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A Nonconvex Framework for Structured Dynamic Covariance Recovery

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Abstract

We propose a flexible, yet interpretable model for high-dimensional data with time-varying second-order statistics, motivated and applied to functional neuroimaging data. Our approach implements the neuroscientific hypothesis of discrete cognitive processes by factorizing covariances into sparse spatial and smooth temporal components. Although this factorization results in parsimony and domain interpretability, the resulting estimation problem is nonconvex. We design a two-stage optimization scheme with a tailored spectral initialization, combined with iteratively refined alternating projected gradient descent. We prove a linear convergence rate up to a nontrivial statistical error for the proposed descent scheme and establish sample complexity guarantees for the estimator. Empirical results using simulated data and brain imaging data illustrate that our approach outperforms existing baselines.

Keywords: dynamic covariance, structured factor model, alternating projected gradient descent, time series data, functional connectivity

1. Introduction

We propose and evaluate a model for dynamic functional brain network connectivity, defined as the time-varying covariance of associations between brain regions (Fox and Raichle, 2007). Understanding the variation in brain connectivity between individuals is believed to be a crucial step towards uncovering the mechanisms of neural information processing (Sakoğlu et al., 2010; Chang et al., 2016), with potentially transformative applications in understanding and treating neurological and neuropsychiatric disorders (Calhoun et al., 2014).

In the neuroscience literature, estimators for time-varying covariances range from sliding window methods to hidden Markov models. The commonly used sliding-window sample covariance estimator is computationally efficient (Preti et al., 2017). However, this estimate is sensitive to the length of the selected window and spurious correlations may occur when the underlying window length is not specified correctly (Leonardi and Ville, 2015). Discrete-state hidden Markov models construct interpretable estimates of brain connectivity in terms of recurring connectivity patterns (Vidaurre et al., 2017), yet fail to capture the smooth nature of brain dynamics (Shine et al., 2016a,b). These shortcomings motivate a new approach. Specifically, our proposed approach implements the neuroscientific hypothesis that brain functions are interactions between cognitive processes (Posner et al., 1988), which we model as weighted combinations of low-rank components (Andersen et al., 2018). Beyond neuroscientific foundations, high-dimensional data often have a low-dimensional representation (Udell and Townsend, 2019), and low rank can help prevent overfitting (Udell et al., 2016). Specifically, we propose a structured and smooth low-rank time-varying covariance model inspired by the observed sparsity of brain factors (Eavani et al., 2012), and temporal dynamics of brain activity (Shine et al., 2016a,b). Hence, we constrain the temporal components to be smoothly varying via projection to a temporal kernel and restrict the sparsity of the spatial components via hard-thresholding, respectively.

We estimate the parameters of the resulting model using a first-order optimization scheme that is analogous to a Burer-Monteiro factorization (Burer and Monteiro, 2003, 2005). While the first-order approach reduces computational complexity as compared to semidefinite programming, the resulting optimization program is nonconvex, and special care is needed to design and analyze an optimization scheme that avoids converging to bad local optima. To this end, we build on the growing literature studying matrix estimation problems (Candès et al., 2015; Chi et al., 2019) using a two-stage algorithm. First, spectral initialization is used to find an initial point within a local region where the objective satisfies local regularity conditions. Next, the projected gradient descent is used to refine the estimate and find a stationary point of the objective.

In summary, our contributions include a novel dynamic covariance model motivated by neuroscientific models of functional brain connectivity networks. We provide an efficient procedure for the estimation along with convergence analysis and sample complexity. Specifically, under the assumption that spatial components are shared across time, we develop a structured spectral initialization method, which effectively uses available samples and provides a better spatial estimate than separate initialization per individual. We prove linear convergence of the factored gradient method to an estimate with a nontrivial statistical error and provide a non-asymptotic bound on the statistical error when data are Gaussian. Experiments show that the model successfully recovers temporal smoothness and detects temporal changes induced by task activation. The code to implement our procedure is available at: `https://github.com/koyejo-lab/dynamicCov.git`.

2. Background

In this section, we first introduce the notation used throughout the manuscript. Then, we introduce the problem formulation and related work.

2.1 Notation

The inner product of two matrices is denoted as $\langle X, Y\rangle = \mathrm{tr}(X^T Y)$. For a matrix X, $\sigma_k(X)$ denotes the kth largest singular value, $\|X\|_F^2 = \mathrm{tr}(X^T X)$ denotes the Frobenius norm, $\|X\|_2 = \sigma_1(X)$ denotes the spectral norm, and $\|X\|_\infty = \max_{i,j} |X_{i,j}|$ denotes the maximum norm. For two symmetric matrices X and Y, $X \preceq Y$ means that $Y - X$ is positive semidefinite. The pseudoinverse of X is denoted by $X^\dagger$. The set of $K \times K$ rotation matrices is denoted as $\mathcal{O}(K)$. Let $x \in \mathbb{R}^d$, $\|x\|_0 = \sum_{i=1}^d \mathbf{1}_{\{x_i \neq 0\}}$ denotes the ℓ_0 norm and $\|x\|_2 = (\sum_{i=1}^d x_i^2)^{1/2}$ denotes the ℓ_2 norm. We use $\kappa(\cdot,\cdot)$ to denote a positive definite kernel function. The function $\mathrm{diag} : \mathbb{R}^K \to \mathbb{R}^{K\times K}$ converts a K-dimensional vector to a $K \times K$ diagonal matrix. For scalars a and b, $a \vee b$ denotes $\max(a, b)$ and $a \wedge b$ denotes $\min(a, b)$. We use $a \gtrsim b$ ($a \lesssim b$) to denote that there exists a constant $C > 0$ such that $a \geq Cb$ ($a \leq Cb$). We use $a \asymp b$ to denote $a \gtrsim b$ and $a \lesssim b$. We use $[J]$ to denote the index set $\{1, \ldots, J\}$.

2.2 Problem Statement

Given samples from N subjects recorded at J time points, denoted $x_j^{(n)} \in \mathbb{R}^P$, $n \in [N]$, $j \in [J]$, let $S_{N,j} = N^{-1} \sum_{n=1}^N x_j^{(n)} x_j^{(n)T}$ be the sample covariance across subjects at time j. We assume that the population covariance takes a factorized form as

$$\mathbb{E}(S_{N,j}) = \Sigma_j^\star + E_j = V^\star \mathrm{diag}(a_j^\star) V^{\star T} + E_j, \quad j \in [J], \tag{1}$$

where $\mathbb{E}(\cdot)$ denotes the expectation, $\Sigma_j^\star$ has a rank that is at most K, E_j is a noise matrix such that the largest singular value of E_j is strictly smaller than the smallest nonzero singular value of $\Sigma_j^\star$. This structured assumption allows us to separate the components of interest $\Sigma_j^\star$ from the nuisance components E_j. This factorization employs spatial components $V^\star = (v_1^\star, \ldots, v_K^\star) \in \mathbb{R}^{P\times K}$ that are time invariant and column-wise orthonormal. The matrix V corresponds to the top-K eigenvectors of the set of J covariance matrices: $\{\mathbb{E}(S_{N,j})\}_{j\in[J]}$. Analogously, $A^\star = (a_1^\star, \ldots, a_J^\star) \in \mathbb{R}^{K\times J}$ represents the temporal components. To facilitate the estimation in a high-dimensional setting, we further assume that the columns of $V^\star$ are sparse and belong to $\mathcal{C}_V(s^\star) = \{v \in \mathbb{R}^P : \|v\|_0 \leq s^\star, \|v\|_2 = 1\}$. Let $G \in \mathbb{R}^{J\times J}$ be a positive semidefinite kernel matrix whose entries are $G_{x,y} = \kappa(x, y)$ for $x, y \in [J]$, where the kernel κ is known a priori. Denote $G^\dagger = Q\Lambda Q^T$ as the eigendecomposition of $G^\dagger$, the generalized inverse of G, with $Q \in \mathbb{R}^{J\times J}$ being a matrix with orthonormal columns and $\Lambda \in \mathbb{R}^{J\times J}$ being a diagonal matrix with nonnegative entries. The rows of $A^\star$, denoted $A^\star_{k\cdot}$, $k \in [K]$, are smooth, bounded, and belong to $\mathcal{C}_A(c^\star, \gamma^\star) = \{\alpha = Qu \in \mathbb{R}^J : 0 \le \alpha_j \le c^\star, u^T \Lambda u \le \gamma^\star\}$. The kernel κ is used to model the temporal smoothness of the rows of $A^\star$ and the box constraint ensures that $\alpha_j \ge 0$, so the covariance model is positive semidefinite and is upper bounded by a positive constant for $j \in [J]$.

The eigenvalues of the kernel matrix G may decay quickly, which can result in numerically unstable algorithms when projecting onto the set $\mathcal{C}_A$. For example, the eigenvalues of a kernel matrix corresponding to the Sobolev kernel decay at a polynomial rate, while for the Gaussian kernel they decay at an exponential polynomial rate (Schölkopf and Smola, 2001). Instead of working with the kernel matrix G, we construct a low-rank approximation, $\tilde{G}$, of G by truncating small eigenvalues. Write $Q = (\tilde{Q}, Q_1)$, where the columns of $\tilde{Q}$ are eigenvectors of G corresponding to eigenvalues greater than or equal to δ_A, and $\tilde{\Lambda}^{-1} = \operatorname{diag}(\Lambda^{-1}_{jj} \ge \delta_A \mid j \in [J])$. Then $\tilde{G}^\dagger = \tilde{Q}\tilde{\Lambda}\tilde{Q}^T$. We define $\tilde{\mathcal{C}}_A(c, \gamma) = \{\alpha = \tilde{Q}u : 0 \le \alpha_j \le c, u^T \tilde{\Lambda} u \le \gamma\}$ and the rank of $\tilde{G}$ is denoted as $r(\tilde{G})$.

Under the model (1), we estimate the parameters $Z^\star = (V^{\star T}, A^\star)^T$ by minimizing the following objective

$$\min_Z f_N(Z) = \min_{\substack{v_k \in \mathcal{C}_V(s),\, k \in [K] \\ A_{k\cdot} \in \tilde{\mathcal{C}}_A(c,\gamma),\, k \in [K]}} \frac{1}{J} \sum_{j=1}^J \frac{1}{2} \| S_{N,j} - V \operatorname{diag}(a_j) V^T \|_F^2, \tag{2}$$

where $A_{k\cdot}$ is the kth row of A. Although f_N is nonconvex with respect to $Z = (V^T, A)^T$, the corresponding covariance loss $\ell_{N,j}(\Sigma_j) = \frac{1}{2}\|S_{N,j} - \Sigma_j\|_F^2$ is m-strongly convex and L-smooth with $m = L = 1$ (Nesterov, 2013). We use alternating projected gradient descent to update V and A. The selection of tuning parameters of $\mathcal{C}_V$ and $\tilde{\mathcal{C}}_A$ is discussed in Section 3.2.

2.3 Related Work

Dynamic covariance models are common for analyzing time series data in applications ranging from computational finance and economics (Engle et al., 2019) to epidemiology (Fox and Dunson, 2015) and neuroscience (Foti and Fox, 2019). Dynamic covariance models can be fully nonparametric with kernel functions encoding the temporal dependencies (Wu and Pourahmadi, 2003; Chen and Leng, 2016). In practice, however, it is common to impose an additional structure on the covariance model. For example, one can assume that the inverse covariance matrix is sparse and furthermore, follows a particular temporal dynamic. Such an approach is called dynamic graphical modeling as nonzero entries in the inverse covariance matrix encode the structure of a Markov network when data are Gaussian. In the dynamic graphical model setting, one can either impose temporal smoothness using regularization approaches (Kolar et al., 2010b; Kolar and Xing, 2012; Monti et al., 2014; Hallac et al., 2017; Gibberd and Nelson, 2017; Zhu and Koyejo, 2018; Geng et al., 2019) or using kernel smoothing (Song et al., 2009a; Kolar and Xing, 2009; Zhou et al., 2010; Kolar et al., 2010a; Kolar and Xing, 2011; Wang and Kolar, 2014; Qiu et al., 2016; Lu et al., 2018; Geng et al., 2020). We emphasize that the methods for dynamic graphical models assume that the data are sampled independently at different time points but are generated by related distributions. In contrast, functional graphical models treat the data as multivariate random functions (Li and Solea, 2018; Qiao et al., 2019; Zhao et al., 2019; Qiao et al., 2020; Zapata et al., 2021; Zhao et al., 2022, 2021). Wang et al. (2020) focused on the estimation and inference of a graph that underlies the data from a point process. Another popular approach to modeling dynamic covariance models is via factor models. Factor models can encode the temporal structure using latent kernel regularization (Paciorek, 2003; Kastner et al., 2017). For example, Andersen et al. (2018) encoded smooth temporal dynamics by introducing a latent Gaussian process prior. Li (2019) also used piecewise Gaussian process factors to capture combinations of gradual and abrupt changes. Along similar lines, our approach implements temporal and spatial structure through projection onto suitable constraint sets.

Our work is also related to dictionary learning (Olshausen and Field, 1997; Mairal et al., 2010), which can be viewed as a type of factorization where the signal is decomposed into atoms and coefficients. In this factorization, the sparsity is controlled through a sparse penalty on the coefficients. Mishne and Charles (2019) extended this approach to encode temporal data by constructing time-trace atoms with spatial coefficients. In comparison, our model has shared spatial structure and individual temporal structure.

Autoregressive models have been applied to model dynamic connectivity in fMRI (Song et al., 2009b; Qiu et al., 2016; Liégeois et al., 2019). Although autoregressive models employ modeling assumptions different from ours, they can capture smooth temporal dynamics of signals. However, autoregressive model forecasts can become unreliable in high-dimensional settings (Bańbura et al., 2010). To this end, various implementations of structured transition matrices (Davis et al., 2016; Ahelegbey et al., 2016; Skripnikov and Michailidis, 2019) have been proposed and shown to improve computational efficiency and prediction accuracy.

The optimization problem in (2) is nonconvex and is optimized by alternating minimization. Recent literature has established a linear convergence rate to global optima (Jain et al., 2013; Hardt, 2014; Gu et al., 2016; Chen et al., 2021; Yu et al., 2020a, 2018; Na et al., 2021, 2020). In particular, our work builds on Bhojanapalli et al. (2016), who showed linear convergence in V when the underlying objective function is strongly convex with respect to $X = VV^T$. Subsequently, Park et al. (2018) and Yu et al. (2020c) proved a linear convergence rate for nonsymmetric matrices. Unlike previous work, our factorization scheme $V\text{diag}(a_j)V^T$ imposes additional structure on eigenvalues, which has potential applications in regularizing graph-structured models (Kumar et al., 2020).

In nonconvex optimization, finding a good initialization in a local region is often useful to avoid convergence to bad local optima (e.g., $Z = 0$ is a trivial stationary point in our model). One common approach is to use the minimizer of a convex relaxation of the original problem as a starting point (Yu et al., 2020c). For some problems, spectral methods also provide good initialization (Chen and Candès, 2015). We employ a problem-specific spectral approach to develop a novel initialization method. After initialization, a first-order gradient descent method is sufficient to ensure convergence to the desired optima (Candès et al., 2015). Combining with structured constraints, Chen and Wainwright (2015) provided a theoretical framework for the projected gradient descent method when the constraint sets are convex. In our work, the iterates are projected onto a nonconvex set, which might increase the distance $\|V - V^\star R\|_F^2$. Therefore, we need a problem-specific analysis to quantify the expansion coefficient.

3. Methodology

We first introduce the proposed two-stage algorithm. Subsequently, we discuss the selection of the tuning parameters.

3.1 Two-stage Algorithm

We develop a two-stage algorithm to solve the optimization problem in (2). As the objective is nonconvex, a local iterative procedure may converge to bad local optima or saddle points. In the first stage of the algorithm, spectral decomposition is used to find an initialization point. In the second stage, projected gradient descent is used to locally refine the initial estimate and find a stationary point that is within the statistical error of the population parameters. Algorithm 1 summarizes our initialization procedure. Here, the eigendecomposition of $\{S_{N,j}\}_{j\in[J]}$ is performed to obtain initial estimates of $V^\star$ and $A^\star$. Specifically, the initialization uses the shared spatial structure of $\{\Sigma_j^\star\}_{j\in[J]}$ to increase the effective sample size. That is, the initial estimate V^0 is obtained from the eigenvectors corresponding to the largest K eigenvalues of the covariance matrix pooled over time, $M_N = J^{-1}\sum_{j=1}^J S_{N,j}$. The initial estimate of the temporal coefficients, A^0, is obtained by projecting $\{S_{N,j}\}_{j\in[J]}$ onto V^0.

Set $M_N = (NJ)^{-1}\sum_{j=1}^J \sum_{n=1}^N x_j^{(n)} x_j^{(n)T}$
Set $V^0 = (v_1^0, v_2^0, \ldots, v_k^0) \leftarrow$ top K eigenvectors of M_N
For $j = 1$ to $j = J$ and $k = 1$ to $k = K$
    $a_{k,j}^0 \leftarrow v_k^{0T} S_{N,j} v_k^0$
Set $A^0 = (a_{k,j}^0)_{k\in[K],j\in[J]}$
Output V^0, A^0

Algorithm 1: Spectral initialization

After initialization, we iteratively refine the estimates of V and A using an alternating projected gradient descent. In each iteration, the iterates V and A are updated using the gradient of f_N, where η denotes the step size. Note that we scale down the step size for the V update by J to balance the magnitude of the gradient. After a gradient update, we project the iterates onto the constraint sets $\mathcal{C}_V$ and $\tilde{\mathcal{C}}_A$ to enforce sparsity in V and smoothness in A. Details are given in Algorithm 2.

Set $V^0, A^0 =$ Spectral initialization$(\{x_j^{(n)}\}_{n\in[N],j\in[J]})$
While $|f_N(Z^{i-1}) - f_N(Z^{i-2})| > \varepsilon$
 $\widehat{A}^i \leftarrow A^{i-1} - \eta\nabla_A f_N(Z^{i-1})$
 $A^i \leftarrow$ Project rows of $\widehat{A}^i$ to $\tilde{\mathcal{C}}_A$
 $\widehat{V}^i \leftarrow V^{i-1} - \frac{\eta}{J}\nabla_V f_N(Z^{i-1})$
 $V^i \leftarrow$ Project columns of $\widehat{V}^i$ to $\mathcal{C}_V$
Output V, A

Algorithm 2: Dynamic covariance estimation

Although $\mathcal{C}_V$ is a nonconvex set, projection onto this set can be computed efficiently by picking the top-s largest entries in magnitude and then projecting the constructed vector to the unit sphere. Despite projecting onto a nonconvex set, we are able to show that the gradient and projection step jointly result in a contraction (see Appendix A). On the other hand, the projection onto the convex set $\tilde{\mathcal{C}}_A$ can be computed efficiently via convex programming: we project onto $\tilde{\mathcal{C}}_A$ by iteratively projecting onto $\{\alpha \in \mathbb{R}^J : 0 \le \alpha_j \le c, j \in [J]\}$ and $\{\alpha = \tilde{Q}u : u^T\tilde{\Lambda}u \le \gamma\}$, which gives us a point in the intersection of the sets by von Neumann's theorem (Escalante and Raydan, 2011).

3.2 Selection of Tuning Parameters

The parameters of the proposed model include the sparsity level s, the rank K, the kernel length scale l, the smoothness coefficient γ, the truncation level δ_A, and the upper bound c for the constraint. For some kernels, such as the Gaussian kernel, the Matérn five-half kernel, and other radial basis function kernels, one must also select the length scale parameter l, which captures the smoothness of the curves (i.e., $\{A^\star_{k\cdot}\}_{k\in[K]}$); for example, a Gaussian kernel function is $\kappa_l(x,y) = \sigma^2\exp\{-(x-y)^2/(2l^2)\}$, where l affects the slope of decay of the eigenvalues. We denote such kernel functions as κ_l rather than κ. Our theory suggests that δ_A should be upper bound by the magnitude of $\min_{j\in[J]}\sigma_K^2(\Sigma_j^\star)$ to obtain a good statistical error. Furthermore, δ_A is selected for numerical stability. In experiments, we find that $\delta_A = 10^{-5}$ is a good empirical choice and satisfies the sufficient conditions. In principle, we do not want to cut off any important signals, so we choose c as a value greater than $\max_{j\in[J]}\|S_{N,j}\|_2$ and $c^\star = \max_{j\in[J]}\|\Sigma_j^\star\|_2$. In terms of estimation performance, we observe that the selection of s and K has a greater effect than the selection of γ and l. Although the underselection of s and K leads to poor evaluation scores, the improper selection of l and γ has a relatively minor influence. Therefore, we adopt a two-stage approach to selecting parameters. In the first stage, we perform a grid search on s, K, γ, l and find the configuration that minimizes the Bayesian information criterion BIC $= \log N \sum_{k=1}^{K} \|v_k\|_0 - 2\widehat{L}_N$, where $\widehat{L}_N$ is the maximized Gaussian log-likelihood function. We notice that varying γ and l have a subtle influence on BIC. Consequently, in the second stage, we fix s, K with values selected in the first stage and select γ and l using a 5-fold cross-validation with the Gaussian log-likelihood, which is motivated by prior work on nonparametric dynamic covariances (Yin et al., 2010). Empirically, we find that tuning the length scale parameter l is more effective than tuning γ in producing globally smooth temporal structures (see Appendix G.5).

4. Theory

We provide theoretical guarantees on the algorithm described in the previous section. We show that given enough samples, that is, $N \gtrsim KP(\log P + \log J)$, the estimate converges linearly to a statistically good point with high probability. A statistically good point is one that is close to the population parameters as quantified by the statistical error.

4.1 Preliminaries

Before presenting our main theoretical results, we introduce two tools that will help us establish the results.

First, we discuss orthogonalization. The spatial component V produced by Algorithm 2 is not necessarily orthonormal. However, $V^\star$ has full rank, and if $\min_{Y \in \mathcal{O}(K)} \|V - V^\star Y\|_2^2 < 1$ is guaranteed at each iteration, then V also has full rank. As a result, the subspace spanned by columns of V is equal to the subspace spanned by columns of the orthogonalized version of it. To simplify the analysis of Algorithm 2, we add a QR decomposition step that orthogonalizes V after projection onto $\mathcal{C}_V$. That is, in each iteration, we compute

$$V^i_{ortho} \leftarrow V^i (L^i)^{-1} \quad \text{(QR decomposition)},$$

where L^i is the upper triangular matrix, with diagonal entries less than or equal to 1. Note that orthogonalization of V in each iteration of Algorithm 2 is not needed in practice and is only used to establish theoretical properties. This approach is commonly used in the literature (Jain et al., 2013; Zhao et al., 2015). We further note that the addition of QR decomposition only increases the distance of the iterate V^i to $V^\star R$ by a mild constant (Stewart, 1977; Zhao et al., 2015) (see Appendix A.3). Furthermore, QR decomposition increases the number of nonzero elements of the iterate V to at most Ks. As we consider the rank K to be fixed and $P \gtrsim s$, the effect of QR decomposition is mild. Our experiments further demonstrate that optimization with and without the QR decomposition step results in comparable performance.

Next, we introduce the notion of statistical error, which allows us to quantify the distance of the population parameters from the stationary point to which the optimization algorithm converges. Note that the notion of statistical error was used in the context of M-estimation (Loh and Wainwright, 2015). Let $\mathcal{B}_t = \{v \in \mathbb{R}^P \mid \|v\|_0 \le t, \|v\|_2 \le 1\}$ and

$$\Upsilon(r,t,h,\delta_A) = \{\{\Delta_j = V\mathrm{diag}(a_j)W^T\}_{j\in J} \mid v_k \in \mathcal{B}_t, w_k \in \mathcal{B}_t, A_{k\cdot}^T \tilde{G}^\dagger A_{k\cdot} \le h, k \in [r]\},$$

where $\tilde{G}$ is the truncation of G at the level of δ_A. We define the statistical error as

$$\varepsilon_{stat} = \varepsilon_{stat}(2K, 2s+s^\star, 2\gamma, \delta_A) = \max_{\{\Delta_j\}_{j\in[J]}\in\Upsilon(2K,2s+s^\star,2\gamma,\delta_A)} \frac{\sum_{j=1}^J \langle \nabla \ell_{N,j}(\Sigma_j^\star), \Delta_j\rangle}{\left(\sum_{j=1}^J \|\Delta_j\|_F^2\right)^{1/2}}.$$

The statistical error describes the geometric landscape around the optimum—it quantifies the magnitude of gradient of the empirical loss function evaluated at the population parameter in the directions constrained to the set Υ.

4.2 Assumptions and Main Results

We begin by stating the assumptions needed to establish the main results. Note that V in this section is used to denote an iterate in after the QR factorization step.

An upper bound on the step size is required for convergence of Algorithm 2. Let $Z_j^0 = (V^{0T}, \mathrm{diag}(a_j^0))^T$, $j \in [J]$, denote the output of Algorithm 1.

Assumption 1 *The step size satisfies* $\eta \le \min_{j\in[J]} \; J^{1/2}/(64\|Z_j^0\|_2^2)$.

Note that the step size depends on the initial estimate, but remains constant throughout the iterations. Let $\beta = 1 - \eta/(4J\xi^2) < 1$, $\chi = 4\beta^{1/2}(1-2I_0/\sqrt{J})^{-2}(1+32\|A^\star\|_\infty^2)$, and $\tau = J^{-1}\{9/2 + (1/2 \vee K/8)\}$, where

$$I_0^2 = \left\{\frac{1}{16\xi^2}\frac{1}{(1+\|A^\star\|_\infty^2 J^{-1})} \wedge \frac{J}{4}\right\}, \quad \xi^2 = \max_{j\in[J]}\left\{\frac{16}{\sigma_K^2(\Sigma_j^\star)} + \left(1 + \frac{8c}{\sigma_K(\Sigma_j^\star)}\right)^2\right\}. \tag{3}$$

We also require that the tuning parameters be selected appropriately.

Assumption 2 *We have* $c \ge c^\star$, $\gamma \ge \gamma^\star$, $s \ge [\{4(1/\chi - 1)^{-2} + 1\} \vee 2]s^\star$. *The matrix* $\tilde{G}$ *is obtained with the truncation level* $\delta_A \le (16\gamma^\star)^{-1}\min_{j\in[J]}\sigma_K^2(\Sigma_j^\star)$.

Note that the condition on δ_A is mild. It guarantees that we do not truncate too much of the signal. Finally, we require an assumption on the statistical error.

Assumption 3 *We have* $\varepsilon_{stat}^2 \le JI_0^2\{(\beta^{1/2}-\beta)/(\tau\eta) \wedge \min_{j\in[J]} 3\|Z_j^\star\|_2^2\}$.

Assumption 3 is essentially a requirement on the sample size N, since for a sufficiently large N the assumption will be satisfied with high probability. Note that as the sample size increases, the statistical error becomes smaller, while the radius of the local region of convergence, I_0, remains constant. Furthermore, if Assumption 3 is not satisfied, this implies that the initialization point is already close enough to the population parameters and that the subsequent refinement by Algorithm 2 is not needed.

With these assumptions, we are ready to state the main result, which tells us how far the estimate obtained by Algorithms 1 and 2 is from the population parameter. Let $\Sigma_j^I = V^I \mathrm{diag}(a_j^I)(V^I)^T$, $j \in [J]$, denote the estimate of the population covariance after the Ith iteration.

Theorem 4 *Suppose Assumptions 1—3 are satisfied and $J \geq 4$. Furthermore, for a sufficiently large constant C_0, suppose that there are $N = C_0 KP \log(PJ/\delta_0)$ independent samples such that $\|x_j^{(n)}\|_2^2 \leq P\|A^\star\|_\infty$ almost surely, $j \in [j]$, with zero mean and covariance as in (1). Then, with probability at least $1-\delta_0$, the estimate obtained by Algorithm 1 and Algorithm 2 satisfies*

$$\sum_{j=1}^{J} \|\Sigma_j^I - \Sigma_j^\star\|_F^2 \leq \beta^{I/2}(4\mu^2\xi^2) \sum_{j=1}^{J} \|\Sigma_j^0 - \Sigma_j^\star\|_F^2 + \frac{2\tau\mu^2\eta}{\beta^{1/2}-\beta}\varepsilon_{stat}^2 + 2K\gamma^\star\delta_A, \tag{4}$$

where $\mu = \max_{j\in[J]}(17/8)\|Z_j^\star\|_2$.

The first term on the right-hand side of (4) corresponds to the optimization error, and we observe a linear convergence rate. The second and third terms of (4) correspond to the statistical and approximation errors due to the truncation of the kernel matrix, respectively. From the bound we observe a trade-off between ε_{stat} and the truncation error δ_A: if δ_A decreases, ε_{stat} increases.

The proof of Theorem 4 is given in two steps. First, we establish the convergence rate of the iterates obtained by Algorithm 2 when the initial points V^0 and A^0 lie in a neighborhood around $V^\star$ and $A^\star$ (see §4.3). Subsequently, we show in Theorem 7 that Algorithm 1 provides suitable V^0 and A^0 with high probability (see §4.5).

To give an example of Theorem 4, we consider the case where data are generated from a multivariate Gaussian distribution and for a Gaussian kernel.

Proposition 5 *Let $x_j^{(n)} \in \mathbb{R}^P$ be independent Gaussian samples with mean zero and covariance as in (1) with $J \geq 4$ and $N \gtrsim K(P + \log J/\delta_0)$. Suppose that G is a Gaussian kernel matrix whose eigenvalue decays at the rate $\exp(-l^2 j^2)$ for some length scale $l > 0$. Let $\delta_A \asymp (\gamma^\star l N)^{-1}\{\log(\gamma^\star l N)\}^{1/2}$. Suppose that Assumptions 1—2 hold, $s^\star \log(P/s^\star) < PJ$ and $\max_{j\in[J]} \|E_j\|_2 \lesssim I_0$. Then after $I \gtrsim \log(1/\delta_1)$ iterations of Algorithm 2, with probability at least $1-\delta_0$, we have*

$$\sum_{j=1}^{J} \|\Sigma_j^I - \Sigma_j^\star\|_F^2 \lesssim \delta_1 + \frac{1}{N}\left[K\left\{\frac{1}{l}(\log\gamma^\star l N)^{1/2} + s^\star \log\frac{P}{s^\star}\right\} + \log\delta_0^{-1}\right] + J\max_{j\in[J]}\|E_j\|_2^2.$$

Condition $s^\star \log(P/s^\star) < PJ$ is mild, since $s^\star \lesssim P$, and condition $\max_{j\in[J]} \|E_j\|_2 \lesssim I_0$ is mild, since $\|E_j\|_2 < \sigma_K(\Sigma_j^\star)$, $j \in [J]$. Under the Gaussian distribution, the sample complexity is improved to $N \gtrsim K(P + \log J)$ from $N \gtrsim KP(\log P + \log J)$ in Theorem 4. Proposition 5 provides an explicit bound on the estimator that can be obtained under an assumption on the eigenvalue decay. The statistical error is comprised of two terms that correspond to errors when estimating smooth temporal components and sparse spatial components. In our choice of δ_A, the truncation error is in the same order as the statistical error induced by the smooth temporal components.

4.3 Linear Convergence

We establish the linear convergence rate of Algorithm 2 when it is appropriately initialized. Recall that the rows of $A^\star$ belong to $\mathcal{C}_A(c^\star, \gamma^\star) \subseteq \mathcal{C}_A(c, \gamma)$, while the projected gradient descent is implemented on the set $\tilde{\mathcal{C}}_A(c, \gamma) \subset \mathcal{C}_A(c, \gamma)$. Let

$$\tilde{A}^\star = \operatorname*{argmin}_{B_{k\cdot} \in \tilde{\mathcal{C}}_A(c,\gamma), k\in[K]} \|B - A^\star\|_F^2,$$

be the best approximation of $A^\star$ in $\tilde{\mathcal{C}}_A(c, \gamma)$. See Appendix E for details on the construction of $\tilde{A}^\star$. We define $\tilde{\Sigma}_j^\star = V^\star \operatorname{diag}(\tilde{a}_j^\star) V^{\star T}$, $j \in [J]$, and $\tilde{Z}^{\star T} = (V^{\star T}, \tilde{A}^\star)$. With these definitions, we establish the linear rate of convergence of the iterates to $\tilde{\Sigma}_j^\star$ and $\tilde{Z}^\star$. The convergence rate in Theorem 4 will then follow by combining the results with the truncation error.

Observe that the covariance factorization is not unique since, for any $R \in \mathcal{O}(K)$, we have $\Sigma_j = V \operatorname{diag}(a_j) V^T = V R_j {R_j}^T \operatorname{diag}(a_j) R_j {R_j}^T V^T$, $j \in [J]$. By the triangle inequality, we have

$$\sum_{j=1}^J \|\Sigma_j - \tilde{\Sigma}_j^\star\|_F^2 \le \sum_{j=1}^J \alpha_{V,j} \|V - V^\star R\|_F^2 + \alpha_A \|\operatorname{diag}(a_j) - R^T \operatorname{diag}(\tilde{a}_j^\star) R\|_F^2, \tag{5}$$

where $\alpha_{V,j} = 3\{\|V \operatorname{diag}(a_j)\|_2^2 + \|V^\star \operatorname{diag}(\tilde{a}_j^\star)\|_2^2\}$, $j \in [J]$, and $\alpha_A = 3\|V^\star\|_2^2 \|V\|_2^2$. This implies that if $\|V - V^\star R\|_F^2 + \|\operatorname{diag}(a_j) - R^T \operatorname{diag}(\tilde{a}_j^\star) R\|_F^2$ is small for some rotation matrix R and every $j \in [J]$, then the left-hand side will also be small. To this end, our goal is to show that the following distance metric contracts in each iteration of Algorithm 2. Let

$$R = \operatorname*{argmin}_{Y \in \mathcal{O}(K)} \|V - V^\star Y\|_F^2, \quad \operatorname{dist}^2(Z, \tilde{Z}^\star) = \sum_{j=1}^J d^2(Z_j, \tilde{Z}_j^\star); \tag{6}$$

$$d^2(Z_j, \tilde{Z}_j^\star) = \|V - V^\star R\|_F^2 + \|\operatorname{diag}(a_j) - R^T \operatorname{diag}(\tilde{a}_j^\star) R\|_F^2,$$

where $Z_j^T = (V^T, \operatorname{diag}(a_j))$ and $\tilde{Z}_j^{\star T} = (V^{\star T}, \operatorname{diag}(\tilde{a}_j^\star))$. The metric first finds the rotation matrix that aligns two subspaces and then computes the transformation of $\operatorname{diag}(\tilde{a}_j^\star)$ along the rotation R. This metric is similar to the distance metric commonly used in matrix factorization problems (Anderson and Rubin, 1956; ten Berge, 1977), but in our model the choice of R depends only on V.

To show the convergence of $\mathrm{dist}^2(Z, \tilde{Z}^\star)$, we need the following assumptions.

Assumption 6 *Suppose that Z_j^0 satisfies $d^2(Z_j^0, Z_j^\star) \le I_0^2$, for $j \in [J]$, where I_0 is defined in* (3)*. Assume that $\|V^0 - V^\star R\|_F^2 \le I_0^2/J$ and $\|\mathrm{diag}(a_j^0) - R^T \mathrm{diag}(a_j^\star) R\|_F^2 \le (J-1)I_0^2/J$.*

Since $d^2(Z_j^0, \tilde{Z}_j^\star) \le d^2(Z_j^0, Z_j^\star)$ for $j \in [J]$, Assumption 6 ensures that the distance between initial estimates and the population parameters is bounded within the ball of radius I_0. Furthermore, $I_0^2 \le J$ ensures that $\|V - V^\star R\|_2 \le 1$, so that V is full-rank. Intuitively, we assume that the squared distance for V is $1/(J-1)$ times smaller than the squared distance for A, because we have J times more samples to estimate V compared to A.

Theorem 7 *Assume that Assumptions 1—3, and Assumption 6 hold. After I iterations of Algorithm 2, we have*

$$\mathit{dist}^2(Z^I, \tilde{Z}^\star) \le \beta^{I/2} \mathit{dist}^2(Z^0, \tilde{Z}^\star) + \frac{\tau \eta \varepsilon_{stat}^2}{\beta^{1/2} - \beta}.$$

Theorem 7 establishes a linear rate of convergence in $\mathrm{dist}^2(Z, \tilde{Z}^\star)$. The second term on the left-hand side denotes the constant multiple of the statistical error, which depends on the distribution of the data and the sample size. Combining with (5) gives us a linear rate of convergence in $\sum_{j=1}^J \|\Sigma_j - \tilde{\Sigma}_j^\star\|_F^2$.

4.4 Statistical Error

Theorem 7 shows the linear convergence of the algorithm to a region around the population parameters characterized by statistical error. One may wonder how large the statistical error can be. While Assumption 3 provides a condition under which convergence is guaranteed, this bound is loose, as it does not depend on the sample size. We establish a tighter bound under the Gaussian distribution.

Proposition 8 (Statistical Error for Gaussian Data) *Let $x_j^{(n)} \in \mathbb{R}^P$ be independent Gaussian samples with mean zero and covariance as in* (1)*. Then, with probability at least $1-\delta$,*

$$\varepsilon_{stat}(2K, (2m+1)s^*, 2m'\gamma^\star, \delta_A) \le (\nu \vee \nu^2) + \sqrt{J} \max_{j \in [J]} \|E_j\|_2,$$

where

$$\nu = \frac{\|A^\star\|_\infty}{e_0} \left[\frac{1}{N} \left\{ \log \frac{1}{\delta} + K r(\tilde{G}) + K s^\star \log \frac{P}{s^\star} \right\} \right]^{\frac{1}{2}},$$

m, m' are positive integers, e_0 is an absolute constant depending on m and m', and $r(\tilde{G})$ is the rank of the δ_A-truncated kernel matrix $\tilde{G}$.

We interpret ε_{stat} as follows. The first term corresponds to the error in estimating the low-rank matrix, while the second term corresponds to the essential error incurred from approximating the covariance matrix by a low-rank matrix. The low-rank matrix can be estimated with the rate that converges to zero as $[K\{r(\tilde{G}) + s^\star \log P\}/N]^{-1/2}$, which corresponds to the rate of convergence of temporal and spatial components. We also highlight that truncation of G simplifies the statistical analysis because we can view the projection to $\tilde{\mathcal{C}}_A$ as restricting rows of A to a subset of a $r(\tilde{G})$-dimensional smooth subspace with $r(\tilde{G})$ much smaller than J, the original dimension.

4.5 Sample Complexity of Spectral Initialization

We discuss the sample complexity required to satisfy Assumption 6. That is, we characterize the sample size needed for Algorithm 1 to give a good initial estimate, so that Algorithm 2 outputs a solution characterized in Theorem 7. We consider a general case of a bounded distribution.

Theorem 9 (Sample Complexity of Spectral Initialization) *Let $x_j^{(n)} \in \mathbb{R}^P$ be independent zero mean samples with $\|x_j^{(n)}\|_2^2 \leq P\|A^\star\|_\infty$ almost surely, $n \in [N]$, $j \in [J]$, $J \geq 4$. Let $M^\star = J^{-1}\sum_{j=1}^J \mathbb{E}(S_{N,j})$ and $g = \sigma_K(M^\star) - \sigma_{K+1}(M^\star) > 0$ be the eigengap. Then, with probability at least $1-\delta$,*

$$dist^2(Z^0, Z^\star) \leq \phi(g, A^\star)\left\{\frac{KJP^2}{N^2}\left(\log\frac{4JP}{\delta}\right)^2 + \frac{KJP}{N}\log\frac{4JP}{\delta}\right\}; \qquad (7)$$

$$\phi(g, A^\star) = 4\|A^\star\|_\infty^2\left\{\frac{5(1+16\varphi^2\|A^\star\|_\infty^2)}{g^2 J} \vee 8\varphi^2\right\},$$

where $\varphi^2 = \max_{j\in[J]}\{1 + 4\sqrt{2}\|A^\star\|_\infty/\sigma_K(\Sigma_j^\star)\}$.

From (7) we note that if $N \gtrsim P\log(PJ/\delta)$, then Assumption 6 will be satisfied with high probability. The eigengap g must be greater than 0 for the bound in (7) to be nontrivial. Moreover, since $g \leq \|A^\star\|_\infty$, the first term of $\phi(g, A^\star)$ dominates when J is small. Combining results from (5), Theorem 7, and Theorem 9, we can establish Theorem 4.

5. Simulations

We evaluate the algorithm described in Section 3 using the metric in (6) and the average log-Euclidean metric (Arsigny et al., 2006) over a variety of temporal dynamics. Table 1 collects competing methods.

We generate synthetic samples from the following Gaussian distribution: $x_j^{(n)} \sim \mathcal{N}(0, \Sigma_j^\star + \sigma I)$, $n \in [N]$, $j \in [J]$, where $\Sigma_j^\star = \sum_{k=1}^K a_{k,j}^\star v_k^\star v_k^{\star T}$ and σI is additive noise. Unless stated otherwise, we use the Matérn five-half kernel (Minasny and McBratney, 2005) as the smoothing kernel for all simulations.

Method	Model	low-rank	smooth A	sparse V
M1	Sliding window principal component analysis	✓	✓	✗
M2	Hidden Markov model	✗	✗	✗
M3	Autoregressive hidden Markov model (Poritz, 1982)	✗	✓	✗
M4	Sparse dictionary learning (Mairal et al., 2010)	✓	✗	✓
M5	Bayesian structured learning (Andersen et al., 2018)	✓	✓	✓
M6	Lasso and kernel regularization (Daubechies et al., 2010)	✓	✓	✓
M7	Slinding window shrunk covariance (Ledoit and Wolf, 2004)	✗	✓	✗
MS	Spectral initialization (Algorithm 1)	✓	✗	✗
MR	Proposed model with random initialization (Algorithm 2)	✓	✓	✓
M**	Proposed model (Algorithm 1—2)	✓	✓	✓
MQ**	Proposed model (Algorithm 1—2) with QR decomposition	✓	✓	✓

Table 1: List of competing methods. The implementation details of M6 and MR are presented in Appendix G.1 and Appendix G.2.

5.1 Ground-truth recovery and linear convergence

We demonstrate the performance of the proposed algorithm under different smooth temporal structures. The tuning parameters are selected as described in Section 3.2. Figure 1 shows the temporal dynamics of the ground truth together with the results. The upper row corresponds to the setting of mixing temporal weights, where we have sine functions, a constant function, and a ramp function. The bottom row corresponds to the setting with different sine functions. Figure 2 shows how distance $\mathrm{dist}^2(Z, Z^\star)$ changes with the number of subjects $N \in \{1, 5, 15, 200\}$. We see linear convergence of the distance up to some statistical error, which decreases as the sample size increases, as predicted by Theorem 7.

To compare with other methods, we use the average log-Euclidean metric (Arsigny et al., 2006): $J^{-1}\sum_{j=1}^{J} \|\log(\Sigma_j) - \log(\Sigma_j^\star)\|_F$, where $\log(\Sigma_j) = U_j \log(\Lambda_j) U_j^T$, U is the matrix of eigenvectors, and Λ is the diagonal matrix of eigenvalues of Σ_j. In practice, we truncate the eigenvalues whose magnitude is less than 10^{-5} to maintain the stability of the evaluation. Table 2 reports the average log-Euclidean metric, while Table 3 reports the average running time over 20 independent runs. The simulations under discrete switching dynamics are presented in Appendix G.3 and simulations with varying K are presented in Appendix G.4.

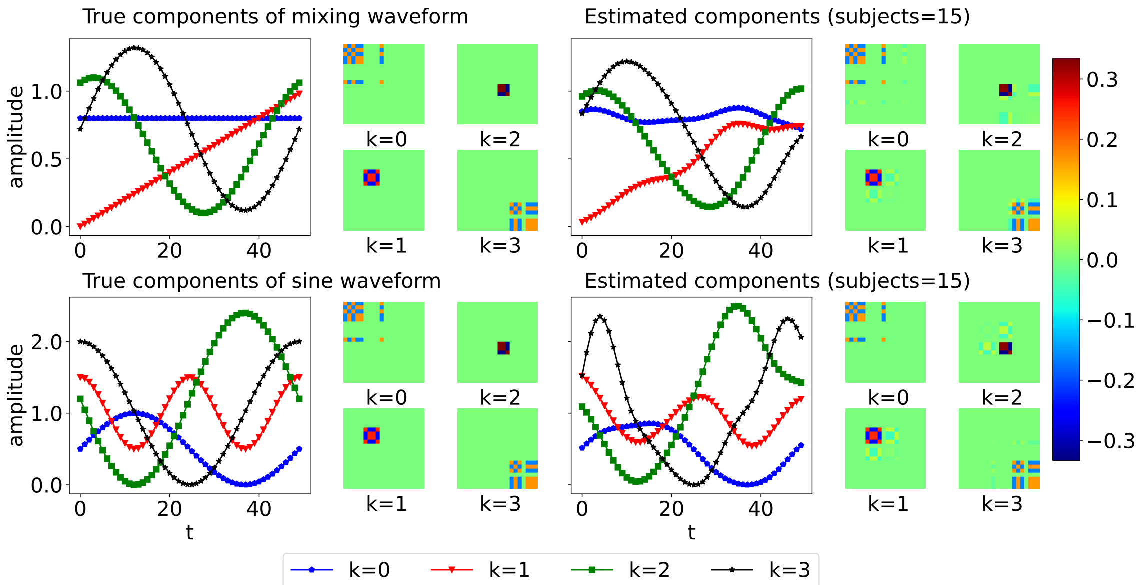

Figure 1: Covariance recovery with $K = 4, P = 20, J = 50$ and $\sigma = 0$. The left two columns show the ground truth and the right two columns show the recovery with $N = 15$. The results indicate good spatial and temporal recovery.

Method	Mixing waveform $N=1$	Mixing waveform $N=5$	Mixing waveform $N=10$	Sine waveform $N=1$	Sine waveform $N=5$	Sine waveform $N=10$
M1	0.45 ± 0.01	0.40 ± 0.01	0.38 ± 0.01	0.67 ± 0.03	0.63 ± 0.01	0.63 ± 0.01
M2	6.58 ± 0.31	0.68 ± 0.02	0.62 ± 0.01	6.22 ± 1.02	0.82 ± 0.01	0.80 ± 0.01
M3	6.91 ± 1.39	0.75 ± 0.02	0.65 ± 0.01	7.36 ± 0.24	0.87 ± 0.02	0.80 ± 0.01
M4	0.46 ± 0.01	0.43 ± 0.02	0.39 ± 0.02	0.64 ± 0.01	0.58 ± 0.03	0.54 ± 0.04
M5	0.41 ± 0.01	0.36 ± 0.01	0.34 ± 0.00	0.58 ± 0.01	0.54 ± 0.01	0.53 ± 0.01
M6	0.51 ± 0.03	0.39 ± 0.02	0.37 ± 0.02	0.67 ± 0.03	0.58 ± 0.02	0.57 ± 0.03
MS	0.89 ± 0.05	0.41 ± 0.01	0.38 ± 0.01	0.94 ± 0.06	0.58 ± 0.02	0.57 ± 0.01
MR	0.42 ± 0.01	0.41 ± 0.02	0.41 ± 0.00	0.62 ± 0.00	0.62 ± 0.00	0.62 ± 0.00
M**	0.41 ± 0.03	0.29 ± 0.03	0.30 ± 0.04	0.59 ± 0.02	0.51 ± 0.03	0.50 ± 0.02
MQ**	0.37 ± 0.02	0.31 ± 0.03	0.29 ± 0.03	0.58 ± 0.02	0.56 ± 0.02	0.56 ± 0.01

Table 2: Log-Euclidean metric averaged over 20 independent runs. Temporal dynamics are given in Figure 1 with $K = 4, P = 20, J = 50$ and $\sigma = 0.5$. For M1, we set the window length as $W = 20$. For both task cases, M** and MQ** outperform the competing methods under varying sample size. When $N = 1$, M2 and M3 have sufficiently large average log-Euclidean. This is because M2 and M3 are not designed to be low-rank models, whereas the ground truth is low-rank. Hence, when the estimated covariance matrices are not low-rank, they have many small trailing nonzero eigenvalues, which results in large average log-Euclidean metric.

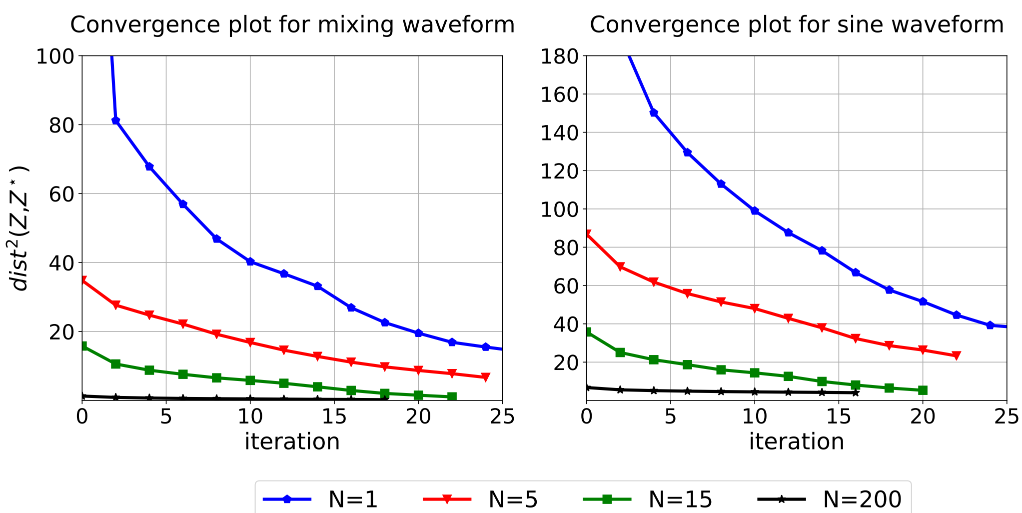

Figure 2: Convergence rate for different sample sizes with $K = 4, P = 20, J = 50$ and $\sigma = 0$. The data generating mechanism is given in Figure 1. Irrespective of the sample size, we observe linear convergence of the algorithm up to a neighborhood of the population parameters. The radius of the neighborhood is characterized by the statistical error, which depends on the sample size.

	Method	Number of subjects (N) 1		5		10	
Mixing waveform	M1	0.5	$\pm$ 0.1	0.4	$\pm$ 0.0	0.6	$\pm$ 0.0
	M2	300.0	$\pm$ 4.0	790.0	$\pm$ 10.0	1000.0	$\pm$ 40.0
	M3	190.0	$\pm$ 30.0	2840.0	$\pm$ 10.0	2450.0	$\pm$ 70.0
	M4	60.0	$\pm$ 30.0	520.0	$\pm$ 290.0	1080.0	$\pm$ 510.0
	M5	350.0	$\pm$ 80.0	360.0	$\pm$ 130.0	380.0	$\pm$ 60.0
	M6	620.0	$\pm$ 50.0	630.0	$\pm$ 40.0	620.0	$\pm$ 60.0
	MS	0.1	$\pm$ 0.0	0.2	$\pm$ 0.0	0.2	$\pm$ 0.0
	MR	90.0	$\pm$ 3.7	220.0	$\pm$ 0.0	220.0	$\pm$ 3.5
	M**	8.2	$\pm$ 1.6	4.5	$\pm$ 0.2	5.9	$\pm$ 0.6
	MQ**	18.1	$\pm$ 2.2	14.0	$\pm$ 0.6	13.4	$\pm$ 0.3
Sine waveform	M1	0.5	$\pm$ 0.1	0.4	$\pm$ 0.0	0.5	$\pm$ 0.0
	M2	30.0	$\pm$ 4.0	800.0	$\pm$ 10.0	830.0	$\pm$ 7.9
	M3	180.0	$\pm$ 20.0	2840.0	$\pm$ 20.0	290.0	$\pm$ 4.1
	M4	70.0	$\pm$ 30.0	320.0	$\pm$ 150.0	780.0	$\pm$ 330.0
	M5	350.0	$\pm$ 80.0	360.0	$\pm$ 110.0	380.0	$\pm$ 90.0
	M6	570.0	$\pm$ 30.0	570.0	$\pm$ 30.0	570.0	$\pm$ 30.0
	MS	0.1	$\pm$ 0.0	0.2	$\pm$ 0.0	0.2	$\pm$ 0.0
	MR	73.9	$\pm$ 11.4	35.9	$\pm$ 3.2	42.9	$\pm$ 2.0
	M**	6.7	$\pm$ 1.7	4.5	$\pm$ 0.8	4.3	$\pm$ 1.2
	MQ**	15.3	$\pm$ 2.4	12.9	$\pm$ 0.8	12.0	$\pm$ 0.6

Table 3: Running time ($\times 10^{-2} s$) averaged over 20 independent runs. Temporal dynamics are given in Figure 1 with $K = 4, P = 20, J = 50$ and $\sigma = 0.5$. For M1, we set the window length as $W = 20$. When P is small, MS is the most efficient method as it only computes eigendecomposition once. M1 computes eigendecomposition multiple times and, as a result, is slower. The running time of M* is composed of the running time of MS and the running time of Algorithm 2. MQ** is slower than M** because it requires additional QR decomposition at each step.

	Method	Number of subjects (N) 1	5	15	200	1000
Mixing waveform	MS	208.12 ± 41.82	45.22 ± 2.62	15.15 ± 1.10	1.17 ± 0.10	0.23 ± 0.01
	MR	29.67 ± 0.39	29.16 ± 0.21	28.86 ± 0.12	28.68 ± 0.04	28.43 ± 0.06
	M**	18.99 ± 5.00	6.72 ± 2.14	2.22 ± 1.13	0.19 ± 0.05	0.04 ± 0.01
Sine waveform	MS	477.54 ± 95.10	103.24 ± 9.64	36.64 ± 3.39	3.53 ± 1.08	1.53 ± 1.46
	MR	95.60 ± 0.63	94.05 ± 0.36	92.86 ± 0.26	91.39 ± 0.12	90.85 ± 0.12
	M**	51.06 ± 12.28	22.27 ± 5.99	9.85 ± 4.38	2.41 ± 3.22	1.44 ± 2.13

Table 4: Average $\text{dist}^2(Z, Z^\star)$ over 20 independent runs. The data generating mechanism is given in Figure 1 with $K = 4, P = 20, J = 50$ and $\sigma = 0$. M** performs the best under varying sample sizes. MR outperforms MS for small sample sizes. However, MS outperforms MR in the large sample size settings.

5.2 Importance of Spectral Initialization

We compare the spectral initialization method in Algorithm 1 (MS), random initialization method with iterative refinement (MR) (see Appendix G.1 for details), and the proposed model (M**) to demonstrate the importance of proper initialization and iterative refinement. The data are generated as in the previous experiment. For each setting, we run the simulation 20 times with $N \in \{1, 5, 15, 200, 1000\}$ and average $\mathrm{dist}^2(Z, Z^\star)$ at convergence. Table 4 indicates that the error of the MS decreases with increasing sample size, which corresponds to the result of Theorem 9. We further observe that M** outperforms MR and MS, indicating that the two-stage algorithm works better than the single-stage algorithms (MR or MS) under varying sample sizes. Figure 3 shows that the MR method converges to poor local optima that correspond to large $\mathrm{dist}^2(Z, Z^\star)$. We also visually observe that the recovered temporal dynamics is far from the population ground truth. When the sample size is small, both MR and M** outperform MS, implying that iterative refinement helps improve estimation in addition to spectral initialization. When the sample size is larger, that is, $N \in \{15, 200, 1000\}$, MS outperforms MR, implying that spectral initialization is a better option than random initialization.

5.3 Simulations with increasing P and K

We increase both the dimension of the data P and the number of components K to demonstrate the effectiveness of the proposed algorithm in a high-dimensional setting. For the data generation process, we randomly generate a sparse orthogonal matrix $V^\star \in \mathbb{R}^{P\times K}$. Specifically, we generate a diagonal block matrix, denoted $\tilde{V}^\star$, with four blocks, and the size of each block is $\lceil K/4 \rceil$. Each block matrix is generated as a random sparse matrix where the probability that an entry is nonzero is 0.4 and each nonzero entry is drawn from $\mathrm{unif}([0, 1])$. Finally, we obtain $V^\star$ from the QR decomposition of $\tilde{V}^\star = QR$ as $V^\star = Q$. Note that the spatial components are partially overlapping in this setting, unlike in the previous setting, making the task more challenging. We generate the temporal components $A_{k\cdot}$ as follows. We select 6 knots uniformly at random in $[0, T]$ and draw the corresponding y value from $\mathrm{unif}([0, 1])$. These values are interpolated with a cubic spline function.

Table 5 reports the results with $N = 100$ and $J = 100$, while $P \in \{50, 100, 150, 200, 300\}$ and $K \in \{10, 20, 30, 40, 50\}$. The results indicate that with a fixed N, the distance increases with rank, which can be expected since there are more parameters to estimate. Moreover, we also find that dimension P has a small influence on performance.

We compare our algorithm with competing methods in a setting where $P = 100$, since some of the other methods are not scalable to higher-dimensional problems. We set $J = 100$, $K = 10$, and the noise level to $\sigma = 0.5$. Results for $\sigma \in \{0.1, 0.2\}$ are reported in Appendix G.5. For all methods, we set the number of components to estimate as 10, that is, all methods know the true number of components in the data generation process. For the Bayesian model (M5), we draw 30 samples from the posterior distribution and compute the estimated covariance. We report the results averaged over 20 independent

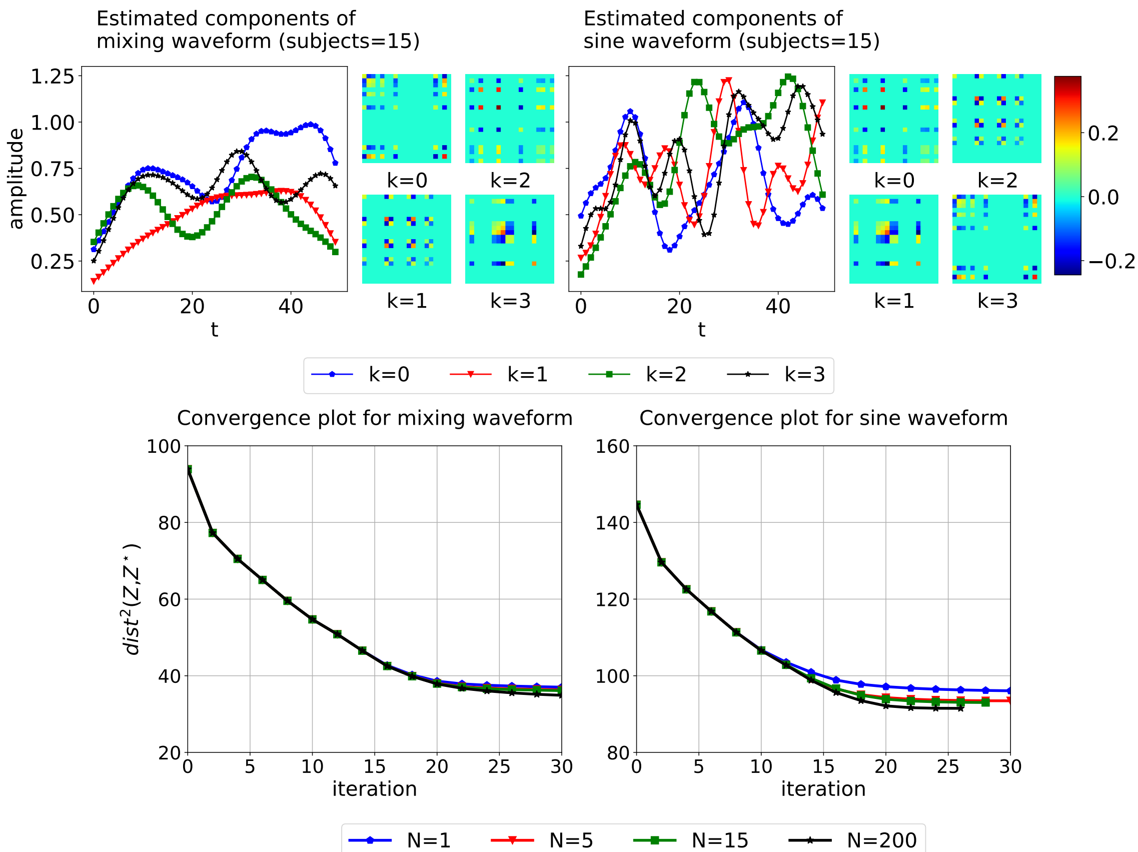

Figure 3: Performance of random initialization method (MR). The data generating mechanism is given in Figure 1. The left two columns in the top row correspond to recovery of mixing waveform, while the right two columns correspond to recovery of sine waveform. The bottom row displays the convergence with respect to the number of iterations.

simulation runs. Table 6 summarizes the results. The proposed algorithm performs the best compared to the alternatives. We also observe that the log-Euclidean metric decreases as the number of samples increases for M**, which is predicted by the results of Theorem 4 and Proposition 8. When comparing MS and M**, we see a decrease in the log-Euclidean metric resulting from Algorithm 2. The results of M** and MQ** are similar, implying that the QR decomposition does not affect the estimation much, supporting the theory that V and V_{ortho} span the same subspace. Our method delivers performance comparable to M5, although M5 uses variational inference in a Bayesian framework. This can be expected, as the underlying models are similar. Table 7 reports the running times for different methods. The running time of the proposed method remains relatively stable as the number of subjects increases. On the other hand, the running time of the M2, M3, and M4 methods increases as N increases. Our method remains efficient even in a high-dimensional setting, while many other methods become slow as the dimension increases. Finally, although the M5 and M6 methods make the same structural assumptions and achieve comparable performance, our method is more computationally efficient. In particular, when the sample size N increases, our method provides the best practical choice.

5.4 Selections of kernel functions

We vary the number of knots to see how the choice of kernel length scale affects the estimation. Moreover, we investigate the effect of the kernel function. We average simulation results over 20 independent runs with $N = 50$, $K = 10$, $P = 100$, $J = 100$ and $\sigma = 0.5$. The components are generated by the same data generation process described in Section 5.3 and we only vary the number of knots. Table 8 shows that as the number of knots increases, indicating that the temporal signal fluctuates more intensively, the optimal choice of length scale decreases. This behavior is observed with all three kernel functions.

6. Experiment on neuroimaging data

To investigate the proposed model on real data, we focus on (i) the interpretability of the model and (ii) the out-of-sample prediction. We use motor task data from the Human Connectome Project functional magnetic resonance imaging (fMRI) data (Van Essen et al., 2013). Data are preprocessed using the existing pipeline (Van Essen et al., 2013), and an additional high-pass filter with a cutoff frequency $0.015Hz$ to remove physiological noise as recommended by Smith et al. (1999). The data consist of five motor tasks: tapping the right hand, tapping the left foot, wagging the tongue, tapping the right foot, and tapping the left hand. During a session, each task is activated twice. See the activation sequence in Figure 4.

For the model interpretation experiment, we select $N = 20$ subjects. For each subject, the preprocessed time series of length $J = 284$ were extracted from $P = 375$ cortical and subcortical parcels, following (Shine et al., 2019). The regions include 333 cortical parcels

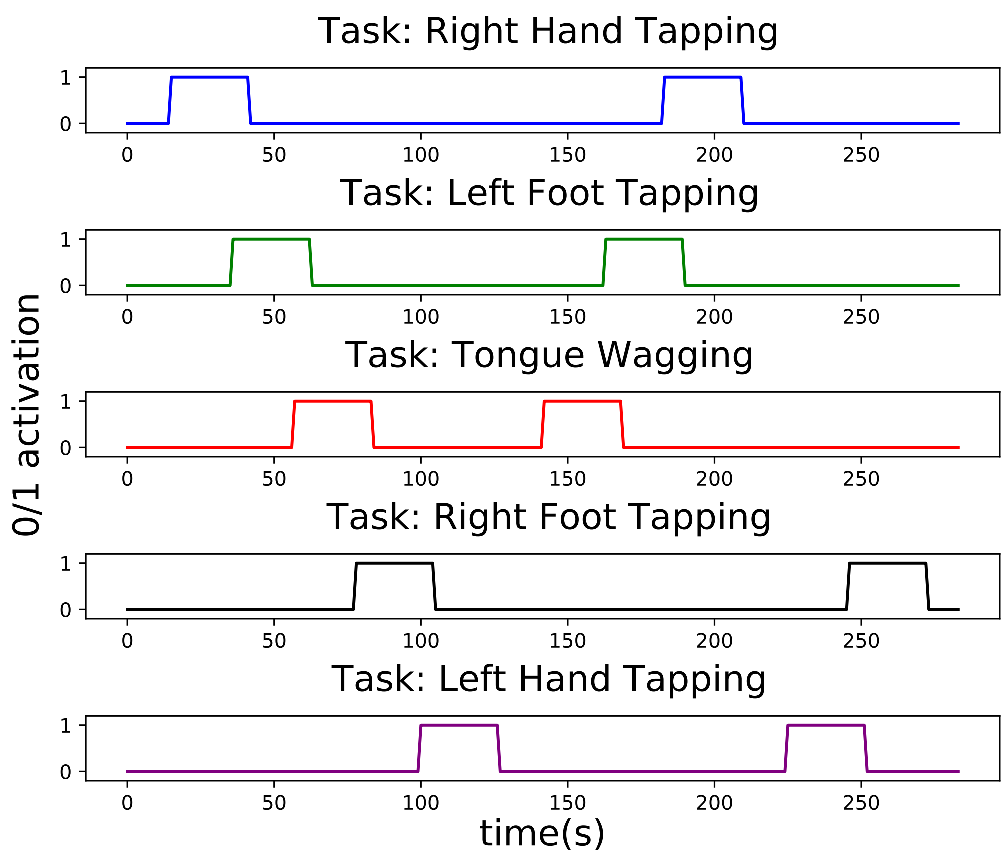

Figure 4: The activation map for the Human Connectome Project motor data set (Van Essen et al., 2013). The activation time of each task is partially overlapping with the other tasks.

(161 and 162 regions from the left and right hemispheres, respectively) using the Gordon atlas (Gordon et al., 2014), 14 subcortical regions from the Harvard–Oxford subcortical atlas (bilateral thalamus, caudate, putamen, ventral striatum, globus pallidus, amygdala, and hippocampus), and 28 cerebellar regions from the SUIT atlas 54 (Diedrichsen et al., 2009).

The goal is to analyze the corresponding dynamic connectivity. To investigate the temporal and spatial components, we compute the correlation of each weight $A_{k \cdot}$ with the onset task activation, and select the component that has the highest correlation. Figure 5 shows the results for three tasks: tapping the left foot, wagging the tongue, and tapping the left hand. Figure 12 in Appendix G.6 shows the results for tapping the right foot

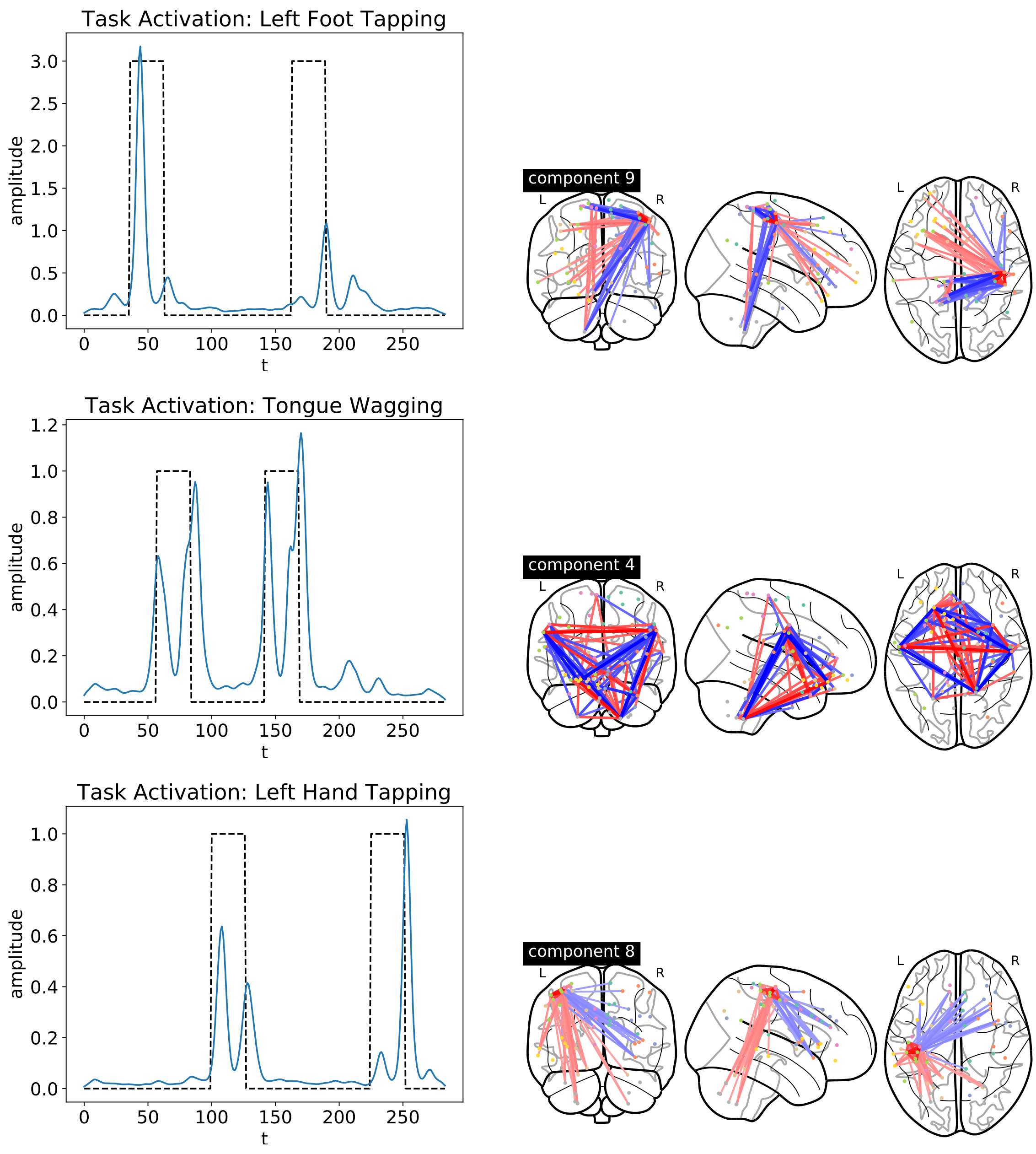

Figure 5: The left column shows the temporal components (blue solid lines) whose correlations are the largest with respect to the task activation (black dotted lines). The right column shows the corresponding brain connectivity patterns (spatial components) for the tasks. The red lines denote positive connectivity and blue lines denote negative connectivity.

Rank (K)	Dimension (P) 50	100	150	200	300
$K = 10$	0.35 ± 0.01	0.35 ± 0.02	0.41 ± 0.02	0.35 ± 0.01	0.34 ± 0.03
$K = 20$	0.66 ± 0.02	0.66 ± 0.02	0.69 ± 0.02	0.66 ± 0.03	0.66 ± 0.01
$K = 30$	0.82 ± 0.01	0.82 ± 0.01	0.82 ± 0.02	0.80 ± 0.01	0.79 ± 0.01
$K = 40$	0.97 ± 0.01	0.93 ± 0.01	0.93 ± 0.01	0.88 ± 0.01	0.87 ± 0.01
$K = 50$	1.11 ± 0.01	1.10 ± 0.01	1.04 ± 0.01	0.99 ± 0.01	0.99 ± 0.01

Table 5: Log-Euclidean metric averaged over 20 independent simulation runs ($\sigma = 0.5$). The data generating mechanism is described in Section 5.3. For a fixed sample size and increasing K increases, the log-Euclidean metric increases due to large number of parameters that need to be estimated. The log-Euclidean metric is only mildly affected by the dimension P, which is due to the number of nonzero entries being the same for different values of P.

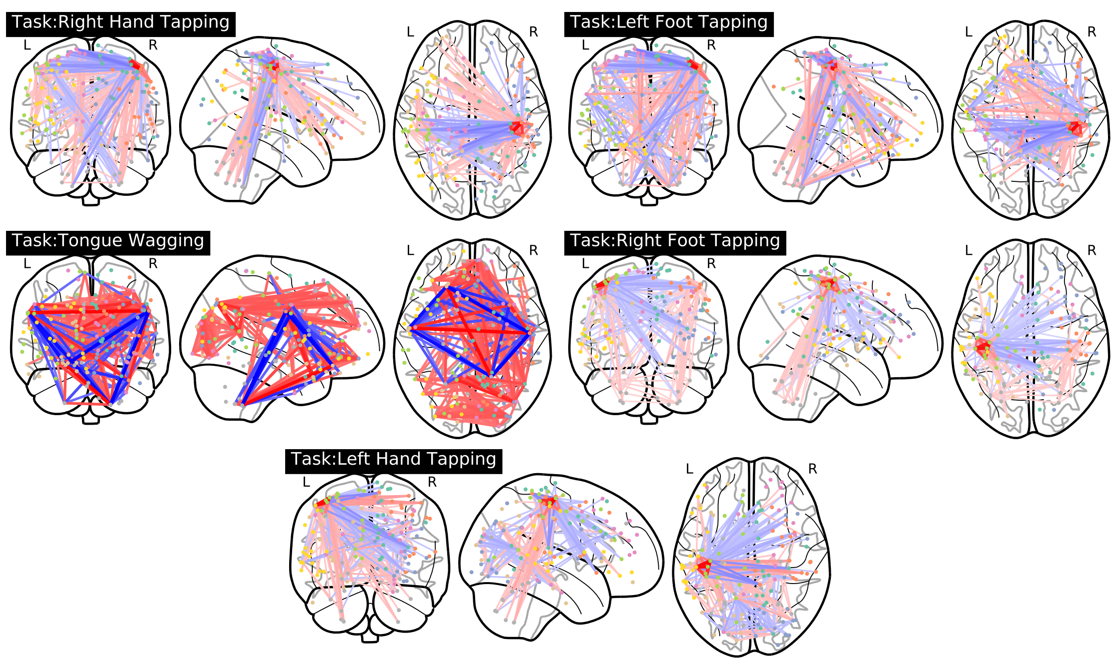

Figure 6: Each connectome is the superposition of the top three spatial components. The spatial hubs in the connectivity matrices closely match with the expected motor regions (hands, feet, tongue) as defined in the cortical homunculus (Marieb and Hoehn, 2018).

	Number of subjects (N)				
Methods	10	20	30	40	50
M1	0.49 ± 0.01	0.46 ± 0.01	0.45 ± 0.01	0.45 ± 0.01	0.44 ± 0.01
M2	1.22 ± 0.01	1.04 ± 0.01	1.00 ± 0.01	0.98 ± 0.01	0.97 ± 0.01
M3	71.50 ± 6.46	1.90 ± 0.26	1.12 ± 0.01	1.14 ± 0.01	1.12 ± 0.01
M4	0.94 ± 0.03	0.46 ± 0.01	0.41 ± 0.01	0.39 ± 0.01	0.38 ± 0.01
M5	0.51 ± 0.01	0.46 ± 0.01	0.43 ± 0.01	0.42 ± 0.01	0.41 ± 0.01
M6	0.43 ± 0.01	0.41 ± 0.01	0.40 ± 0.01	0.40 ± 0.01	0.39 ± 0.01
MS	0.43 ± 0.01	0.40 ± 0.01	0.40 ± 0.01	0.39 ± 0.01	0.39 ± 0.01
M**	0.42 ± 0.03	0.35 ± 0.05	0.36 ± 0.04	0.33 ± 0.04	0.32 ± 0.02
MQ**	0.40 ± 0.04	0.40 ± 0.01	0.35 ± 0.04	0.32 ± 0.03	0.32 ± 0.02

Table 6: Log-Euclidean metric averaged over 20 independent simulation runs. The data generating mechanism is described in Section 5.3 and $\sigma = 0.5$. For M1, we set the window length to be $W = 20$.

	Number of subjects (N)				
Methods	10	20	30	40	50
M1	0.8 ± 0.4	0.5 ± 0.3	1.2 ± 0.6	1.5 ± 0.5	1.8 ± 0.5
M2	151.9 ± 17.9	222.6 ± 51.0	376.7 ± 53.6	498.8 ± 40.0	635.5 ± 58.2
M3	422.0 ± 33.3	729.7 ± 161.6	1154.4 ± 53.6	1427.4 ± 64.4	1751.7 ± 52.0
M4	267.0 ± 112.1	378.4 ± 148.5	846.2 ± 358.8	872.0 ± 440.8	1841.5 ± 697.2
M5	2243.8 ± 33.3	2263.3 ± 38.4	2273.7 ± 36.0	2259.8 ± 34.2	2278.9 ± 35.1
M6	84.9 ± 37.5	195.2 ± 62.6	201.8 ± 31.9	191.6 ± 51.6	218.3 ± 15.6
MS	0.1 ± 0.0	0.1 ± 0.1	0.2 ± 0.1	0.2 ± 0.1	0.2 ± 0.1
M**	1.2 ± 0.6	1.3 ± 0.7	2.9 ± 1.4	3.9 ± 0.8	3.6 ± 0.7
MQ**	2.2 ± 1.3	1.4 ± 0.7	2.5 ± 1.0	3.8 ± 0.7	3.5 ± 0.6

Table 7: Running time in *seconds* averaged over 20 independent simulation runs. The data generating mechanism is described in Section 5.3 and $\sigma = 0.5$. For M1, we set the window length to be $W = 20$.

and tapping the right hand. Our results show that the temporal fluctuations of the top components coincide with the task activation. Following the hypothesis that neural activity is the consequence of multiple components rather than a single component (Posner et al., 1988), for each task, we select three components with the highest correlations and plot the connectivity patterns in Figure 6. The spatial hubs in the connectivity matrices closely match the expected motor regions as defined in the cortical homunculus (Marieb and Hoehn, 2018). Thus, the results indicate that the proposed algorithm can separate and identify the components of each task and that each task has a unique connectivity pattern.

As the ground truth is unknown and motivated by the hypothesis that each task has a different activation pattern, we design a classification task as a surrogate experiment to evaluate the algorithm. Previous work also indicated that task fMRI data share similar connectivity patterns among test subjects (Zalesky et al., 2012; Calhoun et al., 2014).

Methods	Number of knots in $J = 100$ 5	10	15	20
Radial-basis function ($l = 5$)	0.16 ± 0.04	0.26 ± 0.05	0.52 ± 0.07	0.91 ± 0.17
Radial-basis function ($l = 10$)	0.12 ± 0.04	0.18 ± 0.05	0.33 ± 0.07	0.69 ± 0.18
Radial-basis function ($l = 50$)	0.09 ± 0.04	0.16 ± 0.05	0.76 ± 0.09	1.17 ± 0.11
Radial-basis function ($l = 200$)	0.08 ± 0.04	0.30 ± 0.05	0.84 ± 0.08	1.27 ± 0.16
Matérn five-half ($l = 5$)	0.15 ± 0.04	0.23 ± 0.05	0.44 ± 0.06	0.80 ± 0.18
Matérn five-half ($l = 10$)	0.13 ± 0.04	0.20 ± 0.05	0.36 ± 0.06	0.70 ± 0.18
Matérn five-half ($l = 50$)	0.10 ± 0.04	0.15 ± 0.05	0.31 ± 0.06	0.59 ± 0.12
Matérn five-half ($l = 200$)	0.09 ± 0.04	0.14 ± 0.05	0.31 ± 0.07	0.62 ± 0.19
Rational quadratic ($l = 5$)	0.14 ± 0.04	0.24 ± 0.05	0.51 ± 0.07	0.89 ± 0.18
Rational quadratic ($l = 10$)	0.17 ± 0.04	0.29 ± 0.05	0.61 ± 0.07	1.03 ± 0.18
Rational quadratic ($l = 50$)	0.10 ± 0.04	0.13 ± 0.05	0.43 ± 0.07	1.07 ± 0.18
Rational quadratic ($l = 200$)	0.08 ± 0.04	0.22 ± 0.05	0.81 ± 0.08	1.26 ± 0.16

Table 8: Average distance $\mathrm{dist}^2(Z, Z^\star)/J$ over 20 independent runs with $\sigma = 0.5$. Results are comparable for different kernel functions when the number of knots is smaller than $J = 15$. When $J = 20$, Matérn five-half kernel function is more effective in capturing the temporal smoothness compared to the other two kernels.

Therefore, if we can recover the functional connectivity patterns of the training subjects, then similar patterns exist in the test subjects. We partition 103 subjects in the Human Connectome Project motor task data set (Van Essen et al., 2013) randomly into a training and testing set. The duration of each task is identical, 27 time points for each activation, and 2 activations in each session. Since each task partially overlaps with others (see Figure 4), we predict the task based on activation blocks rather than on a single time point. We group the estimated covariances $\{\Sigma_j\}_{j\in[J]}$ and the test data based on the task activation map and perform a nearest-neighbor search. Clustered covariances are denoted as $\Sigma_{\text{task},i}$, where task $\in$ {tapping the right hand, tapping the left foot, wagging the tongue, tapping the right foot, tapping the left hand} and $i \in [54]$. The task score for each test data block is defined as

$$score_{task}(\{x_i\}_{i\in[54]}) = \sum_{i=1}^{54} \|x_i x_i^T - \Sigma_{task,i}\|_F^2,$$

where $\{x_i\}_{i\in[54]}$ is a block of test data. We predict the task of the block data by choosing the task with the minimum score. We repeat the experiment 10 times. In each run, we randomly split the data into a training and testing set. We select the number of training subjects to be $N \in \{10, 20, 30, 40, 50\}$ and set the remaining subjects as test sets. The results are shown in Table 9. Note that the Markov model (M2) performs worst even if we increase the number of states to 60. The dictionary learning model (M4) performs similarly to our model when the sample size is large, but our model performs better with small sample sizes.

7. Discussion

Several directions are worthy of further investigation. We plan to explore a more flexible spatial structure. Previous work (Gibberd and Nelson, 2017; Hallac et al., 2017) applied fused graphical lasso and group graphical lasso to encourage similar sparse structures for time-varying graphical models. These approaches did not restrict the spatial components to be identical but only similar, and thus are more flexible compared to the proposed model. To this end, one idea is to build factor models that encourage similar, but not identical, spatial structures while retaining low rank. Our work has focused on modeling data sampled at fixed intervals. We also plan to explore models with samples obtained at irregular time intervals (Tank et al., 2019; Qiao et al., 2020), as this setting is common in multimodal data (Tsai et al., 2022). Finally, our work has focused on the estimation of parameters in a flexible covariance model, while the question of how to quantify the statistical uncertainty remains open. There has been a growing literature on inference for parameters in high-dimensional models, including linear models (Zhang and Zhang, 2014; van de Geer et al., 2014; Javanmard and Montanari, 2014; Zhao et al., 2014; Bradic and Kolar, 2017; Dai and Kolar, 2021; Wang et al., 2021), nonparametric models (Kozbur, 2021; Lu et al., 2020), and graphical models (Ren et al., 2015; Wasserman et al., 2014; Janková and van de Geer, 2015, 2017; Barber and Kolar, 2018; Wang and Kolar, 2016; Yu et al., 2016, 2020b; Xia et al., 2015; Kim et al., 2021). The model considered in our paper is comprised of a sparse spatial and a smooth nonparametric temporal component that will require the development of new inferential techniques.

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Appendix A. Projection to Constraint Sets

We describe the algorithms used to project iterates to the constraints $\mathcal{C}_V$, $\tilde{\mathcal{C}}_V$, and $\tilde{\mathcal{C}}_A$. Next, we characterize the expansion coefficient induced by projecting to nonconvex sets.

A.1 Projections to $\mathcal{C}_V$ and $\tilde{\mathcal{C}}_V$

Recall that $\mathcal{C}_V(s) = \{v \in \mathbb{R}^P : \|v\|_0 \leq s, \|v\|_2 = 1\}$. To project a vector v onto $\mathcal{C}_V(s)$, we want to solve the following problem

$$\arg\min_{x \in \mathcal{C}_V} \|v - x\|_2^2. \tag{8}$$

Let $\mathcal{S}(x) = \{i : x_i \neq 0\}$ be the support of x. Given a support $E \subset [P]$, let $[x]_E \in \mathbb{R}^P$ be a vector whose ith entry is equal to x_i if $i \in E$ and 0 otherwise. Let

$$d(E) = \min_x \|v - x\|_2^2 \quad \text{subject to} \quad \mathcal{S}(x) \subseteq E,\ \|x\|_2 = 1,$$

and observe that

$$\begin{aligned} d(E) &= \min_x \|v\|_2^2 + \|x\|_2^2 - 2\langle x, v\rangle \\ &= \|v\|_2^2 + 1 - 2\max_x \langle x, v\rangle \\ &= \|v\|_2^2 + 1 - 2\|[v]_E\|_2. \end{aligned}$$

Then we can conclude that

$$\hat{E} = \arg\min_{E:|E| \leq s} d(E) = \arg\max_{E:|E|\leq s} \|[v]_E\|_2, \tag{9}$$

which can be solved by finding the top-s entries of v in magnitude. This can be done with computational complexity $O(P \log P)$. After finding the support in (9), we can obtain (8) by projecting $[x]_{\hat{E}}$ onto the unit sphere. Algorithm 3 summarizes the procedure.

Input: $v \in \mathbb{R}^P$
$v_S \leftarrow$ Pick the top-s entries of v in magnitude and set the rest of entries to 0
$\hat{v} \leftarrow$ Project v_S to the unit sphere S^{P-1}
Output $\hat{v}$

Algorithm 3: Projection to $\mathcal{C}_V(s)$

Next, we discuss the projection procedure when we additionally orthogonalize the estimate via QR decomposition. Let $\hat{V} = \Pi_{\mathcal{C}_V}(V)$ and $\hat{V} = BL$ be the QR decomposition of $\hat{V}$, where B has orthonormal columns, and L is an upper triangular matrix. We define

$$\Pi_{\tilde{\mathcal{C}}_V}(V) = B. \tag{10}$$

A.2 Projection to $\tilde{\mathcal{C}}_A$

Recall that $\tilde{\mathcal{C}}_A(c,\gamma) = \{\alpha = \tilde{Q}u : 0 \le \alpha_j \le c, u^T\tilde{\Lambda}u \le \gamma\}$, where $\tilde{G}^\dagger = \tilde{Q}\tilde{\Lambda}\tilde{Q}^T$ is the eigendecomposition of $\tilde{G}^\dagger$ and λ_j denotes the jth diagonal entry of $\tilde{\Lambda}$. To project to the convex set $\tilde{\mathcal{C}}_A$, we use an alternating projection method. While the convergence rate of the alternating projection method is not our focus, in the experiments, we observe that often one iteration of the alternating projection results in an iterate that satisfies both constraints. Algorithm 4 summarizes the alternating projection procedure.

Input: $\alpha \in \mathbb{R}^J$
While $\alpha \notin \tilde{\mathcal{C}}_A(c,\gamma)$
  $\hat{\alpha} \leftarrow$ Project α to the hypercube $[0,c]^J$
  $\alpha \leftarrow$ Project $\hat{\alpha}$ to the set $\{\alpha = \tilde{Q}u : u^T\tilde{\Lambda}u \le \gamma\}$ using Algorithm 5
Output α

Algorithm 4: Projection to $\tilde{\mathcal{C}}_A(c,\gamma)$

Next, we provide an algorithm to project to the ellipsoid $\{\alpha = \tilde{Q}u : u^T\tilde{\Lambda}u \le \gamma\}$, which is one of the steps in Algorithm 4. It is easy to see that a vector y belonging to $\{\alpha = \tilde{Q}u : u^T\tilde{\Lambda}u \le \gamma\}$ lies in the range space of $\tilde{Q}$, which has dimension $r(\tilde{G})$. Let Q_1 be the matrix whose orthonormal columns form the subspace orthogonal to the columns of $\tilde{Q}$, which has dimension $J - r(\tilde{G})$. In this case, we have $Q_1^T y = 0$.

Input: $\hat{\alpha} \in \mathbb{R}^J$
If $\hat{\alpha}^T\tilde{G}^\dagger\hat{\alpha} \le \gamma$
  $a \leftarrow \tilde{Q}\tilde{Q}^T\hat{\alpha}$
Else
  $u \leftarrow \tilde{Q}^T\hat{\alpha}$
  $D \leftarrow$ Find the roots of x: $3x^2\sum_{j=1}^{r(\tilde{G})}\lambda_j^3u_j^2 - 2x\sum_{j=1}^{r(\tilde{G})}\lambda_j^2u_j^2 + \sum_{j=1}^{r(\tilde{G})}\lambda_ju_j^2 - \gamma = 0$
  $\hat{x} \leftarrow$ Pick the largest nonnegative value in the set D
  $a \leftarrow \tilde{Q}\tilde{\Lambda}^{-1}(\hat{x}I + \tilde{\Lambda}^{-1})^{-1}\tilde{Q}^T\hat{\alpha}$
Output a

Algorithm 5: Projection to $\{\alpha = \tilde{Q}u : u^T\tilde{\Lambda}u \le \gamma\}$

The projection of $\hat{\alpha}$ to the ellipsoid $\{\alpha = \tilde{Q}u : u^T\tilde{\Lambda}u \le \gamma\}$ is performed by solving the following constrained optimization problem

$$\arg\min_y \|\hat{\alpha} - y\|_2^2, \quad \text{subject to } y^T\tilde{G}^\dagger y \le \gamma,\ y \in \mathcal{R}(\tilde{Q}), \tag{11}$$

where $\mathcal{R}(\tilde{Q})$ denotes the range space of $\tilde{Q}$. We can find the solution by finding the Karush-Kuhn-Tucker condition of the Lagrangian function. The following proposition characterizes the solution, which justifies Algorithm 5.

Proposition 10 *The solution to* (11) *is*

$$\begin{cases} \tilde{Q}\tilde{Q}^T\hat{\alpha}, & \text{if } \hat{\alpha}^T\tilde{G}^\dagger\hat{\alpha} \leq \gamma; \\ \tilde{Q}\tilde{\Lambda}^{-1}(\hat{x}I + \tilde{\Lambda}^{-1})^{-1}\tilde{Q}^T\hat{\alpha}, & \text{otherwise,} \end{cases}$$

where $\tilde{G}^\dagger = \tilde{Q}\tilde{\Lambda}\tilde{Q}^T$ is the eigendecomposition of $\tilde{G}^\dagger$ and λ_j denotes the jth diagonal entry of $\tilde{\Lambda}$, $\hat{x}$ is the largest nonnegative solution to

$$3x^2\sum_{j=1}^{r(\tilde{G})}\lambda_j^3u_j^2 - 2x\sum_{j=1}^{r(\tilde{G})}\lambda_j^2u_j^2 + \sum_{j=1}^{r(\tilde{G})}\lambda_ju_j^2 - \gamma = 0,$$

and $u = \tilde{Q}^T\hat{\alpha}$.

Proof of Proposition 10: Let $(\tilde{Q},\ Q_1)$ be a unitary matrix with $\tilde{Q}^TQ_1 = 0$. Let $\hat{u} = (\tilde{Q},\ Q_1)^T\hat{\alpha}$, and $\hat{z} = (\tilde{Q},\ Q_1)^Ty$. Since $(\tilde{Q},\ Q_1)$ is unitary, we have

$$\|\hat{\alpha} - y\|_2^2 = \|(\tilde{Q}\ Q_1)^T(\hat{\alpha} - y)\|_2^2 = \|\hat{u} - \hat{z}\|_2^2.$$

Let $u = \tilde{Q}^T\hat{\alpha}$ and $z = \tilde{Q}^Ty$. Since $Q_1^T\tilde{Q} = 0$ and we must have $Q_1^Ty = 0$, the problem (11) is equivalent to the following

$$\arg\min_z \|u - z\|_2^2, \quad \text{subject to } z^T\tilde{\Lambda}z \leq \gamma. \tag{12}$$

Letting $w = \tilde{\Lambda}^{1/2}z$, we can rewrite the objective function (12) as follows

$$\arg\min_w \|u - \tilde{\Lambda}^{-1/2}w\|_2^2, \quad \text{subject to } w^Tw \leq \gamma.$$

Let the corresponding Lagrangian function be

$$\mathcal{L}(w, x) = \|u - \tilde{\Lambda}^{-1/2}w\|_2^2 + x(w^Tw - \gamma).$$

Condition $\nabla_w\mathcal{L} = 0$ implies that

$$w = (xI + \tilde{\Lambda}^{-1})^{-1}\tilde{\Lambda}^{-1/2}u.$$

By the Karush–Kuhn–Tucker condition, if $w^Tw < \gamma$, then $y^* = \tilde{Q}\tilde{Q}^T\hat{\alpha}$. Otherwise, $w^Tw = \gamma$. This implies that

$$\sum_{j=1}^{r(\tilde{G})}\frac{(u_j^2\lambda_j)}{(1 + x\lambda_j)^2} = \gamma. \tag{13}$$

Using the second-order Taylor expansion, we write (13) as

$$3x^2 \sum_{j=1}^{r(\tilde{G})} \lambda_j^3 u_j^2 - 2x \sum_{j=1}^{r(\tilde{G})} \lambda_j^2 u_j^2 + \sum_{j=1}^{r(\tilde{G})} \lambda_j u_j^2 - \gamma = 0. \tag{14}$$

Then, finding x is equivalent to finding the roots of the above polynomial function. Finally, we plug $\hat{x}$, the largest nonnegative solution to (14), into $y^* = \tilde{Q}\tilde{\Lambda}^{-1}(\hat{x}I + \tilde{\Lambda}^{-1})^{-1}\tilde{Q}^T\hat{\alpha}$ and complete the proof. ∎

A.3 Expansion Coefficients of Projections to $\mathcal{C}_V$ and $\tilde{\mathcal{C}}_V$

Let v be a column of V, $v^\star$ be a column of $V^\star$, and let $\hat{v}$ denote projection of v to $\mathcal{C}_V$. Since $\mathcal{C}_V$ is a nonconvex set, $\hat{v}$ may be further away from $v^\star$ compared to v. We denote $\Pi_{\mathcal{C}_V}(V)$ as the projection operator that projects columns of V to $\mathcal{C}_V$. We characterize ρ such that

$$\|\Pi_{\mathcal{C}_V}(V) - V^\star R\|_F^2 \leq \rho \|V - V^\star R\|_F^2.$$

Lemma 13 characterizes ρ by combining results from Lemma 11 and Lemma 12. Lemma 15 provides a bound on ρ when an additional step is performed to orthogonalize V by QR decomposition.

The following lemma shows the expansion coefficient of the hard thresholding operator, which corresponds to the first step in Algorithm 3.

Lemma 11 (Lemma 4.1 in Li et al. (2016)) *Suppose that $u \in \mathbb{R}^P$ is a sparse vector such that $\|u\|_0 \leq s^\star$. Let $\Pi_s(\cdot) : \mathbb{R}^P \to \mathbb{R}^P$ be the hard thresholding operator, which outputs a vector by selecting the top-s entries of the input vector in absolute value and setting the rest of the entries to 0. Given $s > s^\star$, for any vector $v \in \mathbb{R}^P$, we have*

$$\|\Pi_s(v) - u\|_2^2 \leq \left\{1 + \frac{2\sqrt{s^\star}}{\sqrt{(s - s^\star)}}\right\} \|v - u\|_2^2.$$

The following results characterize the expansion coefficient for the second step in Algorithm 3.

Lemma 12 *Assume that $v^T u \geq 0$, $\|u\|_2 = 1$, and $\|v\|_2 \leq 1$. Then*

$$2\|v - u\|_2^2 \geq \left\| \frac{v}{\|v\|_2} - u \right\|_2^2.$$

Proof of Lemma 12: Showing $2\|v - u\|_2^2 \geq \|v/\|v\|_2 - u\|_2^2$ is equivalent to showing

$$2\|v\|_2^2 + 2v^T u \left(\frac{1}{\|v\|_2} - 2 \right) \geq 0.$$

Let $\cos\theta = (v^T u)/(\|v\|_2 \|u\|_2)$. Then we need to show that

$$2\|v\|_2^2 + 2\cos\theta - 4\|v\|_2 \cos\theta \geq 0. \tag{15}$$

Since $(a+b) \geq 2\sqrt(ab)$, for $a \geq 0$ and $b \geq 0$, and $\cos^{1/2}\theta \geq \cos\theta$ for $\cos\theta \geq 0$, we have established (15). ∎

Combining Lemma 11 and Lemma 12, we obtain the following result.

Lemma 13 *Consider two matrices $U, V \in \mathbb{R}^{P\times K}$, and assume that $v_k^T u_k \geq 0$ and $\|u_k\|_0 \leq s^\star$ for $k \in [K]$. Assume that $s > s^*$. Let $\Pi_{\mathcal{C}_V} : \mathbb{R}^{P\times K} \to \mathbb{R}^{P\times K}$ be the projection operator that projects columns of the matrix onto the set $\mathcal{C}_V$, defined in Section A.1. Then*

$$\|\Pi_{\mathcal{C}_V}(V) - U\|_F^2 \leq 2\left\{1 + \frac{2\sqrt{s^\star}}{\sqrt{(s-s^\star)}}\right\}\|V-U\|_F^2. \tag{16}$$

Proof of Lemma 13: Lemma 11 states the expansion coefficient of the first projection in Algorithm 3. Similarly, Lemma 12 states the expansion coefficient of the second projection in Algorithm 3 when the vector before the projection has a norm smaller than 1. If the vector before projection has the norm greater than or equal to 1, then the projection to the unit sphere is equivalent to the projection to the unit ball, which is a convex set. Then, the resulting projection is a contraction. Multiplying the results of two lemmas, we can obtain the expansion coefficient for the projection to $\mathcal{C}_V$ for each column vector. Stacking all the column vectors together, we obtain the result (16). ∎

The following lemmas characterize the expansion coefficient ρ when an additional QR decomposition step is used. First, we state a result from the perturbation theory of QR decomposition (Stewart, 1977).

Lemma 14 (Adapted from Theorem 1 in Stewart (1977)) *Let $A \in \mathbb{R}^{P\times K}$ be a matrix with rank K and let $A^\dagger$ denote its pseudo-inverse. Suppose that $E \in \mathbb{R}^{P\times K}$ and $\|E\|_2\|A^\dagger\|_2 < 1$. Then, given a QR decomposition of $(A+E) = BL$, there exists a decomposition of $A = B^\star L^\star$, such that $B^\star$ has orthonormal columns and $L^\star$ is a nonsingular upper triangular matrix and*

$$\|B - B^\star\|_F \leq \frac{\sqrt{2}\|A^\dagger\|_2\|E\|_F}{1 - \|E\|_2\|A^\dagger\|_2}.$$

Next, we apply Lemma 14 to our setting and establish the following lemma.

Lemma 15 *Let U be a matrix with orthonormal columns. Let $V \in \mathbb{R}^{P\times K}$ be a rank K matrix with unit norm columns and $\|V-U\|_2 \leq r' < 1$. Let $V = BL$ be the QR* *decomposition of* V*, where* $B \in \mathbb{R}^{P\times K}$ *has orthonormal columns and* $L \in \mathbb{R}^{K\times K}$ *is an upper triangular matrix. Then*

$$\|B - B^\star\|_F^2 \leq \frac{2}{(1-r')^2}\|V - U\|_F^2 \,.$$

Proof of Lemma 15: Let $E = V - U$ and $A = U$. We have $\|E\|_2 \leq r'$, $\|A^\dagger\|_2 = 1$. The result then follows from Lemma 14 as $\|B - B^\star\|_F \leq \sqrt{2}\|V - U\|_F/(1-r')$. ■

Appendix B. Linear Convergence and Statistical Error

We prove Theorem 7. The proof is divided into several parts. In Section B.1, we discuss the relationship between $\mathrm{dist}^2(Z, \tilde{Z}^\star)$ and $\{\|\Sigma_j - \tilde{\Sigma}_j^\star\|_F^2\}_{j\in[J]}$. The main result is given in Section B.2, while the supporting lemmas for the proof are given in Sections B.3—B.5.

B.1 Upper Bound for the Distance Metric

We establish an upper bound on $\mathrm{dist}^2(Z, \tilde{Z}^\star)$ in terms of $\{\|\Sigma_j - \tilde{\Sigma}_j^\star\|_F^2\}_{j\in[J]}$, which serves as an important ingredient in the analysis of linear convergence.

Lemma 16 *For two matrices $V, V^\star \in \mathbb{R}^{P\times K}$ with orthonormal columns, let*

$$R = \operatorname*{argmin}_{Y\in\mathcal{O}(K)} \|V - V^\star Y\|_F^2.$$

Let $\Sigma_j = V\mathrm{diag}(a_j)V^T$, $\tilde{\Sigma}_j^\star = V^\star\mathrm{diag}(\tilde{a}_j^\star)V^{\star T}$, $\Sigma_j^\star = V^\star\mathrm{diag}(a_j^\star)V^{\star T}$, and c be a positive constant such that $\|\mathrm{diag}(a_j)\|_2 \le c$ for $j \in [J]$. Suppose that $\sigma_K(\Sigma_j^\star - \tilde{\Sigma}_j^\star) \le 1/4\sigma_K(\Sigma_j^\star)$ for $j \in [J]$, then

$$\sum_{j=1}^J \|V - V^\star R\|_F^2 + \|\mathrm{diag}(a_j) - R^T\mathrm{diag}(\tilde{a}_j^\star)R\|_F^2 \le \xi^2 \sum_{j=1}^J \|\Sigma_j - \tilde{\Sigma}_j^\star\|_F^2,$$

where

$$\xi^2 = \max_{j\in[J]} \left\{ \frac{16}{\sigma_K^2(\Sigma_j^\star)} + \left(1 + \frac{8c}{\sigma_K(\Sigma_j^\star)}\right)^2 \right\}.$$

Proof of Lemma 16: We establish the result for a single $j \in [J]$. The bound can easily be extended to the sum of all $j \in [J]$.

Since $\tilde{\Sigma}_j^\star$ is of rank K, we have $\sigma_{K+1}(\tilde{\Sigma}_j^\star) = 0$ for $j \in [J]$. Consequently,

$$\sigma_K(\tilde{\Sigma}_j^\star) - \sigma_{K+1}(\tilde{\Sigma}_j^\star) = \sigma_K(\tilde{\Sigma}_j^\star) > 0,$$

for $j \in [J]$ and we can use Lemma 29 to obtain

$$\|V - V^\star R\|_F^2 \le \frac{8}{\sigma_K^2(\tilde{\Sigma}_j^\star)} \|\Sigma_j - \tilde{\Sigma}_j^\star\|_F^2. \tag{17}$$

Moreover, since

$$\sigma_K(\tilde{\Sigma}_j^\star) \ge \sigma_K(\Sigma_j^\star) - \sigma_K(\Sigma_j^\star - \tilde{\Sigma}_j^\star) \ge \frac{3}{4}\sigma_K(\Sigma_j^\star),$$

we have

$$\|V - V^\star R\|_F^2 \le \frac{16}{\sigma_K^2(\Sigma_j^\star)}\|\Sigma_j - \tilde{\Sigma}_j^\star\|_F^2.$$

Next, by the triangular inequality, we have

$$\begin{aligned}
\|\Sigma_j - \tilde{\Sigma}_j^\star\|_F &= \|V\mathrm{diag}(a_j)V^T - V^\star RR^T\mathrm{diag}(\tilde{a}_j^\star)RR^TV^{\star T}\|_F \\
&\ge \|V^\star R\{\mathrm{diag}(a_j) - R^T\mathrm{diag}(\tilde{a}_j^\star)R\}R^TV^{\star T}\|_F \\
&\quad - \|(V - V^\star R)\mathrm{diag}(a_j)V^T\|_F - \|V^\star R\mathrm{diag}(a_j)(V - V^\star R)^T\|_F.
\end{aligned}$$

Since

$$\|V^\star R\{\mathrm{diag}(a_j) - R^T\mathrm{diag}(\tilde{a}_j^\star)R\}R^TV^{\star T}\|_F = \|\mathrm{diag}(a_j) - R^T\mathrm{diag}(\tilde{a}_j^\star)R\|_F$$

and

$$\|V\|_2 + \|V^\star R\|_2 = 2,$$

we further have

$$\begin{aligned}
\|\Sigma_j - \tilde{\Sigma}_j^\star\|_F &\ge \|\mathrm{diag}(a_j) - R^T\mathrm{diag}(\tilde{a}_j^\star)R\|_F \\
&\quad - \|(V - V^\star R)\|_F\left\{\|\mathrm{diag}(a_j)V^T\|_2 + \|V^\star R\mathrm{diag}(a_j)\|_2\right\} \\
&\ge \|\mathrm{diag}(a_j) - R^T\mathrm{diag}(\tilde{a}_j^\star)R\|_F - 2\|\mathrm{diag}(a_j)\|_2\|V - V^\star R\|_F.
\end{aligned}$$

Therefore,

$$\|\mathrm{diag}(a_j) - R^T\mathrm{diag}(\tilde{a}_j^\star)R\|_F \le \|\Sigma_j - \tilde{\Sigma}_j^\star\|_F + 2\|\mathrm{diag}(a_j)\|_2\|V - V^\star R\|_F.$$

Combining (17) and $\|\mathrm{diag}(a_j)\|_2 \le c$, we have

$$\|\mathrm{diag}(a_j) - R^T\mathrm{diag}(\tilde{a}_j^\star)R\|_F \le \left(1 + \frac{8c}{\sigma_K(\Sigma_j^\star)}\right)\|\Sigma_j - \tilde{\Sigma}_j^\star\|_F. \tag{18}$$

The proof is complete by combining (17) and (18). ■

B.2 Proof of Theorem 7

We prove Theorem 7 in several steps. First, we show that given a current iterate Z, which satisfies suitable assumptions, the subsequent iterate Z^+ obtained by Algorithm 2 with a suitable step size satisfies

$$\mathrm{dist}^2(Z^+, \tilde{Z}^\star) \le \beta^{1/2}\mathrm{dist}^2(Z, \tilde{Z}^\star) + C_1\varepsilon_{stat}^2,$$

with $0 < \beta^{1/2} < 1$ and some constant C_1. Second, we show that the step size can be chosen in a way that does not depend on the specific iterate. Finally, the lemma follows by applying the first step of the proof I times starting from Z^0.

We start by introducing some additional notation for simplicity of presentation. We define

$$Z = \begin{pmatrix} V \\ A^T \end{pmatrix}, \quad Z^+ = \begin{pmatrix} V^+ \\ A^{+T} \end{pmatrix}, \quad \tilde{Z}^\star = \begin{pmatrix} V^\star \\ \tilde{A}^{\star T} \end{pmatrix},$$

where Z is the current iterate, Z^+ is the iterate obtained by one step of Algorithm 2 starting from Z, and $\tilde{Z}^\star$ is the truncated version of the ground truth parameter $Z^\star$. Furthermore, let

$$R = \operatorname*{argmin}_{Y \in \mathcal{O}(K)} \|V - V^\star Y\|_F^2, \quad R^+ = \operatorname*{argmin}_{Y \in \mathcal{O}(K)} \|V^+ - V^\star Y\|_F^2.$$

be the optimal rotation matrices in the current and subsequent step.

Let $\Pi_{\tilde{\mathcal{C}}_V}(X)$ be the projection operator defined in (10). Let $\Pi_{\tilde{\mathcal{C}}_A}(Y)$ be the projection operator that projects rows of Y to $\tilde{\mathcal{C}}_A$, given in Algorithm 4. One update of Algorithm 2 can be written as

$$V^+ = \Pi_{\tilde{\mathcal{C}}_V}(V - \eta_V \nabla_V f_N), \quad A^+ = \Pi_{\tilde{\mathcal{C}}_A}(A - \eta_A \nabla_A f_N),$$

where

$$\nabla_V f_N(Z) = \frac{2}{J} \sum_{j=1}^{J} \nabla \ell_{N,j}(\Sigma_j) V \operatorname{diag}(a_j), \quad \nabla_A f_N(Z) = \frac{1}{J} W(V), \tag{19}$$

with

$$W(V) = [w_{kj}(v_k)] \in \mathbb{R}^{K \times J}, \quad w_{kj}(v_k) = v_k^T \nabla \ell_{N,j}(\Sigma_j) v_k, \quad \nabla \ell_{N,j}(\Sigma_j) = \Sigma_j - S_{N,j}.$$

Similarly, we define

$$W^\star(V) = [w^\star_{kj}(v_k)] \in \mathbb{R}^{K \times J}, \quad w^\star_{kj}(v_k) = v_k^T \nabla \ell_{N,j}(\Sigma^\star_j) v_k.$$

Let $\mathcal{S}_U = \mathcal{S}(V) \cup \mathcal{S}(V^+) \cup \mathcal{S}(V^\star)$ and note that $|\mathcal{S}_U| \leq 2s + s^\star$. Given the index set $\mathcal{S}_U$, we write $[X]_{\mathcal{S}_U}$ to denote the projection of X to the support $\mathcal{S}_U$

$$[X]_{\mathcal{S}_U} = \begin{cases} X_{ij} & (i,j) \in \mathcal{S}_U \\ 0 & (i,j) \notin \mathcal{S}_U \end{cases}.$$

With some abuse of notation, given a matrix Y with the factored form $Y = XX^T$, we write

$$[Y]_{\mathcal{S}_U, \mathcal{S}_U} = [X]_{\mathcal{S}_U} [X]_{\mathcal{S}_U}^T.$$

With this notation, we have

$$V^+ = \Pi_{\tilde{\mathcal{C}}_V}(V - \eta_V \nabla_V f_N) = \Pi_{\tilde{\mathcal{C}}_V}(V - \eta_V \left[\nabla_V f_N\right]_{\mathcal{S}_U}). \tag{20}$$

Furthermore, recall that $\tilde{Q}$ is the matrix whose columns are eigenvectors of $\tilde{G}$. Then $\tilde{Q}\tilde{Q}^T$ is the projection operator to the subspace spanned by the columns of $\tilde{Q}$. Since the output of Algorithm 5 is in the range space of $\tilde{Q}$, we have that $A\tilde{Q}\tilde{Q}^T = A$ and

$$A^+ = \Pi_{\tilde{\mathcal{C}}_A}(A - \eta_A \nabla_A f_N) = \Pi_{\tilde{\mathcal{C}}_A}\{(A - \eta_A \nabla_A f_N)\tilde{Q}\tilde{Q}^T\} = \Pi_{\tilde{\mathcal{C}}_A}\{A - \eta_A(\nabla_A f_N \tilde{Q}\tilde{Q}^T)\}. \tag{21}$$

For later convenience, we also note that for a rotation matrix $R \in \mathcal{O}(K)$, we have

$$\langle \nabla_V f_N(Z), V - V^\star R\rangle = \frac{2}{J}\sum_{j=1}^{J} \langle \nabla \ell_{N,j}(\Sigma_j), V \text{diag}(a_j) V^T - V^\star R \text{diag}(a_j) V^T \rangle \tag{22}$$

and

$$\begin{aligned}\langle \text{diag}\{(\nabla_A f_N)_j\}, &\text{diag}(a_j) - R^T \text{diag}(\tilde{a}_j^\star) R\rangle \\ &= \frac{1}{J}\langle \nabla \ell_{N,j}(\Sigma_j), V\text{diag}(a_j)V^T - VR^T\text{diag}(\tilde{a}_j^\star)RV^T\rangle.\end{aligned} \tag{23}$$

With this notation, we are ready to state the result of the first step of the proof.

Lemma 17 *Suppose that Z satisfies*

$$d^2(Z_j, Z_j^\star) \le I_0^2, \quad \|V - V^\star R\|_F \le I_0^2/J, \quad \|\text{diag}(a_j) - R^T\text{diag}(a_j^\star)R\|_F^2 \le (J-1)I_0^2/J, \tag{24}$$

where I_0^2 is given in (3)*. Furthermore, suppose that Assumptions 1, 2, and 3 hold. Then*

$$\mathit{dist}^2(Z^+, \tilde{Z}^\star) \le \beta^{1/2}\mathit{dist}^2(Z, \tilde{Z}^\star) + \tau\beta^{-1/2}\eta\varepsilon_{stat}^2,$$

where Z^+ is obtained with one iteration of Algorithm 2 starting from Z and $\tau = J^{-1}\{9/2 + (1/2 \vee K/8)\}$.

Note that $d^2(Z_j \tilde{Z}_j^\star) \le d^2(Z_j, Z_j^\star)$ for $j \in [J]$ and δ_A quantifies the proximity of $Z^\star$ to $\tilde{Z}^\star$. Therefore, we do not need additional assumptions for $\tilde{Z}^\star$.

Starting from Z^0, which satisfies Assumption 6 (which is also restated in (24)), we show that Z^1 also satisfies (24). Therefore, we can apply Lemma 17 over I iterations to obtain Theorem 7.

Proof of Theorem 7: When we apply one iteration of Algorithm 2, Lemma 17 gives us

$$\text{dist}^2(Z^+, \tilde{Z}^\star) \le \beta^{1/2}\text{dist}^2(Z, \tilde{Z}^\star) + \tau\beta^{-1/2}\eta\varepsilon_{stat}^2. \tag{25}$$

Under Assumption 3, the right-hand side of (25) is bounded by JI_0^2. This implies that the new estimate is still in a good region where we can apply Lemma 17. That is, Z^+ satisfies

(24). Consequently, since Z^0 satisfies Assumption 6 and, therefore, the equation (24), we can apply the result of Lemma 17 for I iterations to obtain

$$\mathrm{dist}^2(Z^I, \tilde{Z}^\star) \leq \beta^{I/2}\mathrm{dist}^2(Z^0, \tilde{Z}^\star) + \frac{\tau\eta\varepsilon_{stat}^2}{\beta^{1/2} - \beta},$$

which completes the proof. ■

B.3 Proof of Lemma 17

Recall that

$$\mathrm{dist}^2(Z^+, \tilde{Z}^\star) = \sum_{j=1}^{J} \|V^+ - V^\star R^+\|_F^2 + \|\mathrm{diag}(a_j^+) - R^{+T}\mathrm{diag}(\tilde{a}_j^\star)R^+\|_F^2. \tag{26}$$

We bound the two terms on the right-hand side of (26) separately. From the triangle inequality, we have

$$\begin{aligned}\|\mathrm{diag}(a_j^+) - R^{+T}\mathrm{diag}(\tilde{a}_j^\star)R^+\|_F^2 \leq 2\|\mathrm{diag}(a_j^+) - R^T\mathrm{diag}(\tilde{a}_j^\star)R\|_F^2 \\ + 2\|R^T\mathrm{diag}(\tilde{a}_j^\star)R - R^{+T}\mathrm{diag}(\tilde{a}_j^\star)R^+\|_F^2.\end{aligned}$$

By Lemma 21, $\|R^T\mathrm{diag}(\tilde{a}_j^\star)R - R^{+T}\mathrm{diag}(\tilde{a}_j^\star)R^+\|_F \leq 4\|\mathrm{diag}(\tilde{a}_j^\star)\|_2\|V^+ - V^\star R\|_F$ and

$$\begin{aligned}&\sum_{j=1}^{J} \|\mathrm{diag}(a_j^+) - R^{+T}\mathrm{diag}(\tilde{a}_j^\star)R^+\|_F^2 \\ &\qquad \leq 32J\|\tilde{A}^\star\|_\infty^2\|V^+ - V^\star R\|_F^2 + 2\sum_{j=1}^{J}\|\mathrm{diag}(a_j^+) - R^T\mathrm{diag}(\tilde{a}_j^\star)R\|_F^2.\end{aligned} \tag{27}$$

Combining (27) with (26) and recalling the definition of V^+ and A^+ from (20) and (21) we have

$$\begin{aligned}\mathrm{dist}^2(Z^+, \tilde{Z}^\star) \leq J\kappa\|\Pi_{\tilde{\mathcal{C}}_V}(V - \eta_V[\nabla_V f_N]_{\mathcal{S}_U}) - V^\star R\|_F^2 \\ + 2\sum_{j=1}^{J}\|\mathrm{diag}\left[\Pi_{\tilde{\mathcal{C}}_A}\{A - \eta_A(\nabla_A f_N \tilde{Q}\tilde{Q}^T)\}_j\right] - R^T\mathrm{diag}(\tilde{a}_j^\star)R\|_F^2.\end{aligned}$$

where $\kappa = (1 + 32\|\tilde{A}^\star\|_\infty^2)$. Next, we define

$$\bar{V}^+ = \Pi_{\mathcal{C}_V}(V - \eta_V[\nabla_V f_N]_{\mathcal{S}_U}),$$

where we recall that $\Pi_{\tilde{\mathcal{C}}_V}(\cdot)$ is the projection operator by first applying $\Pi_{\mathcal{C}_V}(\cdot)$ followed by a QR decomposition step. By Lemma 23, we have $\|\bar{V}^+ - V^\star R\|_2 \le 2I_0/J^{1/2}$. Therefore, we can apply Lemma 15 with $U = V^\star R = B^\star L^\star$, where $B^\star = V^\star R$ and a nonsingular matrix $L^\star = I$. Then,

$$\begin{aligned}\|\Pi_{\tilde{\mathcal{C}}_V}(V - \eta_V[\nabla_V f_N]_{\mathcal{S}_U}) - V^\star R\|_F^2 &\le \frac{2}{(1-\frac{2I_0}{J^{1/2}})^2}\|\bar{V}^+ - V^\star R\|_F^2 \\ &= \frac{2}{(1-\frac{2I_0}{J^{1/2}})^2}\|\Pi_{\mathcal{C}_V}(V - \eta_V[\nabla_V f_N]_{\mathcal{S}_U}) - V^\star R\|_F^2.\end{aligned}$$

By the fact $\|V - V^\star R\|_F \le I_0/J^{1/2}$ and Lemma 22, we have

$$\|V - \eta_V[\nabla_V f_N]_{\mathcal{S}_U} - V^\star R\|_F \le \|V - V^\star R\|_F + \|\eta_V[\nabla_V f_N]_{\mathcal{S}_U}\|_F \le \frac{7I_0}{6J^{1/2}} < 1. \tag{28}$$

Since columns of $V^\star R$ are unit norm, the result in (28) implies that the inner product of the kth column of $V - \eta_V[\nabla_V f_N]_{\mathcal{S}_U}$ and the kth column of $V^\star R$ is nonnegative for every $k \in [K]$. Therefore, we can apply Lemma 13 and the further bound $\|\Pi_{\tilde{\mathcal{C}}_V}(V - \eta_V[\nabla_V f_N]_{\mathcal{S}_U}) - V^\star R\|_F^2$ as

$$\|\Pi_{\tilde{\mathcal{C}}_V}(V - \eta_V[\nabla_V f_N]_{\mathcal{S}_U}) - V^\star R\|_F^2 \le \rho\|V - \eta_V[\nabla_V f_N]_{\mathcal{S}_U} - V^\star R\|_F^2,$$

where $\rho = 4(1-r')^{-2}\{1 + 2\sqrt{s^\star}/\sqrt{(s-s^\star)}\}$ and $r' = 2I_0/J^{1/2}$. By the contraction property of projection to convex sets, we have

$$\begin{aligned}&\sum_{j=1}^J \left\|\mathrm{diag}\left[\Pi_{\tilde{\mathcal{C}}_A}\{A - \eta_A(\nabla_A f_N \tilde{Q}\tilde{Q}^T)\}_j\right] - R^T\mathrm{diag}(\tilde{a}_j^\star)R\right\|_F^2 \\ &\qquad\le \sum_{j=1}^J \|\mathrm{diag}(a_j) - \eta_A \mathrm{diag}\{(\nabla_A f_N \tilde{Q}\tilde{Q}^T)_j\} - R^T\mathrm{diag}(\tilde{a}_j^\star)R\|_F^2,\end{aligned}$$

where $(\nabla_A f_N \tilde{Q}\tilde{Q}^T)_j$ denotes the jth column of $\nabla_A f_N \tilde{Q}\tilde{Q}^T \in \mathbb{R}^{K\times J}$. Combining the last two displays and noting that $\rho' = \rho\kappa > 2$, we have

$$\begin{aligned}\mathrm{dist}^2(Z^+, \tilde{Z}^\star) \le \rho'\Bigg[&J\|V - \eta_V[\nabla_V f_N]_{\mathcal{S}_U} - V^\star R\|_F^2 \\ &+ \sum_{j=1}^J \|\mathrm{diag}(a_j) - \eta_A\mathrm{diag}\{(\nabla_A f_N\tilde{Q}\tilde{Q}^T)_j\} - R^T\mathrm{diag}(\tilde{a}_j^\star)R\|_F^2\Bigg].\end{aligned}$$

Recall that $\eta_V = \eta/J$ and $\eta_A = \eta$. Therefore

$$\mathrm{dist}^2(Z^+, \tilde{Z}^\star) \le \rho'\mathrm{dist}^2(Z, \tilde{Z}^\star) + \eta^2\rho'(B1 + B2) - \eta\rho'(A1 + A2), \tag{29}$$

where

$$
\begin{aligned}
A1 &= 2\langle [\nabla_V f_N]_{\mathcal{S}_U}, V - V^\star R\rangle, \\
A2 &= 2\sum_{j=1}^{J} \langle \operatorname{diag}\{(\nabla_A f_N \tilde{Q}\tilde{Q}^T)_j\}, \operatorname{diag}(a_j) - R^T \operatorname{diag}(\tilde{a}_j^\star) R\rangle, \\
B1 &= J^{-1}\|[\nabla_V f_N]_{\mathcal{S}_U}\|_F^2, \\
B2 &= \sum_{j=1}^{J} \|\operatorname{diag}\{(\nabla_A f_N \tilde{Q}\tilde{Q}^T)_j\}\|_F^2.
\end{aligned}
$$

Next, we upper bound $B = B1 + B2$ and lower bound $A = A1 + A2$ in Lemma 18 and Lemma 19, respectively. With these bounds, we will be able to show contraction $\operatorname{dist}^2(Z^+, \tilde{Z}^\star)$ with respect to $\operatorname{dist}^2(Z, \tilde{Z}^\star)$.

Lemma 18 *Under the conditions of Lemma 17, we have*

$$
\begin{aligned}
A \geq \frac{1}{J}\Bigg\{ &\frac{3}{4}\sum_{j=1}^{J} \|\Sigma_j - \tilde{\Sigma}_j^\star\|_F^2 - \frac{9}{2}\varepsilon_{stat}^2 + \frac{1}{2}\sum_{j=1}^{J} \left\| \left[\nabla \ell_{N,j}(\Sigma_j) - \nabla \ell_{N,j}(\Sigma_j^\star)\right]_{\mathcal{S}_U,\mathcal{S}_U} \right\|_F^2 \\
&- 8\left(1 + \frac{\|A^\star\|_\infty^2}{J}\right) I_0^2\, dist^2(Z, \tilde{Z}^\star) \Bigg\}.
\end{aligned}
$$

Lemma 19 *Under the conditions of 17, we have*

$$
B \leq \frac{16}{J^2}\sum_{j=1}^{J} \left\{ \|\nabla \ell_{N,j}(\Sigma_j) - \nabla \ell_{N,j}(\Sigma_j^\star)\|_F^2 \right\} \|Z_j\|_2^2 + \frac{4(4 \vee K)}{J^2}\varepsilon_{stat}^2 \max_{j\in[J]} \|Z_j\|_2^2.
$$

Using Lemma 18 and Lemma 19, we have

$$
\begin{aligned}
\eta\rho' A - \eta^2\rho' B \geq{}& \frac{\rho'\eta}{J} \underbrace{\left\{ \frac{3}{4}\sum_{j=1}^{J} \|\Sigma_j - \tilde{\Sigma}_j^\star\|_F^2 - 8\left(1 + \frac{\|A^\star\|_\infty^2}{J}\right) I_0^2 \operatorname{dist}^2(Z, \tilde{Z}^\star) \right\}}_{C1} \\
&- \rho'\eta\varepsilon_{stat}^2 \underbrace{\left(\frac{9}{2J} + \frac{4(4\vee K)\eta}{J^2} \max_{j\in[J]} \|Z_j\|_2^2 \right)}_{C2} \\
&+ \frac{\eta\rho'}{J} \underbrace{\left(\frac{1}{2} - 16\frac{\eta}{J} \max_{j\in[j]} \|Z_j\|_2^2 \right)}_{C3} \sum_{j=1}^{J} \|\nabla \ell_{N,j}(\Sigma_j) - \nabla \ell_{N,j}(\Sigma_j^\star)\|_F^2.
\end{aligned}
$$

Under the assumption that $\delta_A \leq (16\gamma^\star)^{-1} \min_{j\in[J]} \sigma_K^2(\Sigma_j^\star)$ and the inequality in (55), we have

$$\sigma_K(\Sigma_j^\star - \tilde{\Sigma}_j^\star) = \sigma_K(V^\star \mathrm{diag}(a_j^\star - \tilde{a}_j^\star) V^{\star T}) = \min_{k\in[K]} \left| a_{jk}^\star - \tilde{a}_{jk}^\star \right| \\ \leq \max_{k\in[K]} \|\tilde{A}_{k\cdot}^\star - A_{k\cdot}^\star\|_2 \leq (\delta_A \gamma^\star)^{1/2} \leq \frac{1}{4}\sigma_K(\Sigma_j^\star)$$

for $j \in [J]$. Therefore, using Lemma 16, we have

$$\sum_{j=1}^J \|\Sigma_j - \tilde{\Sigma}_j^\star\|_F^2 \geq \frac{1}{\xi^2}\mathrm{dist}^2(Z, \tilde{Z}^\star).$$

Furthermore, from the definition of I_0^2 in Assumption 6, we have that

$$8I_0^2\left(1 + \frac{\|A^\star\|_\infty^2}{J}\right) \leq \frac{1}{2\xi^2}.$$

Combining the last two displays, we arrive at

$$C1 \geq \frac{1}{4\xi^2}\mathrm{dist}^2(Z, \tilde{Z}^\star).$$

In fact, we can verify that $C3$ is nonnegative because the step size η that satisfies Assumption 1 is small enough such that the following inequality holds.

Lemma 20 *Under the conditions of Lemma 17 we have* $\|Z_j\|_2^2 \leq 2\|Z_j^0\|_2^2$.

Therefore,

$$\eta \leq \min_{j\in[J]} \frac{J}{32\|Z_j\|_2^2}$$

and $C3 \geq 0$ can be omitted, while $C2 \leq \tau$. Then

$$\eta\rho' A - \eta^2 \rho' B \geq \frac{\rho'\eta}{J}\frac{1}{4\xi^2}\mathrm{dist}^2(Z, \tilde{Z}^\star) - \varepsilon_{stat}^2 \tau \rho' \eta.$$

Under Assumption 2, it is easy to verify that $\rho' \leq \beta^{-1/2}$, where $\beta = 1 - \eta/(4J\xi^2)$. Plugging into (29), we have

$$\mathrm{dist}^2(Z^+, \tilde{Z}^\star) \leq \beta^{1/2}\mathrm{dist}^2(Z, \tilde{Z}^\star) + \tau\beta^{-1/2}\eta\varepsilon_{stat}^2,$$

which completes the proof.

B.4 Proofs of Lemma 18—20

In this section, we provide the proofs of Lemma 18—20.
Proof of Lemma 18: Using (22) and $\langle [\nabla \ell_{N,j}(\Sigma_j) V \mathrm{diag}(a_j)]_{\mathcal{S}_U^c}, [V - V^\star R]_{\mathcal{S}_U} \rangle = 0$, we have

$$
\begin{aligned}
A1 &= \frac{4}{J} \sum_{j=1}^{J} \langle [\nabla \ell_{N,j}(\Sigma_j) V \mathrm{diag}(a_j)]_{\mathcal{S}_U}, [V - V^\star R]_{\mathcal{S}_U} \rangle \\
&= \frac{4}{J} \sum_{j=1}^{J} \langle [\nabla \ell_{N,j}(\Sigma_j) V \mathrm{diag}(a_j)]_{\mathcal{S}_U} + [\nabla \ell_{N,j}(\Sigma_j) V \mathrm{diag}(a_j)]_{\mathcal{S}_U^c}, [V - V^\star R]_{\mathcal{S}_U} \rangle \\
&= \frac{4}{J} \sum_{j=1}^{J} \langle \nabla \ell_{N,j}(\Sigma_j) V \mathrm{diag}(a_j), [V - V^\star R]_{\mathcal{S}_U} \rangle \\
&= \frac{4}{J} \sum_{j=1}^{J} \langle \nabla \ell_{N,j}(\Sigma_j), [V - V^\star R]_{\mathcal{S}_U} \mathrm{diag}(a_j) V^T \rangle \\
&= \frac{4}{J} \sum_{j=1}^{J} \langle \nabla \ell_{N,j}(\Sigma_j), [V - V^\star R]_{\mathcal{S}_U} [\mathrm{diag}(a_j) V^T]_{\mathcal{S}_U} \rangle \\
&= \frac{4}{J} \sum_{j=1}^{J} \langle \nabla \ell_{N,j}(\Sigma_j), [V \mathrm{diag}(a_j) V^T - V^\star R \mathrm{diag}(a_j) V^T]_{\mathcal{S}_U, \mathcal{S}_U} \rangle.
\end{aligned}
$$

Furthermore, we can write $A1$ as

$$
A1 = A13 + \frac{4}{J} \sum_{j=1}^{J} \langle \nabla \ell_{N,j}(\Sigma_j), [\overbrace{(V - V^\star R)}^{\Delta V} R^T \mathrm{diag}(\tilde{a}_j^\star) R V^T]_{\mathcal{S}_U, \mathcal{S}_U} \rangle; \tag{30}
$$

$$
A13 = \frac{4}{J} \sum_{j=1}^{J} \langle \nabla \ell_{N,j}(\Sigma_j), [\overbrace{(V - V^\star R)}^{\Delta V} \overbrace{\{\mathrm{diag}(a_j) - R^T \mathrm{diag}(\tilde{a}_j^\star) R\}}^{\Delta a_j} V^T]_{\mathcal{S}_U, \mathcal{S}_U} \rangle.
$$

We also write $A2$ in a suitable way. Note that $\operatorname{diag}\{R^T \operatorname{diag}(\tilde{a}_j^\star) R\} = \operatorname{diag}(H a_j^\star)$, where $H = (h_{ij})_{i\in[K], j\in[K]}$ with $h_{ij} = r_{ji}^2$ and r_{ij} is the ijth entry of R. Then

$$\begin{aligned}
A2 &= 2\sum_{j=1}^{J} \langle \operatorname{diag}\{(\nabla_A f_N \tilde{Q}\tilde{Q}^T)_j\}, \operatorname{diag}(a_j) - R^T \operatorname{diag}(\tilde{a}_j^\star) R \rangle \\
&= 2\langle \nabla_A f_N \tilde{Q}\tilde{Q}^T, A - H\tilde{A}^\star \rangle \\
&= 2\langle \nabla_A f_N, A - H\tilde{A}^\star \rangle \\
&= 2\sum_{j=1}^{J} \langle \operatorname{diag}\left\{(\nabla_A f_N)_j\right\}, \operatorname{diag}(a_j) - R^T \operatorname{diag}(\tilde{a}_j^\star) R \rangle,
\end{aligned}$$

since rows of A and $\tilde{A}^\star$ belong to the subspace spanned by eigenvectors of $\tilde{G}$ and $A\tilde{Q}\tilde{Q}^T = A$ and $\tilde{A}^\star \tilde{Q}\tilde{Q}^T = \tilde{A}^\star$. Finally, using (23), we have

$$\begin{aligned}
A2 &= \frac{2}{J}\sum_{j=1}^{J} \langle \nabla \ell_{N,j}(\Sigma_j), [V \operatorname{diag}(a_j) V^T - V R^T \operatorname{diag}(\tilde{a}_j^\star) R V^T]_{\mathcal{S}_U, \mathcal{S}_U} \rangle \\
&= A11 + A12 + \frac{2}{J}\sum_{j=1}^{J} \langle \nabla \ell_{N,j}(\Sigma_j), [\tilde{\Sigma}_j^\star - V R^T \operatorname{diag}(\tilde{a}_j^\star) R V^T]_{\mathcal{S}_U, \mathcal{S}_U} \rangle \qquad (31) \\
A11 &= \frac{2}{J}\sum_{j=1}^{J} \langle \nabla \ell_{N,j}(\Sigma_j) - \nabla \ell_{N,j}(\Sigma_j^\star), [\Sigma_j - \tilde{\Sigma}_j^\star]_{\mathcal{S}_U, \mathcal{S}_U} \rangle; \\
A12 &= \frac{2}{J}\sum_{j=1}^{J} \langle \nabla \ell_{N,j}(\Sigma_j^\star), [\Sigma_j - \tilde{\Sigma}_j^\star]_{\mathcal{S}_U, \mathcal{S}_U} \rangle.
\end{aligned}$$

Combining (30) and (31), we obtain

$$\begin{aligned}
&A = A11 + A12 + A13 + A14; \\
&A14 = \frac{2}{J}\sum_{j=1}^{J} \langle \nabla \ell_{N,j}(\Sigma_j), [\overbrace{(V - V^\star R)}^{\Delta V} R^T \operatorname{diag}(\tilde{a}_j^\star) R \overbrace{(V - V^\star R)^T}^{\Delta V^T}]_{\mathcal{S}_U, \mathcal{S}_U} \rangle,
\end{aligned}$$

where we have used that

$$\frac{4}{J}\sum_{j=1}^{J}\langle\nabla\ell_{N,j}(\Sigma_j),[\Delta VR^T\mathrm{diag}(\tilde{a}_j^\star)RV^T]_{\mathcal{S}_U,\mathcal{S}_U}\rangle$$
$$=\frac{2}{J}\sum_{j=1}^{J}\langle\nabla\ell_{N,j}(\Sigma_j),[\Delta VR^T\mathrm{diag}(\tilde{a}_j^\star)RV^T]_{\mathcal{S}_U,\mathcal{S}_U}\rangle$$
$$+\frac{2}{J}\sum_{j=1}^{J}\langle\nabla\ell_{N,j}(\Sigma_j),[VR^T\mathrm{diag}(\tilde{a}_j^\star)R\Delta V^T]_{\mathcal{S}_U,\mathcal{S}_U}\rangle,$$

since $\nabla\ell_{N,j}(\Sigma_j)$ for $j\in[J]$ is symmetric. Next, we lower bound $A11$, $A12$, $A13$, and $A14$ separately.

Recall that $\Sigma_j=V\mathrm{diag}(a_j)V^T$ and $\tilde{\Sigma}_j^\star=V^\star\mathrm{diag}(\tilde{a}_j^\star)V^{\star T}$. Additionally, since $[V]_{\mathcal{S}_U^c}=[V^\star]_{\mathcal{S}_U^c}=0$,

$$[\Sigma_j-\tilde{\Sigma}_j^\star]_{\mathcal{S}_U,\mathcal{S}_U^c}=[\Sigma_j-\tilde{\Sigma}_j^\star]_{\mathcal{S}_U,\mathcal{S}_U^c}=[\Sigma_j-\tilde{\Sigma}_j^\star]_{\mathcal{S}_U^c,\mathcal{S}_U^c}=0,$$

for every $j\in[J]$, and therefore $[\Sigma_j-\tilde{\Sigma}_j^\star]_{\mathcal{S}_U,\mathcal{S}_U}=\Sigma_j-\tilde{\Sigma}_j^\star$. Then

$$A11=\frac{2}{J}\sum_{j=1}^{J}\langle\nabla\ell_{N,j}(\Sigma_j)-\nabla\ell_{N,j}(\Sigma_j^\star),\Sigma_j-\tilde{\Sigma}_j^\star\rangle$$
$$\geq\frac{1}{J}\sum_{j=1}^{J}\left\{\|\Sigma_j-\tilde{\Sigma}_j^\star\|_F^2+\|\nabla\ell_{N,j}(\Sigma_j)-\nabla\ell_{N,j}(\Sigma_j^\star)\|_F^2\right\},$$

where we applied Lemma 28 with $m=L=1$. For $A12$, we have

$$\mathrm{A12}\geq-\frac{2}{J}\left|\sum_{j=1}^{J}\langle\nabla\ell_{N,j}(\Sigma_j^\star),[\Sigma_j-\tilde{\Sigma}_j^\star]_{\mathcal{S}_U,\mathcal{S}_U}\rangle\right|.$$

Since $\{[\Sigma_j-\tilde{\Sigma}_j^\star]_{\mathcal{S}_U,\mathcal{S}_U}\}_{j\in[J]}\in\Upsilon(2K,2s+s^\star,2\gamma,\delta_A)$, we have

$$A12\geq-\frac{2}{J}\varepsilon_{stat}\left(\sum_{j=1}^{J}\|[\Sigma_j-\tilde{\Sigma}_j^\star]_{\mathcal{S}_U,\mathcal{S}_U}\|_F^2\right)^{1/2}\geq-\frac{2}{J}\left(\frac{\varepsilon_{stat}^2}{2e_1}+\frac{e_1}{2}\sum_{j=1}^{J}\|\Sigma_j-\tilde{\Sigma}_j^\star\|_F^2\right),$$

where the last inequality follows by Young's inequality $ab \leq a^2/(2\varepsilon) + (\varepsilon b^2)/2$ for every $\varepsilon > 0$. We will use this bound with $e_1 = 1/4$. For $A13$, we have

$$A13 \geq -\frac{4}{J}\left|\sum_{j=1}^{J}\langle\nabla\ell_{N,j}(\Sigma_j^\star), [\Delta V \Delta a_j V^T]_{\mathcal{S}_U,\mathcal{S}_U}\rangle\right| \\ - \frac{4}{J}\sum_{j=1}^{J}\left|\langle\nabla\ell_{N,j}(\Sigma_j) - \nabla\ell_{N,j}(\Sigma_j^\star), [\Delta V \Delta a_j V^T]_{\mathcal{S}_U,\mathcal{S}_U}\rangle\right|. \quad (32)$$

We first bound the second term on the right-hand side of (32). Applying the Cauchy-Schwarz inequality and using $\|V\|_2 = 1$, we have

$$-\frac{4}{J}\sum_{j=1}^{J}|\langle\nabla\ell_{N,j}(\Sigma_j) - \nabla\ell_{N,j}(\Sigma_j^\star), [\Delta V \Delta a_j V^T]_{\mathcal{S}_U,\mathcal{S}_U}\rangle| \\ \geq -\frac{4}{J}\sum_{j=1}^{J}\|\nabla\ell_{N,j}(\Sigma_j) - \nabla\ell_{N,j}(\Sigma_j^\star)\|_F\|\Delta a_j\|_F\|\Delta V\|_F.$$

Using the fact that $\|\Delta V\|_F\|\Delta a_j\|_F \leq \frac{1}{2}d^2(Z_j, \tilde{Z}_j^\star)$, the above display can be further lower bounded as

$$\geq -\frac{2}{J}\sum_{j=1}^{J}\|\nabla\ell_{N,j}(\Sigma_j) - \nabla\ell_{N,j}(\Sigma_j^\star)\|_F \mathrm{d}^2(Z_j, \tilde{Z}_j^\star). \quad (33)$$

Since $\{[\Delta V \Delta a_j V^T]_{\mathcal{S}_U,\mathcal{S}_U}\}_{j\in[J]} \in \Upsilon(2K, 2s+s^\star, 2\gamma, \delta_A)$, we can bound the first term of (32) as

$$-\frac{4}{J}\left|\sum_{j=1}^{J}\langle\nabla\ell_{N,j}(\Sigma_j^\star), [\Delta V \Delta a_j V^T]_{\mathcal{S}_U,\mathcal{S}_U}\rangle\right| \geq -\frac{4}{J}\varepsilon_{stat}\left(\sum_{j=1}^{J}\|V\|_2^2\|\Delta a_j\|_F^2\|\Delta V\|_F^2\right)^{1/2} \\ = -\frac{4}{J}\varepsilon_{stat}\left(\sum_{j=1}^{J}\|\Delta a_j\|_F^2\|\Delta V\|_F^2\right)^{1/2} \\ \geq -\frac{4}{J}\varepsilon_{stat}\left(\frac{1}{4}\sum_{j=1}^{J}d^4(Z_j, \tilde{Z}_j^\star)\right)^{1/2}, \quad (34)$$

where the last inequality uses that $\|\Delta V\|_F\|\Delta a_j\|_F \leq \frac{1}{2}d^2(Z_j, \tilde{Z}_j^\star)$. Combining (33) and (34), we have

$$A13 \geq -\frac{2}{J}\sum_{j=1}^{J}\|\nabla\ell_{N,j}(\Sigma_j) - \nabla\ell_{N,j}(\Sigma_j^\star)\|_F \cdot \mathrm{d}^2(Z_j, \tilde{Z}_j^\star) - \frac{4}{J}\varepsilon_{stat}\left(\frac{1}{4}\sum_{j=1}^{J}d^4(Z_j, \tilde{Z}_j^\star)\right)^{1/2}.$$

Applying Young's inequality with $e_2 = 4$, the above display can be bounded as

$$
\begin{aligned}
A13 \geq & -\frac{1}{J}\sum_{j=1}^{J}\left\{\frac{1}{e_2}\|\nabla\ell_{N,j}(\Sigma_j) - \nabla\ell_{N,j}(\Sigma_j^\star)\|_F^2 + e_2 \mathrm{d}^4(Z_j, \tilde{Z}_j^\star)\right\} \\
& - \frac{1}{J}\left\{\frac{1}{e_2}\varepsilon_{stat}^2 + e_2\sum_{j=1}^{J} d^4(Z_j, \tilde{Z}_j^\star)\right\} \\
\geq & -\frac{1}{e_2 J}\left\{\varepsilon_{stat}^2 + \sum_{j=1}^{J}\|\nabla\ell_{N,j}(\Sigma_j) - \nabla\ell_{N,j}(\Sigma_j^\star)\|_F^2\right\} - \frac{2e_2}{J}\sum_{j=1}^{J} I_0^2 \mathrm{d}^2(Z_j, \tilde{Z}_j^\star),
\end{aligned}
$$

where the last inequality follows by $d^2(Z_j, \tilde{Z}_j^\star) \leq d^2(Z_j, Z_j^\star) \leq I_0^2$.

A lower bound for $A14$ can be obtained in a similar way to the one for $A13$. We have

$$
\begin{aligned}
A14 \geq & -\frac{2}{J}\sum_{j=1}^{J}\|\nabla\ell_{N,j}(\Sigma_j) - \nabla\ell_{N,j}(\Sigma_j^\star)\|_F\|\Delta V\|_F^2\|\mathrm{diag}(\tilde{a}_j^\star)\|_F \\
& - \frac{2}{J}\varepsilon_{stat}\left(\sum_{j=1}^{J}\|\Delta V\|_F^4\|\mathrm{diag}(\tilde{a}_j^\star)\|_2^2\right)^{1/2}.
\end{aligned}
$$

Applying Young's inequality with $e_3 = 4 > 0$, the above display can be bounded as

$$
\begin{aligned}
A14 \geq & -\frac{1}{e_3 J}\left\{\varepsilon_{stat}^2 + \sum_{j=1}^{J}\|\nabla\ell_{N,j}(\Sigma_j) - \nabla\ell_{N,j}(\Sigma_j^\star)\|_F^2\right\} - \frac{2e_3}{J}\sum_{j=1}^{J}\|\Delta V\|_F^4\|\mathrm{diag}(\tilde{a}_j^\star)\|_2^2 \\
\geq & -\frac{1}{e_3 J}\left\{\varepsilon_{stat}^2 + \sum_{j=1}^{J}\|\nabla\ell_{N,j}(\Sigma_j) - \nabla\ell_{N,j}(\Sigma_j^\star)\|_F^2\right\} - \frac{2e_3}{J^2}\sum_{j=1}^{J}\|A^\star\|_\infty^2 I_0^2 \mathrm{d}^2(Z_j, \tilde{Z}_j^\star),
\end{aligned}
$$

since $\|\mathrm{diag}(a_j^\star)\|_2 \leq \|A^\star\|_\infty$ and $\|\Delta V\|_F^4 \leq J^{-1} I_0^2 \mathrm{d}^2(Z_j, \tilde{Z}_j^\star)$.

Putting everything together, we have

$$
\begin{aligned}
A \geq \frac{1}{J}\Bigg\{ & \frac{3}{4}\sum_{j=1}^{J}\|\Sigma_j - \tilde{\Sigma}_j^\star\|_F^2 - \frac{9}{2}\varepsilon_{stat}^2 + \frac{1}{2}\sum_{j=1}^{J}\|\nabla\ell_{N,j}(\Sigma_j) - \nabla\ell_{N,j}(\Sigma_j^\star)\|_F^2 \\
& - 8\left(1 + \frac{\|A^\star\|_\infty^2}{J}\right) I_0^2 \mathrm{dist}^2(Z, \tilde{Z}^\star)\Bigg\},
\end{aligned}
$$

which completes the proof. ■

Proof of Lemma 19: We separately bound $B1$ and $B2$. Since $\Sigma_j = V\mathrm{diag}(a_j)V^T$, recalling (19), we have

$$
\begin{aligned}
B1 &= \frac{1}{J}\left\|\frac{2}{J}\sum_{j=1}^{J}[\nabla\ell_{N,j}(\Sigma_j)V\mathrm{diag}(a_j)]_{\mathcal{S}_U}\right\|_F^2 \\
&= \frac{4}{J^3}\left\|\sum_{j=1}^{J}[\nabla\ell_{N,j}(\Sigma_j^\star)V\mathrm{diag}(a_j)]_{\mathcal{S}_U} + \left[\left\{\nabla\ell_{N,j}(\Sigma_j) - \nabla\ell_{N,j}(\Sigma_j^\star)\right\}V\mathrm{diag}(a_j)\right]_{\mathcal{S}_U}\right\|_F^2.
\end{aligned}
$$

Using the Cauchy–Schwarz inequality and along with the fact that $(a+b)^2 \le 2(a^2+b^2)$, the above display can be bounded as

$$
\begin{aligned}
&\le \frac{8}{J^2}\sum_{j=1}^{J}\|[\{\nabla\ell_{N,j}(\Sigma_j) - \nabla\ell_{N,j}(\Sigma_j^\star)\}V\mathrm{diag}(a_j)]_{\mathcal{S}_U}\|_F^2 \\
&\quad + \frac{8}{J^3}\left\|\sum_{j=1}^{J}[\nabla\ell_{N,j}(\Sigma_j^\star)V\mathrm{diag}(a_j)]_{\mathcal{S}_U}\right\|_F^2.
\end{aligned}
\tag{35}
$$

For any $X_{\mathcal{S}_U}$ with $\|X_{\mathcal{S}_U}\|_F = 1$, we have $\{[V\mathrm{diag}(a_j)X^T]_{\mathcal{S}_U,\mathcal{S}_U}\}_{j\in[J]} \in \Upsilon(2K, 2s{+}s^\star, 2\gamma, \delta_A)$. We can bound the second term in (35) as

$$
\begin{aligned}
\left\|\sum_{j=1}^{J}[\nabla\ell_{N,j}(\Sigma_j^\star)V\mathrm{diag}(a_j)]_{\mathcal{S}_U}\right\|_F &= \sup_{\|X_{\mathcal{S}_U}\|_F=1}\sum_{j=1}^{J}\mathrm{tr}([\nabla\ell_{N,j}(\Sigma_j^\star)V\mathrm{diag}(a_j)]_{\mathcal{S}_U}X_{\mathcal{S}_U}^T) \\
&= \sup_{\|X_{\mathcal{S}_U}\|_F=1}\sum_{j=1}^{J}\langle\nabla\ell_{N,j}(\Sigma_j^\star), [V\mathrm{diag}(a_j)]_{\mathcal{S}_U}X_{\mathcal{S}_U}^T\rangle \\
&\le \varepsilon_{stat}\cdot\sup_{\|X_{\mathcal{S}_U}\|_F=1}\left(\sum_{j=1}^{J}\|[V\mathrm{diag}(a_j)X^T]_{\mathcal{S}_U,\mathcal{S}_U}\|_F^2\right)^{1/2} \\
&\le \varepsilon_{stat}J^{1/2}\|V\|_2^2\|A\|_\infty.
\end{aligned}
\tag{36}
$$

Plugging (36) back into (35), we arrive at

$$
\begin{aligned}
B1 &\le \frac{8}{J^2}\sum_{j=1}^{J}\|\nabla\ell_{N,j}(\Sigma_j) - \nabla\ell_{N,j}(\Sigma_j^\star)\|_F^2\|V\|_2^2\|\mathrm{diag}(a_j)\|_2^2 + \frac{8}{J^2}\varepsilon_{stat}^2\|V\|_2^4\|A\|_\infty^2 \\
&\le \frac{8}{J^2}\sum_{j=1}^{J}\|\nabla\ell_{N,j}(\Sigma_j) - \nabla\ell_{N,j}(\Sigma_j^\star)\|_F^2\|\mathrm{diag}(a_j)\|_2^2 + \frac{8}{J^2}\varepsilon_{stat}^2\|A\|_\infty^2,
\end{aligned}
$$

where the last inequality follows since $\|V\|_2 = 1$.

For $B2$, we have

$$
\begin{aligned}
B2 &= \frac{1}{J^2}\|W(V)\tilde{Q}\tilde{Q}^T\|_F^2 = \frac{1}{J^2}\|\{W(V) - W^\star(V) + W^\star(V)\}\tilde{Q}\tilde{Q}^T\|_F^2 \\
&\le \frac{2}{J^2}\|\{W(V) - W^\star(V)\}\tilde{Q}\tilde{Q}^T\|_F^2 + \frac{2}{J^2}\|W^\star(V)\tilde{Q}\tilde{Q}^T\|_F^2.
\end{aligned}
$$

Since $\tilde{Q}\tilde{Q}^T$ is an orthogonal projection operator, for a matrix X, we have that $\|X\tilde{Q}\tilde{Q}^T\|_F \le \|X\|_F$. Then

$$
\begin{aligned}
&\|\{W(V) - W^\star(V)\}\tilde{Q}\tilde{Q}^T\|_F^2 \le \|W(V) - W^\star(V)\|_F^2 \\
&\le \sum_{j=1}^{J} \|V^T\left\{\nabla \ell_{N,j}(\Sigma_j) - \nabla \ell_{N,j}(\Sigma_j^\star)\right\}V\|_F^2 \le \left\{\sum_{j=1}^{J} \|\nabla \ell_{N,j}(\Sigma_j) - \nabla \ell_{N,j}(\Sigma_j^\star)\|_F^2\right\}, \quad (37)
\end{aligned}
$$

since $\|V\|_2 = 1$. Furthermore, we have

$$
\begin{aligned}
\|W^\star(V)\tilde{Q}\tilde{Q}^T\|_F^2 &= \sum_{k=1}^{K} \|\tilde{Q}\tilde{Q}^T W_{k\cdot}^\star(v_k)\|_2^2 \\
&\le \left\{\sum_{k=1}^{K} \|\tilde{Q}\tilde{Q}^T W_{k\cdot}^\star(v_k)\|_2\right\}^2 \\
&= \left\{\sum_{k=1}^{K} \sup_{\|x_k\|_2\le 1} \left\langle x_k, \tilde{Q}\tilde{Q}^T W_{k\cdot}^\star(v_k)\right\rangle\right\}^2 \\
&\le \left\{\sum_{k=1}^{K} \sup_{\|\tilde{Q}\tilde{Q}^T x_k\|_2\le 1} \left\langle \tilde{Q}\tilde{Q}^T x_k, W_{k\cdot}^\star(v_k)\right\rangle\right\}^2 \\
&\le \left\{\sum_{k=1}^{K} \left(\frac{1}{2\delta_A\gamma}\right)^{1/2} \sup_{\|\tilde{Q}\tilde{Q}^T x_k'\|_2^2\le 2\delta_A\gamma} \left\langle \tilde{Q}\tilde{Q}^T x_k', W_{k\cdot}^\star(v_k)\right\rangle\right\}^2.
\end{aligned}
$$

Let $y_k = \tilde{Q}\tilde{Q}^T x_k'$ for $k \in [K]$. Since a ball of radius $(2\delta_A\gamma)^{1/2}$ is contained in the truncated ellipsoid $\{\alpha = \tilde{Q}u : u^T\tilde{\Lambda}u \le 2\gamma\}$, y_k, for $k \in [K]$, lies in the ellipsoid. Therefore, $\{\sum_{k=1}^K y_{kj} v_k v_k^T\}_{j\in[J]}$ is in $\Upsilon(2K, 2s + s^\star, 2\gamma, \delta_A)$ and we can bound the above display as

$$
\begin{aligned}
&\le \varepsilon_{stat}^2 \cdot \frac{1}{2\delta_A\gamma}\left(\sum_{k=1}^{K} \sup_{\|\tilde{Q}\tilde{Q}^T x_k'\|_2^2\le 2\delta_A\gamma} \|\tilde{Q}\tilde{Q}^T x_k'\|_2^2\right) \\
&= \varepsilon_{stat}^2 K. \quad (38)
\end{aligned}
$$

Combining (37)—(38) and noting that $\|V\|_2 = 1$, we have

$$B2 \leq \frac{2}{J^2}\left\{\sum_{j=1}^{J}\|\nabla\ell_{N,j}(\Sigma_j) - \nabla\ell_{N,j}(\Sigma_j^\star)\|_F^2\right\}\|V\|_2^2 + \frac{2}{J^2}K\varepsilon_{stat}^2\|V\|_2^2.$$

Finally, combining $B1$ and $B2$ we arrive at the following

$$\begin{aligned}
B &\leq \frac{8}{J^2}\sum_{j=1}^{J}\{\|\nabla\ell_{N,j}(\Sigma_j) - \nabla\ell_{N,j}(\Sigma_j^\star)\|_F^2\left(\|\mathrm{diag}(a_j)\|_2^2 + \|V\|_2^2\right)\} \\
&\quad + \frac{2(4\vee K)}{J^2}\varepsilon_{stat}^2(\|V\|_2^2 + \|A\|_\infty^2) \\
&\leq \frac{16}{J^2}\sum_{j=1}^{J}\left\{\|\nabla\ell_{N,j}(\Sigma_j) - \nabla\ell_{N,j}(\Sigma_j^\star)\|_F^2\right\}\|Z_j\|_2^2 + \frac{4(4\vee K)}{J^2}\varepsilon_{stat}^2\max_{j\in[J]}\|Z_j\|_2^2,
\end{aligned}$$

where the second inequality comes from $(\|\mathrm{diag}(a_j)\|_2^2 \vee \|V\|_2^2) \leq \|Z_j\|_2^2$. ■

Proof of Lemma 20: We first upper bound $\|Z_j\|_2$ with $\|Z_j^\star\|_2$ for every $j \in [J]$. We have

$$\begin{aligned}
\|Z_j\|_2 &= \left\|\begin{pmatrix} V \\ \mathrm{diag}(a_j)\end{pmatrix} - \begin{pmatrix} V^\star R \\ R^T\mathrm{diag}(a_j^\star)R\end{pmatrix} + \begin{pmatrix} V^\star R \\ R^T\mathrm{diag}(a_j^\star)R\end{pmatrix}\right\|_2 \\
&\leq \left\|\begin{pmatrix} V \\ \mathrm{diag}(a_j)\end{pmatrix} - \begin{pmatrix} V^\star R \\ R^T\mathrm{diag}(a_j^\star)R\end{pmatrix}\right\|_2 + \|Z_j^\star\|_2 \\
&\leq I_0 + \|Z_j^\star\|_2 \\
&\leq \frac{\sigma_K(\Sigma_j^\star)}{16} + \|Z_j^\star\|_2 \\
&\leq \frac{17}{16}\|Z_j^\star\|_2,
\end{aligned} \tag{39}$$

where the third inequality follows from (3), in particular that $I_0 \leq \sigma_K(\Sigma_j^\star)/16$, and the last inequality follows because $\|Z_j^\star\|_2 \geq \sigma_K(\Sigma_j^\star)$.

Next, we lower bound $\|Z_j^0\|_2$ in terms of $\|Z_j^\star\|_2$ for every $j \in [J]$. Similar to (39), we have

$$
\begin{aligned}
\|Z_j^0\|_2 &= \left\| \begin{pmatrix} V^0 \\ \mathrm{diag}(a_j^0) \end{pmatrix} - \begin{pmatrix} V^\star R \\ R^T \mathrm{diag}(a_j^\star) R \end{pmatrix} + \begin{pmatrix} V^\star R \\ R^T \mathrm{diag}(a_j^\star) R \end{pmatrix} \right\|_2 \\
&\geq - \left\| \begin{pmatrix} V^0 \\ \mathrm{diag}(a_j^0) \end{pmatrix} - \begin{pmatrix} V^\star R \\ R^T \mathrm{diag}(a_j^\star) R \end{pmatrix} \right\|_2 + \|Z_j^\star\|_2 \\
&\geq -I_0 + \|Z_j^\star\|_2 \\
&\geq -\frac{\sigma_K(\Sigma_j^\star)}{16} + \|Z_j^\star\|_2 \\
&\geq \frac{15}{16}\|Z_j^\star\|_2. \qquad (40)
\end{aligned}
$$

Combining (39) and (40), we obtain $\|Z_j\|_2^2 \leq 2\|Z_j^0\|_2^2$. ■

B.5 Proofs of Auxiliary Lemmas

Lemma 21 *Suppose $V^\star$ has orthonormal columns. Let*

$$
R = \operatorname*{argmin}_{Y \in \mathcal{O}(K)} \|V - V^\star Y\|_F^2, \quad R^+ = \operatorname*{argmin}_{Y \in \mathcal{O}(K)} \|V^+ - V^\star Y\|_F^2.
$$

Then

$$
\|R^+ - R\|_F \leq 2\|V^+ - V^\star R\|_F
$$

and

$$
\|R^T \mathrm{diag}(a^\star) R - R^{+T} \mathrm{diag}(a^\star) R^+\|_F \leq 4\|\mathrm{diag}(a^\star)\|_2 \|V^+ - V^\star R\|_F.
$$

Proof of Lemma 21: Recall that $V^\star$ has orthonormal columns. This implies that for a matrix X, we have

$$
\|V^\star X\|_F^2 = \mathrm{tr}(X^T V^{\star T} V^\star X) = \mathrm{tr}(X^T X) = \|X\|_F^2.
$$

Using the above property, we have

$$
\begin{aligned}
\|R^+ - R\|_F = \|V^\star R^+ - V^\star R\|_F &= \|(V^+ - V^\star R) + (V^\star R^+ - V^+)\|_F \\
&\leq \|V^+ - V^\star R\|_F + \|V^+ - V^\star R^+\|_F \qquad (41) \\
&\leq 2\|V^+ - V^\star R\|_F,
\end{aligned}
$$

where the last inequality follows as R^+ minimizes the distance $\|V^+ - V^\star R\|_F$. This completes the proof for the first statement.

For the second statement, we have

$$\begin{aligned}\|R^T\mathrm{diag}(a^\star)R - R^{+T}\mathrm{diag}(a^\star)R^+\|_F &= \|(R-R^+)^T\mathrm{diag}(a^\star)R + R^{+T}\mathrm{diag}(a^\star)(R-R^+)\|_F\\ &\leq \|\mathrm{diag}(a^\star)\|_2(\|R\|_2+\|R^+\|_2)\|R^+-R\|_F\\ &\leq 4\|\mathrm{diag}(a^\star)\|_2\|V^+-V^\star R\|_F,\end{aligned}$$

where the last inequality follows from (41) and $\|R\|_2 = \|R^+\|_2 = 1$. ■

Lemma 22 *Assume that V has orthonormal columns and*

$$\|V - V^\star R\|_F \leq I_0^2/J, \quad \|\mathrm{diag}(a_j) - R^T\mathrm{diag}(a_j^\star)R\|_F^2 \leq (J-1)I_0^2/J,$$

and $Z = (V^T, A)^T$. Under Assumptions 1—3, we have

$$\frac{\eta}{J}\|[\nabla_V f_N(Z)]_{\mathcal{S}_U}\|_F \leq \frac{I_0}{6\sqrt{J}},$$

where I_0 is defined in (3).

Proof of Lemma 22: By (19), we have

$$\begin{aligned}\frac{\eta}{J}\|[\nabla_V f_N]_{\mathcal{S}_U}\|_F &= \frac{2\eta}{J^2}\left\|\sum_{j=1}^J[\nabla\ell_{N,j}(\Sigma_j)V\mathrm{diag}(a_j)]_{\mathcal{S}_U}\right\|_F\\ &\leq \frac{2\eta}{J^2}\Bigg\{\sum_{j=1}^J\left\|\left[\left\{\nabla\ell_{N,j}(\Sigma_j)-\nabla\ell_{N,j}(\Sigma_j^\star)\right\}V\mathrm{diag}(a_j)\right]_{\mathcal{S}_U}\right\|_F\\ &\quad + \left\|\sum_{j=1}^J[\nabla\ell_{N,j}(\Sigma_j^\star)V\mathrm{diag}(a_j)]_{\mathcal{S}_U}\right\|_F\Bigg\}.\end{aligned}$$

We can bound the second term of the above display using (36) and obtain

$$\begin{aligned}&\leq \frac{2\eta}{J^2}\left\{\|A\|_\infty\sum_{j=1}^J\|\nabla\ell_{N,j}(\Sigma_j)-\nabla\ell_{N,j}(\Sigma_j^\star)\|_F + \varepsilon_{stat}J^{1/2}\|A\|_\infty\right\}\\ &= \frac{2\eta}{J^2}\left\{\|A\|_\infty\sum_{j=1}^J\|\Sigma_j-\Sigma_j^\star\|_F + \varepsilon_{stat}J^{1/2}\|A\|_\infty\right\}.\end{aligned}$$

From (5), we have

$$\begin{aligned}\|\Sigma_j-\Sigma_j^\star\|_F^2 &\leq 3\left\{(\|A\|_\infty^2+\|A^\star\|_\infty^2)\|V-V^\star R\|_F^2 + \|\mathrm{diag}(a_j)-R^T\mathrm{diag}(a_j^\star)R\|_F^2\right\}\\ &\leq 3\left(\max_{j\in[J]}\|Z_j\|_\infty^2 + \max_{j'\in[J]}\|Z_{j'}^\star\|_\infty^2\right)I_0^2,\end{aligned}$$

where the last inequality uses that $\|A\|_\infty \le \max_{j\in[J]}\|Z_j\|_2$, $\|A^\star\|_\infty \le \max_{j\in[J]}\|Z_j^\star\|_2$, and $1 = \|V\|_2 \le \|Z_j\|_2$ for $j \in [J]$. Then

$$\frac{\eta}{J}\|[\nabla_V f_N]_{\mathcal{S}_U}\|_2 \le \frac{2\eta}{J^2}\max_{j\in[J]}\left\{2JI_0\|Z_j\|_2(\|Z_j\|_2 + \max_{j'\in[J]}\|Z_{j'}^\star\|_2) + \varepsilon_{stat}J^{1/2}\|Z_j\|_2\right\}.$$

Applying (39), the above display can be bounded by

$$\begin{aligned}
&\le \frac{2\eta}{J^2}\max_{j\in[J]}\left\{\frac{11}{5}JI_0\|Z_j^\star\|_2^2 + \frac{17}{16}\varepsilon_{stat}J^{1/2}\|Z_j^\star\|_2\right\}\\
&\le \frac{9I_0\eta}{J}\max_{j\in[J]}\|Z_j^\star\|_2^2\\
&\le \frac{32I_0\eta}{3J}\max_{j\in[J]}\|Z_j^0\|_2^2,
\end{aligned}$$

where the second to last inequality follows by Assumption 3 and the last inequality follows by (40). Then, by Assumption 1, we have

$$\frac{\eta}{J}\|[\nabla_V f_N]_{\mathcal{S}_U}\|_F \le \frac{I_0}{6\sqrt{J}},$$

which completes the proof. ∎

Lemma 23 *Assume that V has orthonormal columns and let*

$$\bar{V}^+ = \Pi_{\mathcal{C}_V}\{V - \eta/J[\nabla_V f_N(Z)]_{\mathcal{S}_U}\}, \quad Z = (V^T, A)^T, \quad R = \operatorname*{argmin}_{Y\in\mathcal{O}(K)}\|V - V^\star Y\|_F^2.$$

Assume that

$$\|V - V^\star R\|_F \le I_0^2/J, \quad \|\mathrm{diag}(a_j) - R^T\mathrm{diag}(a_j^\star)R\|_F^2 \le (J-1)I_0^2/J,$$

then under Assumptions 1—3, we have

$$\|\bar{V}^+ - V^\star R\|_2 \le \frac{2I_0}{J^{1/2}} < 1.$$

Proof of Lemma 23: By definition, we have

$$\|\bar{V}^+ - V^\star R\|_2 \le \|\bar{V}^+ - V\|_2 + \|V - V^\star R\|_2.$$

By Assumption 6, we have $\|V - V^\star R\|_2 \le I_0/J^{1/2}$. Then, it remains to show that

$$\|\bar{V}^+ - V\|_2 \le \|\bar{V}^+ - V\|_F \le I_0/J^{1/2}.$$

The first step is to apply Lemma 13 on $\|\bar{V}^+ - V\|_F$. We first verify that the inner product of the kth column of $V - (\eta/J)[\nabla_V f_N]_{\mathcal{S}_U}$ and v_k are nonnegative for $k \in [K]$. By Lemma 22, we have

$$\left\langle v_k - \left(\frac{\eta}{J}[\nabla_V f_N]_{\mathcal{S}_U}\right)_k, v_k \right\rangle \geq 1 - \left\| \left(\frac{\eta}{J}[\nabla_V f_N]_{\mathcal{S}_U}\right)_k \right\|_2 \geq 1 - \frac{I_0}{6\sqrt{J}} > 0,$$

for every $k \in [K]$, which allows us to apply Lemma 13. The term $1 + 2\sqrt{s^\star}/\sqrt{(s - s^\star)}$ decreases as s increases. Under Assumption 2, $s \geq 2s^\star$ and we have $1+2\sqrt{s^\star}/\sqrt{(s - s^\star)} \leq 3$. Therefore, applying Lemma 13 with $s = 2s^\star$:

$$\|\bar{V}^+ - V\|_2 \leq \|\bar{V}^+ - V\|_F \leq 6 \left\| V - \frac{\eta}{J}[\nabla_V f_N]_{\mathcal{S}_U} - V \right\|_F \leq \frac{I_0}{\sqrt{J}},$$

where the last inequality follows from Lemma 22. ■

Appendix C. Quantification of statistical error

In this section, we present the proof of Proposition 8. We begin with stating the main result in Section C.1, followed by the details of construction of structured ϵ-net and its property in Section C.2 and Section C.3, respectively.

C.1 Proof of Proposition 8

Let $\Omega(s, P)$ be a collection of subsets of $[P]$, each with cardinality s. Let

$$\mathcal{S}_V = \mathcal{S}_V(K, s) = \{\mathcal{S}_{V_k} \in \Omega(s, P)\}_{k\in[K]},$$

be the set of supports, where $\mathcal{S}_{V_k}$ denotes the support of the component k for $k \in [K]$. We first establish a bound on the statistical error for a fixed support $\mathcal{S}_V$ and then take the union bound to establish a bound on the statistical error on the set $\Upsilon(2K, ms^\star, 2m'\gamma^\star, \delta_A)$, for some constant $m, m' > 0$. For some positive semidefinite matrix G, we define the sets

$$\mathcal{V}(\mathcal{S}_V, K) = \{V \in \mathbb{R}^{P\times K} : \|v_k\|_2 = 1, \|V_{\mathcal{S}_V^c}\| = 0, k \in [K]\};$$
$$\mathcal{T}(G, \gamma) = \{\alpha = Gu : u^T \Lambda u \le \gamma\},$$

where $G^\dagger = Q\Lambda Q^T$ is the eigendecomposition. For a positive semidefinite kernel matrix G and a positive scalar γ, we define the seminorm $\|\cdot\|_{G,\gamma}$ as

$$\|x\|^2_{G,\gamma} = \frac{1}{\gamma}(x^T G^\dagger x).$$

Therefore, the set $\mathcal{T}(G, \gamma)$ is a unit ball in $\|\cdot\|_{G,\gamma}$. We use $\mathcal{N}_\mathcal{V}(\epsilon_v)$ to denote the ϵ-net for $\mathcal{N}(\mathcal{V}(\mathcal{S}_V, 2K), \epsilon_v, \|\cdot\|_F)$ and $\mathcal{N}_\mathcal{T}(\epsilon_a)$ to denote the ϵ-net for $\mathcal{N}(\mathcal{T}(\tilde{G}, 2m'\gamma^\star), \epsilon_a, \|\cdot\|_{\tilde{G},2m'\gamma^\star})$. For a matrix $A \in \mathbb{R}^{K\times J}$, we use $A_{k\cdot}$ to denote kth row of A and a_j to denote the jth column of A. We define the following set

$$\mathcal{U}(\mathcal{S}_V, 2m'\gamma^\star) = \{\{U\mathrm{diag}(a_j)V^T\}_{j\in J} : U, V \in \mathcal{V}(\mathcal{S}_V, 2K), A_{k\cdot} \in \mathcal{T}(\tilde{G}, 2m'\gamma^\star), k \in [2K]\},$$

and let $\{\Delta_j = U\mathrm{diag}(a_j)V^T\}_{j\in[J]} \in \mathcal{U}(\mathcal{S}_V, 2m'\gamma^\star)$. Recall that $\nabla\ell_{N,j}(\Sigma^\star_j) = \Sigma^\star_j - S_{N,j}$ for $j \in [J]$. We have

$$\sup_{\{\Delta_j\}\in\mathcal{U}(\mathcal{S}_V,2m'\gamma^\star)} \sum_{j=1}^J \langle\Sigma^\star_j - S_{N,j}, \Delta_j\rangle$$
$$= \sup_{\{\Delta_j\}\in\mathcal{U}(\mathcal{S}_V,2m'\gamma^\star)} \sum_{j=1}^J \langle S_{N,j} - \Sigma^\star_j - E_j, \Delta_j\rangle + \sum_{j=1}^J \langle E_j, \Delta_j\rangle. \quad (42)$$

For the second term in the above display, we have

$$\sum_{j=1}^{J}\langle E_j,\Delta_j\rangle = \frac{1}{2}\sum_{j=1}^{J}\langle E_j,\Delta_j+\Delta_j^T\rangle \le \frac{1}{2}\sum_j \|E_j\|_2\|\Delta_j+\Delta_j^T\|_F \le \left(\max_j \|E_j\|_2\right)\cdot\sum_j \|\Delta_j\|_F, \quad (43)$$

where the first equality follows by the fact that E_j is symmetric.

Using Lemma 25, we have

$$\sup_{\{\Delta_j\}\in\mathcal{U}(\mathcal{S}_V,2m'\gamma^\star)}\left|\sum_{j=1}^{J}\langle S_{N,j}-\Sigma_j^\star-E_j,\Delta_j\rangle\right| \le (1-2\epsilon_v-\epsilon_a)^{-1}\max_{\substack{U,V\in\mathcal{N}_{\mathcal{V}}(\epsilon_v)\\ A_{k\cdot}\in\mathcal{N}_{\mathcal{T}}(\epsilon_a),\,k\in[2K]}}\left|\frac{1}{2}\sum_{j=1}^{J}\langle S_{N,j}-\Sigma_j^\star-E_j,\Delta_j+\Delta_j^T\rangle\right|.$$

For a fixed set of $\{\Delta_j\}_{j=1}^J$ for $j\in[J]$, we let $Y_j^{(n)}=(1/2)\operatorname{tr}\{(\Delta_j+\Delta_j^T)x_j^{(n)}x_j^{(n)T}\}$. Note that $\mathbb{E}\{Y_j^{(n)}\}=(1/2)\operatorname{tr}\left\{(\Delta_j+\Delta_j^T)\left(\Sigma_j^\star+E_j\right)\right\}$. Consequently, we have

$$\begin{aligned}\frac{1}{4}\sum_{j=1}^{J}\operatorname{tr}\left\{\left(\Sigma_j^\star+E_j\right)\left(\Delta_j+\Delta_j^T\right)\left(\Sigma_j^\star+E_j\right)\left(\Delta_j+\Delta_j^T\right)\right\} &\le \frac{1}{4}\sum_{j=1}^{J}\|\Sigma_j^\star+E_j\|_2^2\operatorname{tr}\left\{(\Delta_j+\Delta_j^T)^2\right\}\\ &= \frac{1}{4}\sum_{j=1}^{J}\|\Sigma_j^\star+E_j\|_2^2\|\Delta_j+\Delta_j^T\|_F^2\\ &\le \max_{j\in[J]}\|\Sigma_j^\star+E_j\|_2^2\sum_{j=1}^{J}\|\Delta_j\|_F^2\\ &\le 4\|A^\star\|_\infty^2\sum_{j=1}^{J}\|\Delta_j\|_F^2,\end{aligned}$$

where the last step follows from $\|\Sigma_j^\star+E_j\|_2\le 2\|\Sigma_j^\star\|_2\le\|A^\star\|_\infty$ for all $j\in[J]$.

Then, using Lemma 34,

$$pr\left[\frac{1}{N}\left|\sum_{n=1}^{N}\sum_{j=1}^{J}Y_j^{(n)}-E\left\{Y_j^{(n)}\right\}\right|\ge\frac{\varepsilon}{4}\right] \le 2\exp\left\{-Ne_0\left(\frac{\varepsilon^2}{16\|A^\star\|_\infty^2\sum_{j=1}^{J}\|\Delta_j\|_F^2}\wedge\frac{\varepsilon}{4\|A^\star\|_\infty\max_{j\in[J]}\|\Delta_j\|_2}\right)\right\},$$

where e_0 are some absolute constant.

We choose $\epsilon_v = \epsilon_a = 1/4$. Taking the union bound over $\mathcal{N}_{\mathcal{V}}(\epsilon_v)$, $\mathcal{N}_{\mathcal{T}}(\epsilon_a)$ and the choice of $\Omega(ms^\star, P)$, we have

$$pr\left[\max_{\mathcal{S}_V \in \{\Omega(ms^\star,P)\}^{2K}} \max_{\substack{U,V\in\mathcal{N}_{\mathcal{V}}(\epsilon_v)\\ A_{k\cdot}\in\mathcal{N}_{\mathcal{T}}(\epsilon_a),\, k\in[2K]}} \frac{1}{N}\sum_{n=1}^{N}\left|\sum_{j=1}^{J} Y_j^{(n)} - \mathbb{E}\left\{Y_j^{(n)}\right\}\right| \geq \frac{\varepsilon}{4}\right] \tag{44}$$
$$\leq 2\binom{P}{ms^\star}^{4K} 9^{4Kms^\star+2Kr(\tilde{G})}$$
$$\times \exp\left\{-Ne_0\left(\frac{\varepsilon^2}{16\|A^\star\|_\infty^2\sum_{j=1}^{J}\|\Delta_j\|_F^2} \wedge \frac{\varepsilon}{4\|A^\star\|_\infty \max_{j\in[J]}\|\Delta_j\|_2}\right)\right\},$$

where we applied the metric entropy in Lemma 24.

Given $0<\delta<1$, let

$$\nu \leq \frac{1}{e_0'}\left[\frac{1}{N}\left\{\log\frac{1}{\delta} + Kr(\tilde{G}) + Ks^\star + Ks^\star\log\frac{Pe}{s^\star}\right\}\right]^{1/2}$$

for some constant e_0'. Combining (43) and (44) with (42), we have

$$\varepsilon_{stat}\left(\sum_{j=1}^{J}\|\Delta_j\|_F^2\right)^{1/2} \leq \|A^\star\|_\infty\left(\sum_{j=1}^{J}\|\Delta_j\|_F^2\right)^{1/2}(\nu\vee\nu^2) + \left(\sum_{j=1}^{J}\|\Delta_j\|_F\right)\max_{j\in[J]}\|E_j\|_2,$$

with probability at least $1-\delta$. Using the Cauchy-Schwarz inequality,

$$\sum_{j=1}^{J}\|\Delta_j\|_F \leq J^{1/2}\left(\sum_{j=1}^{J}\|\Delta_j\|_F^2\right)^{1/2}$$

and, therefore,

$$\varepsilon_{stat} \leq \|A^\star\|_\infty(\nu\vee\nu^2) + J^{1/2}\max_{j\in[J]}\|E_j\|_2$$

with probability at least $1-\delta$.

C.2 Metric Entropy of the Structured Set

We find the metric entropy of $\mathcal{N}(\mathcal{V}(\mathcal{S}_V,K),\epsilon_v,\|\cdot\|_F)$ and $\mathcal{N}(\mathcal{T}(G,\gamma),\epsilon_a,\|\cdot\|_{G,\gamma})$.

Lemma 24 *Given a support set $\mathcal{S}_V(K,s) = \{\mathcal{S}_{V_k}\in\Omega(s,P)\}_{k\in[K]}$, let*

$$\mathcal{V}(\mathcal{S}_V,K) = \{V\in\mathbb{R}^{P\times K} : \|v_k\|_2 = 1, \|V_{\mathcal{S}_V^c}\| = 0, k\in[K]\}.$$

The metric entropy of $\mathcal{N}(\mathcal{V}(\mathcal{S}_V, K), \epsilon_v, \|\cdot\|_F)$ *is*

$$\log|\mathcal{N}(\mathcal{V}(\mathcal{S}_V, K), \epsilon_v, \|\cdot\|_F)| \leq Ks\log\left(1+\frac{2}{\epsilon_v}\right).$$

Given G *and* γ*, the metric entropy of* $\mathcal{N}(\mathcal{T}(G,\gamma), \epsilon_a, \|\cdot\|_{G,\gamma})$ *is*

$$\log|\mathcal{N}(\mathcal{T}(G,\gamma), \epsilon_a, \|\cdot\|_{G,\gamma})| \leq r(G)\log\left(1+\frac{2}{\epsilon_a}\right),$$

where $r(G)$ *is the rank of* G*.*

Proof of Lemma 24: The first result directly follows from Lemma 5.2 in Vershynin (2012). For the second reults, we note that the set $\mathcal{T}(G,\gamma)$ is a $r(G)$-dimensional unit ball in the semi-norm $\|\cdot\|_{G,\gamma}$. Therefore, we can again apply Lemma 5.2 in Vershynin (2012). ■

C.3 Inner Product on a ϵ-net

We define the following operator similar to the definition of $W^\star(V)$:

$$\hat{W}^\star(U,V) = [\hat{w}^\star_{kj}(u_k, v_k)] \in \mathbb{R}^{K\times J}, \quad \hat{w}^\star_{kj}(u_k, v_k) = u_k^T \nabla\ell_{N,j}(\Sigma_j^\star) v_k,$$

where $\nabla\ell_{N,j}(\Sigma_j^\star) = \Sigma_j^\star - S_{N,j}$. Recall that $A = [a_1, \ldots, a_j] \in \mathbb{R}^{K\times J}$. Then

$$\sum_{j=1}^{J}\langle\nabla\ell_{N,j}(\Sigma_j^\star), U\mathrm{diag}(a_j)V^T\rangle = \sum_{k=1}^{K} A_{k\cdot}^T \hat{W}^\star_{k\cdot}(u_k, v_k),$$

where $A_{k\cdot}, \hat{W}^\star_{k\cdot}(u_k, v_k) \in \mathbb{R}^J$ denote the kth row of A and $\hat{W}^\star(U,V)$, respectively. Consequently, if every row $A_{k\cdot}$ lies in $\mathcal{T}(G,\gamma)$, we have

$$\sum_{j=1}^{J}\langle\nabla\ell_{N,j}(\Sigma_j^\star), U\mathrm{diag}(a_j)V^T\rangle = \sum_{k=1}^{K} A_{k\cdot}^T QQ^T \hat{W}^\star_{k\cdot}(u_k, v_k),$$

where Q is the matrix whose columns are eigenvectors of G. This is another representation of the statistical error and will help us to simplify the proof steps of the following lemma.

Lemma 25 (Inner product on a net) *Given a support* $\mathcal{S}_V$*, a matrix* G*, and a positive scalar* γ*, we have*

$$\max_{\substack{U,V\in\mathcal{V}(\mathcal{S}_V,K)\\ A_{k\cdot}\in\mathcal{T}(G,\gamma),\ k\in[K]}} \sum_{k=1}^{K} A_{k\cdot}^T QQ^T \hat{W}^\star_{k\cdot}(u_k, v_k)$$

$$\leq (1-2\epsilon_v-\epsilon_a)^{-1} \max_{\substack{U,V\in\mathcal{N}_{\mathcal{V}}(\epsilon_v)\\ A_{k\cdot}\in\mathcal{N}_{\mathcal{T}}(\epsilon_a)\ k\in[K]}} \sum_{k=1}^{K} A_{k\cdot}^T QQ^T \hat{W}^\star_{k\cdot}(u_k, v_k),$$

where Q *denotes the matrix whose columns are eigenvectors of* G*.*

Proof of Lemma 25: Let $\hat{U}$, $\hat{V}$ and $\hat{A}$ be the quantities that maximize

$$\tilde{\varepsilon} = \max_{\substack{U,V \in \mathcal{V}(\mathcal{S}_V, K) \\ A_{k\cdot} \in \mathcal{T}(G,\gamma),\, k \in [K]}} \sum_{k=1}^{K} A_{k\cdot}^T Q Q^T \hat{W}_{k\cdot}^{\star}(u_k, v_k).$$

Then

$$\begin{aligned}
&\left| \sum_{k=1}^{K} A_{k\cdot}^T Q Q^T \hat{W}_{k\cdot}^{\star}(u_k, v_k) \right| \\
&\qquad = \Bigg| \sum_{k=1}^{K} \hat{A}_{k\cdot}^T Q Q^T \hat{W}_{k\cdot}^{\star}(\hat{u}_k, \hat{v}_k) + \sum_{k=1}^{K} (A_{k\cdot} - \hat{A}_{k\cdot})^T Q Q^T \hat{W}_{k\cdot}^{\star}(u_k, v_k) \\
&\qquad\qquad + \sum_{k=1}^{K} \hat{A}_{k\cdot}^T Q Q^T \{\hat{W}_{k\cdot}^{\star}(u_k, v_k) - \hat{W}_{k\cdot}^{\star}(\hat{u}_k, \hat{v}_k)\} \Bigg|.
\end{aligned}$$

Using the triangle inequality, we have

$$\left| \sum_{k=1}^{K} A_{k\cdot}^T Q Q^T \hat{W}_{k\cdot}^{\star}(u_k, v_k) \right| \geq \tilde{\varepsilon} - T_1 - T_2,$$

where

$$T_1 = \left| \sum_{k=1}^{K} (A_{k\cdot} - \hat{A}_{k\cdot})^T Q Q^T \hat{W}_{k\cdot}^{\star}(u_k, v_k) \right|$$

and

$$T_2 = \left| \sum_{k=1}^{K} \hat{A}_{k\cdot}^T Q Q^T \left\{ \hat{W}_{k\cdot}^{\star}(u_k, v_k) - \hat{W}_{k\cdot}^{\star}(\hat{u}_k, \hat{v}_k) \right\} \right|.$$

Let $\theta_k = (A_{k\cdot} - \hat{A}_{k\cdot}) / \|A_{k\cdot} - \hat{A}_{k\cdot}\|_{G,\gamma} \in \mathbb{R}^J$. Then $\|\theta_k\|_{G,\gamma} = 1$ for every $k \in [K]$ and $\theta_k \in \mathcal{T}(G, \gamma)$. Therefore, we have

$$\begin{aligned}
T1 &\leq \max_{k \in [K]} \|A_{k\cdot} - \hat{A}_{k\cdot}\|_{G,\gamma} \left| \sum_{k=1}^{K} \theta_k^T Q Q^T \hat{W}_{k\cdot}^{\star}(u_k, v_k) \right| \\
&= \max_{k \in [K]} \|A_{k\cdot} - \hat{A}_{k\cdot}\|_{G,\gamma} \left| \sum_{k=1}^{K} (Q Q^T \theta_k)^T Q Q^T \hat{W}_{k\cdot}^{\star}(u_k, v_k) \right| \leq \epsilon_a \tilde{\varepsilon},
\end{aligned}$$

where the inequality holds because $QQ^T\theta_k \in \mathcal{T}(G,\gamma)$ for $k \in [K]$. For $T2$, we have

$$\begin{aligned}
T2 &= \left| \sum_{k=1}^{K} \hat{A}_{k\cdot}^T QQ^T \left\{ \hat{W}_{k\cdot}^\star(u_k, v_k) - \hat{W}_{k\cdot}^\star(\hat{u}_k, \hat{v}_k) \right\} \right| \\
&= \left| \sum_{j=1}^{J} \left\langle \nabla \ell_{N,j}(\Sigma_j^\star), \sum_{k=1}^{K} \hat{a}_{kj}(u_k v_k^T - \hat{u}_k \hat{v}_k^T) \right\rangle \right| \\
&\le \tilde{\varepsilon} \max_{k\in[K]} \left\| u_k v_k^T - \hat{u}_k \hat{v}_k^T \right\|_F \\
&\le \tilde{\varepsilon} \max_{k\in[K]} (\|v_k - \hat{v}_k\|_2 + \|u_k - \hat{u}_k\|_2) \le 2\tilde{\varepsilon}\epsilon_v,
\end{aligned}$$

where the second equality follows from $\hat{A}_{k\cdot}^T QQ^T = A_{k\cdot}$. ■

Appendix D. Sample Complexity of Spectral Initialization

We prove Theorem 9 in Section D.1. The proof uses Davis-Kahan $\sin\theta$ theorem stated in Lemma 29. Section D.2 proves supporting lemmas.

D.1 Proof of Theorem 9

Let $M_N = J^{-1}\sum_{j=1}^J S_{N,j}$, $M^\star = E(M_N)$, and $R^0 = \mathrm{argmin}_{Y\in\mathcal{O}(K)}\|V^0 - V^\star Y\|_F^2$. The proof proceeds in two steps. In the first step, we establish that

$$\mathrm{dist}^2(Z^0, Z^\star) \le e_1\|M_N - M^\star\|_2^2 + e_2\sum_{j=1}^J\|S_{N,j} - \Sigma_j^\star\|_2^2, \tag{45}$$

where $e_1 = 5KJg^{-2}(1 + 16\varphi^2\|A^\star\|_\infty^2)$, $e_2 = 8K\varphi^2$, $g = \sigma_K(M^\star) - \sigma_{K+1}(M^\star) > 0$, and $\varphi^2 = \max_{j\in[J]}\{1 + 4\sqrt{2}\|A^\star\|_\infty/\sigma_K(\Sigma_j^\star)\}$. In the second step, we bound $\|M_N - M^\star\|_2$ and $\|S_{N,j} - \Sigma_j^\star\|_2$ for $j \in [J]$ using Lemma 32.

Step 1. We write

$$\mathrm{dist}^2(Z^0, Z^\star) = T1 + T2,$$

where $T1 = \sum_{j=1}^J\|V^0 - V^\star R^0\|_F^2$ and $T2 = \sum_{j=1}^J\|\mathrm{diag}(a_j^0) - R^{0T}\mathrm{diag}(a_j^\star)R^0\|_F^2$. First, we find a bound on $\min_{Y\in\mathcal{O}(K)}\|V^0 - V^\star Y\|_F^2$ that does not depend on R^0. By Lemma 30, we have

$$\min_{Y\in\mathcal{O}(K)}\|V^0 - V^\star Y\|_F^2 \le \frac{1}{2(\sqrt{2}-1)}\|V^0V^{0T} - V^\star V^{\star T}\|_F^2. \tag{46}$$

The following lemma gives us a bound on $T2$ that does not depend on R^0.

Lemma 26 *Let $\varphi^2 = \max_{j\in[J]}\{1 + 4\sqrt{2}\|A^\star\|_\infty/\sigma_K(\Sigma_j^\star)\}$. We have*

$$T2 \le 4K\varphi^2\sum_{j=1}^J\left(5\|\Sigma_j^\star\|_2^2\|V^0V^{0T} - V^\star V^{\star T}\|_2^2 + 2\|S_{N,j} - \Sigma_j^\star\|_2^2\right).$$

Putting (46) and Lemma 26 together, we have

$$\begin{aligned}\mathrm{dist}^2(Z^0, Z^\star) \le{}& \frac{J}{2(\sqrt{2}-1)}\|V^0V^{0T} - V^\star V^{\star T}\|_F^2\\ &+ 4K\varphi^2\sum_{j=1}^J\left(2\|S_{N,j} - \Sigma_j^\star\|_2^2 + 5\|\Sigma_j^\star\|_2^2\|V^0V^{0T} - V^\star V^{\star T}\|_2^2\right). \end{aligned}\tag{47}$$

Using Lemma 29 to bound $\|V^0V^{0T} - V^\star V^{\star T}\|_F$ and $\|V^0V^{0T} - V^\star V^{\star T}\|_2$, and noting that $\|\Sigma_j^\star\|_2 \le \|A^\star\|_\infty$, we obtain (45).

Step 2. We show that $\|M_N - M^\star\|_2^2$ and $\|S_{N,j} - \Sigma_j^\star\|_2^2$ are bounded with high probability when the eigengap g is bounded away from zero. We apply Lemma 32 with $\|M^\star\|_2 \leq \|A^\star\|_\infty$ and obtain

$$\mathrm{pr}\left\{\|M_N - M^\star\|_2 \geq h_M(\delta)\right\} \leq \frac{\delta}{J},$$

where

$$h_M(\delta) = \frac{2P\|A^\star\|_\infty}{NJ}\log\frac{2PJ}{\delta} + \left(\frac{2P\|A^\star\|_\infty^2}{NJ}\log\frac{2PJ}{\delta}\right)^{\frac{1}{2}}.$$

Similarly, for $J \geq 4$ and for every $j \in [J]$, we have

$$\mathrm{pr}\left\{\|S_{N,j} - \Sigma_j^\star\|_2 \geq h_S(\delta)\right\} \leq \frac{J-1}{J}\frac{\delta}{J},$$

where

$$h_S(\delta) = \frac{2P\|A^\star\|_\infty}{N}\log\frac{4PJ}{\delta} + \left(\frac{2P\|A^\star\|_\infty^2}{N}\log\frac{4PJ}{\delta}\right)^{\frac{1}{2}}.$$

Then, collecting results and applying union bound, we have

$$\mathrm{dist}^2(Z^0, Z^\star) \leq \frac{5KJ}{g^2}\left(1 + 16\varphi^2\|A^\star\|_\infty^2\right)h_M^2(\delta) + 8KJ\varphi^2 h_S^2(\delta),$$

with probability at least $1-\delta$. This implies that

$$\begin{aligned}
&\mathrm{dist}^2(Z^0, Z^\star) \leq \phi(g, A^\star)\left\{\frac{KJP^2}{N^2}\left(\log\frac{4PJ}{\delta}\right)^2 + \frac{KJP}{N}\log\frac{4PJ}{\delta}\right\}; \\
&\phi(g, A^\star) = 4\|A^\star\|_\infty^2\left\{\frac{5(1+16\varphi^2\|A^\star\|_\infty^2)}{g^2 J} \vee 8\varphi^2\right\},
\end{aligned} \tag{48}$$

with probability at least $1-\delta$ and $\phi(g, A^\star)$ is a constant that depends on g and $A^\star$. In particular, the bound holds only when the eigengap g is bounded away from zero.

D.2 Proof of Lemma 26

We prove the result for $J = 1$ and drop the subscript j throughout the proof. The proof can be easily extended to the case where $J > 1$. Recall that $R^0 = \mathrm{argmin}_{Y\in\mathcal{O}(K)}\|V^0 - V^\star Y\|_F^2$ and $\varphi^2 = \max_{j\in[J]}\{1 + 4\sqrt{2}\|A^\star\|_\infty/\sigma_K(\Sigma_j^\star)\}$. Similar to (18) in the proof of Lemma 16, we have

$$\begin{aligned}
\left\|\mathrm{diag}(a^0) - R^{0T}\mathrm{diag}(a^\star)R^0\right\|_F^2 &\leq \varphi^2\left\|V^0\mathrm{diag}(a^0)V^{0T} - V^\star\mathrm{diag}(a^\star)V^{\star T}\right\|_F^2 \\
&\leq 2K\varphi^2\left\|\sum_{k=1}^K (v_k^{0T}S_N v_k^0)v_k^0 v_k^{0T} - \Sigma^\star\right\|_2^2,
\end{aligned} \tag{49}$$

since $V^0 \mathrm{diag}(a^0) V^T = \sum_{k=1}^K (v_k^{0T} S_N v_k^0) v_k^0 v_k^{0T}$. Writing $\Sigma^\star = \sum_{k=1}^K (v_k^{\star T} \Sigma^\star v_k^\star) v_k^\star v_k^{\star T}$ and applying the triangle inequality to the right-hand side of (49), we have

$$\left\|\mathrm{diag}(a^0) - R^{0T} \mathrm{diag}(a^\star) R^0\right\|_F^2 \leq 4K\varphi^2 \Bigg\{ \left\| \sum_{k=1}^K (v_k^{\star T} \Sigma^\star v_k^\star)(v_k^0 v_k^{0T} - v_k^\star v_k^{\star T}) \right\|_2^2 + \left\| \sum_{k=1}^K (v_k^{0T} S_N v_k^0 - v_k^{\star T} \Sigma^\star v_k^\star) v_k^0 v_k^{0T} \right\|_2^2 \Bigg\}. \quad (50)$$

Next, we bound the two terms on the right-hand side of (50) separately. We have

$$\left\| \sum_{k=1}^K (v_k^{\star T} \Sigma^\star v_k^\star)(v_k^0 v_k^{0T} - v_k^\star v_k^{\star T}) \right\|_2^2 \leq \|\Sigma^\star\|_2^2 \left\| \sum_{k=1}^K v_k^0 v_k^{0T} - \sum_{k=1}^K v_k^\star v_k^{\star T} \right\|_2^2 .$$

By the uniqueness of projection operators, the above display can be written as

$$= \|\Sigma^\star\|_2^2 \|V^0 V^{0T} - V^\star V^{\star T}\|_2^2. \quad (51)$$

Since $v_1^0, \ldots, v_K^0$ are orthonormal to each other, we have

$$\begin{aligned} \left\| \sum_{k=1}^K \{v_k^{0T} S_N v_k^0 - v_k^{\star T} \Sigma^\star v_k^\star\} v_k^0 v_k^{0T} \right\|_2^2 &\leq \max_{k\in[K]} |v_k^{0T} S_N v_k^0 - v_k^{\star T} \Sigma^\star v_k^\star|^2 \\ &= \max_{k\in[K]} |v_k^{0T} S_N v_k^0 - v_k^{0T} \Sigma^\star v_k^0 + v_k^{0T} \Sigma^\star v_k^0 - v_k^{\star T} \Sigma^\star v_k^\star|^2 \\ &\leq 2\|S_N - \Sigma^\star\|_2^2 + 2 \max_{k\in[K]} |v_k^{0T} \Sigma^\star v_k^0 - v_k^{\star T} \Sigma^\star v_k^\star|^2 \\ &= 2\|S_N - \Sigma^\star\|_2^2 + 2 \max_{k\in[K]} |\langle \Sigma^\star, v_k^0 v_k^{0T} - v_k^\star v_k^{\star T} \rangle|^2. \quad (52) \end{aligned}$$

Next, we upper bound the second term on the right-hand side of (52). We have

$$\begin{aligned} \max_{k\in[K]} |\langle \Sigma^\star, v_k^0 v_k^{0T} - v_k^\star v_k^{\star T} \rangle|^2 &\leq \|\Sigma^\star\|_2^2 \max_{k\in[K]} \|v_k^0 v_k^{0T} - v_k^\star v_k^{\star T}\|_F^2 \\ &\leq 2\|\Sigma^\star\|_2^2 \max_{k\in[K]} \|v_k^0 v_k^{0T} - v_k^\star v_k^{\star T}\|_2^2 \\ &\leq 2\|\Sigma^\star\|_2^2 \|V^0 V^{0T} - V^\star V^{\star T}\|_2^2. \quad (53) \end{aligned}$$

Plugging the result of (53) into (52), we have

$$\left\| \sum_{k=1}^K (v_k^{0T} S_N v_k^0 - v_k^{\star T} \Sigma^\star v_k^\star) v_k^\star v_k^{\star T} \right\|_2^2 \leq 2\|S_N - \Sigma^\star\|_2^2 + 4\|\Sigma^\star\|_2^2 \|V^0 V^{0T} - V^\star V^{\star T}\|_2^2, \quad (54)$$

where the inequality follows by triangle inequality. Combining results from (51) and (54), we have

$$\|\mathrm{diag}(a_j^0) - R^{0T}\mathrm{diag}(a_j^\star)R^0\|_F^2 \le 4K\varphi^2\left(5\|\Sigma_j^\star\|_2^2\|V^0V^{0T} - V^\star V^{\star T}\|_2^2 + 2\|S_{N,j} - \Sigma_j^\star\|_2^2\right),$$

for every $j \in [J]$. Hence, we complete the proof.

Appendix E. Proof of of Theorem 4 and Proposition 5

To prove Theorem 4, we need the following lemma.

Lemma 27 (Linear Convergence Rate) *Suppose that Assumptions 1—6 are satisfied. After I iterations of Algorithm 2, we have*

$$\sum_{j=1}^{J} \|\Sigma_j^I - \tilde{\Sigma}_j^\star\|_F^2 \le \beta^{I/2}(2\mu^2\xi^2) \sum_{j=1}^{J} \|\Sigma_j^0 - \tilde{\Sigma}_j^\star\|_F^2 + C_1\varepsilon_{stat}^2, \qquad C_1 = \frac{2\tau\mu^2\eta}{\beta^{1/2}-\beta},$$

where $\mu = \max_{j\in[J]}(17/8)\|Z_j^\star\|_2$ and $\tau = J^{-1}\{9/2 + (1/2 \vee K/8)\}$.

Since $\|Z_j^\star\|_2$ and μ are bounded, Lemma 27 shows that Algorithm 2 achieves error smaller than $\delta + C_1\varepsilon_{stat}^2$ after $I \gtrsim \log(1/\delta)$ iterations. The second term on the left-hand side denotes the constant multiple of the statistical error, which depends on the distribution of the data and the sample size.

Proof of Lemma 27: Let $Z = (V^T, A)^T$ be an iterate obtained by Algorithm 2, $\Sigma_j = V\mathrm{diag}(a_j)V^T$ for $j \in [J]$, and

$$R = \operatorname*{argmin}_{Y\in\mathcal{O}(K)} \|V - V^\star Y\|_F^2.$$

We have the following decomposition

$$\begin{aligned}\Sigma_j - \tilde{\Sigma}_j^\star &= (V - V^\star R)R^T\mathrm{diag}(\tilde{a}_j^\star)RR^TV^{\star T}\\ &\qquad + V\{\mathrm{diag}(a_j) - R^T\mathrm{diag}(\tilde{a}_j^\star)R\}R^TV^{\star T} + V\mathrm{diag}(a_j)(V - V^\star R)^T.\end{aligned}$$

Then

$$\|\Sigma_j - \tilde{\Sigma}_j^\star\|_F \le \left\{\|\mathrm{diag}(a_j)\|_2 + \|\mathrm{diag}(\tilde{a}_j^\star)\|_2\right\}\|V - V^\star R\|_F + \|\mathrm{diag}(a_j) - R^T\mathrm{diag}(\tilde{a}_j^\star)R\|_F.$$

Since $\|\mathrm{diag}(a_j)\|_2 \le \|Z_j\|_2$ and $\|Z_j\|_2 + \|\tilde{Z}_j^\star\|_2 \le \|Z_j\|_2 + \|Z_j^\star\|_2 \le (17/8)\|Z_j^\star\|_2$ from (39), we have

$$\|\Sigma_j - \tilde{\Sigma}_j^\star\|_F \le \mu\left\{\|V - V^\star R\|_F + \|\mathrm{diag}(a_j) - R^T\mathrm{diag}(\tilde{a}_j^\star)R\|_F\right\}.$$

Combining Theorem 7 with Lemma 16, in the Ith iteration, we have

$$\begin{aligned}\sum_{j=1}^{J}\|\Sigma_j^I - \tilde{\Sigma}_j^\star\|_F^2 &\le 2\mu^2\mathrm{dist}^2(Z^I, \tilde{Z}^\star)\\ &\le 2\mu^2\left\{\beta^{I/2}\mathrm{dist}^2(Z^0, \tilde{Z}^\star) + \frac{\tau\eta\varepsilon_{stat}^2}{\beta^{1/2}(1-\beta)^{1/2}}\right\}\\ &\le 2\mu^2\left\{\beta^{I/2}\xi^2\sum_{j=1}^{J}\|\Sigma_j^0 - \tilde{\Sigma}_j^\star\|_F^2 + \frac{\tau\eta\varepsilon_{\mathrm{stat}}^2}{\beta^{1/2}(1-\beta)^{1/2}}\right\},\end{aligned}$$

which completes the proof. ■

Proof of Theorem 4: We first note that under the assumptions, using Theorem 9, Assumption 6 is satisfied with probability at least $1-\delta_0$. This allows us to use Lemma 27 to bound $\sum_{j=1}^{J}\|\Sigma_j^I-\tilde{\Sigma}_j^\star\|_F^2$. By triangle inequality, we have

$$\sum_{j=1}^{J}\|\Sigma_j^I-\Sigma_j^\star\|_F^2 \leq 2\underbrace{\sum_{j=1}^{J}\|\Sigma_j^I-\tilde{\Sigma}_j^\star\|_F^2}_{\text{Estimation error}} + 2\underbrace{\sum_{j=1}^{J}\|\tilde{\Sigma}_j^\star-\Sigma_j^\star\|_F^2}_{\text{Approximation error}}$$

and, therefore, it remains only to bound the approximation error. We have

$$\begin{aligned}\sum_{j=1}^{J}\|\tilde{\Sigma}_j^\star-\Sigma_j^\star\|_F^2 &= \sum_{j=1}^{J}\|V^\star\mathrm{diag}(\tilde{a}_j^\star-a_j^\star)V^{\star T}\|_F^2\\ &= \sum_{j=1}^{J}\|\mathrm{diag}(\tilde{a}_j^\star-a_j^\star)\|_F^2\\ &= \|\tilde{A}^\star-A^\star\|_F^2.\end{aligned}$$

Recall that each row of $\tilde{A}^\star$ is the projection of the corresponding row of $A^\star$ to the set $\tilde{\mathcal{C}}_A(c,\gamma)$, where $c\geq c^\star$ and $\gamma\geq\gamma^\star$. Recall that columns of $Q=(\tilde{Q},\ Q_1)$ denote eigenvectors of G For any $k\in[K]$, if $Q_1^TA_{k\cdot}^\star=0$, then we have no loss in projecting $A_{k\cdot}^\star$ to $\tilde{\mathcal{C}}_A(c,\gamma)$. When $Q_1^TA_{k\cdot}^\star\neq 0$, we want to quantify the loss of using a truncated ellipsoid. Let $A_{k\cdot}^\star=Qh_k^\star$, $h_k^\star\in\mathbb{R}^J$ for $k\in[K]$. Then $\tilde{A}_{k\cdot}^\star=\tilde{Q}\tilde{h}_k^\star$, where $\tilde{h}_{kj}^\star=h_{kj}^\star$ for $j\leq r(\tilde{G})$, $k\in[K]$. Therefore, for each row $k\in[K]$, we have

$$\|\tilde{A}_{k\cdot}^\star-A_{k\cdot}^\star\|_2^2=\sum_{j=r(\tilde{G})+1}^{J}h_{kj}^{\star 2}\leq\lambda_{r(\tilde{G})}\sum_{j=r(\tilde{G})+1}^{J}\frac{h_{kj}^{\star 2}}{\lambda_j}\leq\lambda_{r(\tilde{G})}\gamma^\star\leq\delta_A\gamma^\star. \tag{55}$$

Then $\|\tilde{A}^\star-A^\star\|_F^2\leq K\delta_A\gamma^\star$, which completes the proof. ■

Next, we proceed to prove Proposition 5.

Proof of Proposition 5: The proof is carried out in three steps. First, we verify that the iterate Z^0 obtained by Algorithm 1 satisfies Assumption 6. In the second step, we bound the statistical and approximation errors. In the final step, we establish a bound on $\sum_{j=1}^{J}\|\Sigma_j^I-\Sigma_j^\star\|_F^2$.

Step 1. The proof is similar to that of Theorem 9. Let $M_N=J^{-1}\sum_{j=1}^{J}S_{N,j}$ and $M^\star=E(M_N)$. Since $\sigma_K(\Sigma_j^\star)>E_j$ for every $j\in[J]$, the eigengap $g=\sigma_K(M^\star)-\sigma_{K+1}(M^\star)>0$

is nonzero and we can apply the Davis-Kahan "$\sin\theta$" theorem, stated in Lemma 29. From (45),

$$\mathrm{dist}^2(Z^0, Z^\star) \le e_1\|M_N - M^\star\|_2^2 + e_2 \sum_{j=1}^J \|S_{N,j} - \Sigma_j^\star\|_2^2.$$

From the definition of $\|\cdot\|_{\psi_2}$ in (59), for every $n \in [N]$ and $j \in [J]$, we have

$$\|x_j^{(n)}\|_{\psi_2}^2 \le \max_{j\in[J]} \|\Sigma_j^\star + E_j\|_2 \le 2\max_{j\in[J]} \|\Sigma_j^\star\|_2 \le 2\|A^\star\|_\infty.$$

Lemma 33 then gives us

$$\|M_N - M^\star\|_2 \lesssim \|A^\star\|_\infty \left[\frac{1}{NJ}\left(2P + \log\frac{2J}{\delta_0}\right) + \left\{\frac{1}{NJ}\left(2P + \log\frac{2J}{\delta_0}\right)\right\}^{\frac{1}{2}}\right], \tag{56}$$

with probability at least $1 - \delta_0/J$. Similarly, for $J \ge 4$, we have, for every $j \in [J]$,

$$\|S_{N,j} - \Sigma_j^\star\|_2 \lesssim \|A^\star\|_\infty \left[\frac{1}{N}\left(2P + \log\frac{4J}{\delta_0}\right) + \left\{\frac{1}{N}\left(2P + \log\frac{4J}{\delta_0}\right)\right\}^{\frac{1}{2}}\right],$$

with probability at least $1 - (J-1)\delta_0/J^2$. A union bound, together with (48), gives us

$$\mathrm{dist}^2(Z^0, Z^\star) \le \phi(g, A^\star)\left[\frac{KJ}{N^2}\left\{8P^2 + 2\left(\log\frac{4J}{\delta_0}\right)^2\right\} + \frac{KJ}{N}\left(4P + 2\log\frac{4J}{\delta_0}\right)\right],$$

with probability at least $1 - \delta_0$.

This shows that $\mathrm{dist}^2(Z^0, Z^\star) \le JI_0^2$ with probability at least $1 - \delta_0$ when $N \gtrsim K\left(P + \log J/\delta_0\right)$. From (46) and (47) we have that $\|V^0 - V^\star R\|_F^2 \le 5Kg^{-2}\|M_N - M^\star\|_2^2$. From (56) we have that $\|V - V^\star R\|_F^2 \le I_0^2/J$ with high probability when $N \gtrsim K\left(P + \log J/\delta_0\right)$. Similarly, $\|\mathrm{diag}(a_j) - R^T\mathrm{diag}(a_j^\star)R\|_F^2 \le (J-1)I_0^2/J$ with high probability. This shows that Assumption 6 is satisfied..

Step 2. Let $r = r(\tilde{G})$. Recall that $G = Q\Lambda^\dagger Q^T$. Setting $\Lambda_{rr}^\dagger = \exp(-l^2r^2) = \delta_A$, we have

$$r(\tilde{G}) = \left\{\left(\frac{1}{l^2}\log\frac{1}{\delta_A}\right)^{\frac{1}{2}} \wedge J\right\}. \tag{57}$$

Proposition 8 then yields

$$\varepsilon_{stat}^2 \le 2(\nu_2 \vee \nu_2^2) + 2J\max_j \|E_j\|_2^2$$

where

$$\nu_2 \lesssim \frac{\|A^\star\|_\infty^2}{e_4 N}\left[\log\frac{1}{\delta_0} + \left\{\left(\frac{1}{\alpha^2}\log\frac{1}{\delta_A}\right)^{\frac{1}{2}} \wedge J\right\} + Ks^\star\log\frac{P}{s^\star}\right]$$

with probability at least $1-\delta_0$.

If $s^\star \log P/s^\star < PJ$, then

$$\log\frac{1}{\delta_0} + Kr(\tilde{G}) + Ks^\star\log\frac{P}{s^\star} \lesssim \left(\log\frac{1}{\delta_0} + KJP\right).$$

Therefore $(\nu_2 \vee \nu_2^2) \lesssim JI_0^2$. Combining with the assumption $\max_{j\in[J]}\|E_j\|_2 \lesssim I_0$, we establish that Assumption 3 holds with probability at least $1-\delta_0$.

Step 3. Similar to the proof of Theorem 4, we combine results from Step 1 and 2 to obtain

$$\sum_{j=1}^J \|\Sigma_j^I - \Sigma_j^\star\|_F^2 \lesssim \delta_1 + \frac{Kr(\tilde{G}) + Ks^\star\log(P/s^\star) + \log\delta_0^{-1}}{N} + J\max_{j\in[J]}\|E_j\|_2^2 + K\gamma^\star\delta_A \quad (58)$$

after $I \gtrsim \log(1/\delta_1)$ iterations. We omit details for brevity.

In the final step, we choose a δ_A that satisfies $\delta_A \le (16\gamma^\star)^{-1}\min_{j\in[J]}\sigma_K^2(\Sigma_j^\star)$ and obtains the optimal error of (58). Plugging (57) into (58), the optimal choice is $\delta_A^\star \asymp (\gamma^\star lN)^{-1}(\log\gamma^\star lN)^{1/2}$. Note that $\delta_A^\star$ is smaller than $(16\gamma^\star)^{-1}\min_{j\in[J]}\sigma_K^2(\Sigma_j^\star)$, given that N is not too small. Then, plugging $\delta_A^\star$ into (58), we establish the result. ■

Appendix F. Known Results

For convenience, we present several known results that are used to prove the main results.

Lemma 28 (Theorem 2.1.12 in Nesterov (2013)) *For any L-smooth and m-strongly convex function h, we have*

$$\langle \nabla h(X) - \nabla h(Y), X - Y \rangle \geq \frac{mL}{m+L}\|X - Y\|^2 + \frac{1}{m+L}\|\nabla h(X) - \nabla h(Y)\|^2.$$

Lemma 29 (Davis-Kahan $\sin\theta$ theorem, adapted from Yu et al. (2015)) *Let $M_N = J^{-1}\sum_{j=1}^{J} S_{N,j}$ and $M^\star = E(M_N)$. $V^\star$ is the matrix whose columns are top-K eigenvectors of $M^\star$, and V is the matrix whose columns are the top-K eigenvectors of M_N. Assume that the eigengap $g = \sigma_K(M^\star) - \sigma_{K+1}(M^\star) > 0$ is bounded away from zero. Then*

$$\|VV^T - V^\star V^{\star T}\|_F \leq \frac{2\sqrt{K}}{g}\|M_N - M^\star\|_2, \quad \|VV^T - V^\star V^{\star T}\|_2 \leq \frac{2}{g}\|M_N - M^\star\|_2.$$

Moreover, we have

$$\min_{Y \in \mathcal{O}(K)} \|V - V^\star Y\|_F \leq \frac{2\sqrt{2}}{g}\|M_N - M^\star\|_F.$$

Lemma 30 (Adapted from Lemma 5.4 in Tu et al. (2016)) *For any $X, U \in \mathbb{R}^{P\times K}$, we have*

$$\min_{Y \in \mathcal{O}(K)} \|U - XY\|_F^2 \leq \frac{1}{2(\sqrt{2}-1)\sigma_K^2(X)}\|UU^T - XX^T\|_F^2.$$

Lemma 31 (Adapted from Theorem 6.6.1 in Tropp (2015)) *Consider a sequence of independent, random Hermitian matrices $X_1, \ldots, X_N$ with dimension $P \times P$. Moreover, assume that for $n \in [N]$, we have almost surely $\|X_n\|_2 \leq L$. Define*

$$Y = \sum_{n=1}^{N} X_n - \mathbb{E}(X_n), \quad \nu(Y) = \left\| \sum_{n=1}^{N} var(X_n) \right\|_2.$$

Then, for every $t \geq 0$, we have

$$pr(\|Y\|_2 \geq t) \leq 2P \exp\left\{ \frac{-t^2}{2(\nu(Y) + Lt)} \right\}.$$

Lemma 32 *Let $x_1, \ldots, x_N$ be independent centered random vectors in $\mathbb{R}^P$ with $\|x_n\|_2^2 \leq L$ almost surely and $S = \mathbb{E}(N^{-1}\sum_{n=1}^{N} x_n x_n^T)$. Then*

$$pr\left\{ \left\| \frac{1}{N}\sum_{n=1}^{N} x_n x_n^T - S \right\|_2 \geq t \right\} \leq 2P \exp\left\{ \frac{-Nt^2}{2L(\|S\|_2 + t)} \right\}.$$

Proof of Lemma 32 For each sample, we have

$$\operatorname{var}\left(x_n x_n^T\right) = \mathbb{E}\left\{\left(x_n x_n^T\right)^2\right\} - \mathbb{E}\left(x_n x_n^T\right)^2 \preceq \mathbb{E}\left\{\|x_n\|_2^2 x_n x_n^T\right\} \preceq LE\left(x_n x_n^T\right).$$

Consequently, we have

$$\left\|\frac{1}{N}\sum_{n=1}^{N} \operatorname{var}\left\{x_n x_n^T\right\}\right\|_2 \leq L\|S\|_2.$$

The result follows from Lemma 31. ■

Let Z be a sub-Gaussian random variable, and we define the sub-Gaussian norm as

$$\|Z\|_{\psi_2} = \sup_{p\geq 1} \frac{1}{\sqrt{p}} (\mathbb{E}|Z|^p)^{\frac{1}{p}}.$$

Let Z be a P dimensional Gaussian random vector, then we define the sub-Gaussian norm as

$$\|Z\|_{\psi_2} = \sup_{x\in\mathcal{S}^{P-1}} \|\langle x, Z\rangle\|_{\psi_2}. \tag{59}$$

Lemma 33 (Adapted from Corollary 5.50 **in Vershynin (2012))** *Consider independent centered random vectors with sub-Gaussian distribution, denoted as $x_1, \ldots, x_N$. Let $\|x_n\|_{\psi_2}^2 \leq L$ for every $n \in [N]$. Then, we have*

$$pr\left\{\left\|\frac{1}{N}\sum_{n=1}^{N} x_n x_n^T - S\right\|_2 \geq t\right\} \leq 2\exp\left\{2P - e_0 N\left(\frac{t^2}{L^2}\wedge\frac{t}{L}\right)\right\},$$

for some absolute constant e_0.

Lemma 34 (Adapted from Proposition 1.1 **in Hsu et al. (2012))** *Let $A \in \mathbb{R}^{P\times P}$ be a matrix, and let $\Sigma = A^T A$. Let $x = (x_1, \ldots, x_P)$ be an isotropic multivariate Gaussian random vector with zero mean. For all $t > 0$. We have*

$$pr\left\{\left|\|Ax\|_2^2 - \mathbb{E}(\|Ax\|_2^2)\right| > t\right\} \leq 2\exp\left[-\left\{\frac{t^2}{4\|\Sigma\|_F^2}\wedge\frac{t}{2\|\Sigma\|_2}\right\}\right].$$

Moreover, consider K matrices $A_1, \ldots, A_K$, with $\Sigma_k = A_k^T A_k$ for $k \in [K]$ and $x_k \in \mathbb{R}^P$ be isotropic multivariate random vectors. Then, for all $t > 0$, we have

$$\begin{aligned} pr&\left\{\left|\sum_{k=1}^{K} \|A_k x_k\|_2^2 - \mathbb{E}(\|A_k x_k\|_2^2)\right| > t\right\} \\ &\leq 2\exp\left\{-\left(\frac{t^2}{4\sum_{k=1}^{K}\|\Sigma_k\|_F^2}\wedge\frac{t}{2\max_{k\in[K]}\|\Sigma_k\|_2}\right)\right\}. \end{aligned}$$

Proof of Lemma 34: The first part is shown in Hsu et al. (2012), we show the second result. Let $V_k \Lambda_k V_k^T$ be the eigendecomposition of Σ_k. define $z_k = V_k^T x_k$ which follows isotropic multivariate Gaussian distribution by the rotation invariance of Gaussian distribution. Then $\|A_k x_k\|_2^2 = \sum_{i=1}^P \lambda_{ki} z_{ki}^2$, where λ_{ki} is the ith diagonal entry of Λ_k. Then, we apply the chi-square tail inequality (Laurent and Massart, 2000) and obtain

$$\operatorname{pr}\left\{\sum_{k=1}^K \sum_{i=1}^P \lambda_{ki} z_{ki}^2 - \sum_{k=1}^K \operatorname{tr}(\Sigma_k) > 2\left(\varepsilon \sum_{k=1}^K \|\Sigma_k\|_F^2\right)^{\frac{1}{2}} + 2\varepsilon \max_{k\in[K]} \|\Sigma_k\|_2\right\} \leq e^{-\varepsilon}.$$

Consequently, for all $t > 0$, we have

$$\begin{aligned} &\operatorname{pr}\left(\left|\sum_{k=1}^K \|A_k x_k\|_2^2 - \mathbb{E}\left(\|A_k x_k\|_2^2\right)\right| > t\right) \\ &\qquad\qquad \leq 2\exp\left\{-\left(\frac{t^2}{4\sum_{k=1}^K \|\Sigma_k\|_F^2} \wedge \frac{t}{2\max_{k\in[K]} \|\Sigma_k\|_2}\right)\right\}. \end{aligned}$$

The proof is complete. ■

Appendix G. Additional Empirical Results

In this section, we provide additional empirical results. For the convenience of the readers, we again provide a list of competing methods. We use the same abbreviations as in the main text.

In Section G.1, we discuss the details of the implementation of M6. In Section G.2, we discuss the details of the implementation of MR. The additional implementation of square temporal weights is presented in Section G.3. Section G.4 provides the results for data with varying ranks. Additional simulation results for different noise levels are presented in Section G.5. Finally, we include additional results for the fMRI task in Section G.6.

G.1 Implementation Details for M6

The M6 method in Table 1 provides an alternative approach to estimating the model parameters in (1). Rather than minimizing (2) using a projected gradient descent, the M6 method minimizes an alternative objective function

$$\min_{V,A} \frac{1}{J}\sum_{j=1}^{J}\frac{1}{2}\|S_{N,j} - V\mathrm{diag}(a_j)V^T\|_F^2 + \lambda_1\sum_{k=1}^{K}\|v_k\|_1 + \lambda_2\sum_{k=1}^{K}\tilde{a}_k^T\tilde{G}^\dagger\tilde{a}_k, \tag{60}$$

where a regularizer is used in place of the constraints. Following Daubechies et al. (2010), the objective in (60) can be reformulated as

$$\begin{aligned}\min h_N(Z,D) = \min_{V,A}\min_{D\in\mathbb{R}_+^{K\times P}} \frac{1}{J}\sum_{j=1}^{J}\frac{1}{2}\|S_{N,j} - V\mathrm{diag}(a_j)V^T\|_F^2 \\ + \lambda_1\sum_{k=1}^{K}\frac{1}{2}\left(\sum_{i=1}^{P}\frac{v_{k,i}^2}{d_{ki}} + d_{ki}\right) + \lambda_2\sum_{k=1}^{K}\tilde{a}_k^T\tilde{G}^\dagger\tilde{a}_k,\end{aligned} \tag{61}$$

where d_{ki} denotes the (k,i)th entry of D. An alternating gradient descent with respect to A, V, and D can be used to optimize (61) efficiently when P is large. With A and D fixed, (61) is a quadratic problem with respect to V and is differentiable everywhere. However, the objective is not continuous at $d_{ki} = 0$. Therefore, for numerical stability, we add a small constant $\vartheta > 0$ to d_{ki} so that $d_{ki} + \vartheta$ is always bounded away from zero. As a result, we have

$$\nabla_V h_N(Z,D) = \frac{-2}{J}\sum_{j=1}^{J}\left\{S_{N,j} - V\mathrm{diag}(a_j)V)^T\right\}V\mathrm{diag}(a_j) + \lambda_1\check{V},$$

where $\check{V} \in \mathbb{R}^{P\times K}$ with the (j,k)th entry equal to $v_{k,j}/d_{kj}$. The update with respect to D can be obtained as $d_{kj} = \sqrt{(v_{k,j})^2 + \vartheta}$, $k \in [K]$, $j \in [P]$. The gradient $\nabla_A h_N(Z,D)$ is

$$\nabla_A h_N(Z,D) = \frac{1}{J}W(V) + 2\lambda_2 A\tilde{G}^\dagger,$$

with

$$W(V) = [w_{kj}(v_k)] \in \mathbb{R}^{K \times J}, \quad w_{kj}(v_k) = (v_k)^T(\Sigma_j - S_{N,j})v_k.$$

To select λ_1, λ_2, we use the Bayesian Information Criterion BIC with 5-fold cross-validation discussed in Section 3.2. Algorithm 6 details the method M6.

Input: $\varepsilon, \vartheta > 0$, and $\{x_j^{(n)}\}_{n\in[N],j\in[J]}$
Set $V^0, A^0 =$ Spectral initialization$(\{x_j^{(n)}\}_{n\in[N],j\in[J]})$
While $|h_N(Z^{i-1}, D^{i-1}) - h_N(Z^{i-2}, D^{i-2})| > \varepsilon$
 $A^i \leftarrow A^{i-1} - \eta_A \nabla_A h_N(Z^{i-1}, D^{i-1})$
 $V^i \leftarrow V^{i-1} - \eta_V \nabla_V h_N(Z^{i-1}, D^{i-1})$
 $D^i \leftarrow \operatorname{argmin} h_N(Z^i, D)$
Output: V, A

Algorithm 6: Estimation Procedure of M6

G.2 Implementation Details for MR

The MR method uses random initialization in place of spectral initialization. First, each entry in v_k, $k \in [K]$, is drawn independently from $\mathcal{N}(0, 1)$. Then, we normalize each v_k to have the unit norm. Each entry of $\tilde{a}_k$, $k \in [K]$, is drawn independently from $\mathcal{N}(0, 1)$. Subsequently, Algorithm 2 is used to refine the initial point. In the simulations, we find that the convergence points depend on the initial points Z^0. Therefore, to make the comparison fair across different sample sizes, we initialize Z^0 using the same pseudo-random number. As a result, all the curves in Figure 3 have the same starting point.

G.3 Simulation with Temporal Dynamics Following a Markov Model

The data are generated from a model where the temporal dynamics follow a discrete switching model shown in Figure 7. The results are shown in Table 10 for log-Euclidean metric and in Table 11 for $\mathrm{dist}^2(Z, Z^\star)$. Despite the fact that the temporal dynamics is not smooth, the proposed method outperforms its counterparts, indicating flexibility in empirically capturing a variety of waveforms.

G.4 The Effect of the Rank on Estimation Accuracy

Figures 8—11 show results as the rank K changes in $K \in \{2, 6, 8, 10\}$. Each figure shows simulation results as the sample size varies in $N \in \{20, 200, 2000\}$. The plots consistently show that as we increase N, the estimation results are closer to the ground truth. When $K \in \{2, 6\}$, we can estimate the temporal components fairly well with a small sample size. When $K \in \{8, 10\}$, additional samples are required to obtain good estimation results.

Methods	Number of training subjects 10	20	30	40	50
M1(a)	49.6 ± 6.4	70.8 ± 3.4	77.3 ± 3.2	79.7 ± 2.8	80.8 ± 1.7
M1(b)	26.5 ± 4.7	34.9 ± 3.3	36.5 ± 3.5	38.4 ± 2.0	39.6 ± 2.4
M2(a)	27.7 ± 3.5	27.1 ± 4.3	25.1 ± 2.0	24.3 ± 4.4	24.8 ± 2.0
M2(b)	37.6 ± 5.3	44.5 ± 4.7	40.9 ± 5.8	38.2 ± 3.9	44.1 ± 6.7
M4	52.6 ± 8.3	77.4 ± 6.8	85.0 ± 3.5	89.0 ± 3.0	90.5 ± 2.5
M7	70.9 ± 2.7	78.3 ± 3.3	81.0 ± 1.6	79.8 ± 3.0	81.2 ± 2.6
M**	61.5 ± 7.4	81.5 ± 5.4	87.4 ± 2.5	90.0 ± 1.7	90.5 ± 2.5

Table 9: Classification accuracy on test data averaged over 10 runs. We use use the covariance parameters to predict tasks in the Human Connectome Project motor data set (%). M1(a) denotes M1 with $K = 15, W = 10$; M1(b) denotes M1 with $K = 15, W = 50$; M2(a) denotes M2 with $K = 15$; M2(b) denotes M2 with $K = 60$. The rank K for M4 and M** is set as $K = 15$. The window length W for M7 is set as $W = 10$.

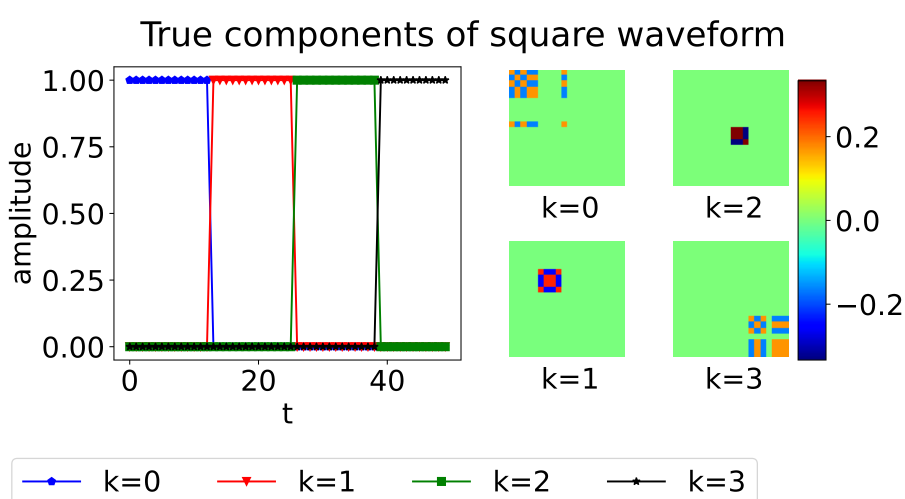

Figure 7: The left plot shows the ground truth of square temporal weights and the right plot shows the corresponding temporal components.

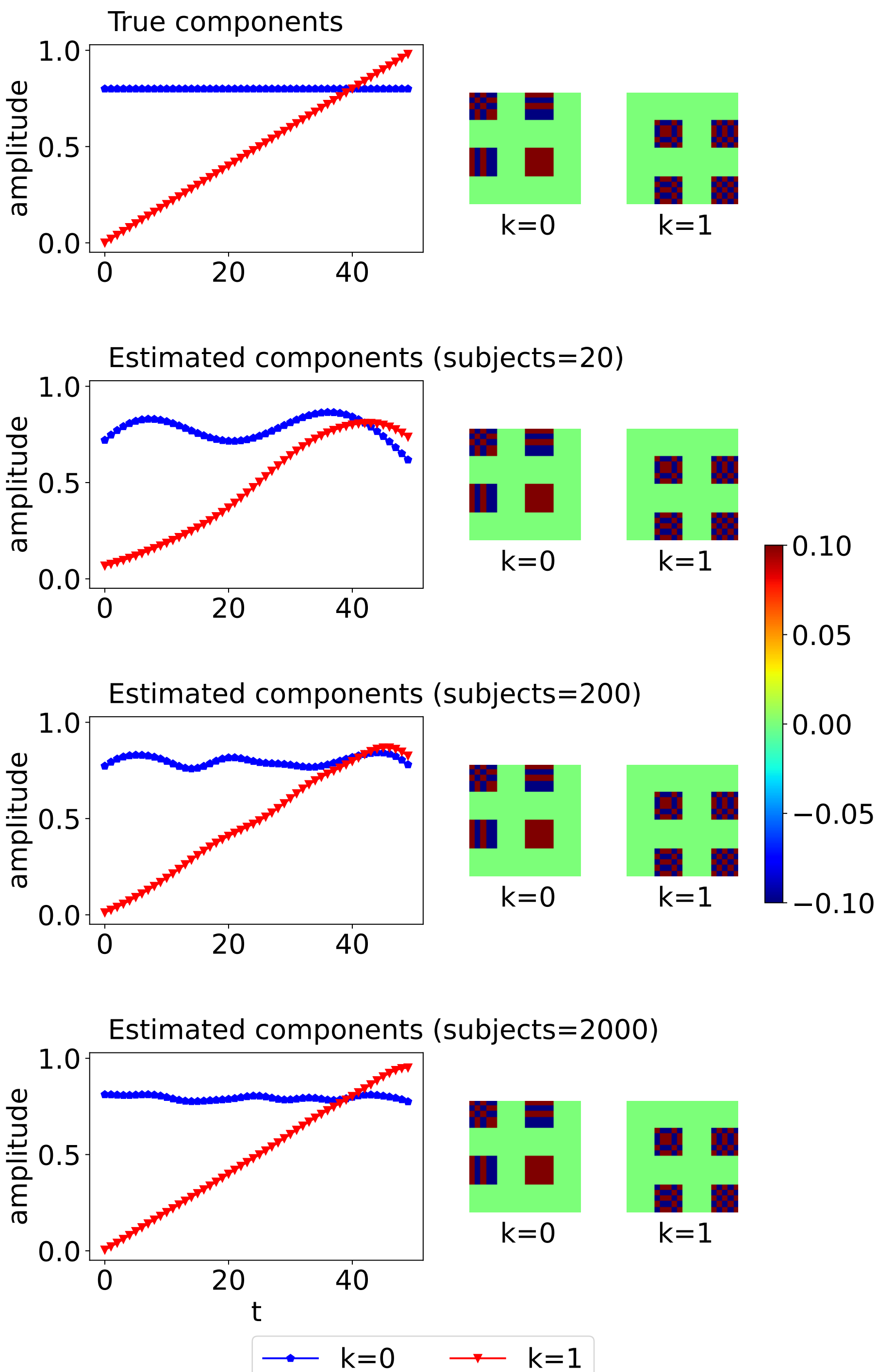

Figure 8: The top row shows the ground truth. The remaining 3 rows present simulation results for $N \in \{20, 200, 2000\}$ when $K = 2, P = 20, J = 50$, and $\sigma = 0$.

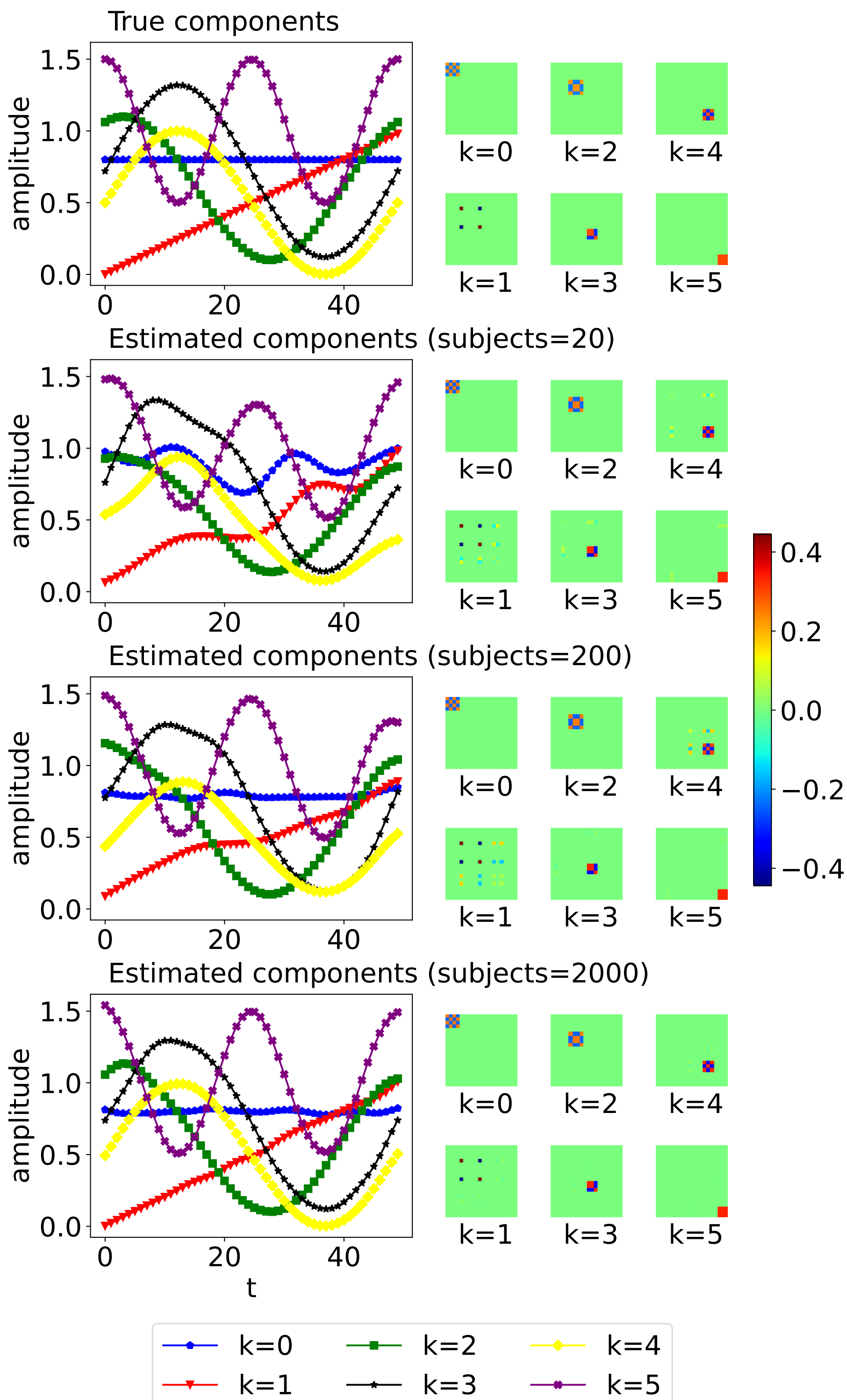

Figure 9: The top row shows the ground truth. The remaining 3 rows present simulation results for $N \in \{20, 200, 2000\}$ when $K = 26, P = 20, J = 50$, and $\sigma = 0$.

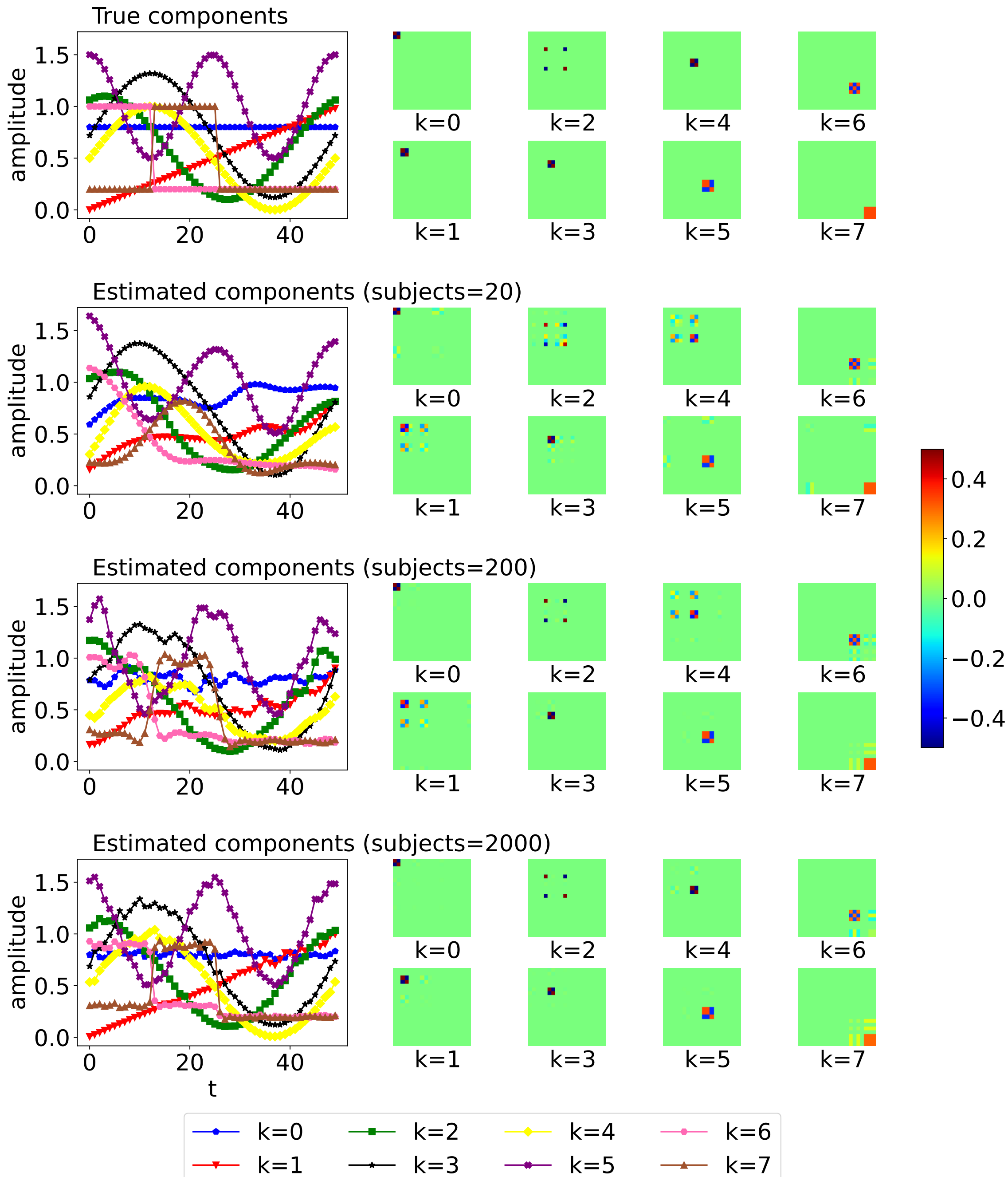

Figure 10: The top row shows the ground truth. The remaining 3 rows present simulation results for $N \in \{20, 200, 2000\}$ when $K = 8, P = 20, J = 50$, and $\sigma = 0$.

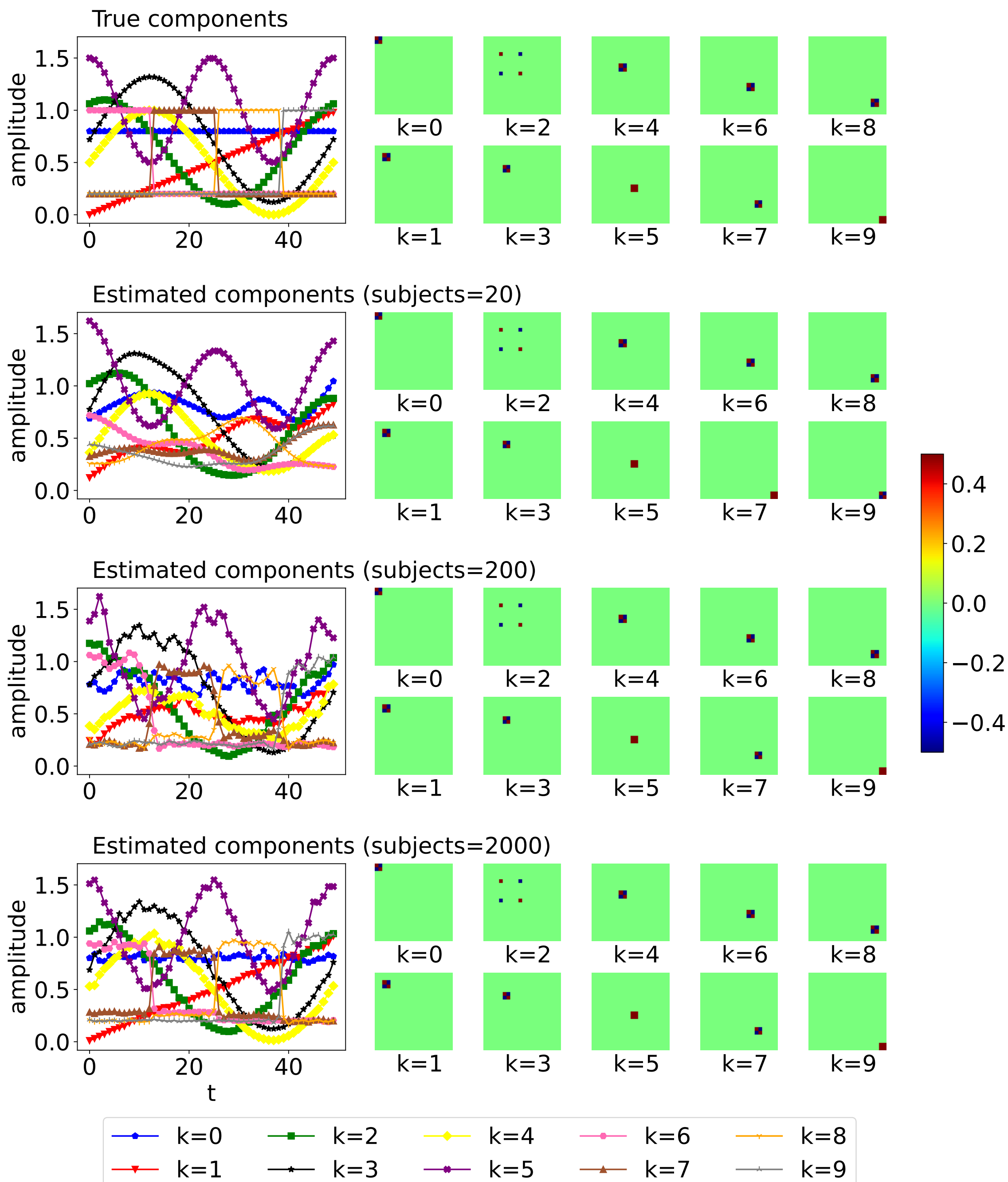

Figure 11: The top row shows the ground truth. The remaining 3 rows present simulation results for $N \in \{20, 200, 2000\}$ when $K = 10, P = 20, J = 50$, and $\sigma = 0$.

Abbr.	Model	low-rank	smooth A	sparse V
M1	Sliding window principal component analysis	✓	✓	✗
M2	Hidden Markov model	✗	✗	✗
M3	Autoregressive hidden Markov model (Poritz, 1982)	✗	✓	✗
M4	Sparse dictionary learning (Mairal et al., 2010)	✓	✗	✓
M5	Bayesian structured learning (Andersen et al., 2018)	✓	✓	✓
M6	Lasso and kernel regularization (Daubechies et al., 2010)	✓	✓	✓
M7	Slinding window shrunk covariance (Ledoit and Wolf, 2004)	✗	✓	✗
MS	Spectral initialization (Algorithm 1)	✓	✗	✗
MR	Proposed model with random initialization (Algorithm 2)	✓	✓	✓
M**	Proposed model (Algorithm 1—2)	✓	✓	✓
MQ**	Proposed model (Algorithm 1—2) with QR decomposition	✓	✓	✓

		Number of subjects (N)		
	Method	1	5	10
Average log-Euclidean metric	M1	0.58 ± 0.02	0.52 ± 0.01	0.50 ± 0.01
	M2	6.41 ± 0.42	0.70 ± 0.01	0.66 ± 0.01
	M3	8.90 ± 3.33	0.81 ± 0.04	0.68 ± 0.01
	M4	0.58 ± 0.01	0.50 ± 0.03	0.43 ± 0.03
	M5	0.50 ± 0.01	0.46 ± 0.01	0.43 ± 0.00
	M6	0.91 ± 0.06	0.48 ± 0.02	0.46 ± 0.02
	MS	0.94 ± 0.11	0.49 ± 0.01	0.46 ± 0.01
	MR	0.68 ± 0.00	0.68 ± 0.00	0.68 ± 0.00
	M**	0.55 ± 0.03	0.42 ± 0.07	0.40 ± 0.09
	MQ**	0.54 ± 0.04	0.42 ± 0.04	0.38 ± 0.06
Running time ($\times 10^{-2}$)	M1	0.4 ± 0.1	0.4 ± 0.0	0.5 ± 0.1
	M2	31.4 ± 2.9	790.8 ± 6.3	829.7 ± 8.4
	M3	173.2 ± 21.0	2834.0 ± 7.3	2909.1 ± 16.6
	M4	44.8 ± 14.2	393.1 ± 225.4	961.6 ± 695.1
	M5	3521.0 ± 123.2	3596.6 ± 98.3	3800.6 ± 113.3
	M6	610.6 ± 48.4	641.6 ± 42.6	634.0 ± 42.7
	MS	0.1 ± 0.0	0.3 ± 0.4	0.2 ± 0.0
	MR	4.4 ± 0.6	4.0 ± 0.2	4.0 ± 0.3
	M**	6.8 ± 1.6	5.9 ± 1.0	5.7 ± 1.5
	MQ**	4.8 ± 0.1	3.8 ± 0.2	3.4 ± 0.3

Table 10: Temporal dynamics are given in Figure 7 with $K = 4, P = 20, J = 50$, and $\sigma = 0$. Log-Euclidean metric and the running time are averaged over 20 independent simulation runs. For M1, we set the window length as $W = 20$. The proposed methods, M** and MQ**, outperform competitors irrespective of the sample size.

	Number of subjects (N)				
Method	1	5	15	200	1000
MS	82.27 ± 27.49	18.86 ± 2.77	6.46 ± 0.89	0.59 ± 0.06	0.10 ± 0.01
MR	43.20 ± 1.17	42.38 ± 0.58	42.07 ± 0.18	41.70 ± 0.03	41.78 ± 0.02
M**	13.22 ± 1.76	5.32 ± 0.90	2.44 ± 0.31	0.42 ± 0.10	0.14 ± 0.01

Table 11: Average distance $\text{dist}^2(Z, Z^\star)$ over 20 independent simulation runs. Temporal dynamics are given in Figure 7 with $K = 4, P = 20, J = 50$, and $\sigma = 0$.

G.5 The Effect of the Noise Level

The data are generated by the data generation process described in Section 5.3. Table 12—13 show the experimental results when $P = 100$, $J = 100$, $K = 10$ and the noise level $\sigma \in \{0.2, 0.1\}$.

	Number of subjects (N)				
Method	10	20	30	40	50
M1	0.40 ± 0.01	0.37 ± 0.01	0.36 ± 0.01	0.36 ± 0.01	0.35 ± 0.01
M2	2.21 ± 0.01	2.07 ± 0.01	2.04 ± 0.01	2.02 ± 0.01	2.02 ± 0.01
M3	78.42 ± 7.46	1.66 ± 0.09	2.15 ± 0.01	2.20 ± 0.01	2.19 ± 0.01
M4	0.96 ± 0.02	0.43 ± 0.02	0.36 ± 0.04	0.36 ± 0.03	0.35 ± 0.03
M5	0.43 ± 0.01	0.39 ± 0.01	0.38 ± 0.01	0.37 ± 0.01	0.36 ± 0.01
M6	0.34 ± 0.01	0.30 ± 0.01	0.29 ± 0.01	0.29 ± 0.01	0.28 ± 0.01
MS	0.35 ± 0.01	0.31 ± 0.01	0.29 ± 0.01	0.29 ± 0.01	0.28 ± 0.01
M**	0.32 ± 0.02	0.29 ± 0.02	0.27 ± 0.01	0.27 ± 0.01	0.26 ± 0.01
MQ**	0.30 ± 0.02	0.28 ± 0.02	0.28 ± 0.02	0.28 ± 0.01	0.27 ± 0.01

Table 12: Log-Euclidean metric averaged over 20 independent simulation runs ($\sigma = 0.2$). The data generating mechanism is described in Section 5.3. For M1, we set the window length to be $W = 20$.

	Number of subjects (N)				
Method	10	20	30	40	50
M1	0.35 ± 0.00	0.33 ± 0.00	0.33 ± 0.00	0.32 ± 0.00	0.32 ± 0.00
M2	3.03 ± 0.01	2.90 ± 0.00	2.87 ± 0.00	2.85 ± 0.00	2.85 ± 0.00
M3	73.62 ± 9.48	2.25 ± 0.15	2.98 ± 0.01	3.03 ± 0.01	3.01 ± 0.00
M4	1.00 ± 0.05	0.48 ± 0.05	0.41 ± 0.05	0.37 ± 0.05	0.39 ± 0.03
M5	0.39 ± 0.01	0.36 ± 0.01	0.35 ± 0.01	0.34 ± 0.01	0.33 ± 0.01
M6	0.31 ± 0.01	0.27 ± 0.02	0.25 ± 0.01	0.25 ± 0.01	0.23 ± 0.01
MS	0.31 ± 0.01	0.27 ± 0.02	0.25 ± 0.01	0.24 ± 0.01	0.24 ± 0.01
M**	0.28 ± 0.02	0.26 ± 0.02	0.24 ± 0.01	0.23 ± 0.01	0.23 ± 0.01
MQ**	0.29 ± 0.02	0.26 ± 0.02	0.25 ± 0.01	0.24 ± 0.01	0.23 ± 0.01

Table 13: Log-Euclidean metric averaged over 20 independent simulation runs ($\sigma = 0.1$). The data generating mechanism is described in Section 5.3. For M1, we set the window length as $W = 20$.

G.6 Experiment on fMRI Data

Figure 12 shows the results for tapping the right foot and tapping the right hand. See Section 6 for a description of the experiment. For each estimated curve, we compute its correlation with the activation map, shown in Figure 4, of each task: tapping the right hand, tapping the left foot, tapping the tongue, tapping the right foot, and tapping the left hand. Then, we rank the components by their correlations with the task activation map (in magnitude). The result is presented in Table 14. Then, for each task, we select the top three components and plot the corresponding combined spatial connectomes in Figure 6.

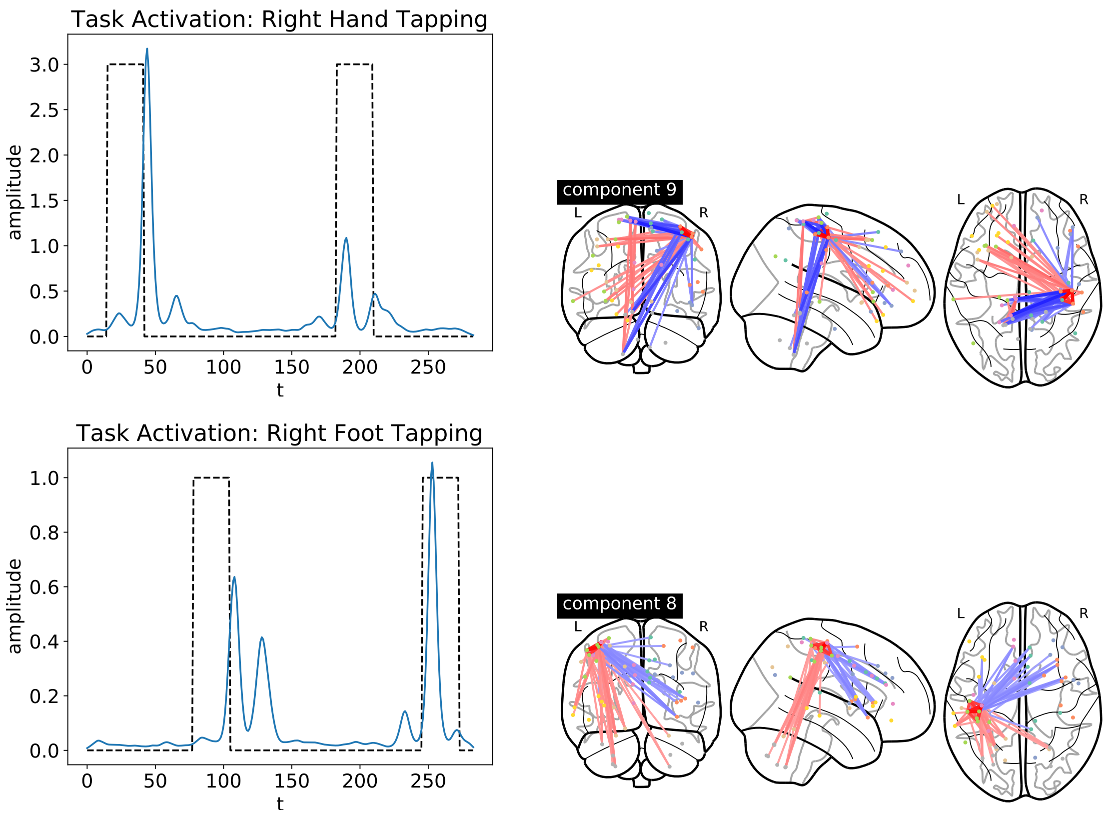

Figure 12: The left column shows the temporal components (blue solid lines) whose correlations are the largest with respect to the task activation (black dotted lines). The right column shows the corresponding brain connectivity patterns (spatial components) for the tasks. The red lines denote positive connectivity and blue lines denote negative connectivity.

Task	Rank of the correlation, order from largest to smallest (component index)														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Right Hand Tapping	9	0	6	5	3	2	14	13	12	11	10	7	1	4	8
Left Foot Tapping	9	4	6	2	14	13	12	11	10	3	1	0	5	7	8
Tongue Wagging	4	1	2	7	14	13	12	11	10	3	9	8	0	5	6
Right Foot Tapping	8	7	6	14	13	12	11	10	3	2	4	1	5	0	9
Left Hand Tapping	8	7	5	3	1	2	6	14	13	12	11	10	0	4	9

Table 14: The correlation of the components with task activation.

References

D. F. Ahelegbey, M. Billio, and R. Casarin. Bayesian graphical models for structural vector autoregressive processes. *Journal of Applied Econometrics*, 31(2):357–386, 2016.

M. R. Andersen, O. Winther, L. K. Hansen, R. A. Poldrack, and O. Koyejo. Bayesian structure learning for dynamic brain connectivity. In *International Conference on Artificial Intelligence and Statistics (AISTATS)*, 2018.

T. W. Anderson and H. Rubin. Statistical inference in factor analysis. In *Berkeley Symposium on Mathematical Statistics and Probability*, 1956.

V. Arsigny, P. Fillard, X. Pennec, and N. Ayache. Log-Euclidean metrics for fast and simple calculus on diffusion tensors. *Magnetic Resonance in Medicine*, 56(2):411–421, 2006.

M. Bańbura, D. Giannone, and L. Reichlin. Large Bayesian vector auto regressions. *Journal of Applied Econometrics*, 25(1):71–92, 2010.

R. F. Barber and M. Kolar. ROCKET: Robust confidence intervals via kendall's tau for transelliptical graphical models. *Annals of Statistics*, 46(6B):3422–3450, 2018.

S. Bhojanapalli, A. Kyrillidis, and S. Sanghavi. Dropping convexity for faster semi-definite optimization. In *Conference on Learning Theory (COLT)*, 2016.

J. Bradic and M. Kolar. Uniform inference for high-dimensional quantile regression: linear functionals and regression rank scores. *arXiv preprint arXiv:1702.06209*, 2017.

S. Burer and R. D. Monteiro. A nonlinear programming algorithm for solving semidefinite programs via low-rank factorization. *Mathematical Programming*, 95(2):329–357, 2003.

S. Burer and R. D. Monteiro. Local minima and convergence in low-rank semidefinite programming. *Mathematical Programming*, 103(3):427–444, 2005.

V. D. Calhoun, R. Miller, G. Pearlson, and T. Adalı. The chronnectome: Time-varying connectivity networks as the next frontier in fMRI data discovery. *Neuron*, 84(2):262–274, 2014.

E. J. Candès, X. Li, and M. Soltanolkotabi. Phase retrieval via Wirtinger flow: theory and algorithms. *IEEE Transactions on Information Theory*, 61(4):1985–2007, 2015.

C. Chang, D. A. Leopold, M. L. Schölvinck, H. Mandelkow, D. Picchioni, X. Liu, F. Q. Ye, J. N. Turchi, and J. H. Duyn. Tracking brain arousal fluctuations with fMRI. *Proceedings of the National Academy of Sciences*, 113(16):4518–4523, 2016.

Y.-L. Chen, M. Kolar, and R. S. Tsay. Tensor canonical correlation analysis with convergence and statistical guarantees. *Journal of Computational and Graphical Statistics*, 30 (3):728–744, 2021.

Y. Chen and M. J. Wainwright. Fast low-rank estimation by projected gradient descent: General statistical and algorithmic guarantees. *arXiv preprint arXiv:1509.03025*, 2015.

Y. Chen and E. J. Candès. Solving random quadratic systems of equations is nearly as easy as solving linear systems. In *Advances in Neural Information Processing Systems (NeurIPS)*, 2015.

Z. Chen and C. Leng. Dynamic covariance models. *Journal of the American Statistical Association*, 111(515):1196–1208, 2016.

Y. Chi, Y. M. Lu, and Y. Chen. Nonconvex optimization meets low-rank matrix factorization: an overview. *IEEE Transactions on Signal Processing*, 67(20):5239–5269, 2019.

R. Dai and M. Kolar. Inference for high-dimensional varying-coefficient quantile regression. *Electronic Journal of Statistics*, 15(2):5696–5757, 2021.

I. Daubechies, R. DeVore, M. Fornasier, and C. S. Güntürk. Iteratively reweighted least squares minimization for sparse recovery. *Communications on Pure and Applied Mathematics*, 63(1):1–38, 2010.

R. A. Davis, P. Zang, and T. Zheng. Sparse vector autoregressive modeling. *Journal of Computational and Graphical Statistics*, 25(4):1077–1096, 2016.

J. Diedrichsen, J. H. Balsters, J. Flavell, E. Cussans, and N. Ramnani. A probabilistic MR atlas of the human cerebellum. *NeuroImage*, 46(1):39–46, 2009.

H. Eavani, R. Filipovych, C. Davatzikos, T. D. Satterthwaite, R. E. Gur, and R. C. Gur. Sparse dictionary learning of resting state fMRI networks. In *International Workshop on Pattern Recognition in NeuroImaging (PRNI)*, 2012.

R. F. Engle, O. Ledoit, and M. Wolf. Large dynamic covariance matrices. *Journal of Business & Economic Statistics*, 37(2):363–375, 2019.

R. Escalante and M. Raydan. *Alternating projection methods*, volume 8 of *Fundamentals of Algorithms*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011.

N. J. Foti and E. B. Fox. Statistical model-based approaches for functional connectivity analysis of neuroimaging data. *Current Opinion in Neurobiology*, 55:48–54, 2019.

E. B. Fox and D. B. Dunson. Bayesian nonparametric covariance regression. *Journal of Machine Learning Research*, 16:2501–2542, 2015.

M. D. Fox and M. E. Raichle. Spontaneous fluctuations in brain activity observed with functional magnetic resonance imaging. *Nature Reviews Neuroscience*, 8(9):700–711, 2007.

S. Geng, M. Yan, M. Kolar, and S. Koyejo. Partially linear additive Gaussian graphical models. In *International Conference on Machine Learning (ICML)*, 2019.

S. Geng, M. Kolar, and O. Koyejo. Joint nonparametric precision matrix estimation with confounding. In *Uncertainty in Artificial Intelligence (UAI)*, 2020.

A. J. Gibberd and J. D. B. Nelson. Regularized estimation of piecewise constant Gaussian graphical models: the group-fused graphical Lasso. *Journal of Computational and Graphical Statistics*, 26(3):623–634, 2017.

E. M. Gordon, T. O. Laumann, B. Adeyemo, J. F. Huckins, W. M. Kelley, and S. E. Petersen. Generation and evaluation of a cortical area parcellation from resting-state correlations. *Cerebral Cortex*, 26(1):288–303, 2014.

Q. Gu, Z. W. Wang, and H. Liu. Low-rank and sparse structure pursuit via alternating minimization. In *International Conference on Artificial Intelligence and Statistics (AISTATS)*, 2016.

D. Hallac, Y. Park, S. Boyd, and J. Leskovec. Network inference via the time-varying graphical lasso. In *ACM SIGKDD International Conference on Knowledge Discovery and Data Mining (SIGKDD)*, 2017.

M. Hardt. Understanding alternating minimization for matrix completion. In *IEEE Symposium on Foundations of Computer Science (FOCS)*, 2014.

D. Hsu, S. M. Kakade, and T. Zhang. A tail inequality for quadratic forms of subgaussian random vectors. *Electronic Communications in Probability*, 17(52):1–6, 2012.

P. Jain, P. Netrapalli, and S. Sanghavi. Low-rank matrix completion using alternating minimization. In *ACM Symposium on Theory of Computing Conference (STOC)*, 2013.

J. Janková and S. van de Geer. Confidence intervals for high-dimensional inverse covariance estimation. *Electronic Journal of Statistics*, 9(1):1205–1229, 2015.

J. Janková and S. van de Geer. Honest confidence regions and optimality in high-dimensional precision matrix estimation. *TEST*, 26(1):143–162, 2017.

A. Javanmard and A. Montanari. Confidence intervals and hypothesis testing for high-dimensional regression. *Journal of Machine Learning Research*, 15:2869–2909, 2014.

G. Kastner, S. Frühwirth-Schnatter, and H. F. Lopes. Efficient Bayesian inference for multivariate factor stochastic volatility models. *Journal of Computational and Graphical Statistics*, 26(4):905–917, 2017.

B. Kim, S. Liu, and M. Kolar. Two-sample inference for high-dimensional Markov networks. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 83(5):939–962, 2021.

M. Kolar and E. P. Xing. Sparsistent estimation of time-varying discrete Markov random fields. *arXiv preprint, arXiv:0907.2337*, 2009.

M. Kolar and E. P. Xing. On time varying undirected graphs. In *International Conference on Artificial Intelligence and Statistics (AISTATS)*, 2011.

M. Kolar and E. P. Xing. Estimating networks with jumps. *Electronic Journal of Statistics*, 6:2069–2106, 2012.

M. Kolar, A. P. Parikh, and E. P. Xing. On sparse nonparametric conditional covariance selection. In *International Conference on Machine Learning (ICML)*, 2010a.

M. Kolar, L. Song, A. Ahmed, and E. P. Xing. Estimating time-varying networks. *The Annals of Applied Statistics*, 4(1):94–123, 2010b.

D. Kozbur. Inference in Additively Separable Models With a High-Dimensional Set of Conditioning Variables. *Journal of Business & Economic Statistics*, 39(4):984–1000, 2021.

S. Kumar, J. Ying, J. V. d. M. Cardoso, and D. P. Palomar. A unified framework for structured graph learning via spectral constraints. *Journal of Machine Learning Research*, 21 (22):1–60, 2020.

B. Laurent and P. Massart. Adaptive estimation of a quadratic functional by model selection. *The Annals of Statistics*, 28(5):1302–1338, 2000.

O. Ledoit and M. Wolf. A well-conditioned estimator for large-dimensional covariance matrices. *Journal of Multivariate Analysis*, 88(2):365–411, 2004.

N. Leonardi and D. V. D. Ville. On spurious and real fluctuations of dynamic functional connectivity during rest. *NeuroImage*, 104:430–436, 2015.

B. Li and E. Solea. A nonparametric graphical model for functional data with application to brain networks based on fMRI. *Journal of the American Statistical Association*, 113 (524):1637–1655, 2018.

R. Li. Multivariate sparse coding of nonstationary covariances with Gaussian processes. In *Advances in Neural Information Processing Systems (NeurIPS)*, 2019.

X. Li, T. Zhao, R. Arora, H. Liu, and J. Haupt. Stochastic variance reduced optimization for nonconvex sparse learning. In *International Conference on Machine Learning (ICML)*, 2016.

R. Liégeois, J. Li, R. Kong, C. Orban, D. Van De Ville, T. Ge, M. R. Sabuncu, and B. T. Yeo. Resting brain dynamics at different timescales capture distinct aspects of human behavior. *Nature Communications*, 10(1):1–9, 2019.

P.-L. Loh and M. J. Wainwright. Regularized M-estimators with nonconvexity: statistical and algorithmic theory for local optima. *Journal of Machine Learning Research*, 16: 559–616, 2015.

J. Lu, M. Kolar, and H. Liu. Post-regularization inference for time-varying nonparanormal graphical models. *Journal of Machine Learning Research*, 18(203):1–78, 2018.

J. Lu, M. Kolar, and H. Liu. Kernel meets sieve: post-regularization confidence bands for sparse additive model. *Journal of the American Statistical Association*, 115(532): 2084–2099, 2020.

J. Mairal, F. Bach, J. Ponce, and G. Sapiro. Online learning for matrix factorization and sparse coding. *Journal of Machine Learning Research*, 11:19–60, 2010.

E. N. Marieb and K. Hoehn. *Human Anatomy & Physiology*. Pearson Education, 2018.

B. Minasny and A. B. McBratney. The Matérn function as a general model for soil variograms. *Geoderma*, 128(3-4):192–207, 2005.

G. Mishne and A. S. Charles. Learning spatially-correlated temporal dictionaries for calcium imaging. In *IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, 2019.

R. P. Monti, P. Hellyer, D. Sharp, R. Leech, C. Anagnostopoulos, and G. Montana. Estimating time-varying brain connectivity networks from functional MRI time series. *NeuroImage*, 103:427–443, 2014.

S. Na, M. Kolar, and O. Koyejo. Estimating differential latent variable graphical models with applications to brain connectivity. *Biometrika*, 108(2):425–442, 2021.

S. Na, Y. Luo, Z. Yang, Z. Wang, and M. Kolar. Semiparametric nonlinear bipartite graph representation learning with provable guarantees. In *International Conference on Machine Learning (ICML)*, 2020.

Y. Nesterov. *Introductory Lectures on Convex Optimization*. Springer, 2013.

B. A. Olshausen and D. J. Field. Sparse coding with an overcomplete basis set: A strategy employed by V1? *Vision Research*, 37(23):3311–3325, 1997.

C. J. Paciorek. *Nonstationary Gaussian processes for regression and spatial modelling.* PhD dissertation, Carnegie Mellon University, 2003.

D. Park, A. Kyrillidis, C. Caramanis, and S. Sanghavi. Finding low-rank solutions via nonconvex matrix factorization, efficiently and provably. *SIAM Journal on Imaging Sciences*, 11(4):2165–2204, 2018.

A. Poritz. Linear predictive hidden Markov models and the speech signal. In *IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP)*, 1982.

M. I. Posner, S. E. Petersen, P. T. Fox, and M. E. Raichle. Localization of cognitive operations in the human brain. *Science*, 240(4859):1627–1631, 1988.

M. G. Preti, T. A. Bolton, and D. Van De Ville. The dynamic functional connectome: State-of-the-art and perspectives. *NeuroImage*, 160:41–54, 2017.

X. Qiao, S. Guo, and G. M. James. Functional graphical models. *Journal of the American Statistical Association*, 114(525):211–222, 2019.

X. Qiao, C. Qian, G. M. James, and S. Guo. Doubly functional graphical models in high dimensions. *Biometrika*, 107(2):415–431, 2020.

H. Qiu, F. Han, H. Liu, and B. Caffo. Joint estimation of multiple graphical models from high dimensional time series. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 78(2):487–504, 2016.

Z. Ren, T. Sun, C.-H. Zhang, and H. H. Zhou. Asymptotic normality and optimalities in estimation of large Gaussian graphical models. *Annals of Statistics*, 43(3):991–1026, 2015.

Ü. Sakoğlu, G. D. Pearlson, K. A. Kiehl, Y. M. Wang, A. M. Michael, and V. D. Calhoun. A method for evaluating dynamic functional network connectivity and task-modulation: application to schizophrenia. *Magnetic Resonance Materials in Physics, Biology and Medicine*, 23(5-6):351–366, 2010.

B. Schölkopf and A. J. Smola. *Learning with kernels.* MIT Press, 2001.

J. M. Shine, P. G. Bissett, P. T. Bell, O. Koyejo, J. H. Balsters, K. J. Gorgolewski, C. A. Moodie, and R. A. Poldrack. The dynamics of functional brain networks: Integrated network states during cognitive task performance. *Neuron*, 92(2):544–554, 2016a.

J. M. Shine, O. Koyejo, and R. A. Poldrack. Temporal metastates are associated with differential patterns of time-resolved connectivity, network topology, and attention. *Proceedings of the National Academy of Sciences*, 113(35):9888–9891, 2016b.

J. M. Shine, M. Breakspear, P. T. Bell, K. A. E. Martens, R. Shine, O. Koyejo, O. Sporns, and R. A. Poldrack. Human cognition involves the dynamic integration of neural activity and neuromodulatory systems. *Nature Neuroscience*, 22(2):289–296, 2019.

A. Skripnikov and G. Michailidis. Regularized joint estimation of related vector autoregressive models. *Computational Statistics & Data Analysis*, 139:164–177, 2019.

A. M. Smith, B. K. Lewis, U. E. Ruttimann, F. Q. Ye, T. M. Sinnwell, Y. Yang, J. H. Duyn, and J. A. Frank. Investigation of low frequency drift in fMRI signal. *NeuroImage*, 9(5):526–533, 1999.

L. Song, M. Kolar, and E. P. Xing. KELLER: estimating time-varying interactions between genes. *Bioinformatics*, 25(12):i128–i136, 2009a.

L. Song, M. Kolar, and E. P. Xing. Time-varying dynamic Bayesian networks. In *Advances in Neural Information Processing Systems (NeurIPS)*, 2009b.

G. W. Stewart. Perturbation bounds for the QR factorization of a matrix. *SIAM Journal on Numerical Analysis*, 14(3):509–518, 1977.

A. Tank, E. B. Fox, and A. Shojaie. Identifiability and estimation of structural vector autoregressive models for subsampled and mixed-frequency time series. *Biometrika*, 106 (2):433–452, 2019.

J. M. F. ten Berge. Orthogonal Procrustes rotation for two or more matrices. *Psychometrika*, 42(2):267–276, 1977.

J. A. Tropp. An introduction to matrix concentration inequalities. *Foundations and Trends in Machine Learning*, 8(1-2):1–230, 2015.

K. Tsai, O. Koyejo, and M. Kolar. Joint Gaussian graphical model estimation: A survey. *WIREs Computational Statistics*, page e1582, 2022.

S. Tu, R. Boczar, M. Simchowitz, M. Soltanolkotabi, and B. Recht. Low-rank solutions of linear matrix equations via Procrustes flow. In *International Conference on Machine Learning (ICML)*, 2016.

M. Udell and A. Townsend. Why are big data matrices approximately low rank? *SIAM Journal on Mathematics of Data Science*, 1(1):144–160, 2019.

M. Udell, C. Horn, R. Zadeh, and S. Boyd. Generalized low rank models. *Foundations and Trends in Machine Learning*, 9(1):1–118, 2016.

S. van de Geer, P. Bühlmann, Y. Ritov, and R. Dezeure. On asymptotically optimal confidence regions and tests for high-dimensional models. *Annals of Statistics*, 42(3): 1166–1202, 2014.

D. C. Van Essen, S. M. Smith, D. M. Barch, T. E. Behrens, E. Yacoub, and K. Ugurbil. The WU-minn human connectome project: An overview. *NeuroImage*, 80:62–79, 2013.

R. Vershynin. Introduction to the non-asymptotic analysis of random matrices. In *Compressed sensing*, pages 210–268. Cambridge Univ. Press, Cambridge, 2012.

D. Vidaurre, S. M. Smith, and M. W. Woolrich. Brain network dynamics are hierarchically organized in time. *Proceedings of the National Academy of Sciences*, 114(48):12827–12832, 2017.

J. Wang and M. Kolar. Inference for sparse conditional precision matrices. *arXiv preprint, arXiv:1412.7638*, 2014.

J. Wang and M. Kolar. Inference for high-dimensional exponential family graphical models. In *International Conference on Artificial Intelligence and Statistics (AISTATS)*, Cadiz, Spain, 2016.

X. Wang, M. Kolar, and A. Shojaie. Statistical inference for networks of high-dimensional point processes. *arXiv preprint, arXiv:2007.07448*, 2020.

Y. S. Wang, S. K. Lee, P. Toulis, and M. Kolar. Robust inference for high-dimensional linear models via residual randomization. In *International Conference on Machine Learning (ICML)*, 2021.

L. Wasserman, M. Kolar, and A. Rinaldo. Berry-Esseen bounds for estimating undirected graphs. *Electronic Journal of Statistics*, 8(1):1188–1224, 2014.

W. B. Wu and M. Pourahmadi. Nonparametric estimation of large covariance matrices of longitudinal data. *Biometrika*, 90(4):831–844, 2003.

Y. Xia, T. Cai, and T. T. Cai. Testing differential networks with applications to the detection of gene-gene interactions. *Biometrika*, 102(2):247–266, 2015.

J. Yin, Z. Geng, R. Li, and H. Wang. Nonparametric covariance model. *Statistica Sinica*, 20(1):469–479, 2010.

M. Yu, V. Gupta, and M. Kolar. Statistical inference for pairwise graphical models using score matching. In *Advances in Neural Information Processing Systems (NeurIPS)*. 2016.

M. Yu, Z. Yang, T. Zhao, M. Kolar, and Z. Wang. Provable Gaussian embedding with one observation. In *Advances in Neural Information Processing Systems (NeurIPS)*, 2018.

M. Yu, V. Gupta, and M. Kolar. Estimation of a low-rank topic-based model for information cascades. *Journal of Machine Learning Research*, 21(71):1–47, 2020a.

M. Yu, V. Gupta, and M. Kolar. Simultaneous inference for pairwise graphical models with generalized score matching. *Journal of Machine Learning Research*, 21(91):1–51, 2020b.

M. Yu, V. Gupta, and M. Kolar. Recovery of simultaneous low rank and two-way sparse coefficient matrices, a nonconvex approach. *Electronic Journal of Statistics*, 14(1):413–457, 2020c.

Y. Yu, T. Wang, and R. J. Samworth. A useful variant of the Davis–Kahan theorem for statisticians. *Biometrika*, 102(2):315–323, 2015.

A. Zalesky, A. Fornito, and E. Bullmore. On the use of correlation as a measure of network connectivity. *NeuroImage*, 60(4):2096–2106, 2012.

J. Zapata, S. Y. Oh, and A. Petersen. Partial separability and functional graphical models for multivariate Gaussian processes. *Biometrika*, 2021.

C.-H. Zhang and S. S. Zhang. Confidence intervals for low dimensional parameters in high dimensional linear models. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 76(1):217–242, 2014.

B. Zhao, Y. S. Wang, and M. Kolar. Direct estimation of differential functional graphical models. In *Advances in Neural Information Processing Systems (NeurIPS)*, 2019.

B. Zhao, S. Zhai, Y. S. Wang, and M. Kolar. High-dimensional functional graphical model structure learning via neighborhood selection approach. *arXiv preprint, arXiv:2105.02487*, 2021.

B. Zhao, Y. S. Wang, and M. Kolar. FuDGE: A method to estimate a functional differential graph in a high-dimensional setting. *Journal of Machine Learning Research*, 23(82):1–82, 2022.

T. Zhao, M. Kolar, and H. Liu. A general framework for robust testing and confidence regions in high-dimensional quantile regression. *arXiv preprint arXiv:1412.8724*, 2014.

T. Zhao, Z. Wang, and H. Liu. A nonconvex optimization framework for low rank matrix estimation. *Advances in Neural Information Processing Systems (NeurIPS)*, 2015.

S. Zhou, J. Lafferty, and L. Wasserman. Time varying undirected graphs. *Machine Learning*, 80(2):295–319, 2010.

Y. Zhu and O. Koyejo. Clustered fused graphical lasso. In *Uncertainty in Artificial Intelligence (UAI)*, 2018.