with confidence  $1 - \delta$ . This completes the proof of Lemma 6.  $\blacksquare$

We then prove Lemma 7 by using Lemma 6.

**Proof of Lemma 7.** Since  $\delta \geq 4 \exp\{2C_1^* \mathcal{B}_{D,\lambda}\}$ , it follows from Lemma 6 that  $\mathcal{R}_{D,\lambda} < \frac{1}{2}$  holds with confidence  $1 - \delta$ . Then,

$$\begin{aligned} & (L_K + \lambda I)^{1/2} (L_{K,D} + \lambda I)^{-1} (L_K + \lambda I)^{1/2} \\ = & (L_K + \lambda I)^{1/2} [(L_{K,D} + \lambda I)^{-1} - (L_K + \lambda I)^{-1}] (L_K + \lambda I)^{1/2} + I = I \\ + & (L_K + \lambda I)^{-1/2} (L_K - L_{K,D}) (L_K + \lambda I)^{-1/2} (L_K + \lambda I)^{1/2} (L_{K,D} + \lambda I)^{-1} (L_K + \lambda I)^{1/2}. \end{aligned}$$

Thus,

$$\begin{aligned} & \| (L_K + \lambda I)^{1/2} (L_{K,D} + \lambda I)^{-1} (L_K + \lambda I)^{1/2} \| \\ \leq & 1 + \frac{1}{2} \| (L_K + \lambda I)^{1/2} (L_{K,D} + \lambda I)^{-1} (L_K + \lambda I)^{1/2} \|. \end{aligned}$$

This implies

$$\| (L_K + \lambda I)^{1/2} (L_{K,D} + \lambda I)^{-1} (L_K + \lambda I)^{1/2} \| \leq 2.$$

Since  $t^{1/2}$  is operator monotone (Bhatia, 1997, Chap.4), we have

$$\mathcal{Q}_{D,\lambda} \leq \| [(L_K + \lambda I)^{1/2} (L_{K,D} + \lambda I)^{-1} (L_K + \lambda I)^{1/2}]^{1/2} \| \leq \sqrt{2}.$$

This completes the proof of Lemma 7. ■

## Appendix B. Training and Testing Flows for DKRR( $\ell$ )

Generally speaking, it is difficult to communicate functions in practice. Thus, the implementation of DKRR( $\ell$ ) requires communications of additional information. In this part, we numerically realize DKRR( $\ell$ ) by communicating the inputs of data for the sake of simplicity, though there are numerous state-of-the-art approaches to approximate the gradient matrix via much less of communication loss.

### Training Flow:

Step 1 (local process). Run KRR (2) on the  $j$ -th local machine with data  $D_j$  and communicate  $D_j(x) = \{x : (x, y) \in D\}$  to the  $k$ -th local machine for  $k = 1, \dots, m$ . Store  $m$  matrices of size  $|D_j| \times |D_k|$ ,  $\mathbb{K}_{D_k, D_j} := \{K(x, x')\}_{x \in D_k(x), x' \in D_j(x)}$  with  $k = 1, \dots, m$ , the matrix  $M_{D_j, \lambda} = (\mathbb{K}_{D_j, D_j} + \lambda |D_j| \mathbb{I}_{D_j})^{-1}$ , and the vector  $\vec{\alpha}_{D_j} = M_{D_j, \lambda} y_{D_j}$ , where  $\mathbb{I}$  is the unit matrix of size  $|D_j| \times |D_j|$ .

Step 2 (synthesization) On the  $j$ -th local machine, communicate  $m$  vectors of size  $|D_k|$ ,  $\mathbb{K}_{D_k, D_j} \vec{\alpha}_{D_j}$  for  $k = 1, \dots, m$  to the global machine. Synthesize  $m$  global vectors  $\vec{f}_{D, \lambda, D_k}^0 = \sum_{j=1}^m \frac{|D_j|}{|D|} \mathbb{K}_{D_k, D_j} \vec{\alpha}_{D_j}$  for  $k = 1, \dots, m$ .

Step 3 (distributing) For  $\ell = 1, 2, \dots$ , distribute  $\vec{f}_{D, \lambda, D_j}^{\ell-1}$  to the  $j$ -th local machine.

Step 4 (local gradients) On the  $j$ -th local machine, compute  $m$  gradient vectors  $\vec{G}_{D_j, \lambda, k, \ell} = \frac{\mathbb{K}_{D_k, D_j}}{|D_j|} (\vec{f}_{D, \lambda, D_j}^{\ell-1} - y_{D_j}) + \lambda \vec{f}_{D, \lambda, D_k}^{\ell-1}$  for  $k = 1, \dots, m$  and communicate these  $m$  vectors to the global machine.

Step 5 (synthesizing gradients) Synthesize  $m$  global gradient vectors,  $\vec{G}_{D, \lambda, k, \ell} := \sum_{j=1}^m \frac{|D_j|}{|D|} \vec{G}_{D_j, \lambda, k, \ell}$  for  $k = 1, \dots, m$ . Distribute  $m$  vectors  $\vec{G}_{D, \lambda, k, \ell}$  with  $k = 1, \dots, m$  to each local machine.

Step 6 (KRR on gradient data) On the  $j$ -th local machine, compute the vector  $\vec{\beta}_{j,\ell} = M_{D_j,\lambda} \vec{G}_{D,\lambda,j,\ell}$  and  $m$  vectors  $\vec{H}_{D_j,\lambda,k,\ell} := \vec{G}_{D,\lambda,k,\ell} - \mathbb{K}_{D_k,D_j} \vec{\beta}_{j,\ell}$  for  $k = 1, \dots, m$ . Communicate  $m$  vectors  $\vec{H}_{D_j,\lambda,k,\ell}$  with  $k = 1, \dots, m$  to the global machine.

Step 7 (final estimator) Generate  $m$  vectors

$$\vec{f}_{D,\lambda,D_k}^\ell = \vec{f}_{D,\lambda,D_k}^{\ell-1} - \frac{1}{\lambda} \sum_{j=1}^m \frac{|D_j|}{|D|} \vec{H}_{D_j,\lambda,k,\ell}, \quad k = 1, \dots, m$$

and transmit  $\vec{f}_{D,\lambda,D_j}^\ell$  to the  $j$ -th local machine with  $j = 1, \dots, m$ .

**Testing Flow:** given vector  $D'(x) := (x'_1, \dots, x'_{|D'|})$  which consists of  $|D'|$  query points.

Step 1 (local estimates) Transmit  $D'(x)$  to  $m$  local machines. On the  $j$ -th local machine, store vectors of size  $|D'|$ ,  $\vec{K}_{D',D_j} \vec{\alpha}_{D_j}$ .

Step 2 (global estimates) Compute test vector of size  $|D'|$ ,

$$\vec{f}_{D,\lambda}^0(D') := \sum_{j=1}^m \frac{|D_j|}{|D|} \vec{K}_{D',D_j} \vec{\alpha}_{D_j}.$$

Step 3 (local gradients) For  $\ell = 1, 2, \dots$ , distribute  $\vec{f}_{D,\lambda}^{\ell-1}(D')$  to  $m$  local machines, and compute

$$G_{D_j,\lambda,\ell}(D') := \frac{\vec{K}_{D',D_j}}{|D_j|} (\vec{f}_{D,\lambda,D_j}^{\ell-1} - y_{D_j}) + \lambda \vec{f}_{D,\lambda}^{\ell-1}(D'),$$

where  $\vec{f}_{D,\lambda,D_j}^{\ell-1}$  is obtained from the training flow.

Step 4 (global gradients) Transmit  $G_{D_j,\lambda,\ell}(D')$  with  $j = 1, \dots, m$  to the global machine and get the global gradient vector

$$G_{D,\lambda,\ell}(D') = \sum_{j=1}^m \frac{|D_j|}{|D|} G_{D_j,\lambda,\ell}(D').$$

Step 5 (final estimator) Generate the vector of estimators

$$\vec{f}_{D,\lambda}^\ell(D') = \vec{f}_{D,\lambda}^{\ell-1}(D') - \frac{1}{\lambda} \sum_{j=1}^m \frac{|D_j|}{|D|} [G_{D,\lambda,\ell}(D') - \vec{K}_{D',D_j} \vec{\beta}_{j,\ell}].$$

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